Maximal flip distances on fancy surfaces

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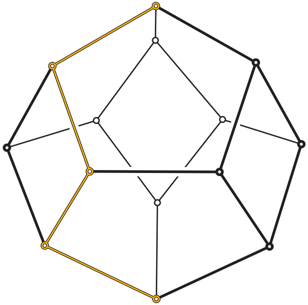


Summary

0. Preliminary definitions

Surfaces, triangulations, and flips.

1. The case of the disc

The diameters of associahedra.

2. Surfaces with 2 or 3 boundaries

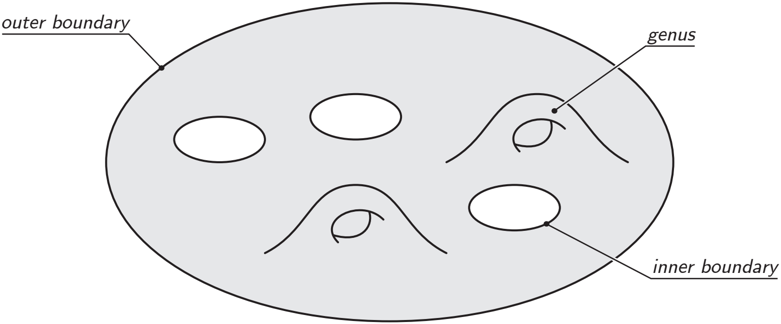
Two examples with large flip distances.

3. Maximal Flip distances on more general surfaces

- A general upper bound,
- Proving lower bounds,

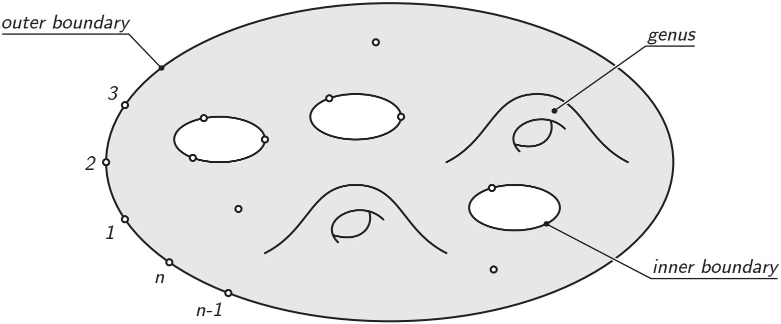
4. Conclusion

Consider an orientable surface of genus g with $r > 0$ boundaries. One of these boundaries will be called the *outer* boundary:



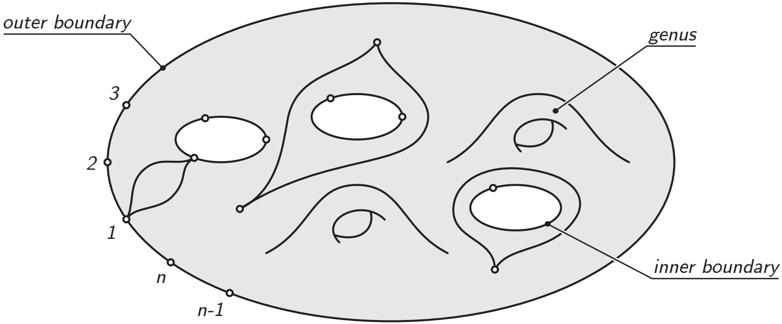
(this is an orientable surface with 4 boundaries and genus 2)

Consider n vertices in the outer boundary, labeled 1 to n clockwise, and other vertices in the surface so that every boundary contains at least one vertex. The set of these vertices will be denoted by $\mathcal{A}$.



(this is an orientable surface with 4 boundaries and genus 2)

Two edges are *isotopic* if they can be continuously deformed into one another. They are non-isotopic if some obstacle lies "between" them (a boundary, a genus, or a vertex). Some edges have only one vertex.

