# Minkowski decompositions of associahedra into faces of a standard simplex

Carsten Lange
UPMC

Séminaire Equipe Modéles Combinatoires - LIX
École Polytechnique
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# Agenda

- What is... a permutahedron/associahedron
- How to... realise associahedra
- What is... a Minkowski decomposition
- How to... decompose associahedra

# Permutahedra

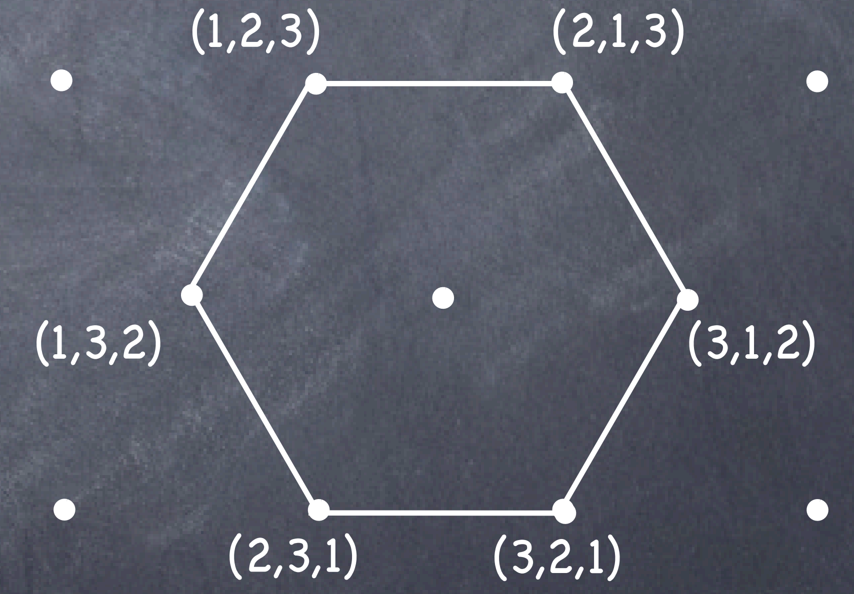
## -- definition --

idea: apply action of $\Sigma_n$ on $R^n$ to generic point

- convex hull of points

$$\{ (\sigma(1),\ldots,\sigma(n)) \mid \sigma \in \Sigma_n \}$$

affine hyperplane
$x_1 + x_2 + x_3 = 6$



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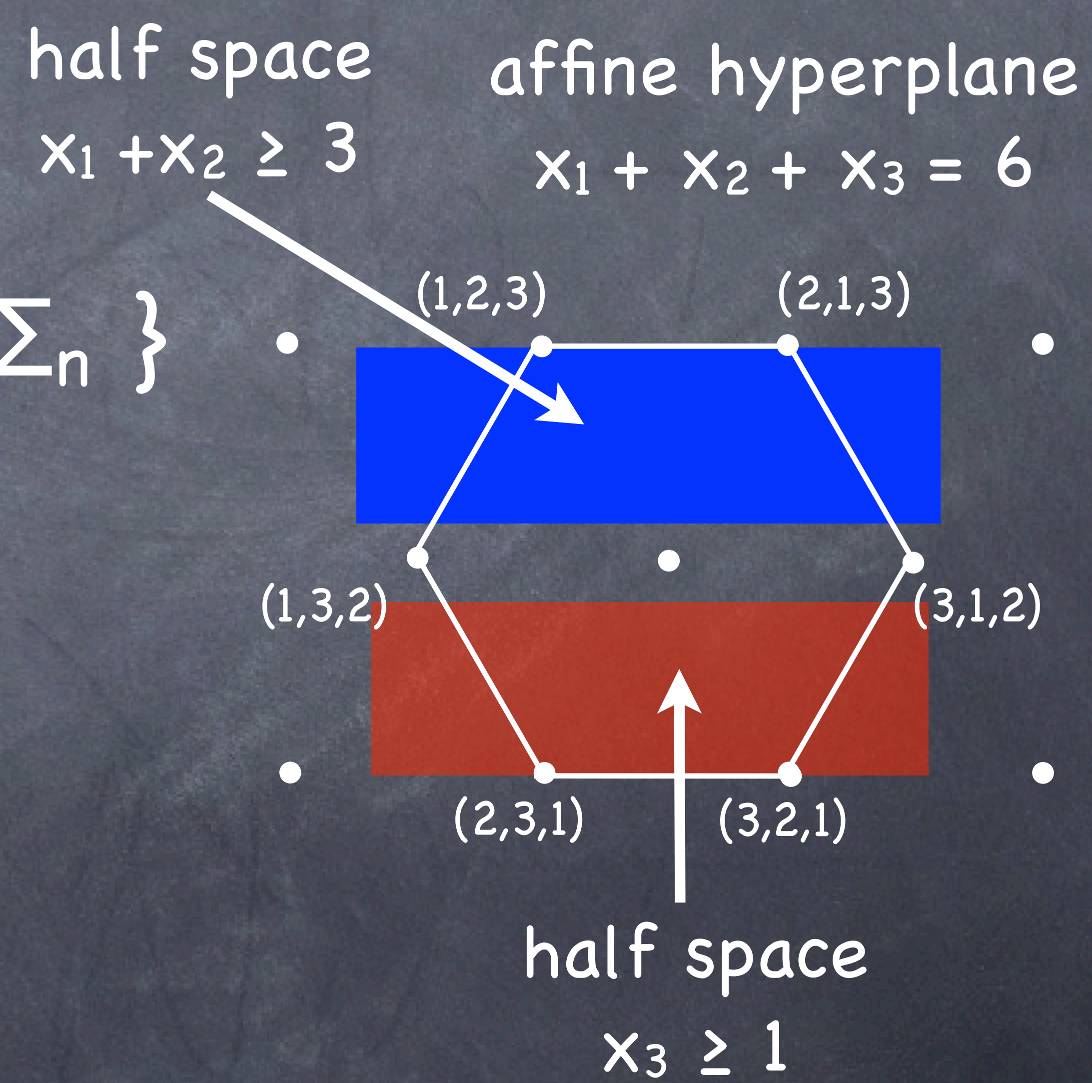
- H-representation

$$\Sigma_{i\in[n]}\ x_i = \tfrac{1}{2}n(n+1)$$

$$\Sigma_{i\in K}\ x_i \geq \tfrac{1}{2}|K|(|K|+1)$$
$$\text{for } \varnothing \neq K \subset [n]$$



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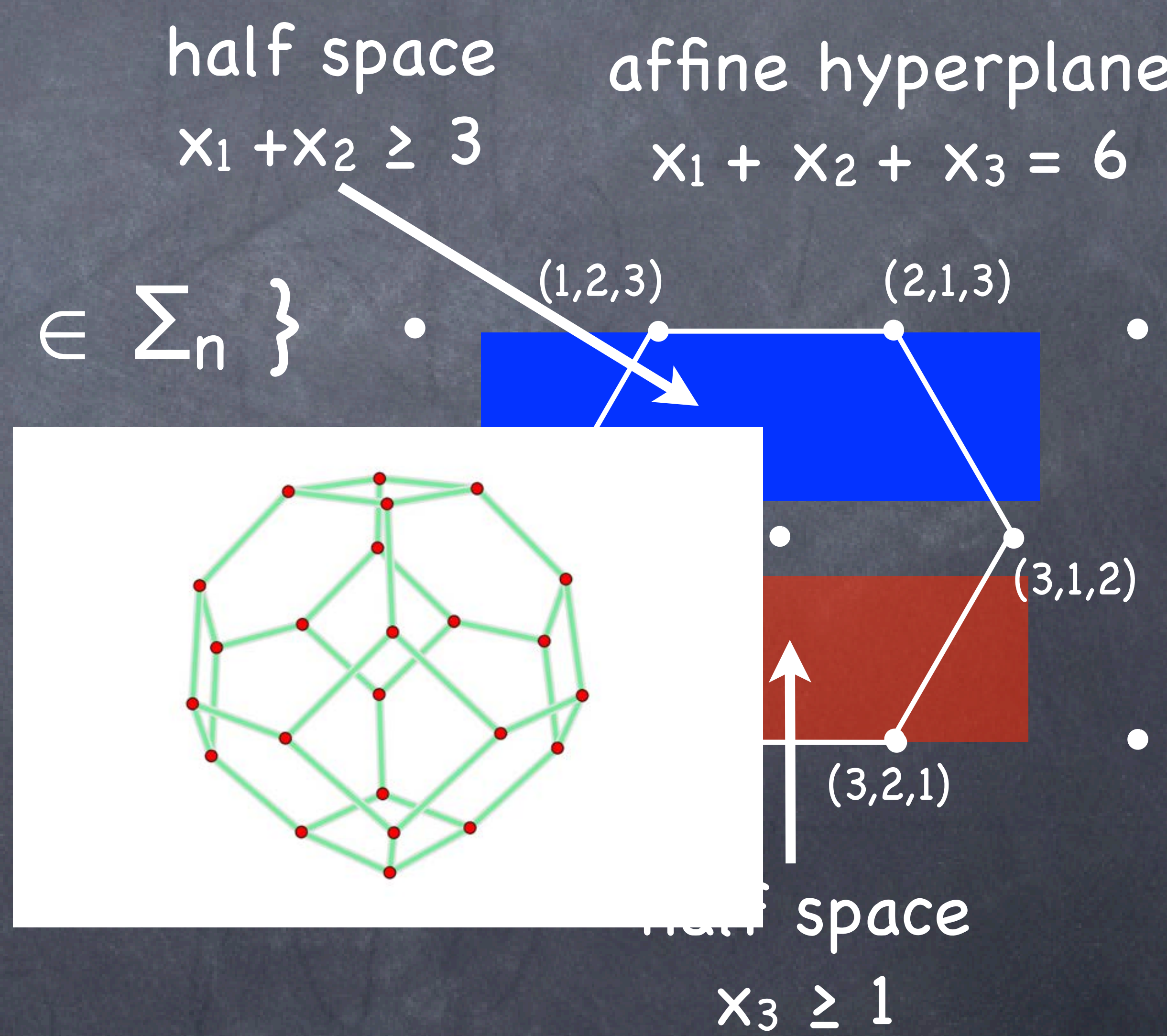

# generalised Permutahedra

## -- definition, $z_I$-coordinates --

idea: change permutahedron's right-hand sides

- H-representation
  $\sum_{i \in [n]} x_i = z_{[n]}$
  $\sum_{i \in I} x_i \geq z_I \quad$ for $\emptyset \neq I \subset [n]$
  (want all "redundant" $z_I$-values tight)

- $z_I$-coordinates
  vector of all $z_I$-values
  (redundancies possible; choose all $z_I$-values tight)

- $P(\{z_I\})$
  generalised permutahedron with given $z_I$-coordinates

# What is... Minkowski sum

-- definition --

- P and Q polytopes

  Minkowski sum P+Q is the polytope

  $$\{ p+q \mid p \in P \text{ and } q \in Q \}$$

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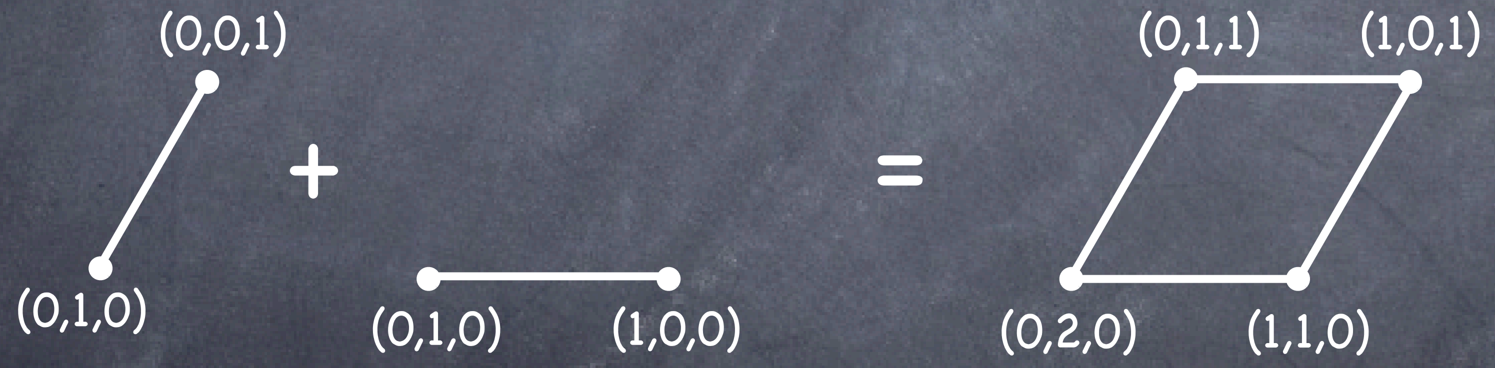
Example (edges of standard simplex):



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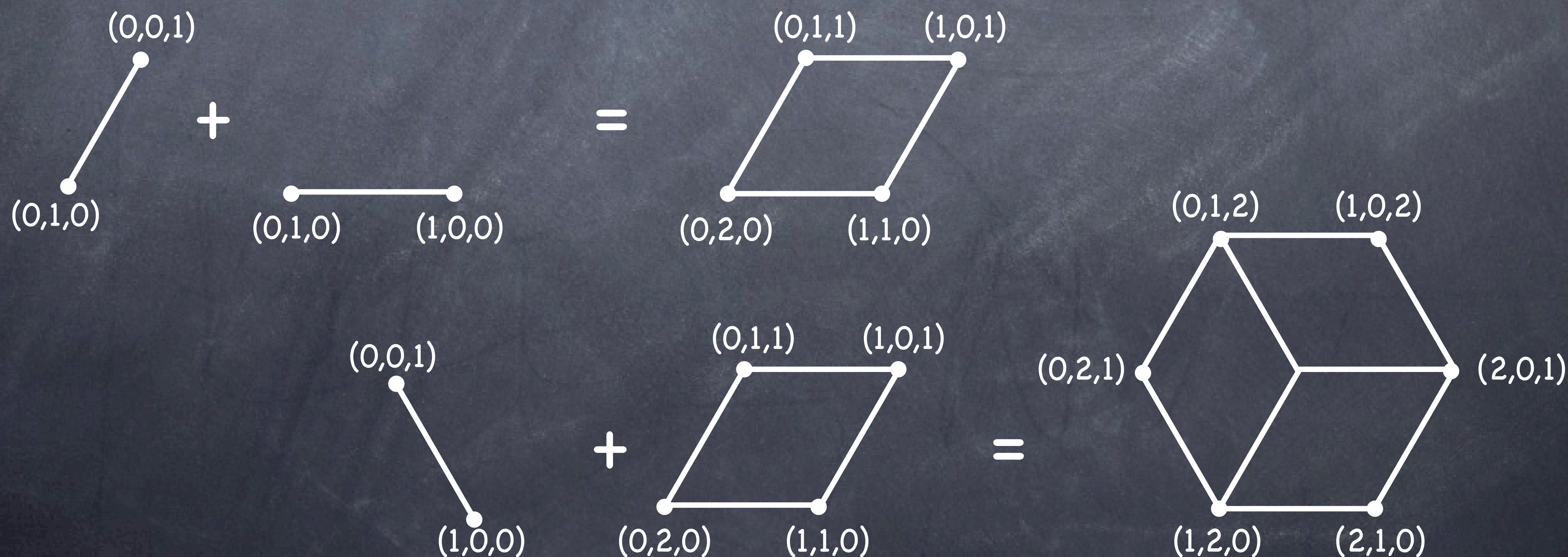

# Generalised Permutahedra

## -- $y_I$-coordinates --

Theorem [Postnikov, 2009]
Every Minkowski sum of dilated faces of a standard simplex yields a generalised permutahedron

$y_I$-coordinates (à la Postnikov)
vector of dilation factors $y_I$ for $\emptyset \subset I \subseteq [n]$
($y_I \geq 0$; $y_I = 0 \Leftrightarrow$ face not used)

Observation: $z_I$- and $y_I$-vectors have same size

Are $z_I$- and $y_I$-coord's related? If yes, how?

# Relate $y_I$- & $z_I$-coordinates

-- $y_I$- & $z_I$-coordinates as functions --

$\{1,2,3\}$
$z_{\{1,2,3\}}$

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$
$z_{\{1,2\}}$ $z_{\{1,3\}}$ $z_{\{2,3\}}$

$\{1\}$ $\{2\}$ $\{3\}$
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[Postnikov, 2009]:
$z_I = \Sigma_{J \subseteq I}\, y_J$

$\longleftarrow$

!! $y_J \geq 0$ !!

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$z_I$-coordinates as
function on Boolean lattice
(geometric constraints on $z_I$)

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!! $y_J \geq 0$ !!

$\longrightarrow$

Möbius inversion
$y_J = \Sigma_{I \subseteq J}\, (-1)^{|J-I|} z_I$

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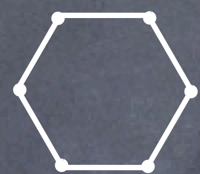

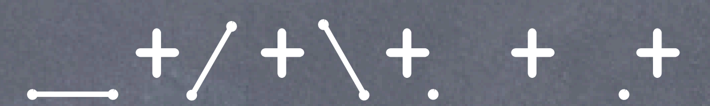

$_{6}\{1,2,3\}$

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Möbius inversion

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$\{1,2,3\}_0$

$\{1,2\}_1$ $\{1,3\}_1$ $\{2,3\}_1$

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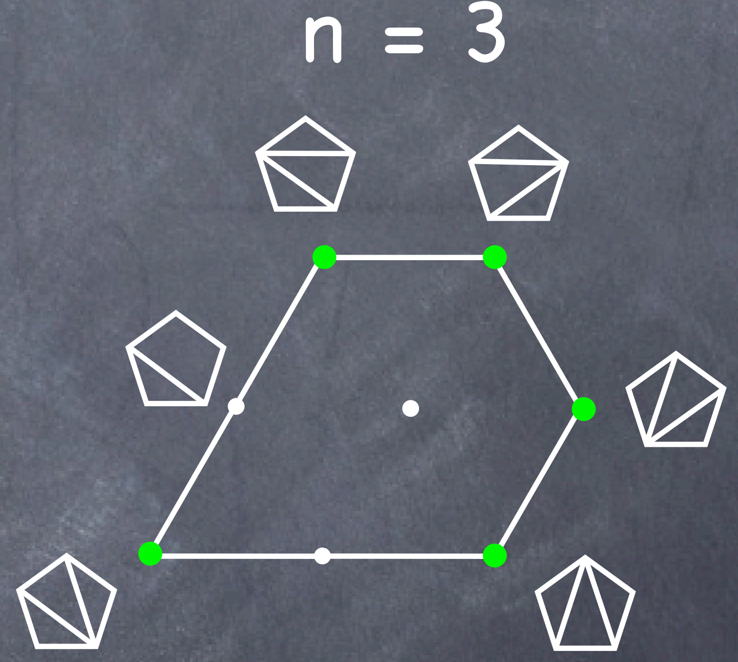
$z_I$-coordinates as function on Boolean lattice (geometric constraints on $z_I$)

$y_I$-coordinates as function on Boolean lattice ($y_I \geq 0$)

# What is... an associahedron

## -- combinatorial description --

- combinatorics of CW-complex (Stasheff)
  - vertices = triangulations of (n+2)-gon
  - k-face = triangulation minus k diagonals
- can be realised as (n-1)-dim polytope
- polytopal realisations were given by Milnor (unpublished), Lee, Haiman, Sternberg&Shnider and Stasheff&Shnider, Loday...

