Happy Endings of noncrossing convex bodies.

joint work with:

Andreas Holmsen and Michael Dobbins

Convex sets with straight lines are equivalent to points with wobbly lines.

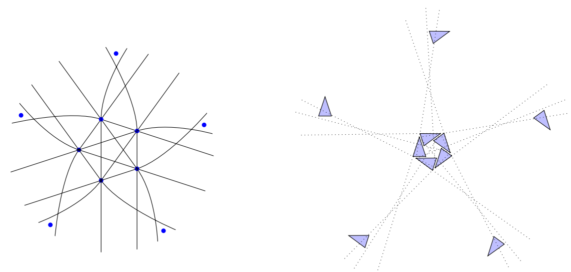
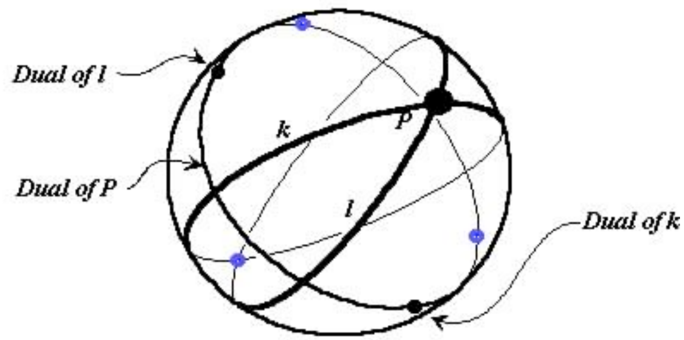


FIGURE 1. Two realizations of the "bad pentagon". **Left:** realization in a topological plane; **Right:** realization by convex sets in the Euclidean plane



Axiomatics.

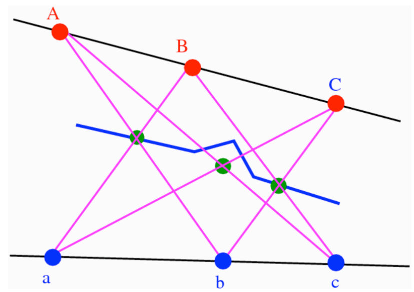
$$\chi(i,j,k) := sign|\hat{x_i}, \hat{x_j}, \hat{x_k}|$$
$$\hat{x} := (x^1, x^2, 1)$$

Counterclockwise

Order Type

$$\chi\colon [n]^3 \to \{+, 0, -\}$$

Chirotopes, abstract order types, signotopes, oriented matroids (uniform of rank 3), CC-systems.



Every five tuple of pseudo lines is stretchable.

P. Erdös
G. Szekeres

A combinatorial problem in geometry

Compositio Mathematica, tome 2 (1935), p. 463-470.

From 5 points of the plane of which no three lie on the same straight line it is always possible to select 4 points determining a convex quadrilateral.

Theorem. *Among any f points in general position in the plane there is a convex polygon with at least* $\log f$ *vertices.*

Definition. *Given a convex hull operator, a set X is convexly independent if for any proper subset $Y \subset X$, $\operatorname{conv} Y \neq \operatorname{conv} X$.*

Theorem. *For any number n, there is a number $f(n)$, such that, among any $f(n)$ points in the plane such that each triple is convexly independent, there is a convexly independent subset with at least n vertices.*

Conjecture (Erdős-Szekeres)**.** $f(n) = 2^{n-2} + 1$.

Theorem. *For any number n, there is a number $g(n)$, such that, among any $g(n)$ points in an* ***abstract order type*** *such that each triple is convexly independent, there is a convexly independent subset with at least n elements.*

Conjecture (Goodman-Pollack)**.** $f(n) = g(n)$.

Theorem. *For any number n, there is a number $h_0(n)$, such that, among any $h_0(n)$ pairwise* ***disjoint convex sets*** *in the plane, such that each triple is convexly independent, there is a convexly independent subset with at least n sets.*

Conjecture (Bisztriczky-Fejes Tóth)**.** $f(n) = h_0(n)$.