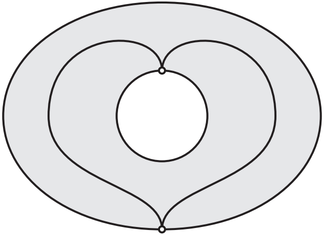


(this is an orientable surface with 4 boundaries and genus 2)

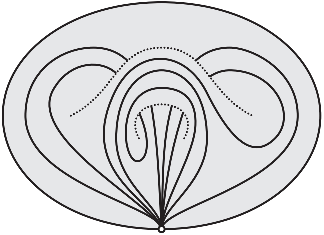
Definition

A triangulation of $\mathcal{A}$ is a maximal set of pairwise non-crossing and pairwise non-isotopic edges with vertices in $\mathcal{A}$.

The edges of a triangulation are considered up to isotopy: they can be deformed under the condition that they remain pairwise non-crossing.



A triangulation of a cylinder with one vertex on each boundary.

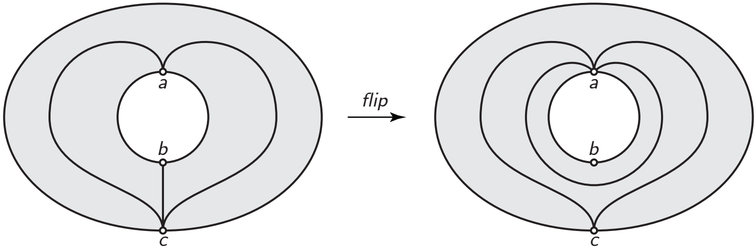


A triangulation of a torus with a boundary containing one vertex.

Any interior edge ε of a triangulation T of $\mathcal{A}$ is the diagonal of some quadrilateral q whose boundary edges belong to T.

Definition

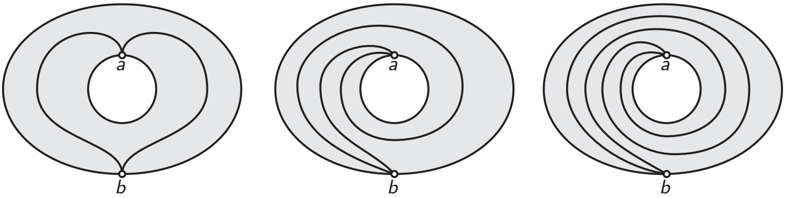
Flipping ε in T consists in replacing ε by the other diagonal of q within T.



Any triangulation of $\mathcal{A}$ can be transformed into any other triangulation of $\mathcal{A}$ by performing a sequence of flips. The *distance* of two triangulations is the minimal number of flips needed to transform one into the other.

What is the maximal distance between any two triangulations of $\mathcal{A}$?

In fact, $\mathcal{A}$ may have an infinite number of triangulations...



It is natural to consider the triangulations of $\mathcal{A}$ **up to homeomorphism**!

Let G be the graph whose:

i. vertices are the different triangulations of $\mathcal{A}$ up to homeomorphism,

ii. edges connect two triangulations of $\mathcal{A}$ if they can be obtained from one another by a flip.

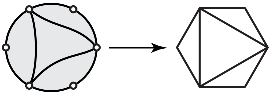
As mentioned above, graph G is connected.

What is the diameter of G?

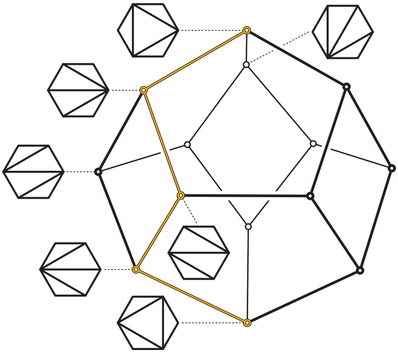
One needs:

- An upper bound on the distance of two triangulations of $\mathcal{A}$,
- Two triangulations U and V of $\mathcal{A}$ (whose flip distance is maximal),
- A proof that U and V indeed have the desired flip distance!

Assume that $\mathcal{A}$ is a set of n vertices in the boundary of a disc. Then, all the edges of a triangulation of $\mathcal{A}$ can be drawn as straight segments.



Moreover, distinct triangulations are never homeomorphic!



Theorem (Lee, 1989)

If the surface is a disc, and $\mathcal{A}$ is a set of n vertices in the boundary of this disc, then G is the graph of the $(n-3)$-dimensional associahedron.