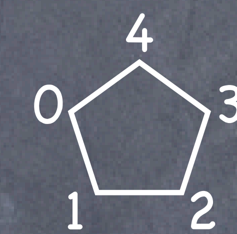


# Realise Associahedra

## -- example (Shnider,Sternberg&Stasheff, Loday) --

Loday: Computes coord's (planar binary trees)

equivalently:
label (n+2)-gon cyclicly
decreasing with {0,...,n+1}

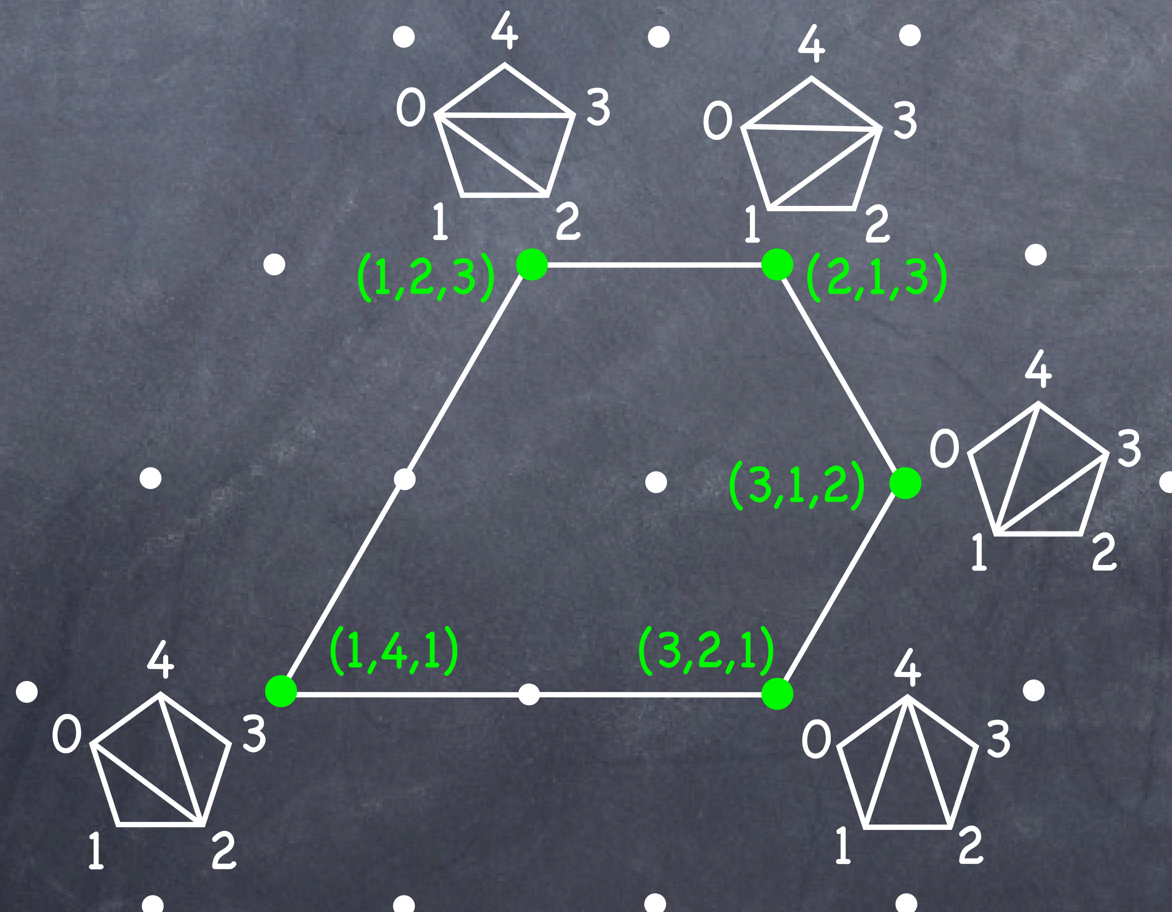



$n=3$

labelled pentagon

2-dim associahedron

realised in $R^3$




affine hyperplane

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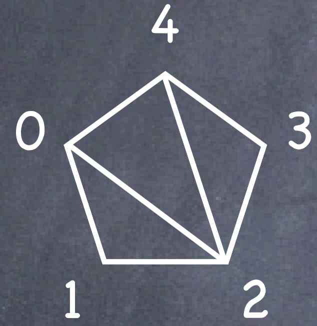


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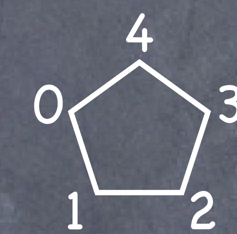


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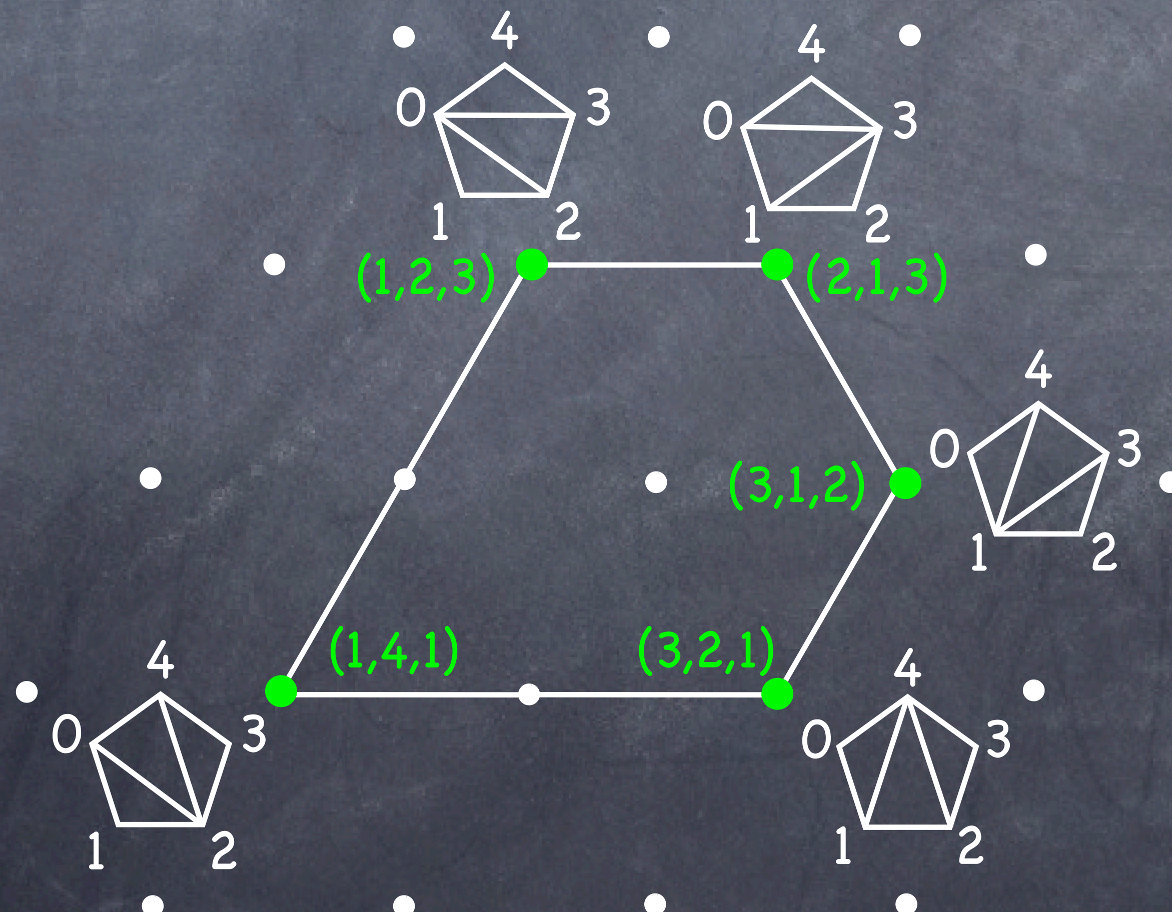


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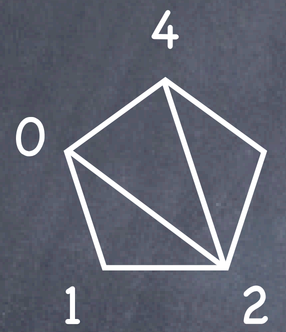


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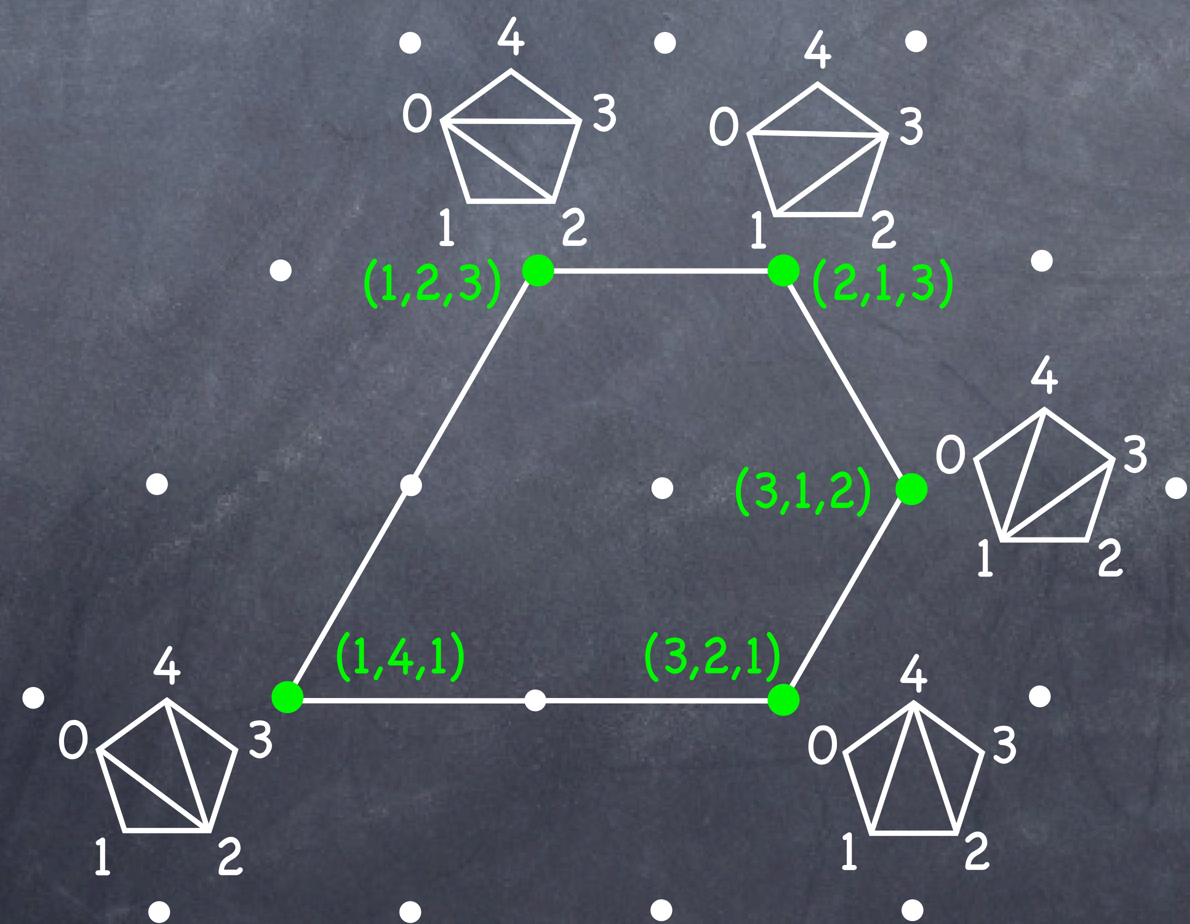


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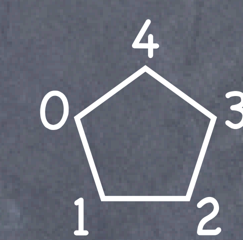


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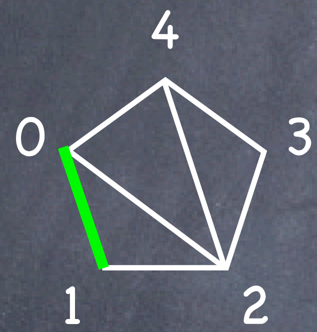


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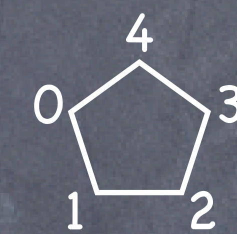


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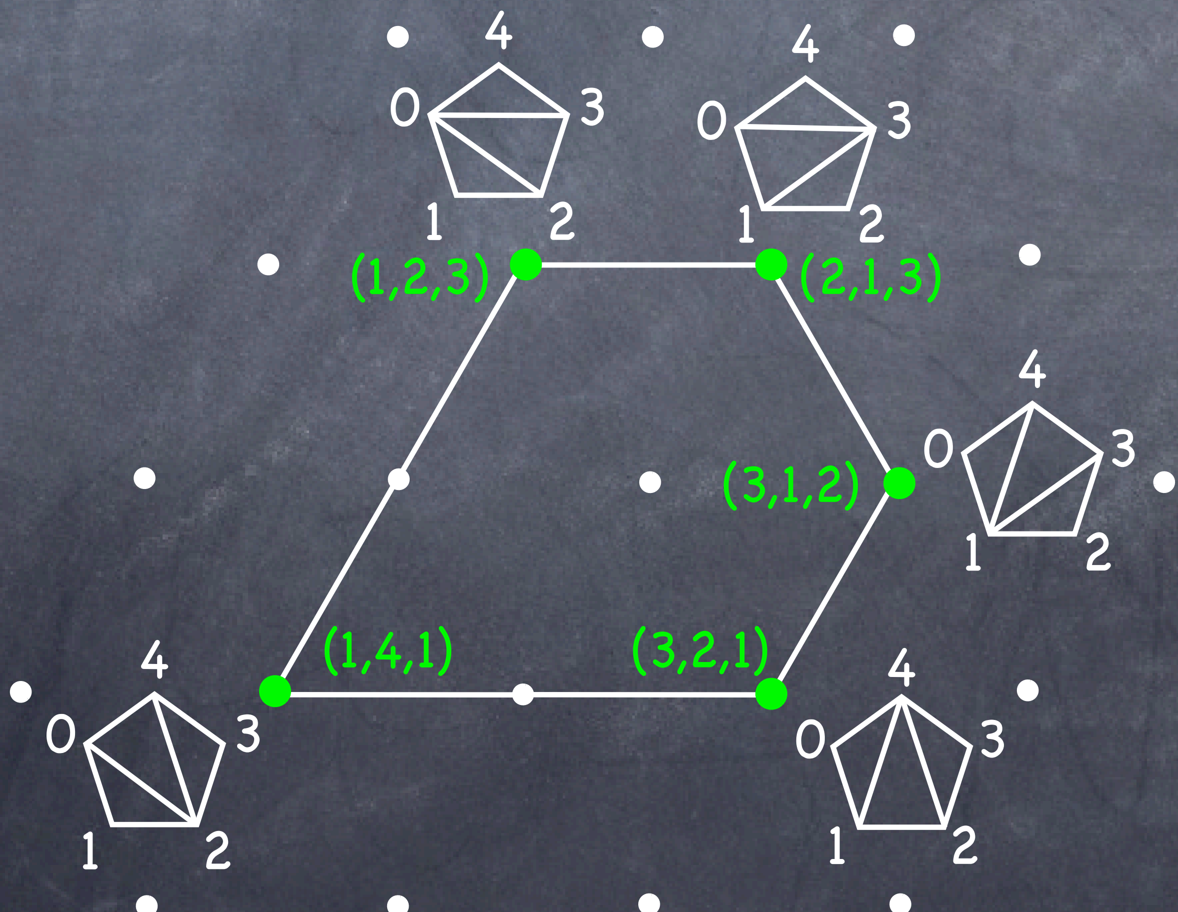


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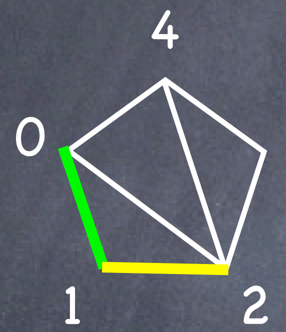


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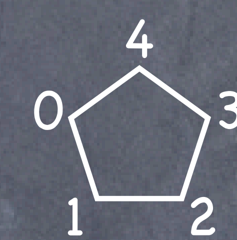


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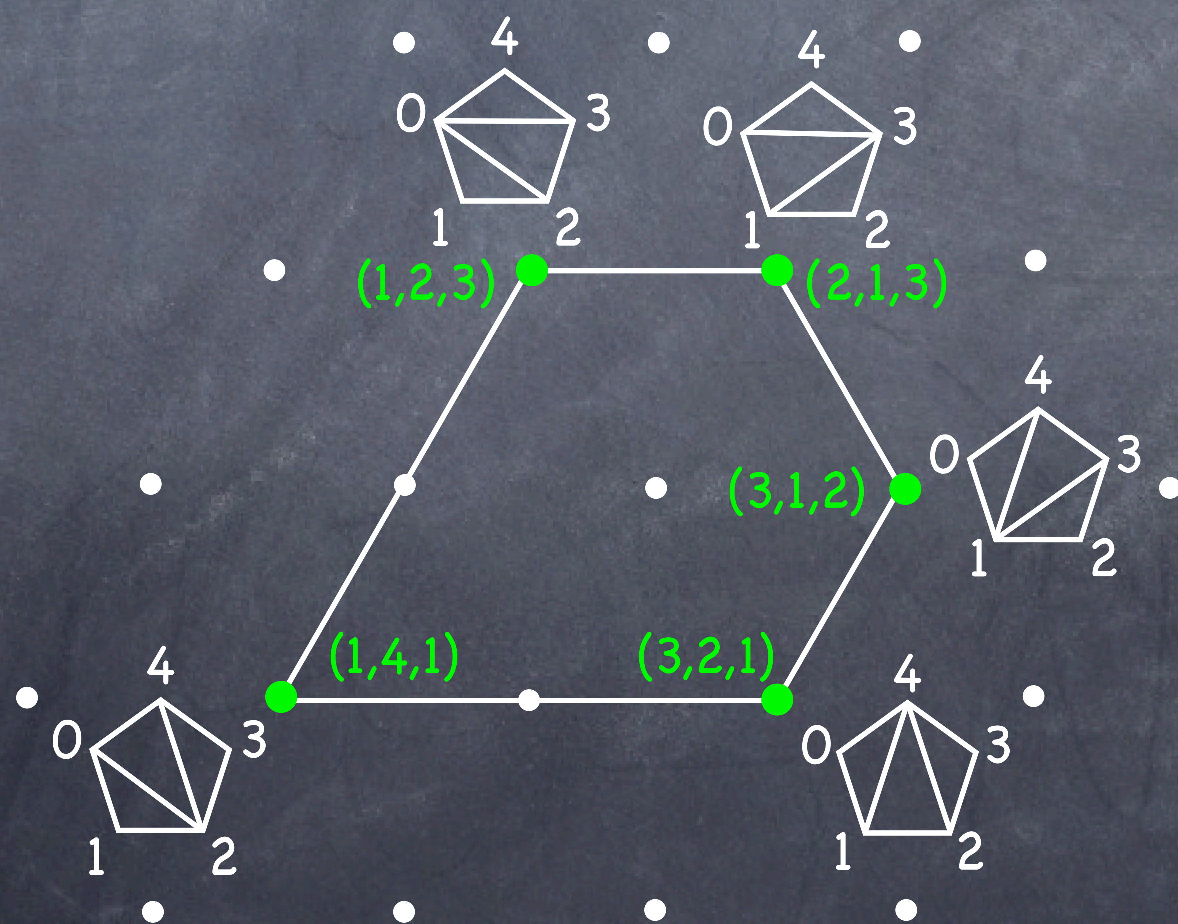


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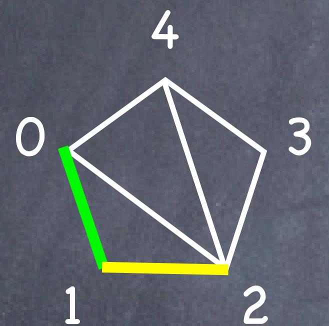


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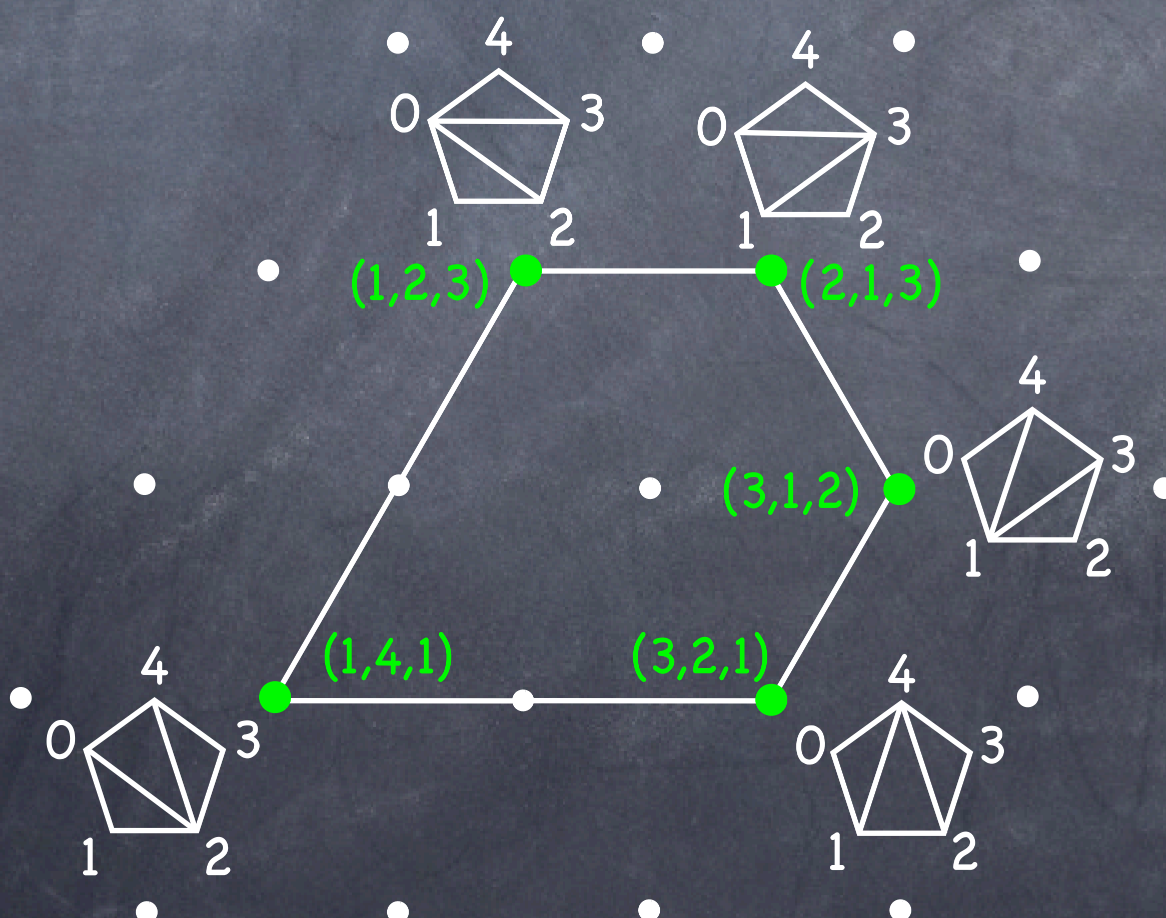


