Éléments totalement commutatifs et chemins du plan

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Fully commutative elements

\((W, S)\) Coxeter group \(W\) given by Coxeter matrix \((m_{st})_{s, t \in S}\).

Relations:
\[
\begin{align*}
    s^2 &= 1 \\
    \underbrace{sts \cdots}_{m_{st}} &= \underbrace{tst \cdots}_{m_{st}} \\
\end{align*}
\]

Braid relations
\(m_{st} = 2\): Commutation relation
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Length \(\ell(w)\) = minimal \(l\) such that

\[w = s_1 s_2 \ldots s_l\] with \(s_i \in S\)

Such a minimal word is a reduced decomposition of \(w\).
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w = s_1 s_2 \ldots s_l \text{ with } s_i \in S
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Such a minimal word is a reduced decomposition of \(w\).

**Matsumoto property** : Given two reduced decompositions of \(w\), there is a sequence of braid relations which can be applied to transform one into the other.
An element $w$ is **fully commutative** if given two reduced decompositions of $w$, there is a sequence of **commutation relations** which can be applied to transform one into the other.
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$\text{ReducedWords}(w)$

$w$ fully commutative
Type $A_{n-1} \rightarrow$ The symmetric group $S_n$

Consider $S = \{s_1, \ldots, s_{n-1}\}$, with relations $s_i^2 = 1$ and

\[
\begin{align*}
&\begin{cases}
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\
s_i s_j = s_j s_i, & |j - i| > 1
\end{cases}
\end{align*}
\]

$\vartheta : s_i \mapsto (i, i + 1)$ is an isomorphism with $S_n$. 

![Diagram](image.png)
Type $A_{n-1} \rightarrow$ The symmetric group $S_n$

Consider $S = \{s_1, \ldots, s_{n-1}\}$, with relations $s_i^2 = 1$ and

$$\begin{cases} s_is_{i+1} = s_{i+1}s_is_{i+1} \\ s_is_j = s_js_i, \quad |j - i| > 1 \end{cases}$$

$\vartheta: s_i \mapsto (i, i + 1)$ is an isomorphism with $S_n$.

**Theorem** [Billey, Jockush, Stanley ’93]

$w$ is fully commutative $\iff \vartheta(w)$ is 321-avoiding.

One can use this to show that FC elements in type $A_{n-1}$ are counted by Catalan numbers, i.e. $|S_{n}^{FC}| = \frac{1}{n+1} \binom{2n}{n}$. 
Previous work

• The seminal papers are \[\textbf{Stembridge '96,'98}\]:
  1. First \textbf{properties};
  2. \textbf{Classification} of $W$ with a \textbf{finite number} of \textbf{FC elements};
  3. \textbf{Enumeration} of these elements in each of these cases.
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● The seminal papers are [Stembridge ’96,’98]:
  1. First properties;
  2. Classification of $W$ with a finite number of FC elements;
  3. Enumeration of these elements in each of these cases.

● [Fan ’95] studies FC elements in the special case where $m_{st} \leq 3$ (the simply laced case).

● [Graham ’95] shows that FC elements in any Coxeter group $W$ naturally index a basis of the (generalized) Temperley-Lieb algebra of $W$.

● Subsequent works [Greene,Shi,Cellini,Papi] relate FC elements (and some related elements) to Kazhdan-Lusztig polynomials.
Today, I will show explain how to enumerate FC elements for any finite or affine Coxeter group $W$.

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Let $W^{FC}(q) = \sum_w q^{\ell(w)}$ (where $w$ runs through FC elements of $W$.)

We can compute $W^{FC}(q)$ for any such $W$. 
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Today I will focus on types $A$ and $\tilde{A}$, corresponding to the finite and affine symmetric groups. The idea is to encode the FC elements in these cases by certain lattice paths.
1. **FC elements and Heaps**
Characterization of FC elements

In general, how can one recognize a FC element? The following is one step in this direction.

**Theorem** [Stembridge] A reduced word represents a FC element if and only no element of its commutation class contains a factor $sts \cdots$ for a $m_{st} \geq 3$.

(Proof: when two words are related by a braid relation with $m_{st} \geq 3$, they do not belong to the same commutation class.)

How to tell if a commutation class verifies the property above?
⇒ Use theory of **heaps**, which are posets which encode commutation classes.
Example of heaps in $A_4(= S_5)$

$s_1 s_3 s_4 s_1 s_2 s_3$
Example of heaps in $A_4(= S_5)$
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Vertex stays above if corresponding generators do not commute.
Example of heaps in $A_4(= S_5)$

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**Heap of a word** = poset $H$ labeled by generators $s_i$ of $W$.
Linear extensions of $H \Leftrightarrow$ Words of the commutation class.

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Linear extensions of $H \iff$ Words of the commutation class.

\[ \text{NOT REDUCED} \quad \text{NOT FC} \quad \text{FC} \]
Characterization of heaps

**Proposition**[Stembridge '95] Heaps $H$ of FC reduced words are characterized by:

(a) No covering relation $i \prec j$ in $H$ such that $s_i = s_j$.