Theorem. *For any number n, there is a number $h_1(n)$, such that, among any $h_1(n)$ pairwise* ***non crossing convex sets*** *in the plane, such that each triple is convexly independent, there is a convexly independent subset with at least n sets.*

$$2^{n-2}+1 \;\le\; f(n) \;\le\; g(n) \;\le\; \binom{2n-5}{n-1}+1 \quad \text{(for } n \ge 7\text{)}$$

Erdős-Szekeres, Goodman-Pollack, Valtr-Toth

$$f(n) \le h_0(n) \le \binom{2n-4}{n-2}^2$$

Bisztrizky-Fejes Toth, Pach-Toth

$$h_0(n) \le \; h_1(n) \;\le\; 2^{O(n^2 \log n)}$$

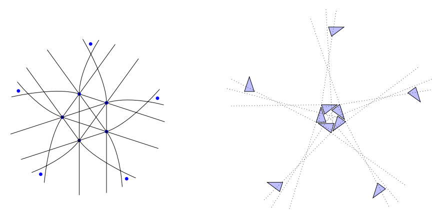
Pach-Toth, Hubard-Montejano-Mora-Suk, Fox-Pach-Sudakov-Suk

$$2^{n-2}+1 = f(n) = g(n) \qquad \text{(for } n \le 6\text{)}$$

$$2^{n-2}+1 = h_0(n) \qquad \text{(for } n \le 5\text{)}$$

Peters-Szekeres, Bisztrizky-Fejes Toth

Theorem 1. *The Erdős-Szekeres problems for generalized configurations and for arrangements of non-crossing bodies are equivalent. In other words, $g(n) = h_1(n)$.*



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$$
\begin{aligned}
2^{n-2}+1 \;\leq\; f(n) \;\leq\; g(n) \;\leq\; \tbinom{2n-5}{n-1}+1 \qquad & (\text{for } n \geq 7) \\
f(n) \;\leq h_0(n) \leq\; \tbinom{2n-4}{n-2}^2 & \\
h_0(n) \leq\; h_1(n) \;\leq\; 2^{O(n^2 \log n)} &
\end{aligned}
$$

$$
\begin{aligned}
2^{n-2}+1 = f(n) = g(n) \qquad & (\text{for } n \leq 6) \\
2^{n-2}+1 = h_0(n) \qquad & (\text{for } n \leq 5)
\end{aligned}
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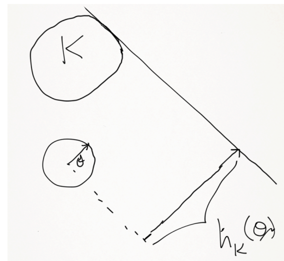
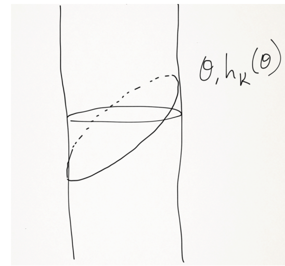
$$
\begin{aligned}
2^{n-2}+1 \leq f(n) \leq h_0(n) \leq h_1(n) = g(n) \leq \tbinom{2n-5}{n-2}+1 \qquad & (\text{for } n \geq 7) \\
2^{n-2}+1 = f(n) = h_0(n) = h_1(n) = g(n) \qquad & (\text{for } n \leq 6)
\end{aligned}
$$