While working on the dynamic optimality conjecture, Daniel Sleator, Robert Tarjan, and William Thurston prove that:

Theorem (Sleator, Tarjan, Thurston, 1988)

When $\mathcal{A}$ is the vertex set of a convex polygon, the diameter of G,

i. is not larger than $2n-10$ when n is greater than 12,

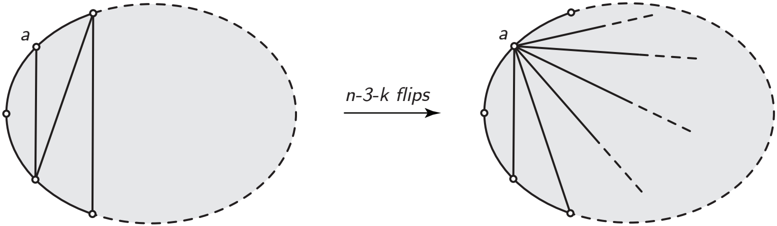
ii. is equal to $2n-10$ when n is **large enough**.

Two problems that remained open for a long time:

- Is there a combinatorial proof (of **ii.**)?
- Are there triangulations at distance $2n-10$ for any $n>12$?

Answers: yes and yes (arXiv:1207.6296, 2012)

In order to obtain the upper bound of $2n - 10$, flip a triangulation T (left) to a canonical triangulation (right) as follows:



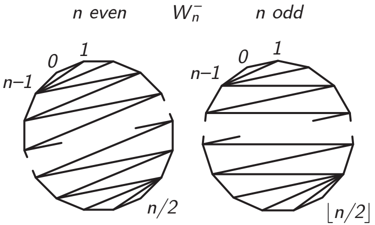
Here, k is the number of interior edges of T incident to vertex a. Call l the number of interior edges incident to a in another trianulation U. One can transform T into U with $2n - 6 - (k + l)$ flips.

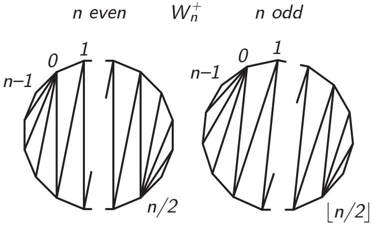
When $n > 12$, a counting argument provide a vertex a so that $k + l \geq 4$.

Hence $\delta(\{T, U\}) \leq 2n - 10$ when $n > 12$.

1. The case of the disc

Two triangulations at distance $2n-10$ when $n>12$ (arXiv:1207.6296, 2012)





- At vertex $n-1$: a comb with 3 teeth,
- At vertex $\lfloor n/2 \rfloor - 1$: a comb with 3 teeth if n is even and 4 teeth if n is odd.

- A comb with 4 teeth at vertex 0,
- At vertex $\lfloor n/2 \rfloor$: a comb with 4 teeth if n is even and 3 teeth if n is odd.

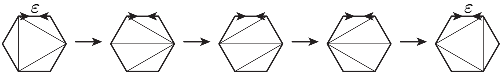
In each triangulation, the two combs are connected by a zigzag.

Call $A_n = \{W_n^-, W_n^+\}$ (can be defined whenever $n \geq 3$).

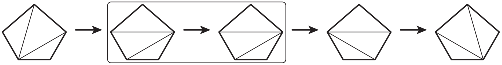
It will be shown that when $n > 12$,

$$\delta(A_n) \geq \min(\delta(A_{n-1}) + 2, \delta(A_{n-2}) + 4, \delta(A_{n-5}) + 10, \delta(A_{n-6}) + 12).$$

Consider a path C of length k between two triangulations T_1 and T_2:



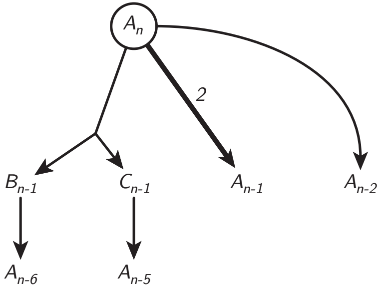
Contracting a boundary edge ε of the polygon in every triangulation along this path results in a path of length $l \leq k$ between two triangulations of a polygon with $n-1$ vertices:

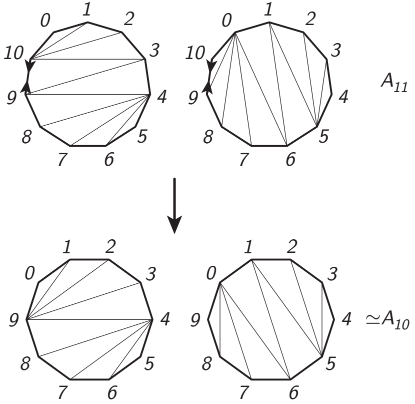


The difference $k - l$ is equal to the number of flips incident to ε.

