Minkowski decompositions of associahedra into faces of a standard simplex

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Agenda

- What is... a permutahedron/associahedron
- How to... realise associahedra
- What is... a Minkowski decomposition
- How to... decompose associahedra
Permutahedra

-- definition --

idea: apply action of $\Sigma_n$ on $\mathbb{R}^n$ to generic point

$\bigcup$ convex hull of points

$\{ (\sigma(1),\ldots,\sigma(n)) \mid \sigma \in \Sigma_n \}$

affine hyperplane

$x_1 + x_2 + x_3 = 6$
Permutahedra

-- definition --

idea: apply action of $\Sigma_n$ on $\mathbb{R}^n$ to generic point

- convex hull of points
  $$\{ (\sigma(1), \ldots, \sigma(n)) \mid \sigma \in \Sigma_n \}$$

- $H$-representation
  $$\sum_{i \in [n]} x_i = \frac{1}{2} n(n+1)$$
  $$\sum_{i \in K} x_i \geq \frac{1}{2} |K|(|K|+1)$$
  for $\emptyset \neq K \subset [n]$
Permutahedra

-- definition --

idea: apply action of Σₙ on Rⁿ to generic point

⊙ convex hull of points

\{ (σ(1),...,σ(n)) | σ ∈ Σₙ \}

⊙ H-representation

\[ Σ_{i∈[n]} x_i = \frac{1}{2}n(n+1) \]

\[ Σ_{i∈K} x_i ≥ \frac{1}{2}|K|(|K|+1) \]

for \( \emptyset ≠ K ⊂ [n] \)

half space

\[ x_1 + x_2 ≥ 3 \]

affine hyperplane

\[ x_1 + x_2 + x_3 = 6 \]

half space

\[ x_3 ≥ 1 \]
generalised Permutahedra

-- definition, $z_I$-coordinates --

idea: change permutahedron's right-hand sides

◊ $H$-representation

$$\sum_{i \in [n]} x_i = z[n]$$
$$\sum_{i \in I} x_i \geq z_I \quad \text{for } \emptyset \neq I \subset [n]$$
(want all "redundant" $z_I$-values tight)

◊ $z_I$-coordinates

vector of all $z_I$-values
(redundancies possible; choose all $z_I$-values tight)

◊ $\mathcal{P}([z_I])$

generalised permutahedron with given $z_I$-coordinates
What is... Minkowski sum

-- definition --

**P and Q polytopes**

Minkowski sum $P+Q$ is the polytope

\[ \{ p+q \mid p \in P \text{ and } q \in Q \} \]
What is... Minkowski sum
-- definition --

P and Q polytopes

Minkowski sum \( P + Q \) is the polytope
\[
\{ p + q \mid p \in P \text{ and } q \in Q \}
\]

Example (edges of standard simplex):
\[
\begin{align*}
(0,0,1) + (0,1,0) + (1,0,0) &= (0,1,1) + (1,0,1) \\
(0,1,0) &+ (0,1,0) + (1,0,0) = (0,2,0) + (1,1,0)
\end{align*}
\]
What is... Minkowski sum

-- definition --

- P and Q polytopes

Minkowski sum $P+Q$ is the polytope

\[ \{ p+q \mid p \in P \text{ and } q \in Q \} \]

Example (edges of standard simplex):

\[
\begin{align*}
(0,0,1) &+ (0,1,0) = (1,1,0) &+ (0,2,0) = (2,1,0) \\
(0,1,0) &+ (1,0,0) = (1,1,0) &+ (1,1,0) = (2,2,0) \\
(0,0,1) &+ (1,0,0) = (1,1,0) &+ (0,2,0) = (2,2,0)
\end{align*}
\]
Generalised Permutahedra

-- $y_I$-coordinates --

Theorem [Postnikov, 2009]

Every Minkowski sum of dilated faces of a standard simplex yields a generalised permutahedron

$y_I$-coordinates (à la Postnikov)

vector of dilation factors $y_I$ for $\emptyset \subset I \subseteq [n]$

($y_I \geq 0; y_I = 0 \iff$ face not used)

Observation: $z_I$- and $y_I$-vectors have same size

Are $z_I$- and $y_I$-coord’s related? If yes, how?
Relate $y_I$- & $z_I$-coordinates

-- $y_I$- & $z_I$-coordinates as functions --

$y_I$-coordinates as function on Boolean lattice

(geometric constraints on $z_I$)

$z_I$-coordinates as function on Boolean lattice

($y_I \geq 0$)

[Postnikov, 2009]:

$$z_I = \sum_{J \subseteq I} y_J$$

!! $y_J \geq 0$ !!