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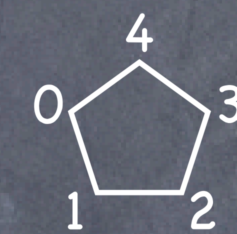


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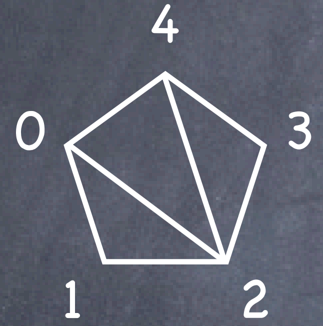


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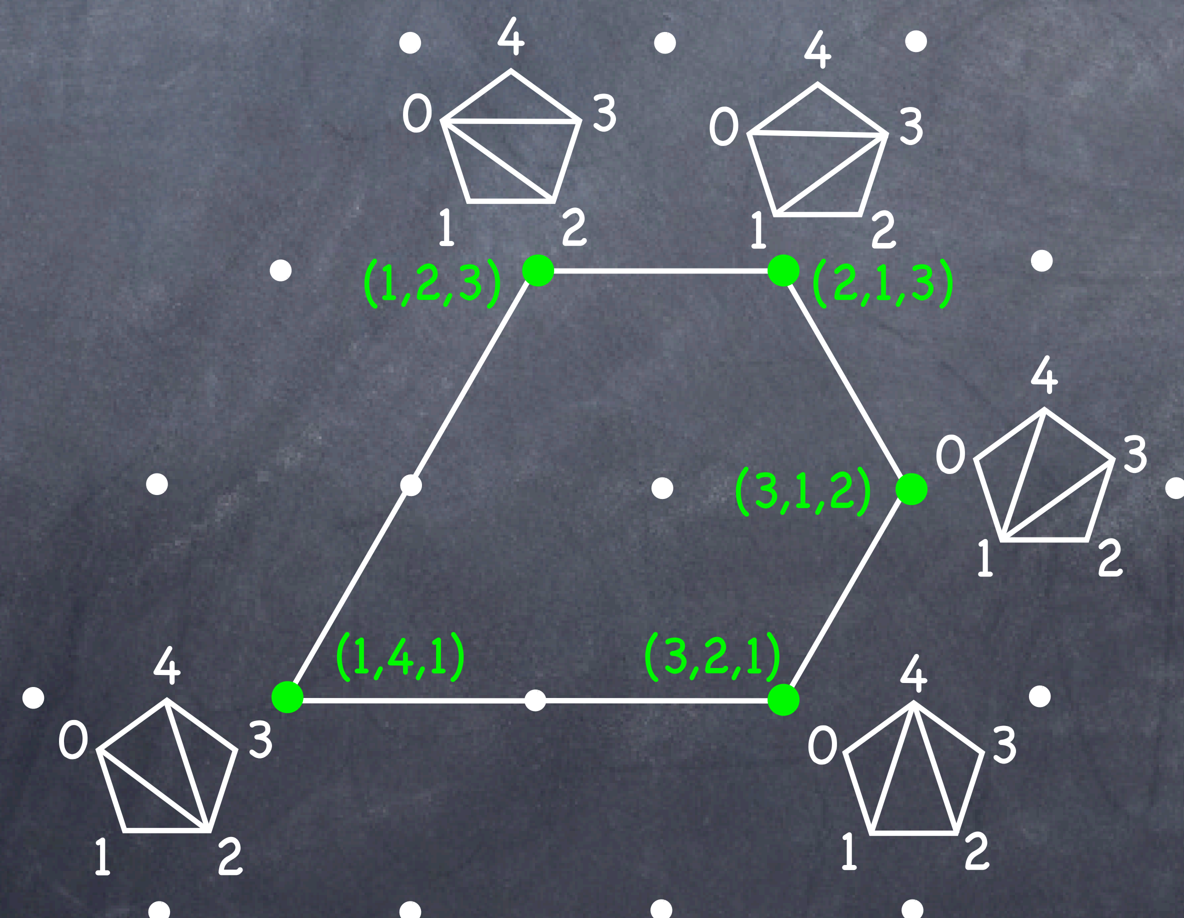


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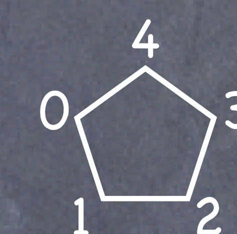


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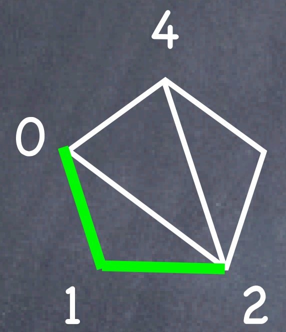


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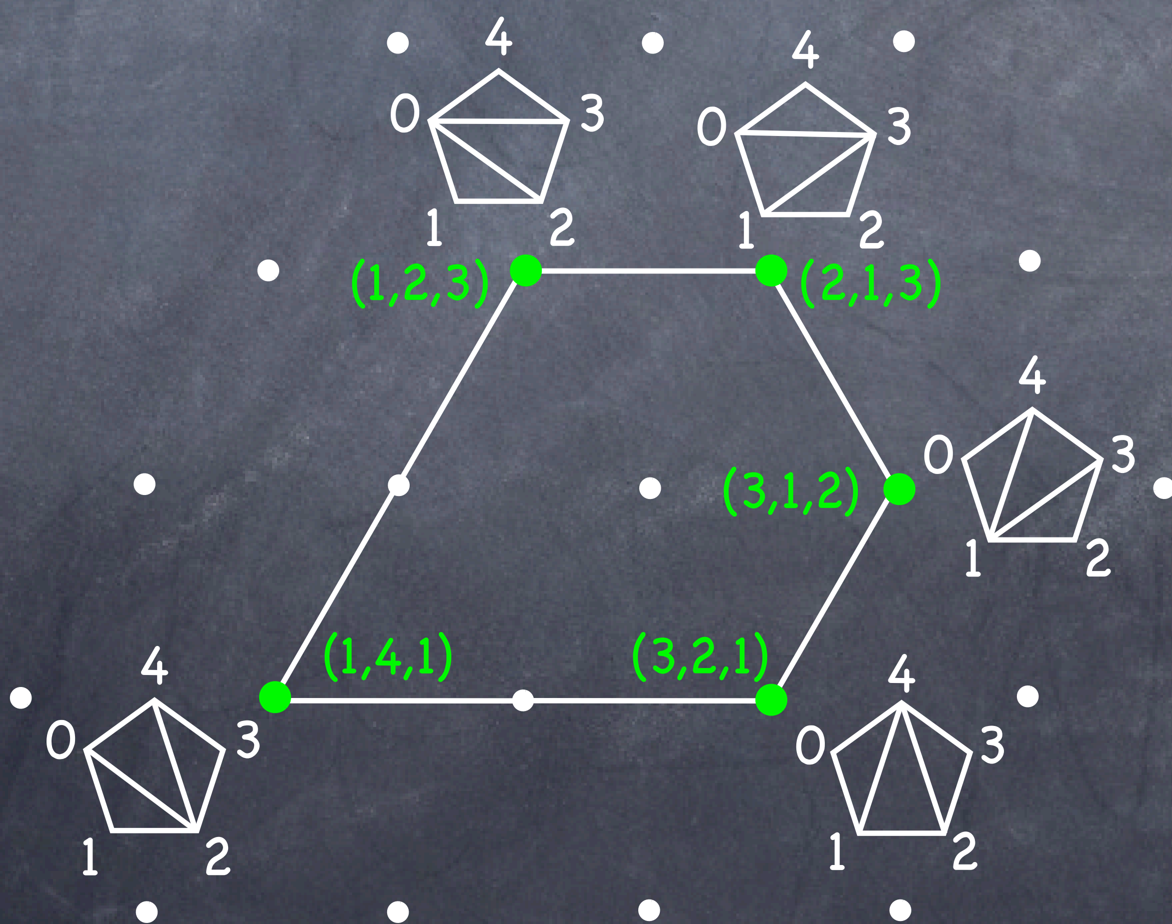


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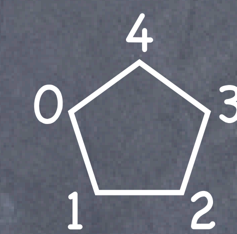


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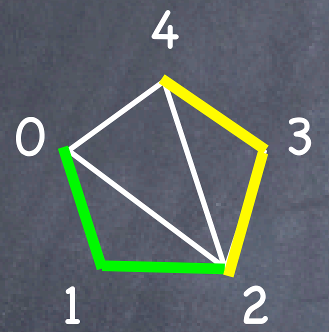


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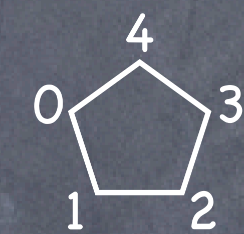


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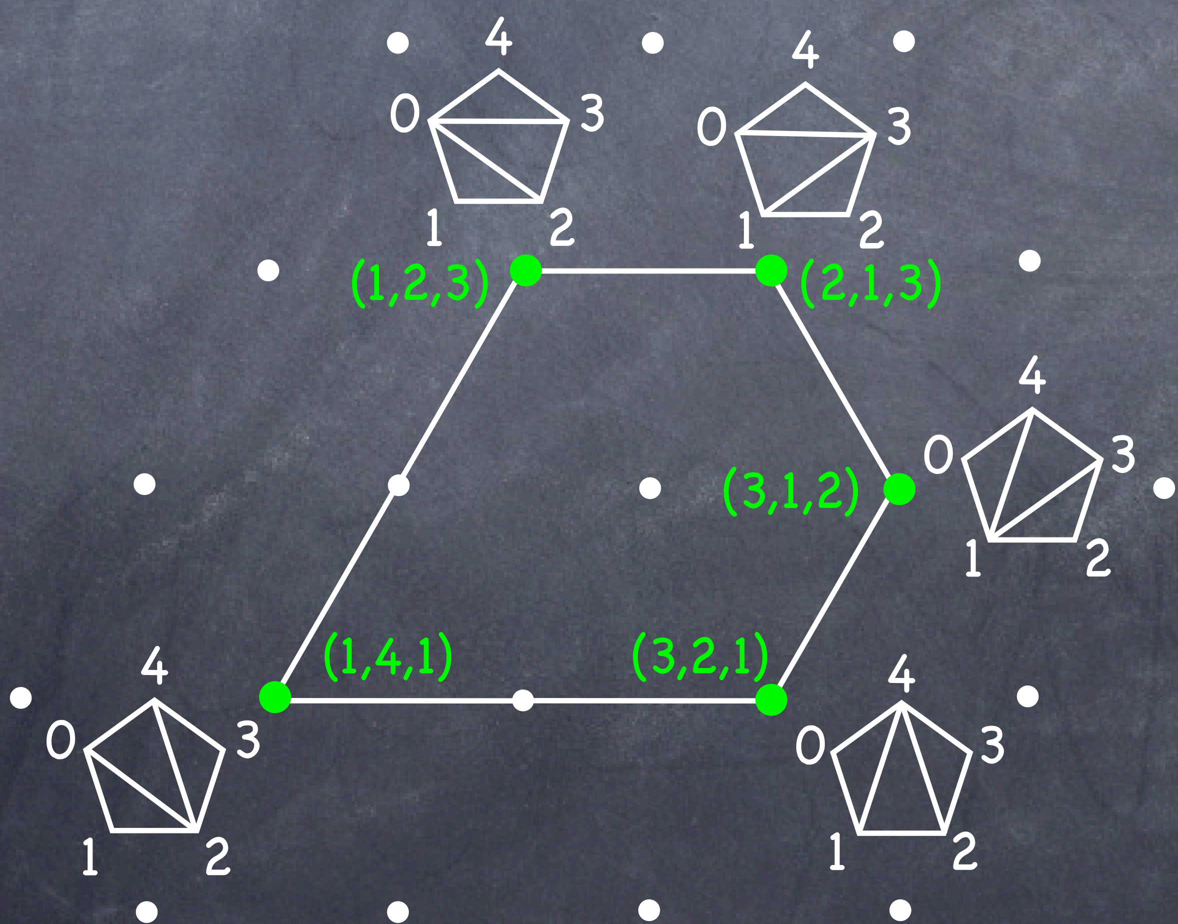


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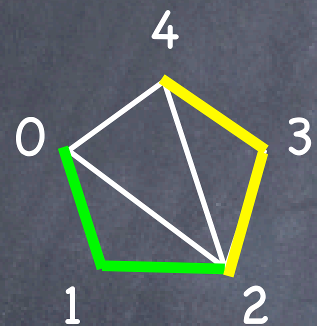


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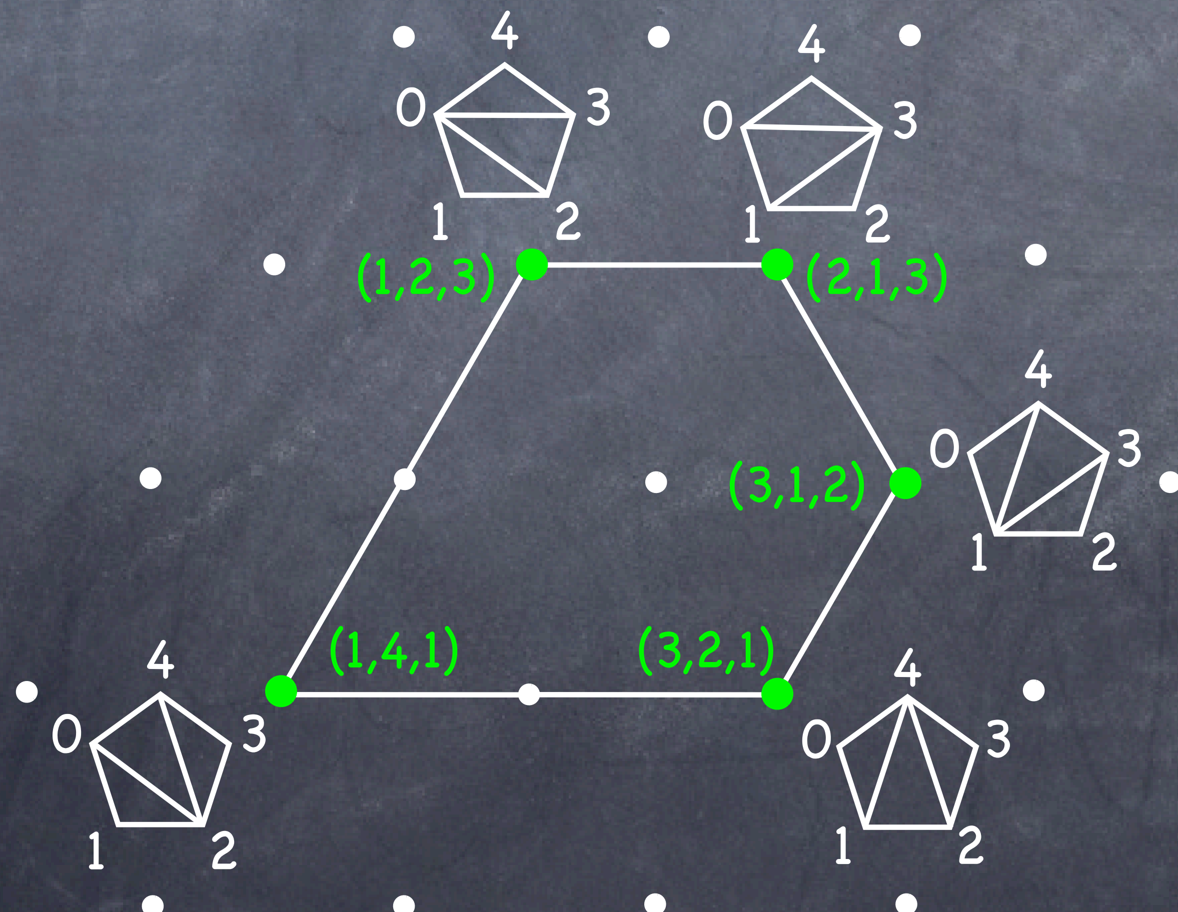


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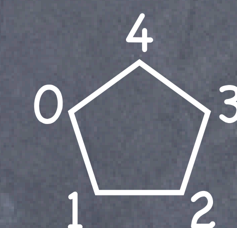


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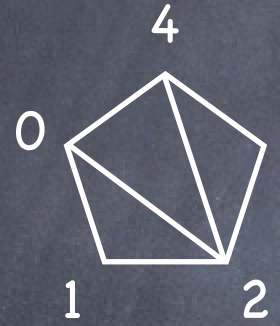


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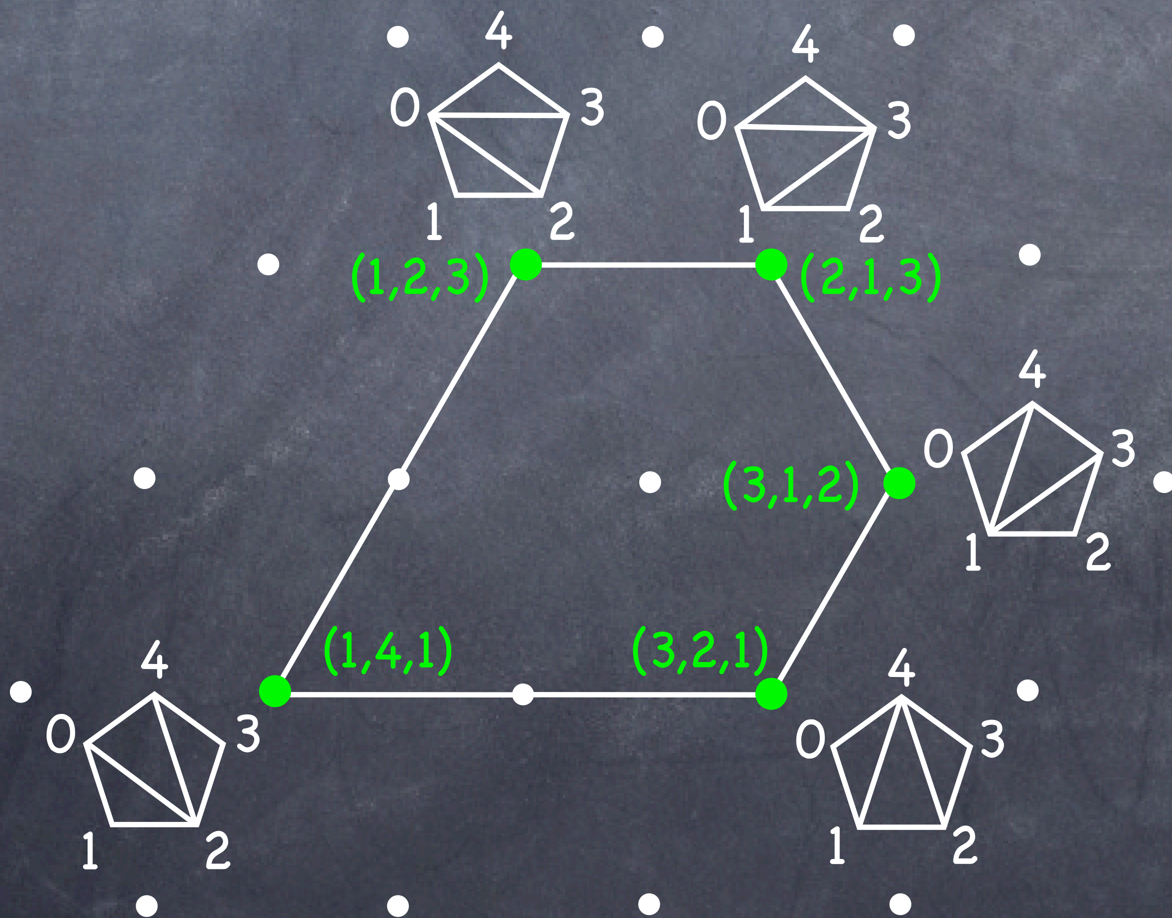