1. The case of the disc

Proof of the lower bound (main ideas)



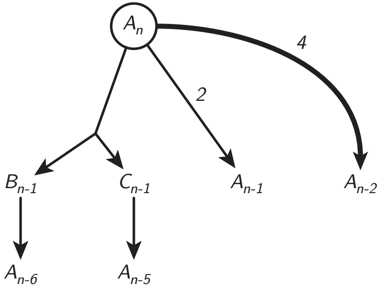


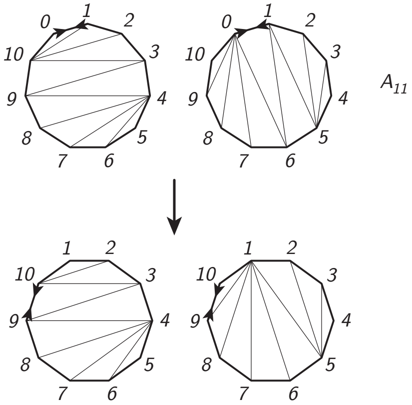
If there exists a minimal path from W_n^- to W_n^+ with at least two flips incident to edge $\{n-2, n-1\}$, then:

$$\delta(A_n) \geq \delta(A_{n-1}) + 2.$$

1. The case of the disc

Proof of the lower bound (main ideas)



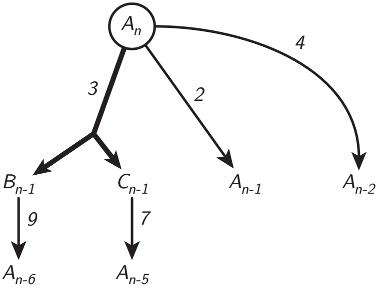


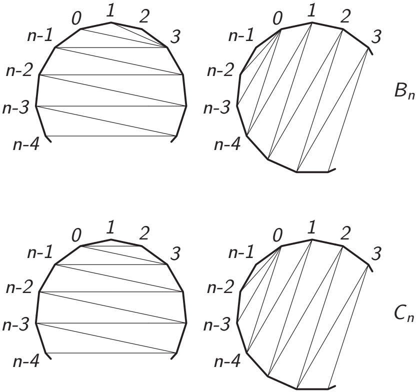
If there exists a minimal path from W_n^- to W_n^+ with at least three flips incident to edge $\{0, 1\}$, then:

$$\delta(A_n) \geq \delta(A_{n-2}) + 4.$$

1. The case of the disc

Proof of the lower bound (main ideas)





The third arc originating at A_n is explored if **every** minimal path,

i. contains at most one flip incident to $\{n-2, n-1\}$,

ii. contains at most two flips incident to $\{0, 1\}$.

Theorem

When n is greater than 12, the following inequality holds:

$$\delta(A_n) \geq \min(\delta(A_{n-1}) + 2, \delta(A_{n-2}) + 4, \delta(A_{n-5}) + 10, \delta(A_{n-6}) + 12).$$

The distance between W_n^- and W_n^+ is at least equal to $2n - 10$ when n ranges from 7 to 12. By induction:

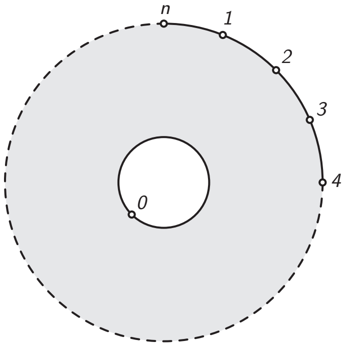
Corollary

Triangulations W_n^- and W_n^+ have flip distance $2n - 10$ when $n > 12$.

In orther words, G has diameter $2n - 10$ when $n > 12$.