$y_I = \{1\}$

$y_I = \{2\}$

$y_I = \{3\}$

$y_I = \{1,2\}$

$y_I = \{1,3\}$

$y_I = \{2,3\}$

$y_I = \{1,2,3\}$

$z_I = \{1,2,3\}$

$z_I = \{1,2\}$

$z_I = \{1,3\}$

$z_I = \{2,3\}$

$z_I = \{1,2,3\}$

$z_I = \{1,2\}$

$z_I = \{1,3\}$

$z_I = \{2,3\}$

$z_I = \{1,2,3\}$
Relate $y_I$- & $z_I$-coordinates

-- $y_I$- & $z_I$-coordinates as functions --

$z_I$-coordinates as function on Boolean lattice (geometric constraints on $z_I$)

$y_I$-coordinates as function on Boolean lattice ($y_I \geq 0$)

[Postnikov, 2009]:

$z_I = \sum_{J \subseteq I} y_J$

Möbius inversion

$y_J = \sum_{I \subseteq J} (-1)^{|J-I|} z_I$

!! $y_J \geq 0$ !!
Relate $y_I$- & $z_I$-coordinates

--- $y_I$- & $z_I$-coordinates as functions ---

$z_I$-coordinates as function on Boolean lattice
(geometric constraints on $z_I$)

$y_I$-coordinates as function on Boolean lattice
($y_I \geq 0$)

[Postnikov, 2009]:

$z_I = \sum_{J \subseteq I} y_J$

Möbius inversion

$y_J = \sum_{I \subseteq J} (-1)^{|J-I|} z_I$

$1$ $2$ $3$

$\{1\} \{2\} \{3\}$

$1$ $1$ $1$

$1$ $1$ $1$

$1$ $1$ $1$

$1$ $1$ $1$

$1$ $1$ $1$

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$1$ $1$ $1$

$1$ $1$ $1$

$0$ $0$ $0$
What is... an associahedron

-- combinatorial description --

- combinatorics of CW-complex (Stasheff)
- vertices = triangulations of (n+2)-gon
- k-face = triangulation minus k diagonals
- can be realised as (n-1)-dim polytope
- polytopal realisations were given by Milnor (unpublished), Lee, Haiman, Sternberg & Shnider and Stasheff & Shnider, Loday...
Realise Associahedra

-- example (Shnider, Sternberg & Stasheff, Loday) --

Loday: Computes coord's (planar binary trees)

equivalently:
label (n+2)-gon cyclicly
decreasing with \{0, ..., n+1\}

n=3
labelled pentagon
2-dim associahedron
realised in R^3

affine hyperplane
\(x_1 + x_2 + x_3 = 6\)
Realise Associahedra

--- example (Shnider, Sternberg & Stasheff, Loday) ---

Loday: Computes coord's (planar binary trees)

\[ x_1 + x_2 + x_3 = 6 \]

equivalently:

label \((n+2)\)-gon cyclicly decreasing with \(\{0, \ldots, n+1\}\)

\[ n=3 \]

labelled pentagon

2-dim associahedron realised in \(\mathbb{R}^3\)
Realise Associahedra

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Realise Associahedra

-- example (Shnider, Sternberg & Stasheff, Loday) --

Loday: Computes coord's (planar binary trees)

affine hyperplane $x_1 + x_2 + x_3 = 6$

equivalently: label $(n+2)$-gon cyclicly decreasing with $\{0, \ldots, n+1\}$

$n=3$
labelled pentagon 2-dim associahedron realised in $\mathbb{R}^3$
Realise Associahedra

-- example (Shnider, Sternberg & Stasheff, Loday) --

Loday: Computes coord's (planar binary trees)

\begin{align*}
  x_1 &= 1 \cdot 1 = 1 \\
  x_2 \\
  x_3 \\
\end{align*}

Equivalent:

- label \((n+2)\)-gon cyclicly decreasing with \(\{0, \ldots, n+1\}\)

For \(n=3\):

- labelled pentagon
- 2-dim associahedron realised in \(\mathbb{R}^3\)
Realise Associahedra

-- example (Shnider, Sternberg & Stasheff, Loday) --

Loday: Computes coord's (planar binary trees)

equivalently:
label (n+2)-gon cyclicly decreasing with \{0, \ldots, n+1\}

n=3
labelled pentagon
2-dim associahedron
realised in \( \mathbb{R}^3 \)
Realise Associahedra

-- example (Shnider, Sternberg & Stasheff, Loday) --

Loday: Computes coord's (planar binary trees)

affine hyperplane
\[ x_1 + x_2 + x_3 = 6 \]

\( x_1 = 1 \cdot 1 = 1 \)
\( x_2 \)
\( x_3 \)

equivalently:
label \( (n+2) \)-gon cyclicly decreasing with \( \{0, \ldots, n+1\} \)
n = 3
labelled pentagon
2-dim associahedron realised in \( \mathbb{R}^3 \)
Realise Associahedra

-- example (Shnider, Sternberg & Stasheff, Loday) --

Loday: Computes coord's (planar binary trees)

equivalently:
label (n+2)-gon cyclicly decreasing with \{0,...,n+1\}

n=3
labelled pentagon
2-dim associahedron realised in \mathbb{R}^3

affine hyperplane
\[ x_1 + x_2 + x_3 = 6 \]

\[ x_1 = 1 \cdot 1 = 1 \]

\[ x_2 \]

\[ x_3 \]
Realise Associahedra

-- example (Shnider, Sternberg & Stasheff, Loday) --

Loday: Computes coord's (planar binary trees)

\[ x_1 = 1 \cdot 1 = 1 \]
\[ x_2 = 2 \cdot 2 = 4 \]
\[ x_3 \]

Equivalently:

label \((n+2)\)-gon cyclicly decreasing with \(\{0, \ldots, n+1\}\)

\[ n=3 \]

labelled pentagon 2-dim associahedron realised in \(\mathbb{R}^3\)

Affine hyperplane
\[ x_1 + x_2 + x_3 = 6 \]
Realise Associahedra

-- example (Shnider, Sternberg & Stasheff, Loday) --

Loday: Computes coord’s (planar binary trees)

affine hyperplane
\[ x_1 + x_2 + x_3 = 6 \]

\[ x_1 = 1 \cdot 1 = 1 \]
\[ x_2 = 2 \cdot 2 = 4 \]
\[ x_3 \]

equivalently:
label (n+2)-gon cyclicly decreasing with \{0, ..., n+1\}

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labelled pentagon
2-dim associahedron
realised in \( \mathbb{R}^3 \)
Realise Associahedra

-- example (Shnider, Sternberg & Stasheff, Loday) --

Loday: Computes coord's (planar binary trees)

\[ x_1 = 1 \cdot 1 = 1 \]
\[ x_2 = 2 \cdot 2 = 4 \]
\[ x_3 \]

affine hyperplane
\[ x_1 + x_2 + x_3 = 6 \]

equivalently:
label \((n+2)\)-gon cyclicly decreasing with \(\{0, \ldots, n+1\}\)
n=3
labelled pentagon
2-dim associahedron realised in \(\mathbb{R}^3\)

\(0 \quad 4 \quad 3\)
\(1 \quad 2 \quad 0\)
\(4 \quad 3 \quad 2\)
\(1 \quad 2 \quad 0\)
\(1 \quad 4 \quad 1\)
Realise Associahedra

-- example (Shnider, Sternberg & Stasheff, Loday) --

Loday: Computes coord's (planar binary trees)

\[ x_1 = 1 \cdot 1 = 1 \]
\[ x_2 = 2 \cdot 2 = 4 \]
\[ x_3 \]

affine hyperplane
\[ x_1 + x_2 + x_3 = 6 \]

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Realise Associahedra
-- example (Shnider, Sternberg & Stasheff, Loday) --

Loday: Computes coord's (planar binary trees)

\[ x_1 = 1 \cdot 1 = 1 \]
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Realise Associahedra