(b) No **convex** chain $i_1 \prec \cdots \prec i_{m_{st}}$ in $H$ such that $s_{i_1} = s_{i_3} = \cdots = s$ and $s_{i_2} = s_{i_4} = \cdots = t$ where $3 \leq m_{st} < \infty$. 
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(the only elements $x$ satisfying $i_1 \leq x \leq i_{m_{st}}$ are the elements $i_j$ of the chain.)
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In type $A$ and $\widetilde{A}$, we will see that the FC heaps above are particularly nice.
1. Type A
FC heaps avoid precisely

Type A

\[ s_i \]

\[ s_i\, s_{i+1}\, s_{i+2} \]

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Type A

FC heaps avoid precisely

So they look like this
FC heaps avoid precisely

So they look like this

Proposition Heaps of type $A$ are characterized by:

(i) At most one occurrence of $s_1$ \((\text{resp. } s_{n-1})\).

(ii) Elements with labels $s_i, s_{i+1}$ form an alternating chain.
Type A: Bijection

FC Heap

\[ s_1 \quad s_2 \quad \quad s_{n-2} \quad s_{n-1} \]
Type A: Bijection

FC Heap

Path

$s_1$  $s_2$

$s_{n-2}$  $s_{n-1}$
Type A: Bijection

Extra information needed!
Type A: Bijection

Extra information needed!

To finish, add initial and final steps to the path.
Type A: Bijection

**Theorem** \([\text{BJN '12, known before?}]\)
This is a bijection between FC heaps of type \(A_{n-1}\) and Motzkin paths of length \(n\) with horizontal steps at height \(h > 0\) (resp. \(h = 0\)) labeled \(L\) or \(R\) (resp. labeled \(L\)).

Size of the heap \(\Leftrightarrow\) **Area** of the path
(Sum of the heights of all vertices)
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Size of the heap $\iff$ **Area** of the path
(Sum of the heights of all vertices)

**Remark**

transforms these paths into Dyck paths $\Rightarrow$ Catalan numbers!
The generating polynomial

We have to count our labeled Motzkin paths with respect to their area.
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→ Use recursive decompositions

(* indicates that horizontal steps at height $h = 0$ must have label $L$)
The generating polynomial

We have to count our labeled Motzkin paths with respect to their area.

→ Use recursive decompositions

\[ \sum_{n \geq 1} A_{n-1}^{FC}(q)x^n. \]

Write the functional equations, and eliminate to get

**Theorem** Define \( A^{FC}(x) = \sum_{n \geq 1} A_{n-1}^{FC}(q)x^n. \) Then
\[
A^{FC}(x) = x + xA^{FC}(x) + qxA^{FC}(x)(A^{FC}(qx) + 1).
\]
2. Type $\tilde{A}$
One can represent this group as the set of permutations \( \sigma \) of \( \mathbb{Z} \) satisfying \( \sigma(i + n) = \sigma(i) + n \), and \( \sum_{i=1}^{n} \sigma(i) = \sum_{i=1}^{n} i \).

\[ \ldots, 17, -12, -14, -1, 17, -8, -10, 3, 21, -4, -6, 7, 25, 0, -2, 11, 29, 4, \ldots \]
One can represent this group as the set of permutations $\sigma$ of $\mathbb{Z}$ satisfying $\sigma(i + n) = \sigma(i) + n$, and $\sum_{i=1}^{n} \sigma(i) = \sum_{i=1}^{n} i$.

Theorem [Green '01] Fully commutative elements of type $\tilde{\mathbb{A}}_{n-1}$ correspond to 321-avoiding permutations.

For instance the permutation above is not FC.

Hanusa and Jones used this representation to enumerate FC elements in type $\tilde{\mathbb{A}}$. 
Generating functions

They computed the generating functions $f_n(q) = \tilde{A}_{n-1}^{FC}(q)$; here are the first ones

\[ f_3(q) = 1 + 3q + 6q^2 + 6q^3 + 6q^4 + \cdots \]
\[ f_4(q) = 1 + 4q + 10q^2 + 16q^3 + 18q^4 + 16q^5 + 18q^6 + \cdots \]
\[ f_5(q) = 1 + 5q + 15q^2 + 30q^3 + 45q^4 + 50q^5 + 50q^6 + 50q^7 + 50q^8 + 50q^9 + \cdots \]
\[ f_6(q) = 1 + 6q + 21q^2 + 50q^3 + 90q^4 + 126q^5 + 146q^6 + 150q^7 + 156q^8 + 152q^9 + 156q^{10} + 150q^{11} + 158q^{12} + 150q^{13} + 156q^{14} + 152q^{15} + 156q^{16} + 150q^{17} + 158q^{18} + \cdots \]

Periodicity $n$ in the coefficients ?
Periodicity

**Theorem** [Hanusa-Jones ’09] The coefficients of $\tilde{A}_{n-1}^{FC}(q)$ are ultimately periodic of period $n$. 
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- In the same article, they also derive a complicated expression for $\tilde{A}_{n-1}^{FC}(q)$.