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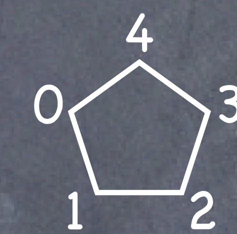


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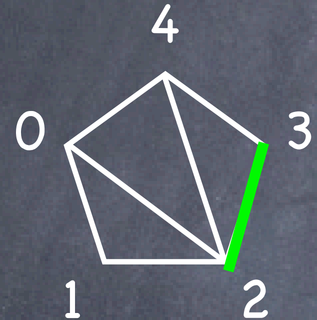


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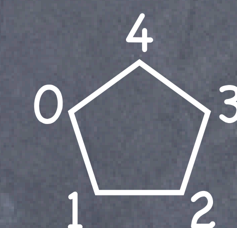
decreasing with {0,...,n+1}

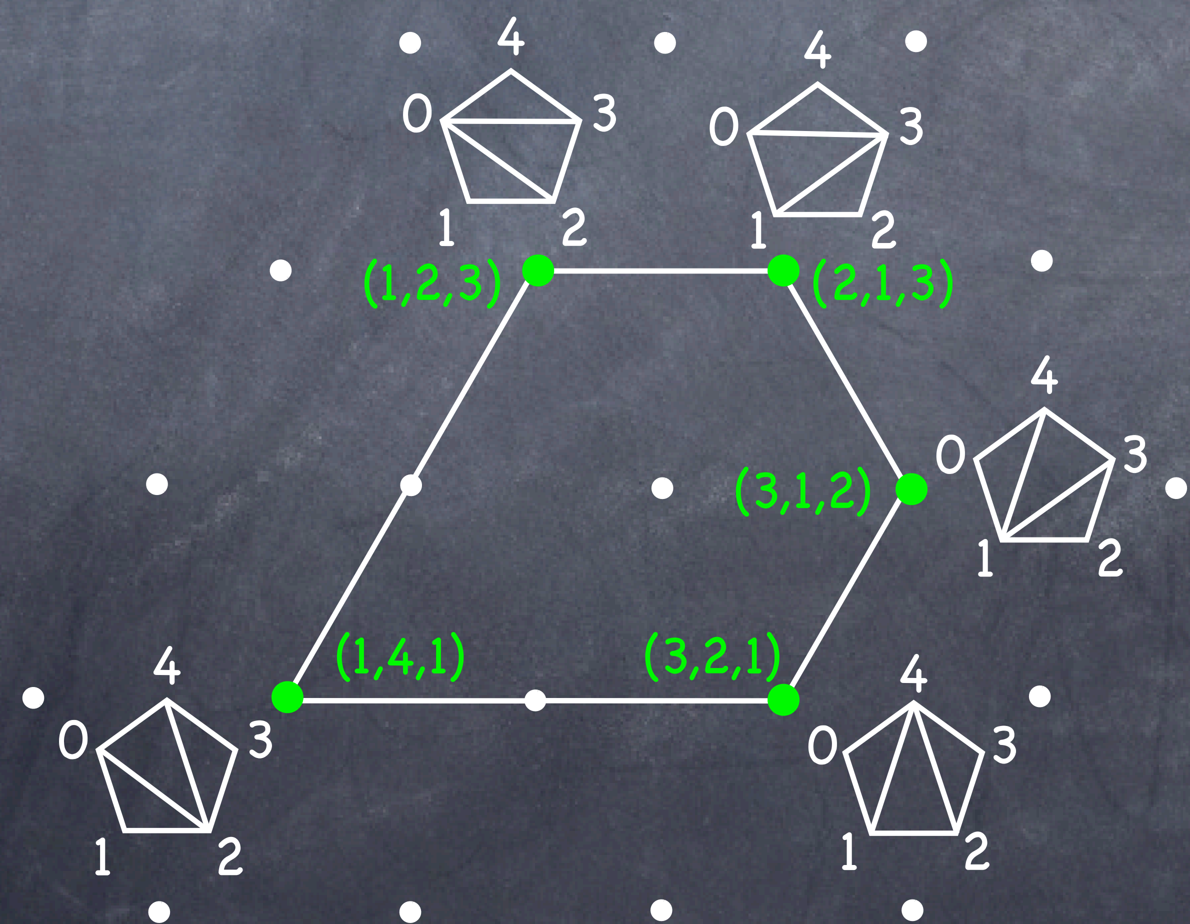



n=3

labelled pentagon

2-dim associahedron

realised in $R^3$



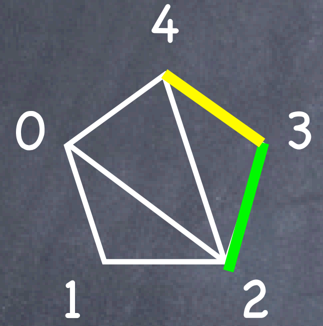

affine hyperplane

$x_1 + x_2 + x_3 = 6$

# Realise Associahedra

## -- example (Shnider,Sternberg&Stasheff, Loday) --

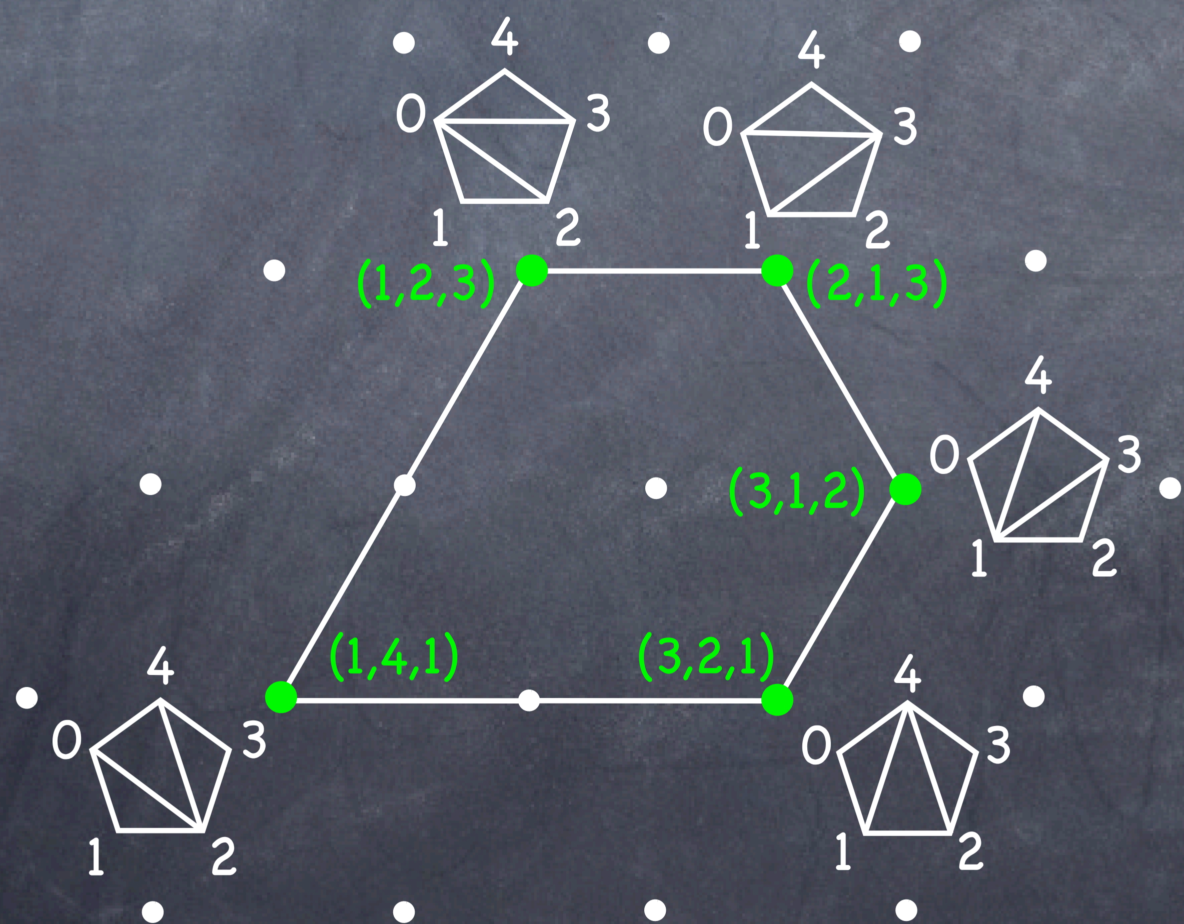
## Loday: Computes coord's (planar binary trees)




$x_1 = 1 \cdot 1 = 1$

$x_2 = 2 \cdot 2 = 4$

$x_3$

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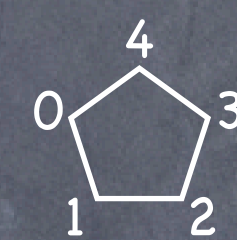


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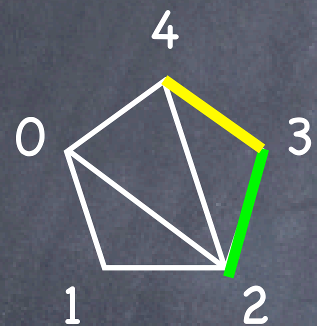


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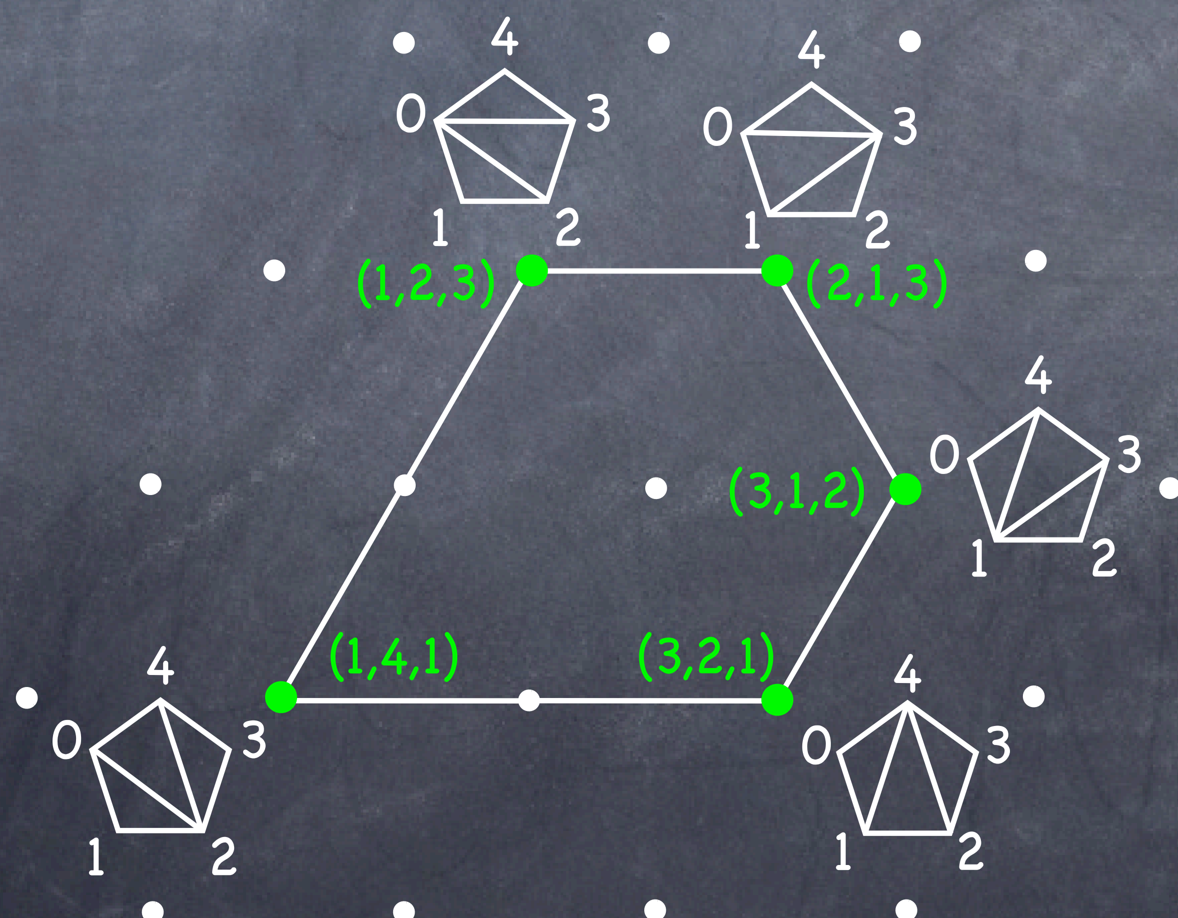


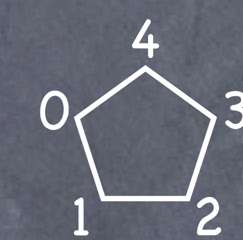


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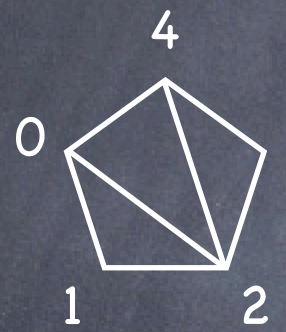


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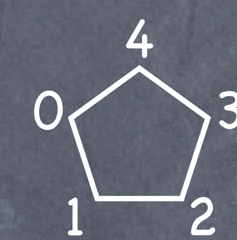


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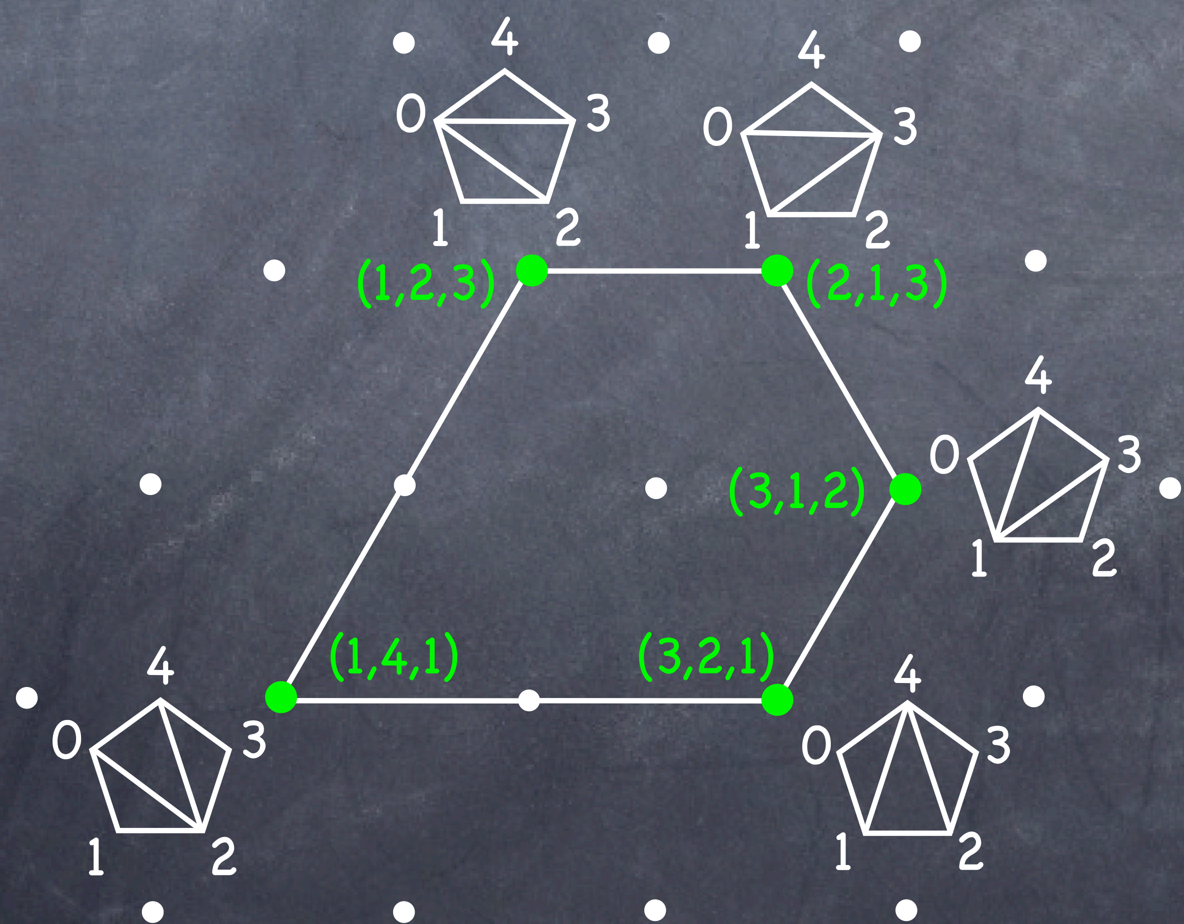


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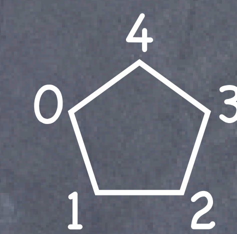


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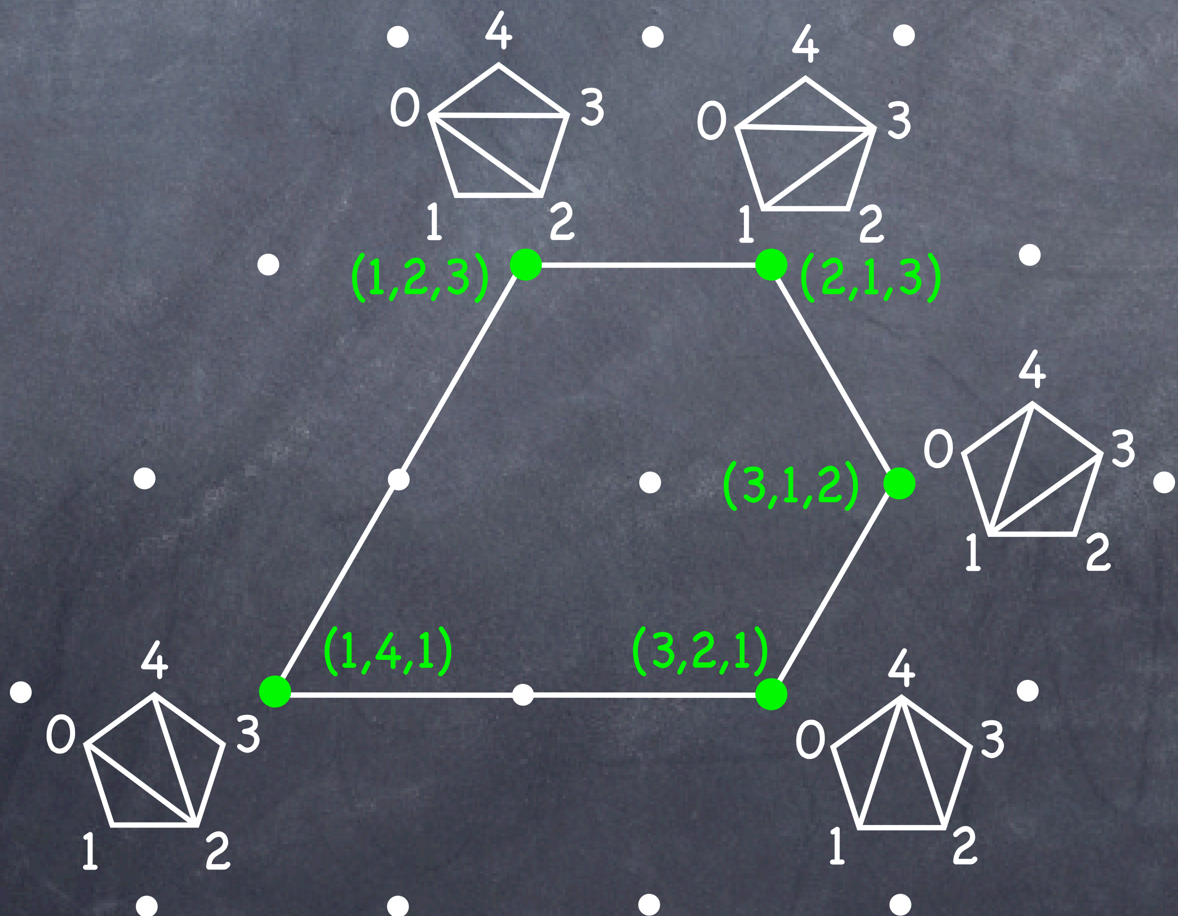