-- example (Shnider, Sternberg & Stasheff, Loday) --

Loday: Computes coord's (planar binary trees)

\[
x_1 = 1 \cdot 1 = 1
\]
\[
x_2 = 2 \cdot 2 = 4
\]
\[
x_3 = 1 \cdot 1 = 1
\]

equivalently:

label \((n+2)\)-gon cyclicly decreasing with \(\{0,\ldots,n+1\}\)

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\[
x_1 + x_2 + x_3 = 6
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Realise Associahedra

--- example (Shnider, Sternberg & Stasheff, Loday) ---

Loday: Computes coord's (planar binary trees)

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affine hyperplane
\( x_1 + x_2 + x_3 = 6 \)
Realise Associahedra

-- example (Shnider, Sternberg & Stasheff, Loday) --

Loday: Computes coord’s (planar binary trees)

half space $x_1 + x_2 \geq 3$

$\equiv$

affine hyperplane $x_1 + x_2 + x_3 = 6$

equivalently:

label $(n+2)$-gon cyclicly decreasing with $\{0, \ldots, n+1\}$

$n=3$

labelled pentagon 2-dim associahedron realised in $\mathbb{R}^3$
Realise Associahedra

-- example: Loday’s associahedra II --

- label \((n+2)\)-gon cyclicly decreasing with \(\{0,\ldots,n+1\}\)

- \(A\) is “good subset” ⇔ strict RHS of oriented diagonal
  
  E.g.: \(A = \{1,2\}\)
  
  \[\sum_{i \in A} x_i \geq z_A := |A|(|A|+1)/2\]

- \(A\) is “bad subset”
  
  ⇔ not RHS of oriented diagonal
  
  E.g.: \(A = \{1,3\}\)
  
  \[\sum_{i \in A} x_i \geq z_A := -\infty\]
Decompose Associahedra

-- example: Loday's associahedra I --

\[ \begin{align*}
    z_I &= \sum_{J \subseteq I} y_J \\
    y_J &= \sum_{I \subseteq J} (-1)^{|J-I|} z_I \\
    \text{Möbius inversion} & \\
    y_J &= \sum_{I \subseteq J} (-1)^{|J-I|} z_I
\end{align*} \]

\( z_{\{1,3\}} = 2 \) is tight value
Decompose Associahedra

-- example: Loday's associahedra I --

\[
\begin{align*}
\{1,2,3\} & \quad 6 \\
\{1,2\} & \quad 3 \\
\{1\} & \quad 1 \\

\{1,3\} & \quad 2 \\
\{2\} & \quad 1 \\
\{1\} & \quad 1 \\

\{2,3\} & \quad 3 \\
\{3\} & \quad 1 \\
\{1\} & \quad 1 \\

\{1,2,3\} & \quad 1 \\
\{1,2\} & \quad 1 \\
\{1\} & \quad 1 \\

\{1,3\} & \quad 0 \\
\{2,3\} & \quad 1 \\
\{3\} & \quad 1 \\

\end{align*}
\]

[Postnikov, 2009]:
\[
z_I = \sum_{J \subseteq I} y_J
\]

Möbius inversion
\[
y_J = \sum_{I \subseteq J} (-1)^{|J| - |I|} z_I
\]

\( y_J \geq 0 \)

\( y_I = 1 \) \iff \text{I good subset}

\( y_I = 0 \) \iff \text{I bad subset}

\( z_{\{1,3\}} = 2 \) is tight value
Decompose Associahedra

--- example: Loday's associahedra II ---

affine hyperplane

\[ x + y + z = 1 \]
Decompose Associahedra

-- example: Loday's associahedra II --

affine hyperplane
\[ x+y+z=2 \]
Decompose Associahedra

-- example: Loday's associahedra II --

affine hyperplane
x+y+z=3

(0,3,0) (2,1,0)
(2,0,1)
(1,0,2) (0,1,2)

(0,1,2) (1,0,2)
(2,0,1)

(0,3,0) (2,1,0)
Decompose Associahedra

-- example: Loday's associahedra II --

affine hyperplane
\[ x+y+z=3 \]

\[(0,3,0) \quad (2,1,0) \quad (2,0,1) \quad (0,1,2) \quad (1,0,2) \]
Decompose Associahedra