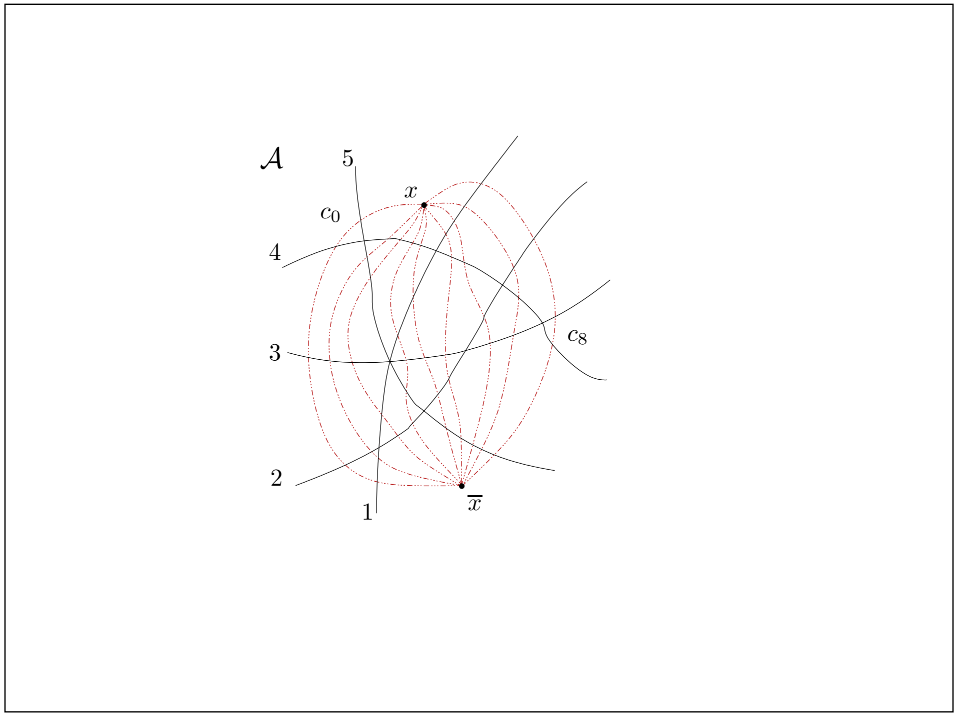
$$h_K(\theta) := \max_{v \in K} \langle \theta, v \rangle$$





The ***dual*** of a convex set is the graph of its support function

$$K^* = \{(x, y) \in \mathbb{S}^1 \times \mathbb{R}^1 : y = h_K(x)\}.$$



$\mathcal{A}$
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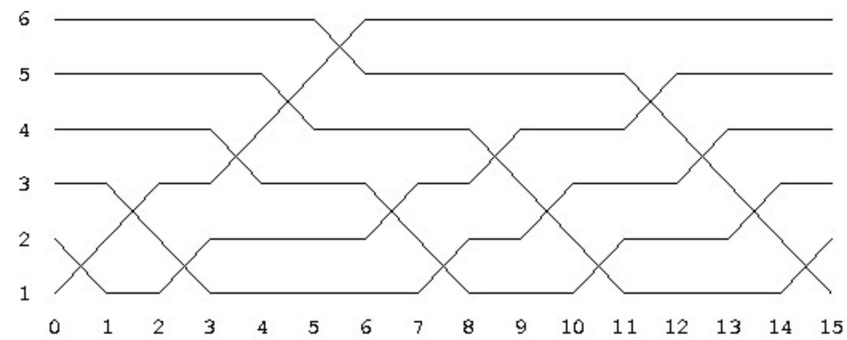
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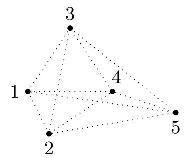
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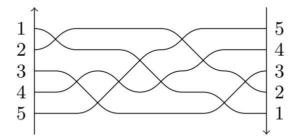
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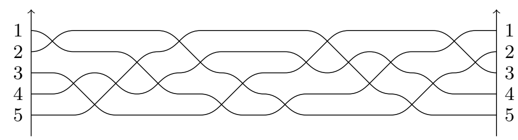
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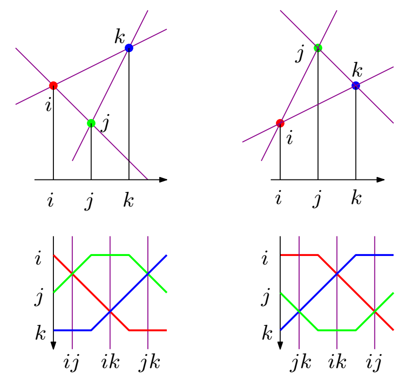