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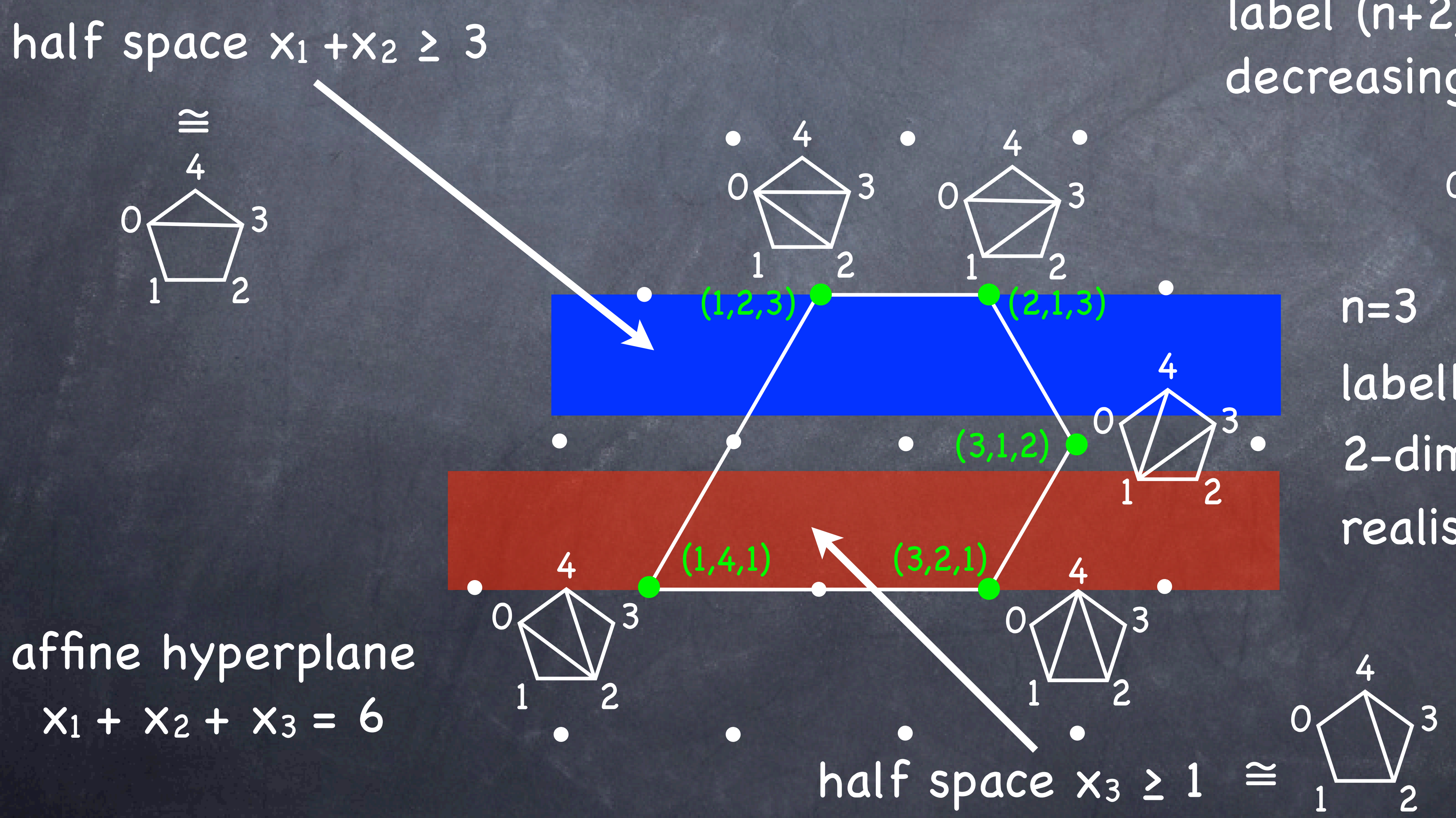


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labelled pentagon

2-dim associahedron

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# Realise Associahedra

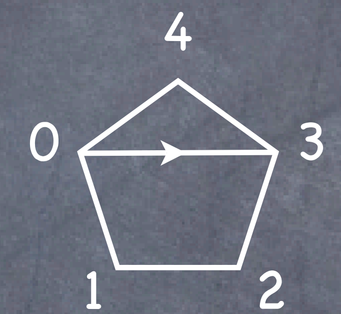
## -- example: Loday's associahedra II --

- label (n+2)-gon cyclicly decreasing with $\{0,\ldots,n+1\}$

- A is "good subset" $\Leftrightarrow$ strict RHS of oriented diagonal

  E.g.: $A = \{1,2\}$

  $\sum_{i\in A} x_i \geq z_A := |A|(|A|+1)/2$




- A is "bad subset"

  $\Leftrightarrow$ not RHS of oriented diagonal

  E.g.: $A = \{1,3\}$

  $\sum_{i\in A} x_i \geq z_A := -\infty$

$6^{\{1,2,3\}}$

$3^{\{1,2\}}$ $-\infty^{\{1,3\}}$ $3^{\{2,3\}}$

$1^{\{1\}}$ $1^{\{2\}}$ $1^{\{3\}}$

# Decompose Associahedra

## -- example: Loday's associahedra I --

$\{1,2,3\}$ 6

$\{1,2\}$ 3 $\quad \{1,3\}$ 2 $\quad \{2,3\}$ 3

$\{1\}$ 1 $\quad \{2\}$ 1 $\quad \{3\}$ 1

[Postnikov, 2009]:

$z_I = \Sigma_{J \subseteq I}\, y_J$

!! $y_J \geq 0$ !!

Möbius inversion

$y_J = \Sigma_{I \subseteq J}\, (-1)^{|J-I|} z_I$

$\{1,2,3\}$

$\{1,2\} \quad \{1,3\} \quad \{2,3\}$

$\{1\} \quad \{2\} \quad \{3\}$

$z_{\{1,3\}} = 2$ is tight value

# Decompose Associahedra

## -- example: Loday's associahedra I --

${}_{6}\{1,2,3\}$

${}_{3}\{1,2\}$ ${}_{2}\{1,3\}$ ${}_{3}\{2,3\}$

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[Postnikov, 2009]:

$z_I = \Sigma_{J\subseteq I}\ y_J$

!! $y_J \geq 0$ !!

Möbius inversion

$y_J = \Sigma_{I\subseteq J}\,(-1)^{|J-I|} z_I$

$\{1,2,3\}_1$

$\{1,2\}_1$ $\{1,3\}_0$ $\{2,3\}_1$

$\{1\}_1$ $\{2\}_1$ $\{3\}_1$

$z_{\{1,3\}} = 2$ is tight value

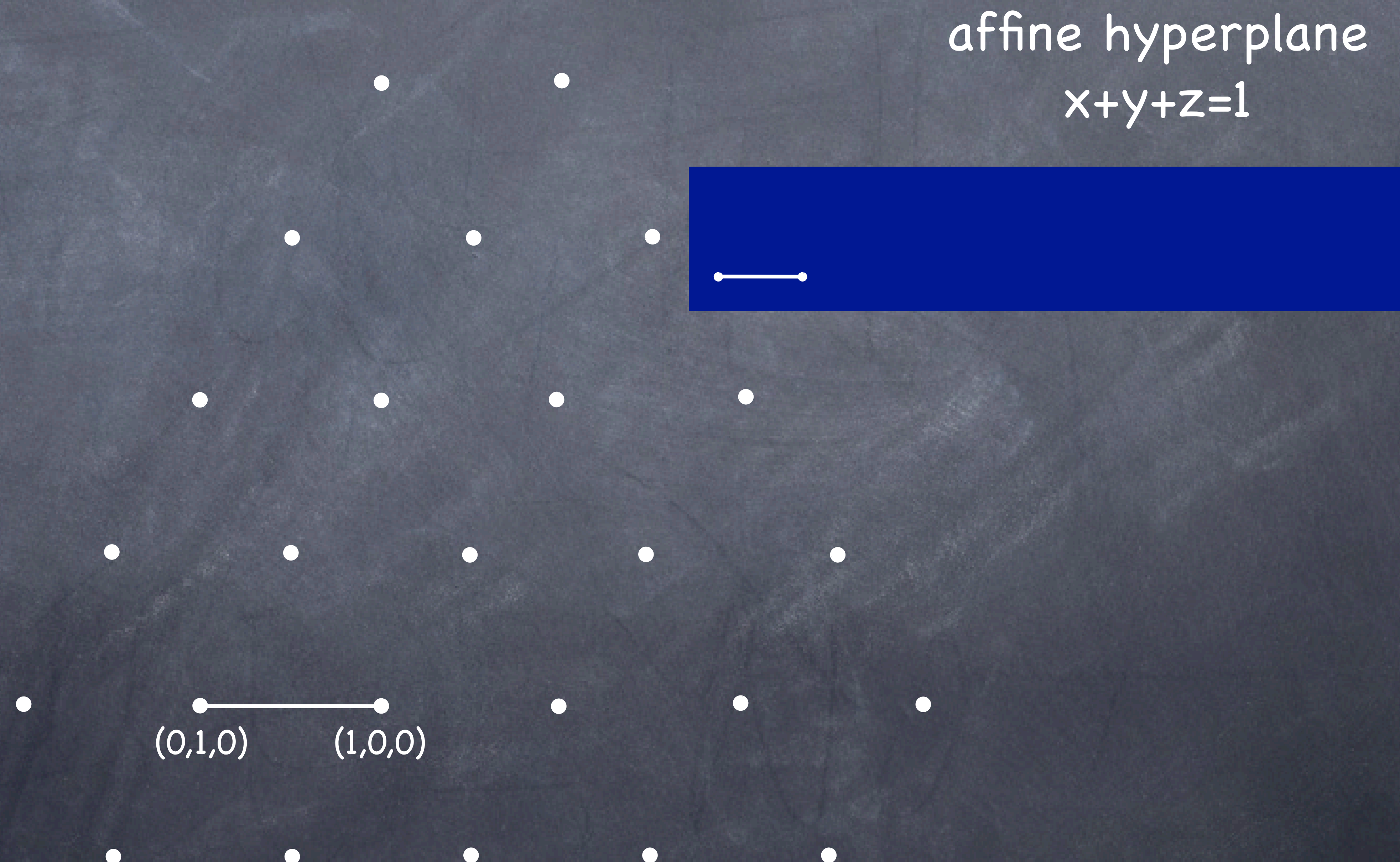
known:

$y_I = 1 \Leftrightarrow I$ good subset

$y_I = 0 \Leftrightarrow I$ bad subset

# Decompose Associahedra

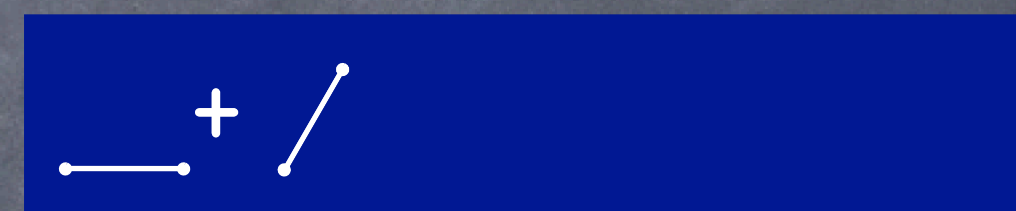
## -- example: Loday's associahedra II --