-- example: Loday’s associahedra II --

affine hyperplane
\[ x + y + z = 6 \]
Associahedra

-- Loday's realisation in dimension 3 --

\[ \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_{1,2} + \Delta_{2,3} + \Delta_{3,4} + \Delta_{1,2,3} + \Delta_{2,3,4} + \Delta_{1,2,3,4} \]
How to ... realise associahedra

-- Loday's realization generalized --

[Hohlweg & L., 2007]

$2^{n-2}$ allowed labellings of $(n+2)$-gon with $\{0,1,\ldots,n+1\}$

A is "good subset" $\iff$ RHS of diagonal

Then

$$\sum x_i = z_{[n]} = n(n+1)/2$$
$$\sum_{i \in A} x_i \geq z_A = |A|(|A|+1)/2 \quad (A \text{ good subset})$$

yields H-description of associahedron.

Furthermore:

V-description generalising Loday's algorithm possible
How to ... realise associahedra

-- allowed labellings of (n+2)-gon --

_partition {1,...,n} into two sets:
“Up” and “Down” with 1,n \in Down

“c-labelling” of (n+2)-gon
label one vertex “0”
label paths starting at 0 by Up and Down
label label remaining vertex “n+1”
How to ... realise associahedra

--- allowed labellings of (n+2)-gon ---

- partition \{1,\ldots,n\} into two sets: “Up” and “Down” with 1,n \in Down
- “c-labelling” of (n+2)-gon
  - label one vertex “0”
  - label paths starting at 0 by Up and Down
  - label label remaining vertex “n+1”