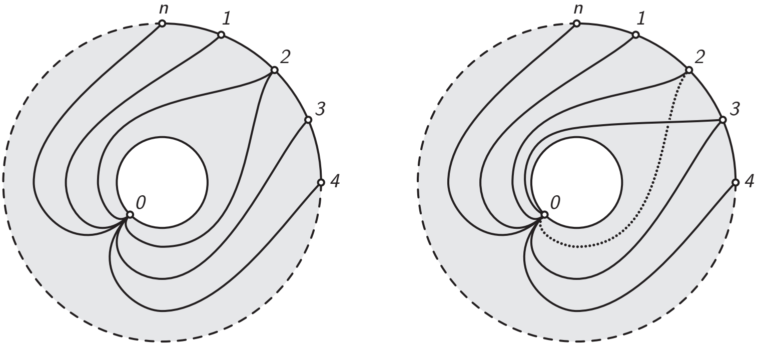
n	3	4	5	6	7	8	9	10	11	12
Ø	0	1	2	4	5	7	9	11	12	15

Consider a cylinder with n vertices in the outer boundary labeled 1 to n clockwise and one vertex labeled 0 in the inner boundary.



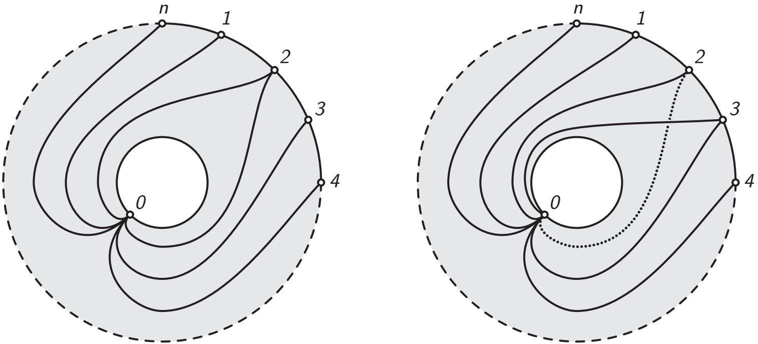
Call $\mathcal{A}$ the set of these $n+1$ vertices.

Using the same trick as in the case of the polygon, a triangulation T of $\mathcal{A}$ can be transformed into a triangulation T' whose interior edges all contain 0. This triangulation contains a unique triangle with vertex set $\{0, x\}$.



To do so, at most $n - 1$ flips are required.

Doing the same from another triangulation U will also result, after at most $n-1$ flips in a triangulation U' whose interior edges all contain 0. This triangulation contains a unique triangle with vertex set $\{0, y\}$.

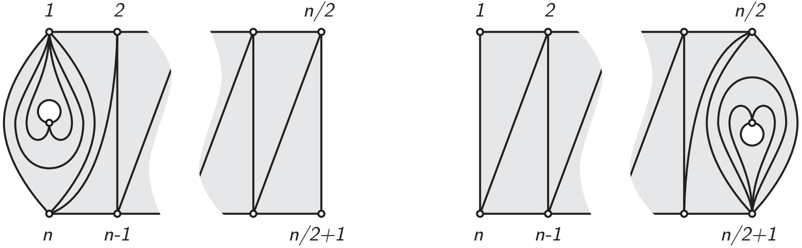


x and y are separated by at most $n/2-1$ vertices along the outer boundary. Bringing them together thus requires at most $n/2$ flips.

Lemma

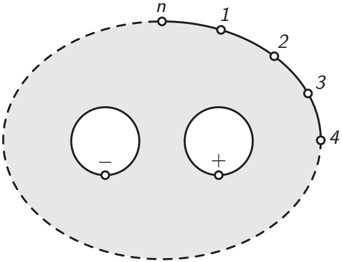
The distance of two triangulations of $\mathcal{A}$ is not greater than $5n/2 - 2$.

It turns out that coefficient $5/2$ is tight in this case!



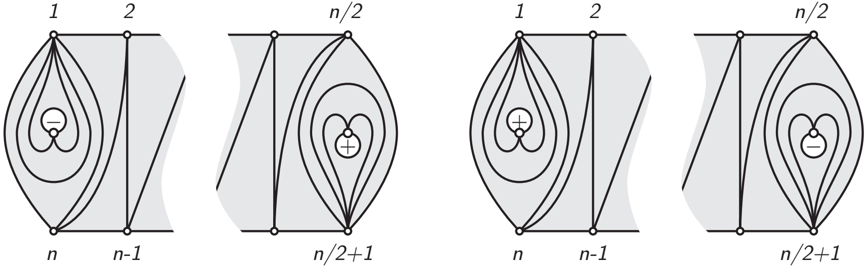
This pair of triangulations, depicted here when n is even, has flip distance $5n/2 + O(1)$. The proof uses the same techniques as in the case of the disc.