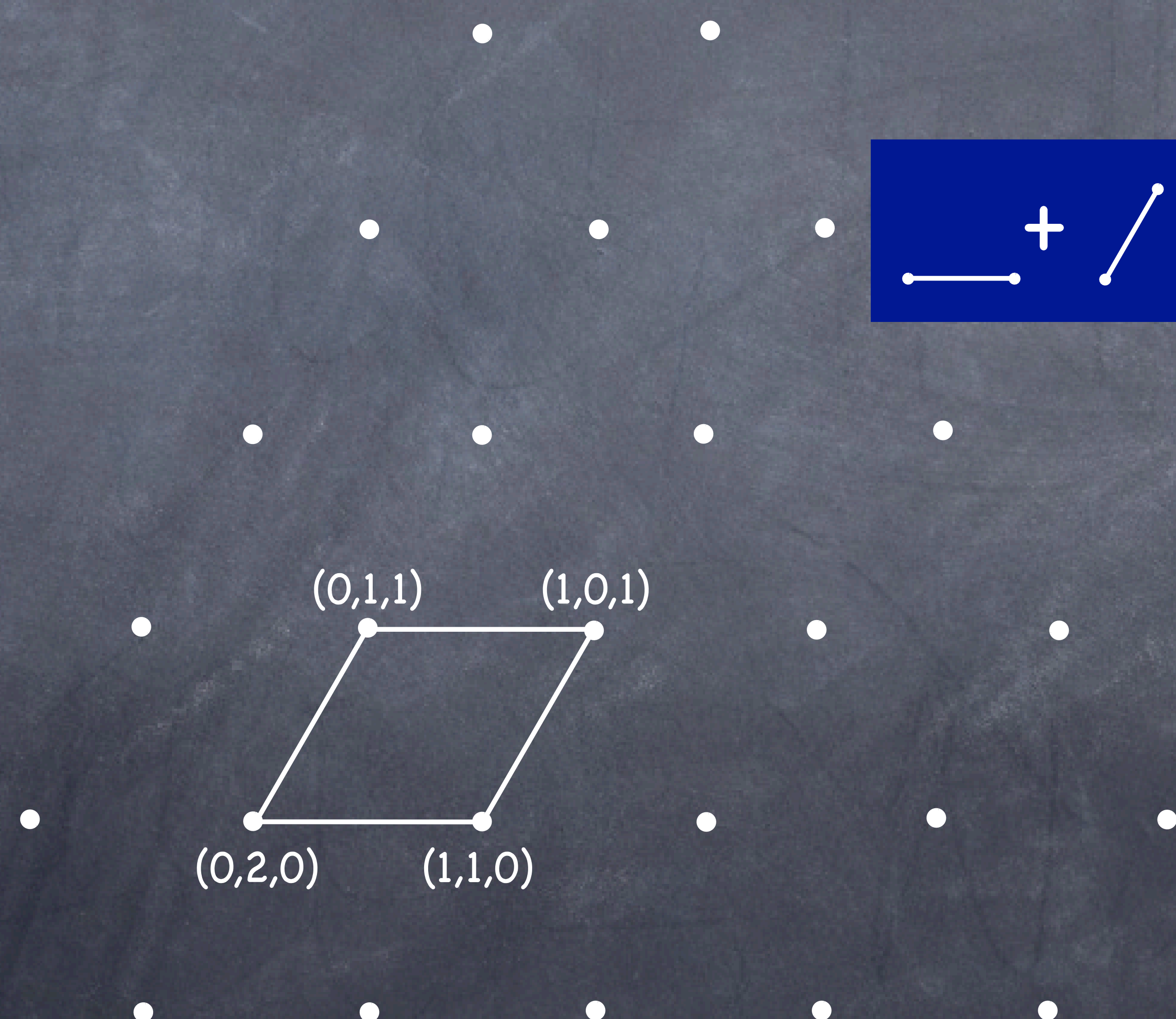


# Decompose Associahedra

## -- example: Loday's associahedra II --

affine hyperplane
x+y+z=2

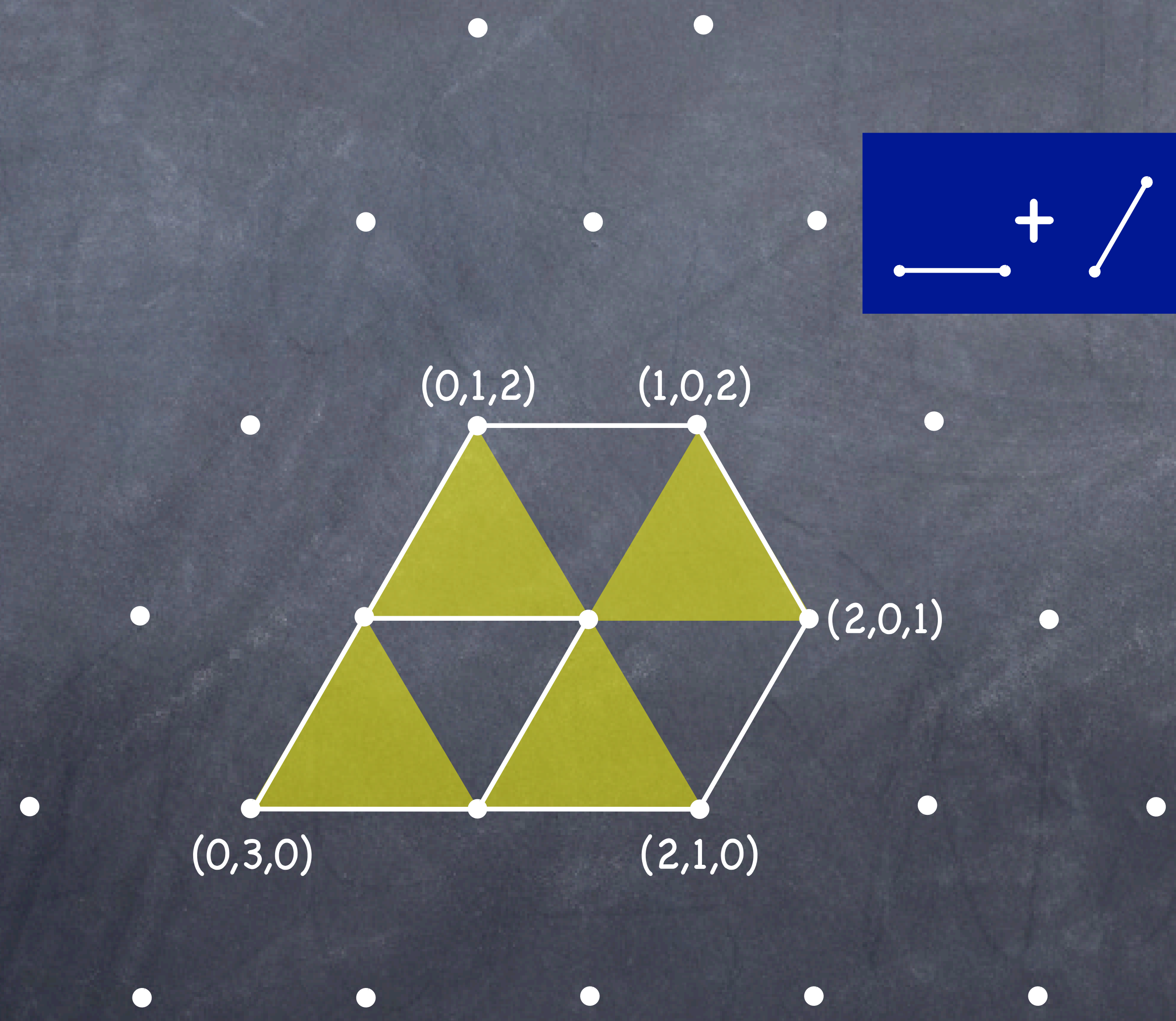


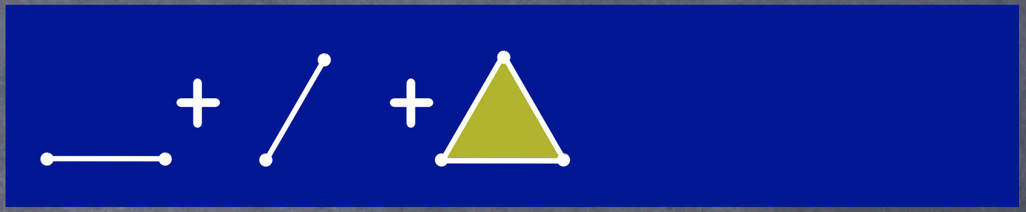


# Decompose Associahedra

## -- example: Loday's associahedra II --

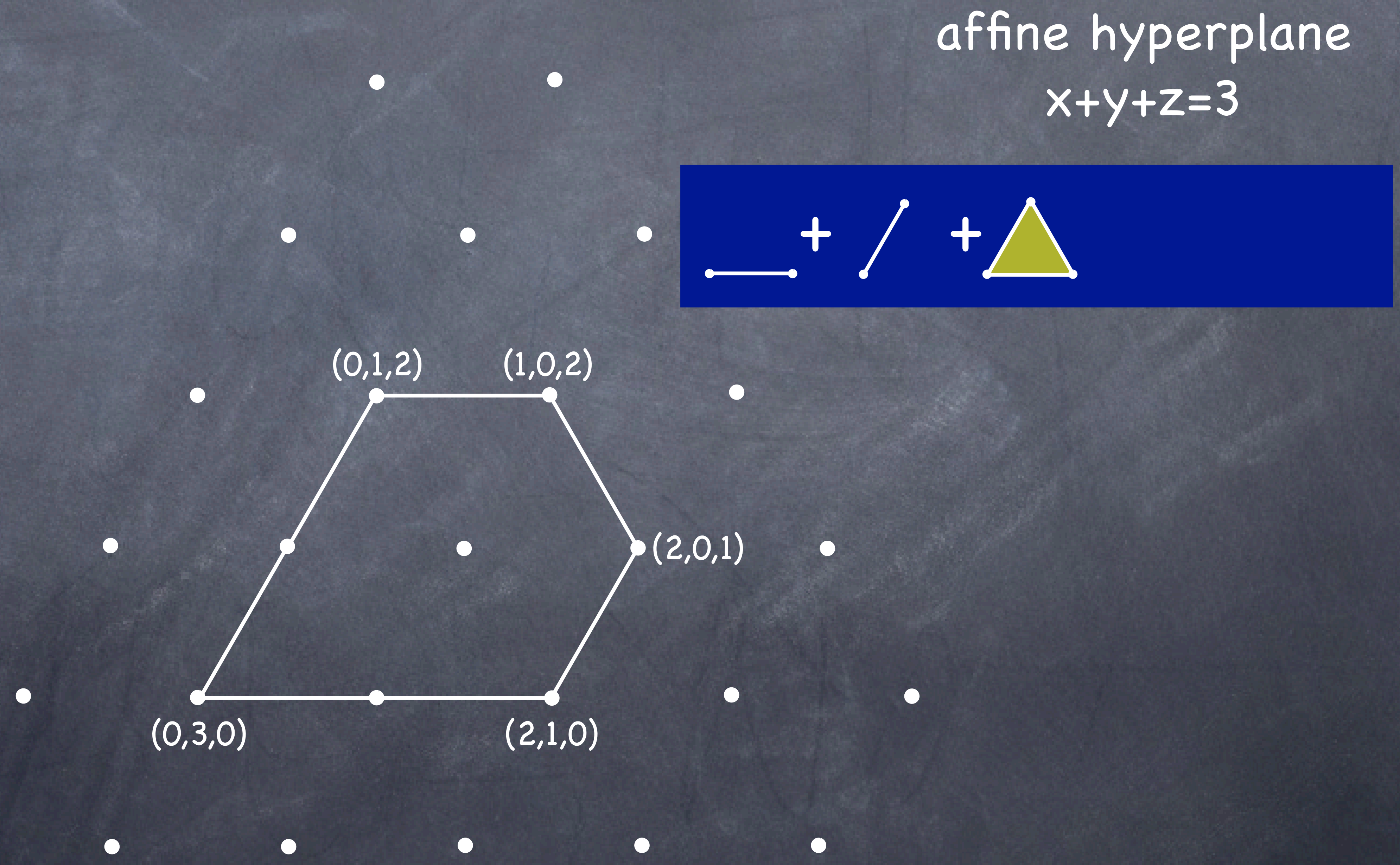
affine hyperplane
x+y+z=3



# Decompose Associahedra

## -- example: Loday's associahedra II --

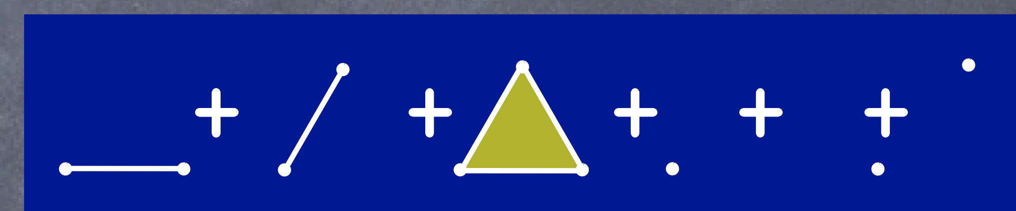


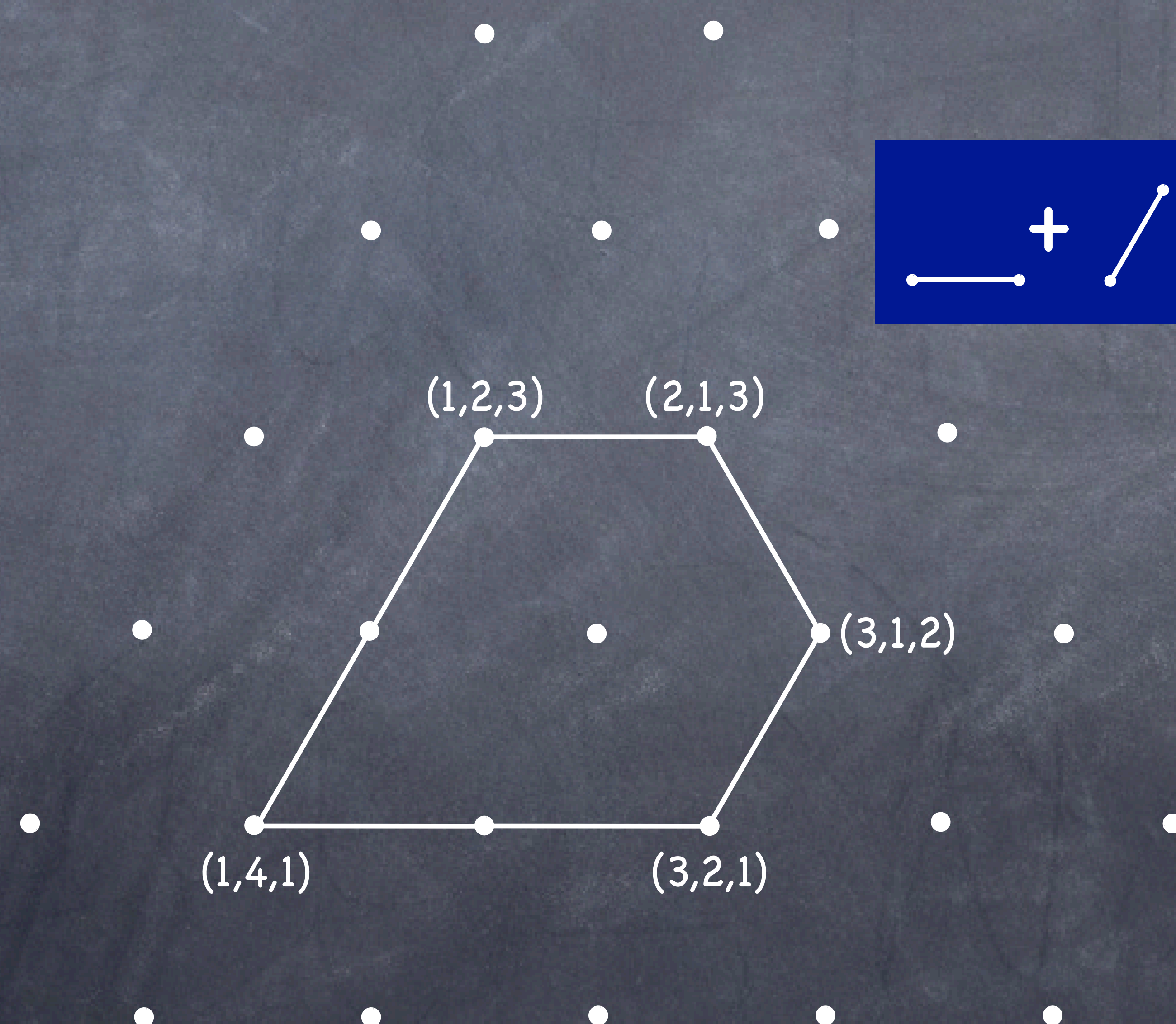
# Decompose Associahedra

## -- example: Loday's associahedra II --

affine hyperplane

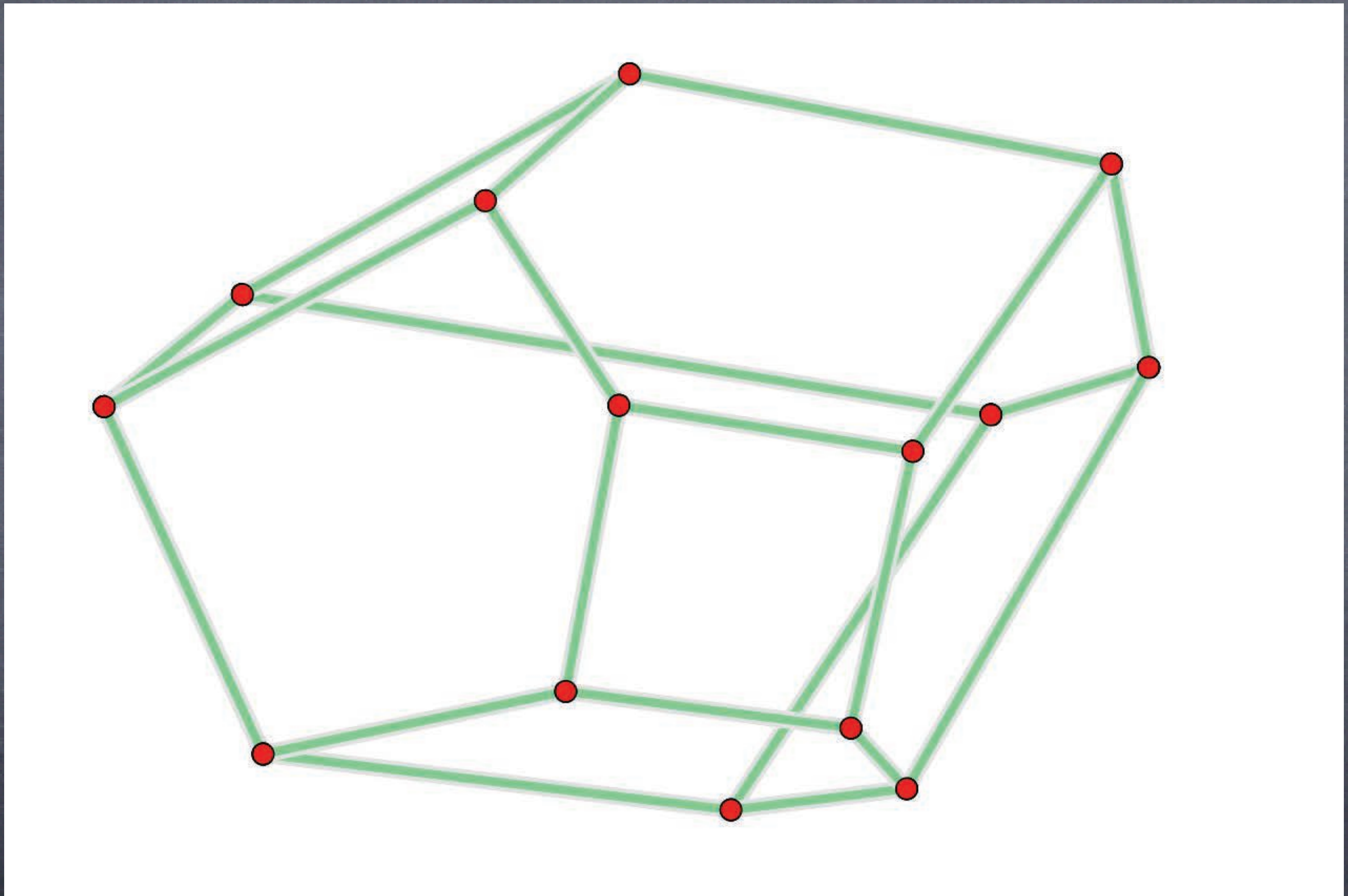
x+y+z=6





# Associahedra

## -- Loday's realisation in dimension 3 --



$$\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_{1,2} + \Delta_{2,3} + \Delta_{3,4}$$
$$+ \Delta_{1,2,3} + \Delta_{2,3,4} + \Delta_{1,2,3,4}$$

# How to … realise associahedra

## -- Loday's realization generalized --

- [Hohlweg&L., 2007]

  $2^{n-2}$ allowed labellings of (n+2)-gon with {0,1,…,n+1}

  A is "good subset" :⇔ RHS of diagonal

  Then

  $$\sum x_i = z_{[n]} = n(n+1)/2$$

  $$\sum_{i\in A} x_i \geq z_A = |A|(|A|+1)/2 \qquad \text{(A good subset)}$$

  yields H-description of associahedron.

  Furthermore:
  V-description generalising Loday's algorithm possible

# How to … realise associahedra

-- allowed labellings of (n+2)-gon --

- partition {1,…,n} into two sets:
  "Up" and "Down" with $1,n \in$ Down
- "c-labelling" of (n+2)-gon

  label one vertex "0"
  label paths starting at 0 by Up and Down
  label label remaining vertex "n+1"

# How to … realise associahedra

-- allowed labellings of (n+2)-gon --

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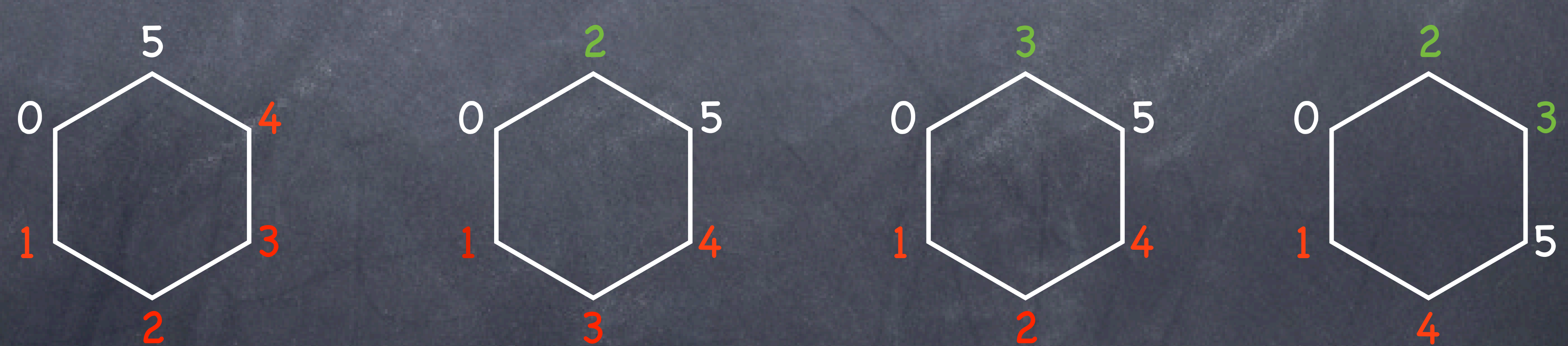
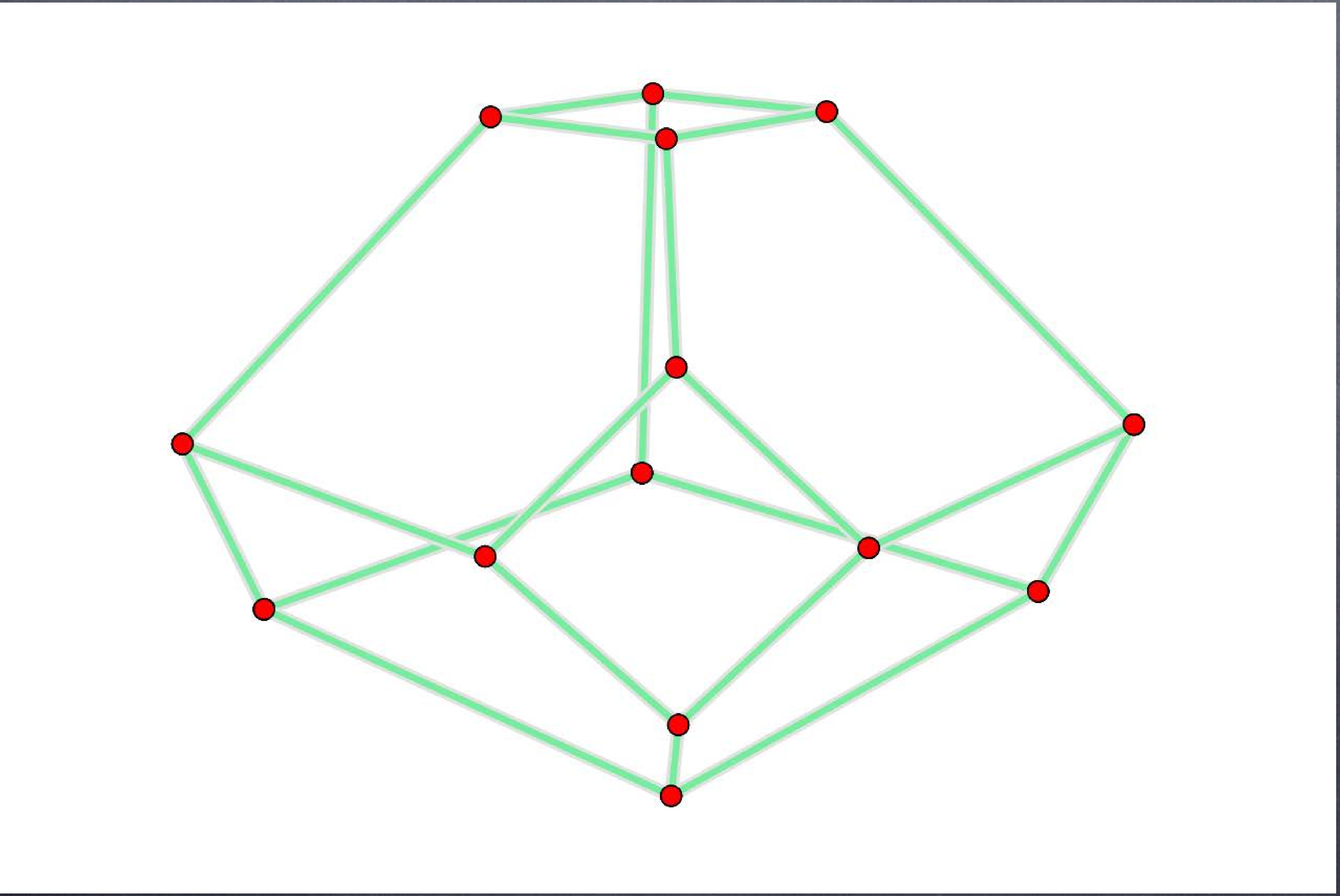
- "c-labelling" of (n+2)-gon
  - label one vertex "0"
  - label paths starting at 0 by Up and Down
  - label label remaining vertex "n+1"
- Example: n=4, $2^{4-2} = 4$ different labellings of hexagon



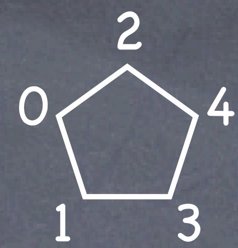

| | | | | |
|---|---|---|---|---|
| "Up" | $\varnothing$ | {2} | {3} | {2,3} |
| "Down" | {1,2,3,4} | {1,3,4} | {1,2,4} | {1,4} |

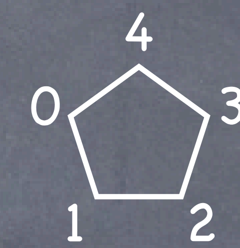
# Associahedra

-- Hohlweg&L. (Down = {1,3,4}) --



# Coordinates revisited

-- Use 

instead of 

--

$\{1,2,3\}$ 6

$\{1,2\}$ 3   $\{1,3\}$ 3   $\{2,3\}$ 3

$\{1\}$ 1   $\{2\}$ 0   $\{3\}$ 1

$z_I = \Sigma_{J \subseteq I}\, y_J$

$y_J = \Sigma_{I \subseteq J}\, (-1)^{|J-I|} z_I$

$\{1,2,3\}$

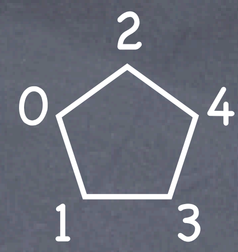
$\{1,2\}$   $\{1,3\}$   $\{2,3\}$

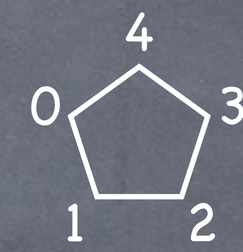
$\{1\}$   $\{2\}$   $\{3\}$

bad subset: $I=\{2\}$

tight value: $z_{\{2\}} = 0$

# Coordinates revisited

-- Use 

instead of 

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$\{1,2,3\}_6$

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$\{1,2,3\}_{-1}$

$\{1,2\}_2$ $\{1,3\}_1$ $\{2,3\}_2$

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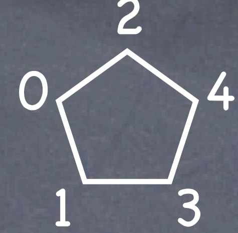
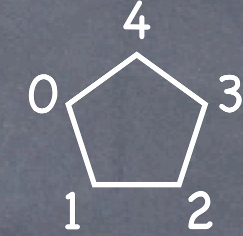
!! $y_{\{1,2,3\}} = -1$ !!

# Coordinates revisited

-- Use 

instead of 

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${}_{6}\{1,2,3\}$

${}_{3}\{1,2\}$ ${}_{3}\{1,3\}$ ${}_{3}\{2,3\}$

${}_{1}\{1\}$ ${}_{0}\{2\}$ ${}_{1}\{3\}$

bad subset: $I=\{2\}$

tight value: $z_{\{2\}} = 0$

$z_I = \Sigma_{J\subseteq I}\, y_J$

$\longleftarrow$

$\longrightarrow$

$y_J = \Sigma_{I\subseteq J}\, (-1)^{|J-I|} z_I$

$\{1,2,3\}_{-1}$

$\{1,2\}_{2}$ $\{1,3\}_{1}$ $\{2,3\}_{2}$

$\{1\}_{1}$ $\{2\}_{0}$ $\{3\}_{1}$

!! $y_{\{1,2,3\}} = -1$ !!

What is the meaning of a negative $y_I$ value??

# Minkowski decomposition

## -- definition --

- P and Q polytopes

  R is Minkowski difference P-Q of P and Q

    There is a polytope R such that R+Q = P
    (pitfall: not always defined!!)