Example: n=4, \(2^{4-2} = 4\) different labellings of hexagon

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</tbody>
</table>
```

“Up” \{1,2,3,4\}, \{2\}, \{3\}, \{2,3\}

“Down” \{1,3,4\}, \{1,3,4\}, \{1,2,4\}, \{1,4\}
Associahedra

-- Hohlweg&L. (Down = \{1,3,4\}) --
Coordinates revisited

\[ z_I = \sum_{J \subseteq I} y_J \]

\[ y_J = \sum_{I \subseteq J} (-1)^{|J-I|} z_I \]

bad subset: \( I = \{2\} \)

tight value: \( z_{\{2\}} = 0 \)
Coordinates revisited

-- Use $\circ$ instead of $\bigcirc$ --

\[ z_I = \sum_{J \subseteq I} y_J \]

\[ y_J = \sum_{I \subseteq J} (-1)^{|J-I|} z_I \]

bad subset: $I = \{2\}$

tight value: $z_{\{2\}} = 0$

$y_{\{1,2,3\}} = -1$!!
Coordinates revisited

-- Use \( \begin{array}{c}
\begin{array}{c}
2 \\
1 \\
3
\end{array}
\end{array} \) instead of \( \begin{array}{c}
\begin{array}{c}
4 \\
1 \\
2
\end{array}
\end{array} \) --

\[ z_I = \sum_{J \subseteq I} y_J \]
\[ y_J = \sum_{I \subseteq J} (-1)^{|J-I|} z_I \]

bad subset: \( I = \{2\} \)

tight value: \( z_{\{2\}} = 0 \)

!! \( y_{\{1,2,3\}} = -1 \) !!

What is the meaning of a negative \( y_I \) value??
Minkowski decomposition

--- definition ---

P and Q polytopes

R is Minkowski difference P−Q of P and Q

There is a polytope R such that R+Q = P

(pitfall: not always defined!!)
Minkowski decomposition

-- definition --

- $P$ and $Q$ polytopes
- $R$ is Minkowski difference $P-Q$ of $P$ and $Q$
  
  There is a polytope $R$ such that $R+Q = P$
  (pitfall: not always defined!!)
- "Minkowski decomposition of $P"$
  
  Write $P$ as Minkowski sums and differences of polytopes $Q_i$
- Idea:

  Use Minkowski decompositions of generalised permutahedra for positive & negative $y_i$-values
Minkowski decomposition

-- associahedron with Down = \{1,3\} --

affine hyperplane
\[ x + y + z = 2 \]

\[
\begin{array}{ccc}
{1} & {2} & {3} \\
{1,2} & {1,3} & {2,3} \\
{1,2,3}
\end{array}
\]
Minkowski decomposition

-- associahedron with Down = \{1,3\} --

affine hyperplane
\[ x + y + z = 4 \]

\[
\begin{array}{ccc}
(0,0,2) & (0,2,2) & (2,0,2) \\
(0,4,0) & (2,2,0) & (2,2,0)
\end{array}
\]

\[
\begin{array}{cccc}
\{1\} & \{2\} & \{3\} & \{1,2,3\} \\
\{1,2\} & \{1,3\} & \{2,3\} & \{-1\} \\
\{1\} & \{2\} & \{3\} & \{0\}
\end{array}
\]

\[
\begin{array}{cccc}
2 & 1 & 0 & 1 \\
1 & 2 & 2 & 1
\end{array}
\]

\[
\begin{array}{cccc}
0 & 2 & 4 & 1 \\
1 & 3 & 3 & 4
\end{array}
\]

\[
2 + 2
\]
Minkowski decomposition

-- associahedron with Down = \{1,3\} --
Minkowski decomposition

-- associahedron with Down = \{1,3\} --

affine hyperplane

\[ x+y+z=7 \]

\((1,2,4)\)
\((3,0,4)\)
\((1,4,2)\)
\((3,0,4)\)
\((2,4,1)\)
\((4,2,1)\)

\(\{1\}\)
\(\{2\}\)
\(\{3\}\)
\(\{1,2\}\)
\(\{1,3\}\)
\(\{2,3\}\)

\(\{1,2,3\}\)

\(-1\)
\(2\)
\(-1\)
\(1\)
\(2\)

\(0\)

\(2\)
Minkowski decomposition

-- associahedron with Down = \{1,3\} --
Minkowski decomposition

-- associahedron with Down = \{1,3\} --

affine hyperplane
x + y + z = 6

\begin{align*}
&{\{1\}} & {\{2\}} & {\{3\}} \\
&{\{1,2\}} & {\{1,3\}} & {\{2,3\}} \\
&{\{1,2,3\}} \\
\end{align*}

\begin{align*}
&-1 \\
&2 \\
&1 \\
&2 \\
&1 \\
&0 \\
&1 \\
\end{align*}

\begin{align*}
&2 + 2 + 1 + \cdot + \cdot - \triangle \\
&0 \quad 1 \quad 3 \quad 4
\end{align*}
Theorem [Ardila,Benedetti&Doker, 2010]

Every generalised permutahedron $P(\{z_I\})$ has a unique Minikowski decomposition

$$P(\{z_I\}) = \sum_{J \subseteq [n]} y_J \Delta_J$$

where $y_J = \sum_{I \subseteq J} (-1)^{|J-I|} z_I$
Theorem [Ardila, Benedetti & Dokter, 2010]
Every generalised permutahedron $P(\{z_i\})$ has a unique Minkowski decomposition

$$P(\{z_i\}) = \sum_{J \subseteq [n]} y_J \Delta_J$$

where

$$y_J = \sum_{I \subseteq J} (-1)^{|J-I|} z_I$$

Proof:
Set $z^-_I := \sum_{J \subseteq I; y_J < 0} (-y_J)$ and $z^+_I := \sum_{J \subseteq I; y_J \geq 0} y_J$.

By inclusion-exclusion $z_I + z^-_I = z^+_I$ which yields

$$P(\{z_I\}) + P(\{z^-_I\}) = P(\{z^+_I\})$$

since $P(\{a_I + b_I\}) = P(\{a_I\}) + P(\{b_I\})$. 