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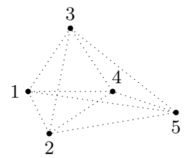
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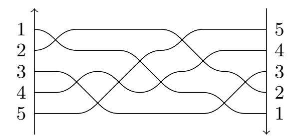
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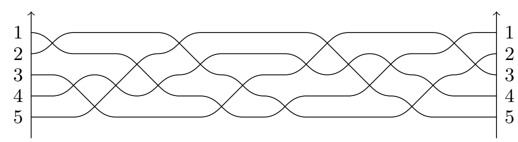
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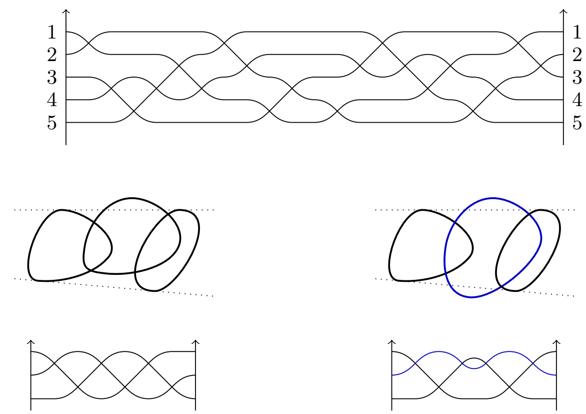
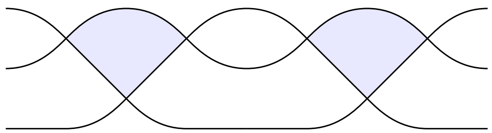
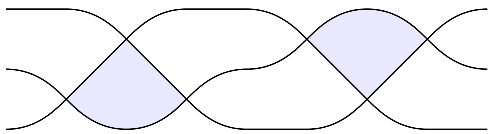
Proposition 1.1. *Every orientable arrangement of bodies gives rise to an abstract order type, and every abstract order type has a realization by an orientable arrangement of bodies.*



Theorem 1. *The Erdős-Szekeres problems for generalized configurations and for arrangements of non-crossing bodies are equivalent. In other words,* $g(n) = h_1(n)$.





Lemma 4. *If $\mathcal{A}$ is not orientable, then $\mathcal{C}(\mathcal{A})$ contains a triangular cell bounded by the support curves of a non-orientable triple.*

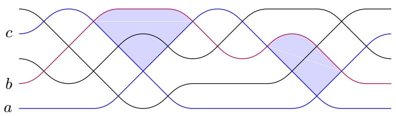


FIGURE 4. The triple $\mathcal{T}^* = \{a, b, c\}$ bounds two zones (shaded) and the top curve is b (red). Neither of the zones of $\mathcal{T}^*$ are empty, but the left one is free.

If $\mathcal{A}$ contains non-orientable triples, then $\mathcal{C}(\mathcal{A})$ contains an empty zone.

Claim 5. *If $\mathcal{C}(\mathcal{A})$ contains an free zone, then $\mathcal{C}(\mathcal{A})$ contains an empty zone.*

Any non orientable arrangement has a free zone.

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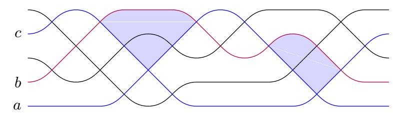


FIGURE 4. The triple $\mathcal{T}^* = \{a, b, c\}$ bounds two zones (shaded) and the top curve is b (red). Neither of the zones of $\mathcal{T}^*$ are empty, but the left one is free.

Each w_i intersects Z_0 in a single connected arc. We may assume w_i enters Z_0 by crossing curve c and exits Z_0 by crossing curve a.

The triangular region in Z_0 bounded by a, w_i, c is a zone.

Distinct curves w_i and w_j cross at most *once* inside Z_0.