Consider a surface with three boundaries. Label n vertices by 1 to n clockwise in the outer boundary. Further consider two vertices $-$ and $+$ in each of the inner boundaries.



Call $\mathcal{A}$ the set of these $n+2$ vertices. There are pairs of triangulations of $\mathcal{A}$ at distance $3n + O(n)$!

Consider a surface with three boundaries. Label n vertices by 1 to n clockwise in the outer boundary. Further consider two vertices $-$ and $+$ in each of the inner boundaries.



These two triangulations, also represented when n is even, have flip distance $3n + O(1)$.The proof uses the same techniques as in the case of the disc.

Let $\mathcal{A}$ be a set of points placed in a surface with $r > 0$ boundaries and genus g. Assume that:

i. the total number of boundary points is b
ii. the total number of interior points is i,

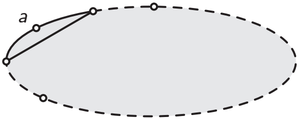
According to Euler's formula, the number e of **interior** edges in any triangulation of $\mathcal{A}$ is:

$$e = 3i + b + 3r + 7g - 6.$$

How does the diameter of G behave when n (the number of points in the outer boundary) grows large at fixed i, $b - n$, r, and g? In this case:

$$e = n + O(1).$$

Each interior edge of a triangulation incident to a vertex $a \in \mathcal{A}$ in the outer boundary can be removed by a flip. Removing every such edge results in an *ear* with apex a.

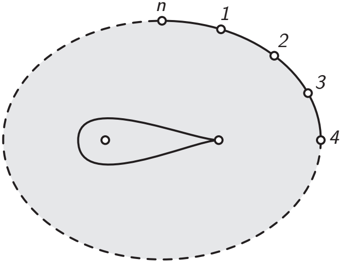


The number of incidences between the interior edges of two triangulations and vertices in the outer boundary is $4e = 4n + O(1)$. If n is large enough, one can introduce the same ear into these triangulations using at most 4 flips. This ear can be cut off from both triangulations, and:

Lemma

The diameter of G is at most $4n + O(1)$.

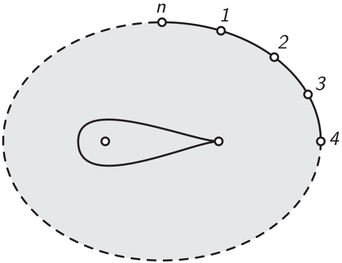
Consider a disc with n vertices in the boundary and two interior vertices. Enclose one of the two interior vertices by an edge ε that has the other vertex as its two ends (and join the two interior vertices with an edge).



Triangulate the cylinder bounded by ε and by the outer boundary using two triangulations at distance $5n/2 + O(1)$. Denote by T and by U the two resulting triangulations of $\mathcal{A}$.

Lemma

Edge ε is never removed along any minimal path between T and U.



As a consequence, graph G has diameter at least $5n/2 + O(1)$ in this case.

Assuming that n grows large, while the topology (r boundaries, genus g) and the number i of interior vertices remain fixed,

i	g	r	Lower bound	Upper bound
1	0	1	$2n + O(1)$	$2n + O(1)$
2	0	1	$5n/2 + O(1)$	$5n/2 + O(1)$
3	0	1	5n/2+O(1)	$4n + O(1)$
≥ 4	0	1	3n+O(1)	$4n + O(1)$
0	1	1	5n/2+O(1)	$4n + O(1)$
1	1	1	5n/2+O(1)	$4n + O(1)$
≥ 2	≥ 1	1	3n+O(1)	$4n + O(1)$
≥ 0	≥ 2	1	3n+O(1)	$4n + O(1)$
≥ 0	≥ 0	2	5n/2+O(1)	$4n + O(1)$
≥ 0	≥ 0	≥ 3	3n+O(1)	$4n + O(1)$