# Minkowski decomposition

-- definition --

- P and Q polytopes

  R is Minkowski difference P-Q of P and Q

    There is a polytope R such that $R+Q = P$
    (pitfall: not always defined!!)

- "Minkowski decomposition of P"

    Write P as Minkowski sums and differences of polytopes $Q_i$

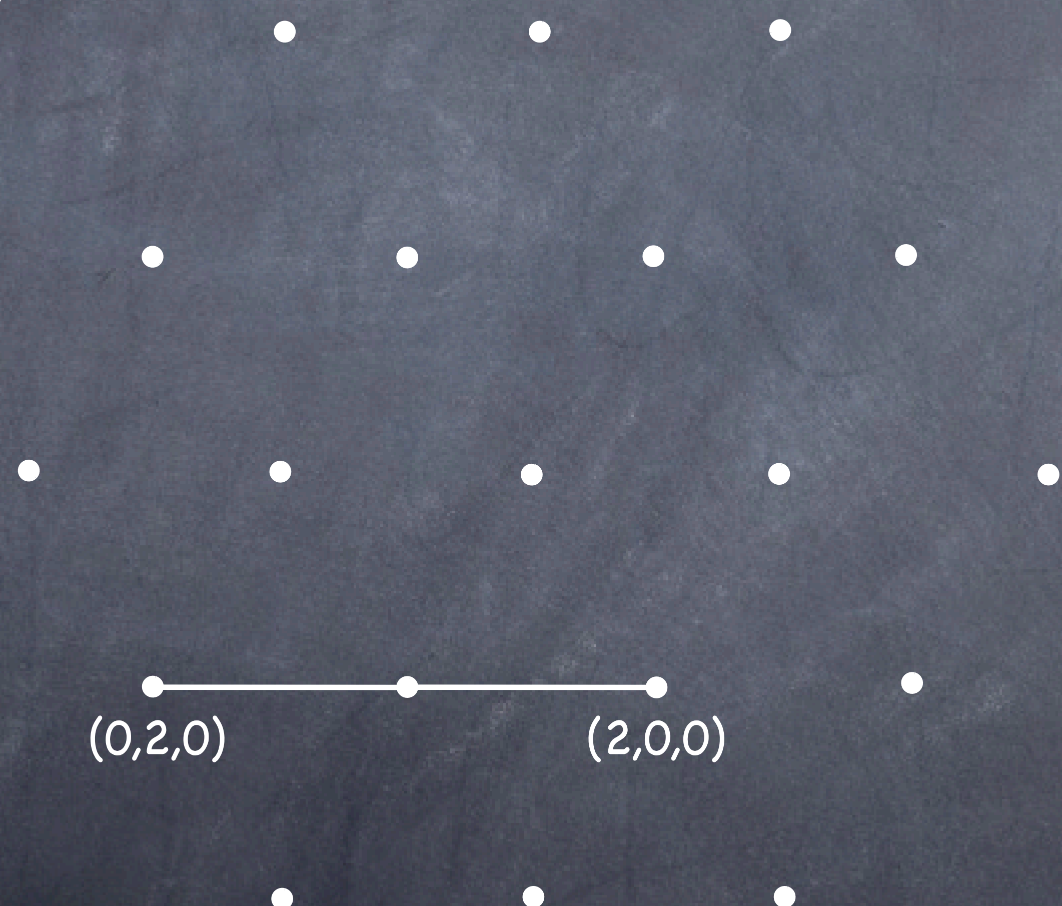
- Idea:

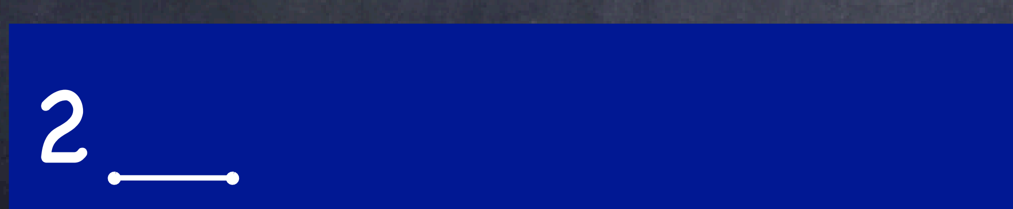
    Use Minkowski decompositions of generalised permutahedra for positive & negative $y_I$-values

# Minkowski decomposition

## -- associahedron with Down = {1,3} --

affine hyperplane
x+y+z=2

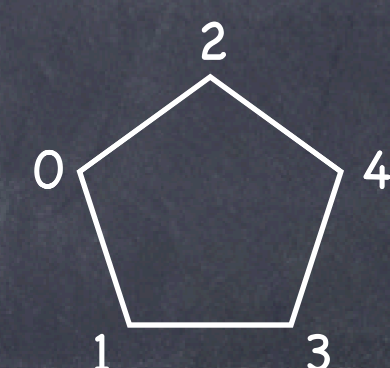



$\{1,2,3\}_{-1}$

$\{1,2\}_{2}$ $\{1,3\}_{1}$ $\{2,3\}_{2}$

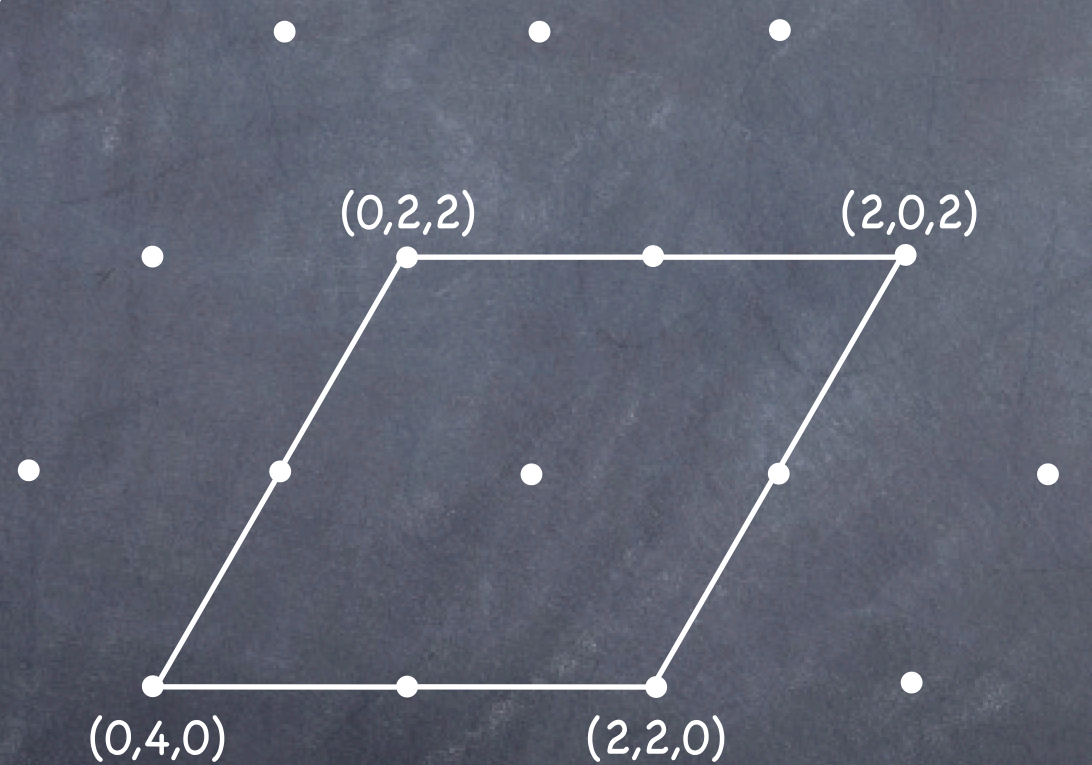
$\{1\}_{1}$ $\{2\}_{0}$ $\{3\}_{1}$



# Minkowski decomposition

-- associahedron with Down = {1,3} --

affine hyperplane
x+y+z=4




$\{1,2,3\}_{-1}$

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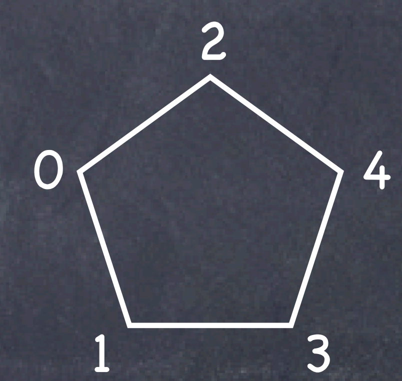
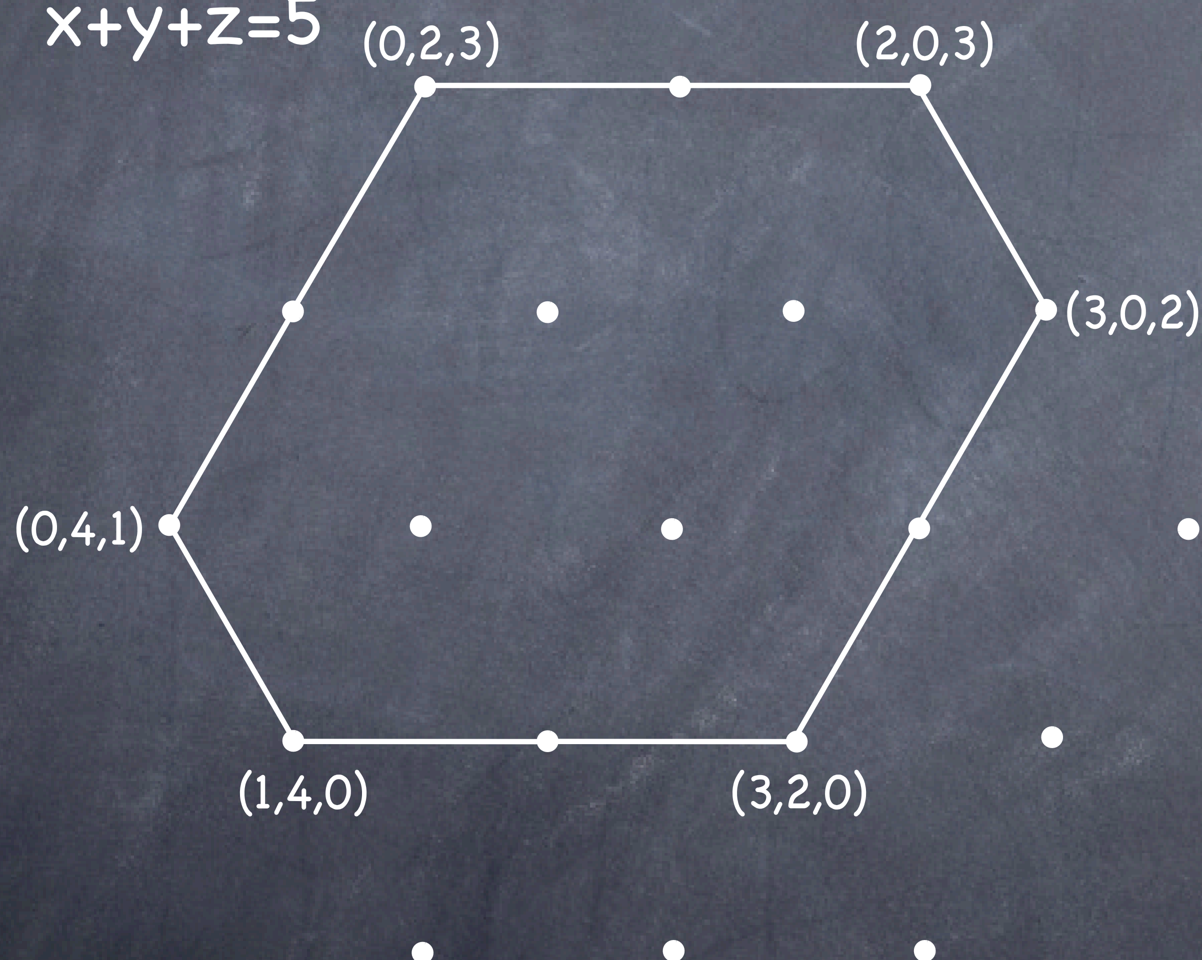