Observation 6. Let Z be a zone bounded by a, b, c where b is the top curve. Suppose w enters Z by crossing c and exits Z by crossing b, then proceeds to cross a. Then one of the triples a, w, b or w, b, c bound a zone. (See Figure 5.)

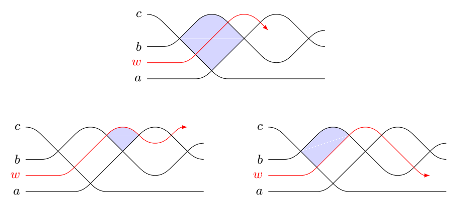


FIGURE 5. **Top:** The zone Z is bounded by a, b, c (shaded). After w leaves Z and crosses a it enters a digon bounded by curves a and b, so it must cross one of them again before crossing c. **Bottom left:** If the next crossing of w is with a, then a, w, b bound a zone (shaded). **Bottom right:** If the next crossing of w is with b then w, b, c bound a zone (shaded).

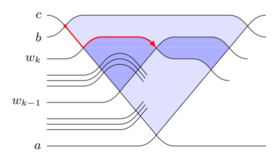


FIGURE 6. Starting at top left corner of Z_0 (light shade) move along the boundary until we meet the first crossing. This is the top corner of a zone bounded by a, w_k, c. Proceed along w_k until we meet the next crossing. By Observation 6 one of the two dark shaded regions must be a zone.

Any non orientable arrangement has a free zone.

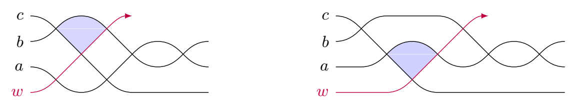
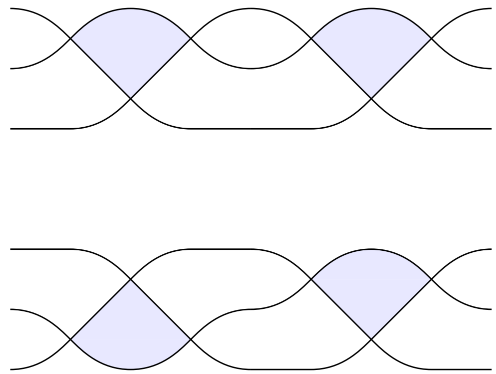


FIGURE 7. Consider w after it leaves Z. **Case (1), left:** If w crosses b before c, then w, b, c bound an empty zone contained in Z. If w crosses c before b, then w, a, c is not in convex position. **Case (2), right:** If w crosses a before c, then w, a, c bound a free zone below Z. If w crosses c before a, then b intersects w again after its two crossings with a, which implies that w, a, b is not in convex position.

Our technique also yields improved bounds of the fractioned happy ending theorem and a proof of the Partitioned happy ending theorem for convex sets.

Theorem 1. *The Erdős-Szekeres problems for generalized configurations and for arrangements of non-crossing bodies are equivalent. In other words,* $g(n) = h_1(n)$.



Theorem 1.4. *For all integers $n > k > 1$, there exists a minimal positive integer $h_k(n)$ such that the following holds: Any arrangement of at least $h_k(n)$ bodies, where the boundaries intersect at most $2k$ times and every m_k-tuple is convexly independent, contains an n-tuple which is convexly independent, where $m_2 = 4$, and $m_k = 5$ for all $k \geq 3$.*

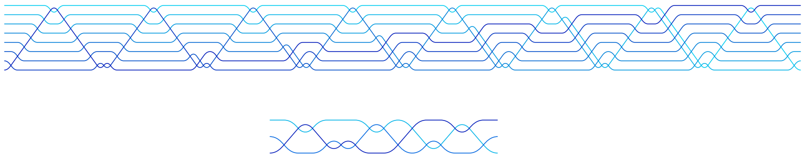


FIGURE 18. **Top:** $\mathcal{F} \in \mathcal{V}_8$; **Bottom:** $\rho_8(\mathcal{F}) \in \mathcal{W}_3$