**P(\{z_I\}) & decompositions**

Theorem [Ardila, Benedetti & Doker, 2010]

Every generalised permutahedron \( P(\{z_I\}) \) has a unique Minkowski decomposition

\[
P(\{z_I\}) = \sum_{J \subseteq [n]} y_J \Delta_J
\]

where \( y_J = \sum_{I \subseteq J} (-1)^{|J-I|} z_I \)

Corollary:

\( y_I \)-values for associahedra of Hohlweg & Lange computable by Möbius inversion from complete set of tight \( z_I \)-values
\( y_I \)-coord's for associahedra

-- Statement of results --

\( z_I \)-values for redundant inequalities computable from "good subsets S" using "Up and Down interval decomposition" of \( I \)

"type" of interval decomposition simplifies \( y_I \)-computation:

- \( I \) of "type \((1, l)\)":
  \[
  y_I = (-1)^{|I-D_1|} (z_{I_1} - z_{I_2} - z_{I_3} + z_{I_4})
  \]

- \( I \) of "type \((k, l)\), \( k > 1 \)":
  \[
  y_I = 0
  \]

Loday-type formula for \( y_I \)-values:

- \( I \neq \{u\} \) of type \((1, l)\):
  \[
  y_I = (-1)^{|I-D_1|} K_Y \cdot K_{\Gamma}
  \]

- \( I = \{u\} \) of type \((1, l)\):
  \[
  y_I = (-1)^{|I-D_1|} (K_Y \cdot K_{\Gamma} - (n+1))
  \]

- \( I \) of type \((k, l)\), \( k > 1 \):
  \[
  y_I = 0
  \]

\( K_Y \) and \( K_{\Gamma} \) : "signed lengths" on boundary of \((n+2)\)-gon
Up&Down intervals

-- Up and Down interval decomposition --

Definition [L., 2011]

- open down interval \((d_i,d_j)\)
  all numbers \(k \in \text{Down} \text{ s.t. } d_i < k < d_j\)

- closed up interval \([u_i,u_j]\)
  all numbers \(k \in \text{Up} \text{ s.t. } u_i \leq k \leq u_j\)

- Up and Down interval decomposition of \(I \subseteq [n]\)
  family of maximal closed up intervals of \(I\) “nested”
  in maximal open (down) intervals of \(I\)

  type of decomposition:
  (#down intervals, #up intervals)
Up&Down intervals

-- examples --

“Up”
∅

“Down”
{1,2,3,4}

no up-intervals

down-intervals:
∅, {1}, {2}, {3}, {4}
{1,2}, {2,3}, {3,4}
{1,2,3}, {2,3,4}
{1,2,3,4}

only up-interval {2}

down-intervals:
∅, {1}, {3}, {4}, {1,3},
{3,4}, {1,3,4}

decomposition type of
{2},{2,3}, {1,4}, {2,4}???
Cyclohedra

-- revisit definition of generalised permutahedra --

Cyclohedra ("type B generalised associahedra") can be realised using certain associahedra.

Minkowski decomposition into dilated faces of standard simplex à la Ardila/Benedetti/Doker?
Cyclohedra

-- revisit definition of generalised permutahedra --

- Cyclohedra ("type B generalised associahedra") can be realised using certain associahedra

- Minkowski decomposition into dilated faces of standard simplex à la Ardila/Benedetti/Doker?

  No! Compute $y_I$-coordinates and compare resulting polytope with cyclohedron

- Postnikov and Postnikov, Reiner & Williams: "generalised permutahedra $P(\{z_I\})$" are in the deformation cone of classical permutahedron!
Open Problems

- Feasible $z_i$- and $y_i$-coordinates?
- Lattice points of associahedra?
- Relation to brick polytopes?
- Minkowski decompositions for other types?
- Implications for cluster algebras?
- Formulae in terms of Coxeter group of type $A$?
[Ardila,Benedetti&Doker]:
Matroid polytopes and their volume,
FPSAC/Discrete&Compututational Geometry, 2009/10

[Hohlweg&L., 2007]:
Realizations of the associahedron and cyclohedron,
Discrete&Computational Geometry, 2007

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[Loday, 2004]
Realization of the Stasheff polytope,
Archiv der Mathematik, 2004

[Postnikov]
Permutahedra, associahedra, and beyond,
International Mathematical Research Notices, 2009

[Postnikov, Reiner, Williams]
Faces of generalised permutahedra,
Documenta Mathematica, 2008

\[ \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_{1,2} + \Delta_{2,3} \\
+ \Delta_{3,4} + \Delta_{1,2,3} + \Delta_{2,3,4} + \Delta_{1,2,3,4} \]

\[ \Delta_1 + \Delta_3 + \Delta_4 + 3\Delta_{1,2} + \Delta_{1,3} \\
+ 2\Delta_{2,3} + \Delta_{3,4} + \Delta_{1,3,4} + 2\Delta_{2,3,4} \\
- (\Delta_2 + \Delta_{1,2,3} + \Delta_{1,2,3,4}) \]