# Minkowski decomposition

## -- associahedron with Down = {1,3} --

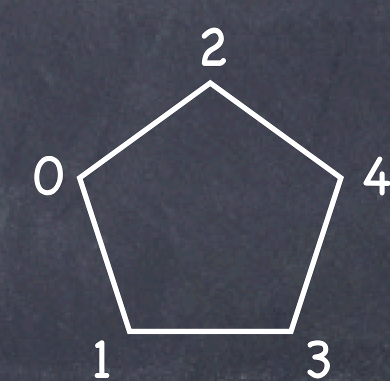
affine hyperplane
x+y+z=5

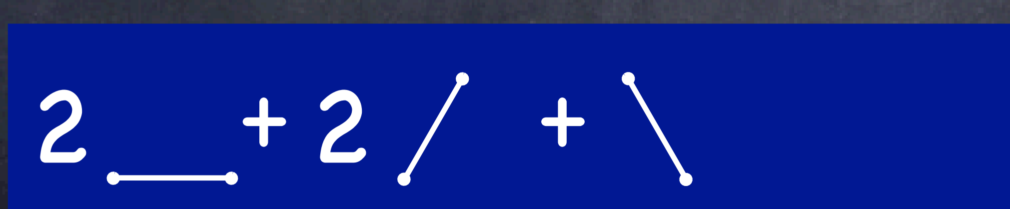



$\{1,2,3\}_{-1}$

$\{1,2\}_{2}$ $\{1,3\}_{1}$ $\{2,3\}_{2}$

$\{1\}_{1}$ $\{2\}_{0}$ $\{3\}_{1}$

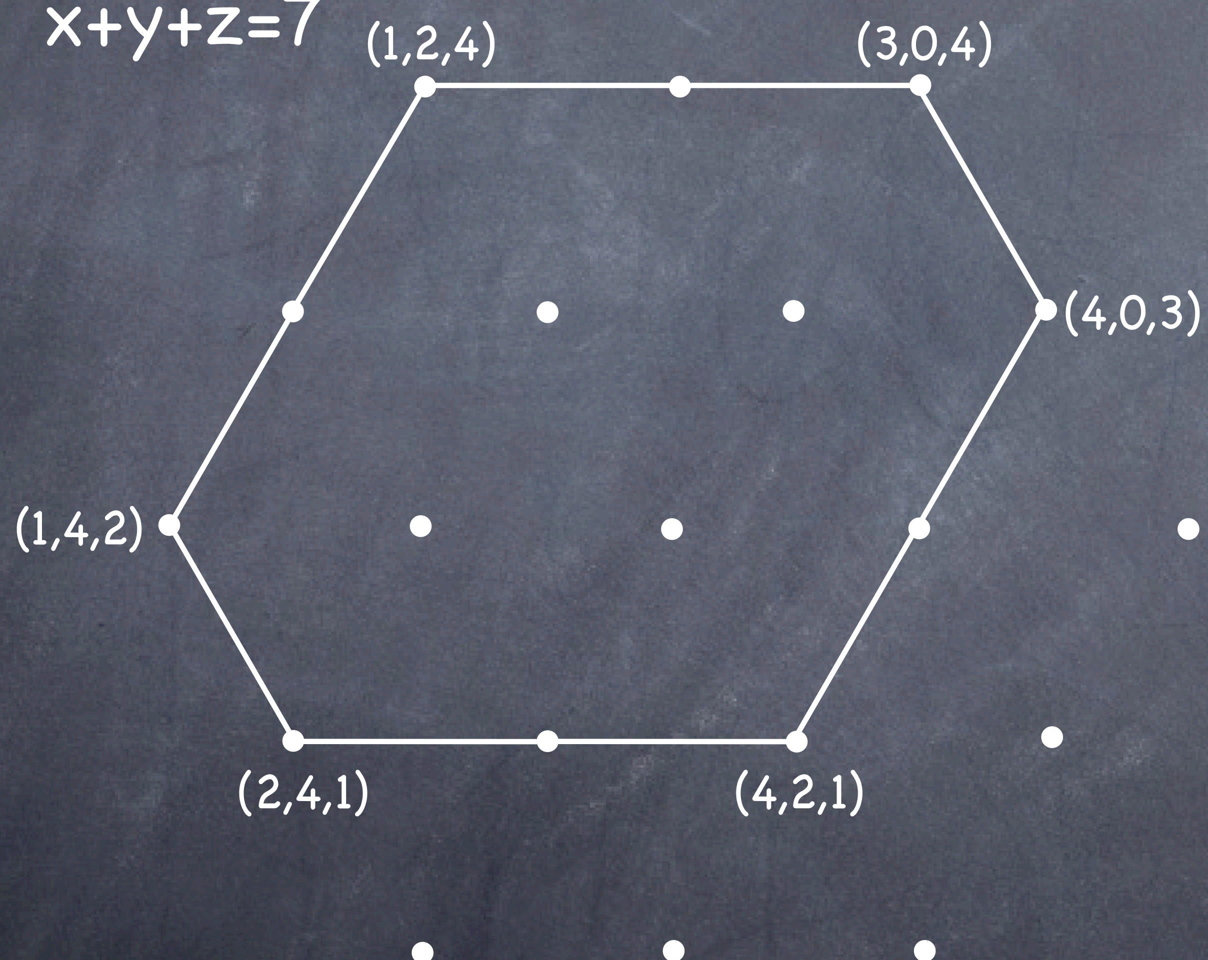


# Minkowski decomposition

## -- associahedron with Down = {1,3} --

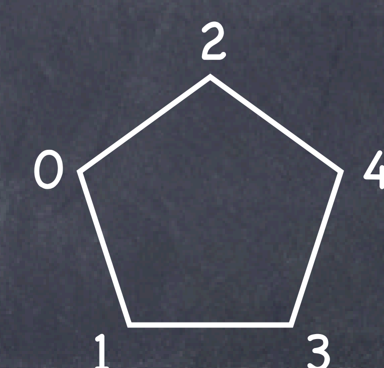
affine hyperplane
$x+y+z=7$

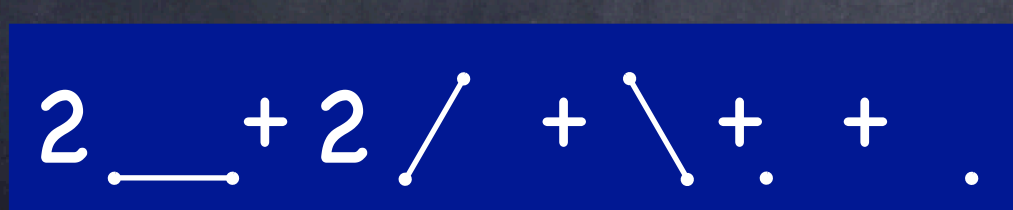



$\{1,2,3\}_{-1}$

$\{1,2\}_{2}$ $\{1,3\}_{1}$ $\{2,3\}_{2}$

$\{1\}_{1}$ $\{2\}_{0}$ $\{3\}_{1}$

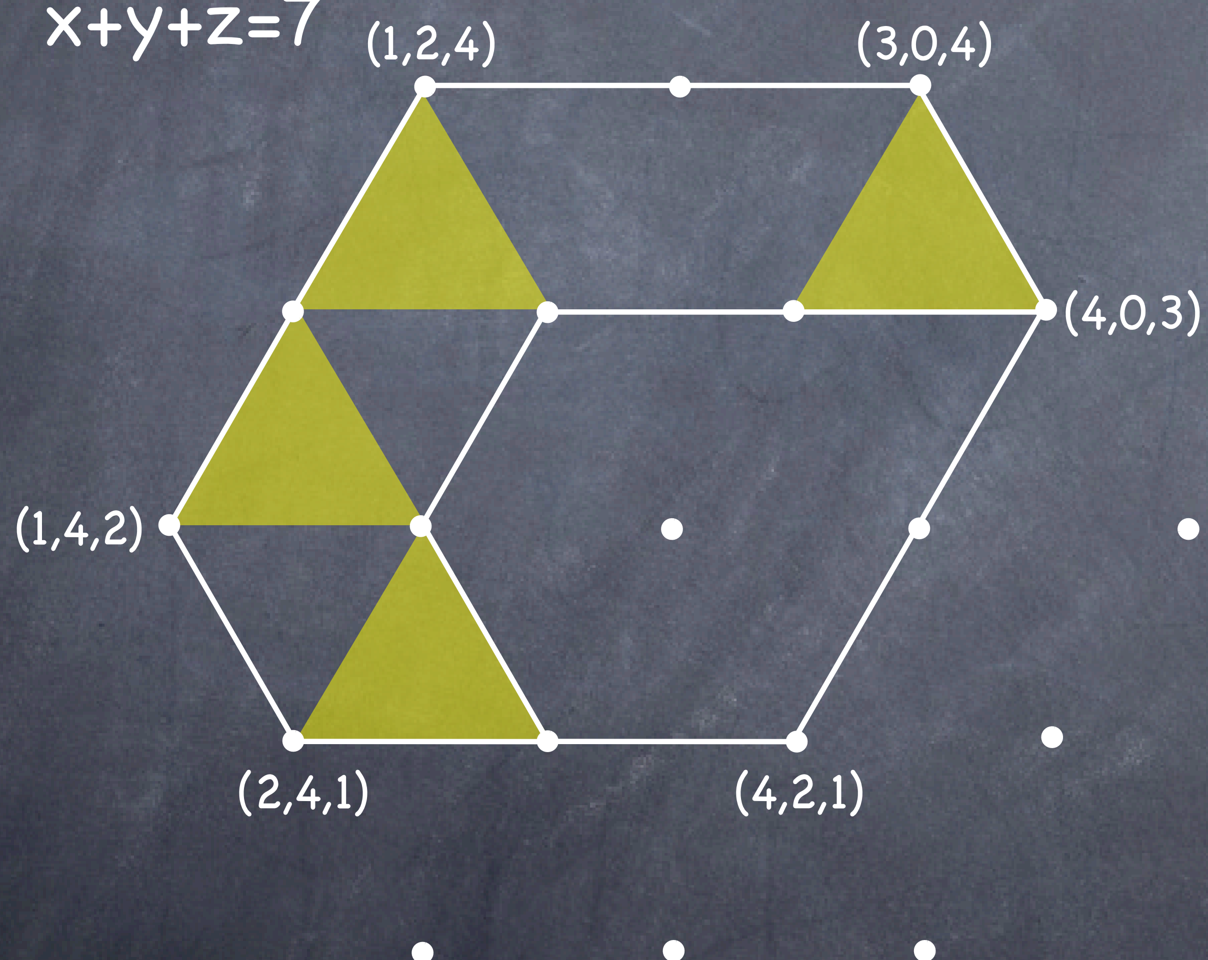


# Minkowski decomposition

## -- associahedron with Down = {1,3} --

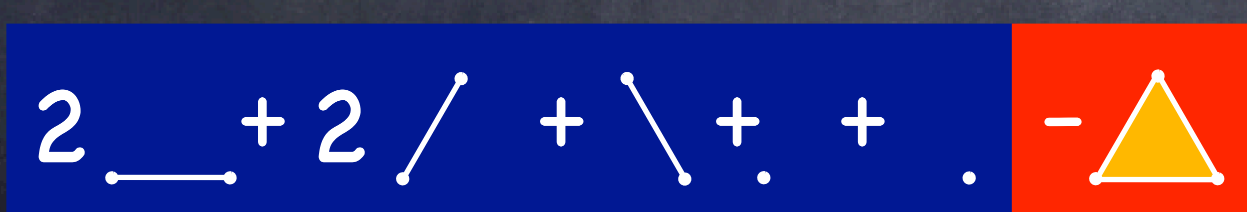
affine hyperplane
x+y+z=7

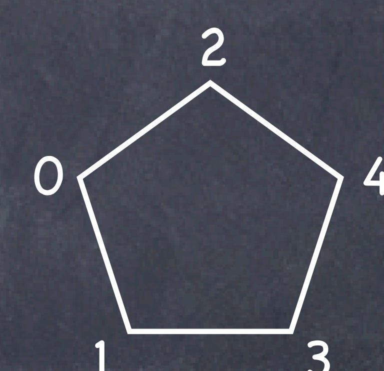



$\{1,2,3\}_{-1}$

$\{1,2\}_{2}$ $\{1,3\}_{1}$ $\{2,3\}_{2}$

$\{1\}_{1}$ $\{2\}_{0}$ $\{3\}_{1}$

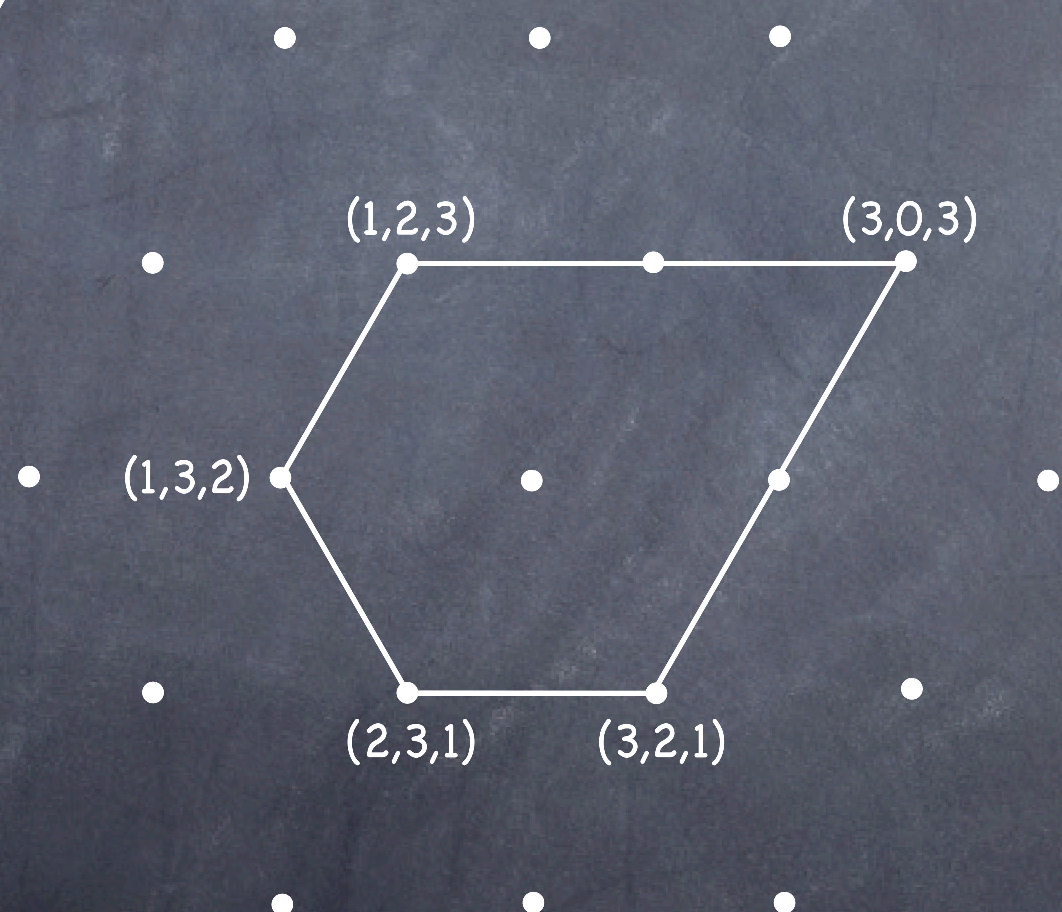


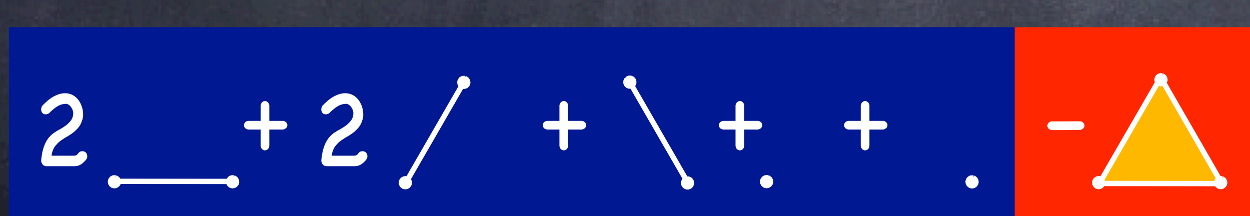



# Minkowski decomposition

## -- associahedron with Down = {1,3} --

affine hyperplane
$x+y+z=6$

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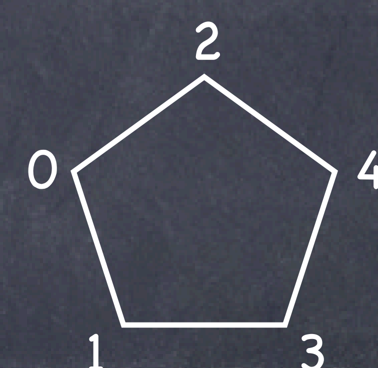

# $P(\{z_I\})$ & decompositions

Theorem [Ardila,Benedetti&Doker, 2010]

Every generalised permutahedron $P(\{z_I\})$ has a unique Minikowski decomposition

$$P(\{z_I\}) = \Sigma_{J\subseteq[n]}\ y_J\Delta_J$$

where $y_J = \Sigma_{I\subseteq J}\,(-1)^{|J-I|}\ z_I$

# $P(\{z_I\})$ & decompositions

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Proof:

Set $z_I^- := \Sigma_{J\subseteq I; y_J<0}\ (-y_J)$ and $z_I^+ := \Sigma_{J\subseteq I; y_J\geq 0}\ y_J$.

By inclusion-exclusion $z_I + z_I^- = z_I^+$ which yields

$$P(\{z_I\}) + P(\{z_I^-\}) = P(\{z_I^+\})$$

since $P(\{a_I+b_I\}) = P(\{a_I\}) + P(\{b_I\})$.

# $P(\{z_I\})$ & decompositions

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Corollary:

$y_I$-values for associahedra of Hohlweg&Lange computable by Möbius inversion from complete set of tight $z_I$-values

# $y_I$-coord's for associahedra

## -- Statement of results --

- $z_I$-values for redundant inequalities computable from "good subsets S" using "Up and Down interval decomposition" of I
- "type" of interval decomposition simplifies $y_I$-computation:

  I of "type (1,l)": $y_I = (-1)^{|I-I_1|} (z_{I_1} - z_{I_2} - z_{I_3} + z_{I_4})$

  I of "type (k,l), k>1": $y_I = 0$

- Loday-type formula for $y_I$-values:

  $I \neq \{u\}$ of type (1,l): $y_I = (-1)^{|I-D_1|} K_\gamma \cdot K_\Gamma$

  $I = \{u\}$ of type (1,l): $y_I = (-1)^{|I-D_1|} (K_\gamma \cdot K_\Gamma - (n+1))$

  I of type (k,l), k>1: $y_I = 0$

  $K_\gamma$ and $K_\Gamma$ : "signed lengths" on boundary of (n+2)-gon

# Up&Down intervals

-- Up and Down interval decomposition --

Definition [L., 2011]

- open down interval $(d_i,d_j)$
  all numbers $k \in$ Down s.t. $d_i < k < d_j$

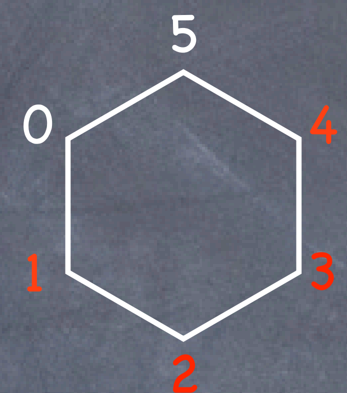
- closed up interval $[u_i,u_j]$
  all numbers $k \in$ Up s.t. $u_i \leq k \leq u_j$

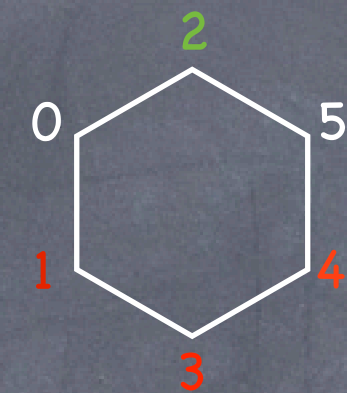
- Up and Down interval decomposition of $I \subseteq [n]$
  family of maximal closed up intervals of $I$ "nested" in maximal open (down) intervals of $I$

type of decomposition:
(#down intervals, #up intervals)

# Up&Down intervals

-- examples --




| | | |
|---|---|---|
| "Up" | $\varnothing$ | $\{2\}$ |
| "Down" | $\{1,2,3,4\}$ | $\{1,3,4\}$ |

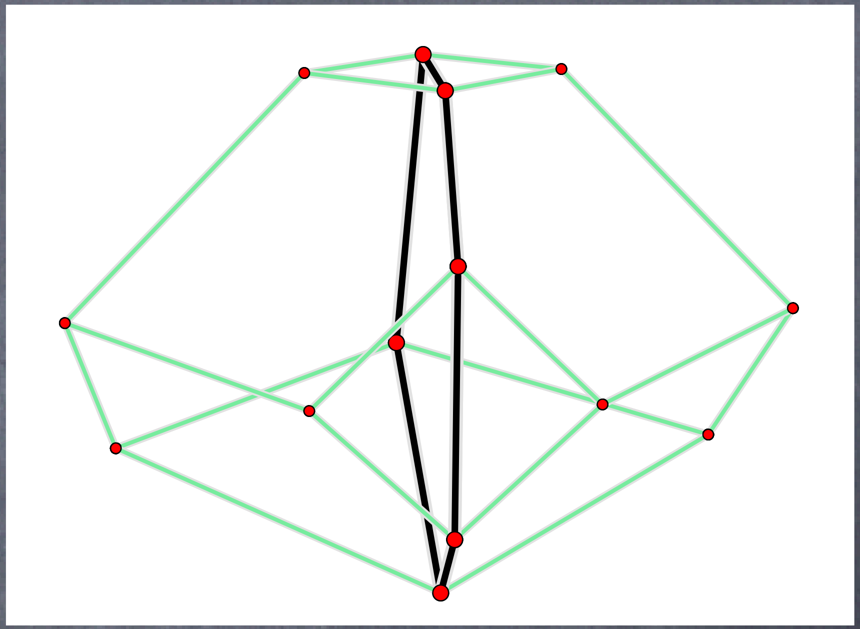
no up-intervals
down-intervals:
$\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$
$\{1,2\}$, $\{2,3\}$, $\{3,4\}$
$\{1,2,3\}$, $\{2,3,4\}$
$\{1,2,3,4\}$

only up-interval $\{2\}$
down-intervals:
$\varnothing$, $\{1\}$, $\{3\}$, $\{4\}$, $\{1,3\}$,
$\{3,4\}$, $\{1,3,4\}$
decomposition type of
$\{2\}$,$\{2,3\}$, $\{1,4\}$, $\{2,4\}$????

# Cyclohedra

## -- revisit defintion of generalised permutahedra --

- Cyclohedra ("type B generalised associahedra") can be realised using certain associahedra
- Minkowski decomposition into dilated faces of standard simplex à la Ardila/Benedetti/Doker?



# Cyclohedra

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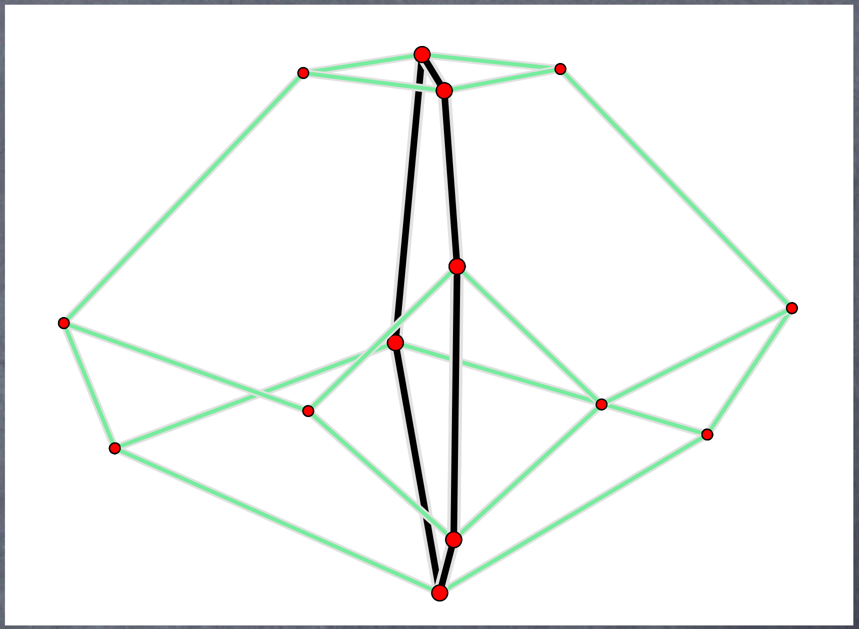


- Minkowski decomposition into dilated faces of standard simplex à la Ardila/Benedetti/Doker?

  No! Compute $y_I$-coordinates and compare resulting polytope with cyclohedron

- Postnikov and Postnikov, Reiner & Williams: "generalised permutahedra $P(\{z_I\})$" are in the deformation cone of classical permutahedron!

# Open Problems

- Feasible $z_I$- and $y_I$-coordinates?
- Lattice points of associahedra?
- Relation to brick polytopes?
- Minkowski decompositions for other types?
- Implications for cluster algebras?
- Formulae in terms of Coxeter group of type A?

- [Ardila,Benedetti&Doker]:
Matroid polytopes and their volume,
FPSAC/Discrete&Compututational Geometry, 2009/10

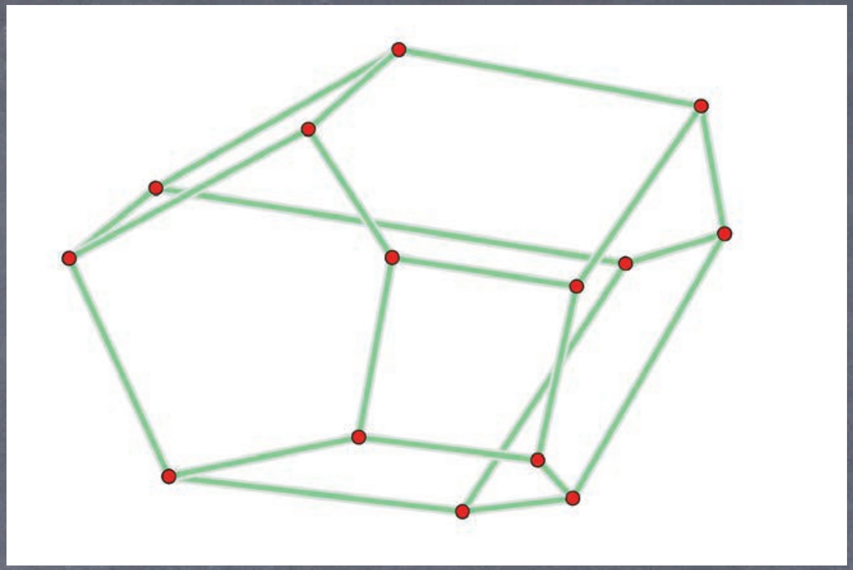
- [Hohlweg&L., 2007]:
Realizations of the associahedron and cyclohedron,
Discrete&Computational Geometry, 2007

- [L., 2011]
Minkowski decompositions of associahedra and the computation of Möbius inversion,
arXiv (abstracts: FPSAC 2011 & CCCG 2011)

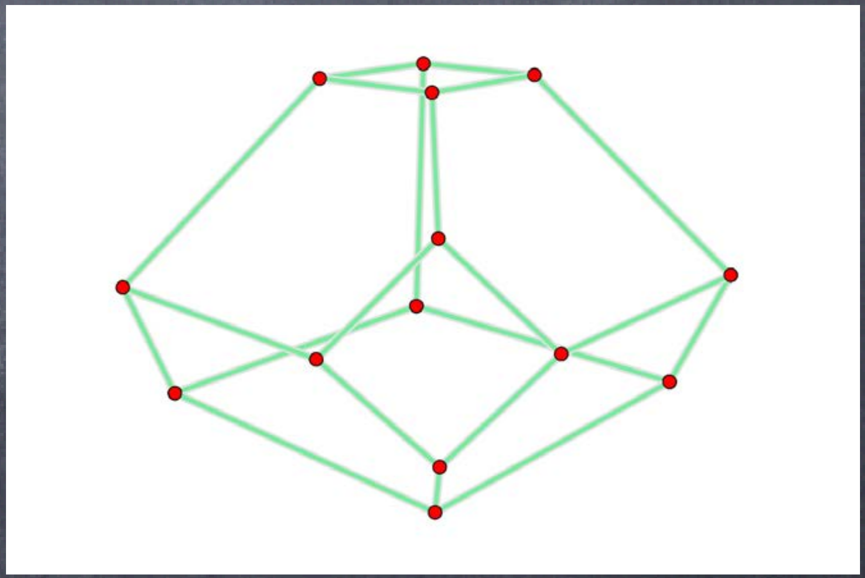
- [Loday, 2004]
Realization of the Stasheff polytope,
Archiv der Mathematik, 2004

- [Postnikov]
Permutahedra, associahedra, and beyond,
International Mathematical Research Notices, 2009

- [Postnikov, Reiner, Williams]
Faces of generalised permutahedra,
Documenta Mathematica, 2008



$$\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_{1,2} + \Delta_{2,3} + \Delta_{3,4} + \Delta_{1,2,3} + \Delta_{2,3,4} + \Delta_{1,2,3,4}$$



$$\Delta_1 + \Delta_3 + \Delta_4 + 3\Delta_{1,2} + \Delta_{1,3} + 2\Delta_{2,3} + \Delta_{3,4} + \Delta_{1,3,4} + 2\Delta_{2,3,4} - (\Delta_2 + \Delta_{1,2,3} + \Delta_{1,2,3,4})$$