Moreover they can prove that one has periodicity starting from the length(degree) $2 \lceil n/2 \rceil \lfloor n/2 \rfloor$ but conjecture that $1 + \lceil (n - 1)/2 \rceil \lfloor (n + 1)/2 \rfloor$ is enough.
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- We will prove this conjecture using heaps/paths.

In the process, we will get much simpler rules to compute the generating functions $\tilde{A}_{n-1}^{FC}(q)$. 
FC elements in type $\tilde{A}$

FC heap satisfy the same local conditions as in finite type $A$.

→ The heaps must avoid

$$s_i \quad s_i \quad s_{i+1} \quad s_{i+2}$$

Difference: the cyclic shape of the Coxeter diagram

→ The labels above must be taken with index modulo $n$; the heaps must be thought of as “drawn on a cylinder”.

$$s_1 \quad s_2 \quad \cdots \quad s_{n-1}$$
Heaps become Paths

We can form a path as before from a heap: because of the cyclic diagram, our paths will **start and end at the same height**.
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Example:

Heap

Note that there is just one minimal element
Heaps become Paths

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Example:

Heap

Path

Note that there is just one minimal element

The area does not take into account the final height.
Starting from an FC element in $\tilde{A}_{n-1}$, we thus obtain a path in $O_n^*$, the set of length $n$ paths with starting and ending point at the same height.
Bijection

Starting from an FC element in $\tilde{A}_{n-1}$, we thus obtain a path in $O_n^*$, the set of length $n$ paths with starting and ending point at the same height.

**Theorem** [BJN '12] This is a bijection between
1. FC elements of $\tilde{A}_{n-1}$ and 
2. $O_n^*$
Starting from an FC element in $\tilde{A}_{n-1}$, we thus obtain a path in $O^*_n$, the set of length $n$ paths with starting and ending point at the same height.

**Theorem**[BJN ’12] This is a bijection between

1. FC elements of $\tilde{A}_{n-1}$ and
2. $O^*_n \setminus \{\text{paths at constant height } h > 0 \text{ with all steps having the same label } L \text{ or } R\}$.

Indeed such paths clearly cannot correspond to FC elements.
Bijection

Starting from an FC element in $\tilde{A}_{n-1}$, we thus obtain a path in $O_n^*$, the set of length $n$ paths with starting and ending point at the same height.

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Indeed such paths clearly cannot correspond to FC elements.

**Corollary** $\tilde{A}^{FC}_{n-1}(q) = O_n^*(q) - \frac{2q^n}{1 - q^n}$
Periodicity revisited

• For a large enough degree, the series $O_n^*(q)$ has periodic coefficients with period $n$: just shift the path up by 1 unit.
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“Large enough”? As soon as the degree $k$ is such that no path with area $k$ can have a horizontal step at height $h = 0$ → $k = 1 + \lceil(n - 1)/2\rceil\lfloor(n + 1)/2\rfloor$ is optimal.

This proves the conjecture of Hanusa and Jones.
Periodicity revisited

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This proves the conjecture of Hanusa and Jones.

• We still have to compute the generating function $O_n^*(q)$.

I will leave it to you as an (interesting) exercise in generating functions (maybe you have a better solution than ours).
3. Other finite and affine Coxeter groups
Other finite types

- The remaining “classical types”

\[ B_n \]

\[ D_{n+1} \]

were also enumerated by Stembridge
→ we can reinterpret his proof in terms of paths and give the length generating polynomials in these cases also.
Other finite types

• The remaining “classical types”

\[ B_n \]

\[ t \overset{4}{\longrightarrow} s_1 \overset{\cdots}{\longrightarrow} s_{n-1} \]

\[ D_{n+1} \]

\[ t_1 \overset{t_2}{\rightarrow} s_1 \overset{\cdots}{\rightarrow} s_{n-1} \]

were also enumerated by Stembridge
→ we can reinterpret his proof in terms of paths and give the length generating polynomials in these cases also.

• Exceptional types \( I_2(m), H_3, H_4, F_4, E_6, E_7, \) and \( E_8 \)
→ Computer assisted (a proof by hand is also possible).
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• Exceptional types \( I_2(m) \), \( H_3 \), \( H_4 \), \( F_4 \), \( E_6 \), \( E_7 \), and \( E_8 \) → Computer assisted (a proof by hand is also possible).

\[
E^\text{FC}_8(q) = 15q^{29} + 30q^{28} + 43q^{27} + 56q^{26} + 69q^{25} + 83q^{24} + 113q^{23} + 143q^{22} + 171q^{21} + 205q^{20} + 259q^{19} + 319q^{18} + 387q^{17} + 457q^{16} + 527q^{15} + 609q^{14} + 701q^{13} + 794q^{12} + 867q^{11} + 924q^{10} + 936q^9 + 897q^8 + 796q^7 + 631q^6 + 427q^5 + 238q^4 + 105q^3 + 35q^2 + 8q + 1.
\]
Other affine types

There are 3 classical types

\[
\begin{align*}
\tilde{C}_n & \quad \tilde{D}_{n+2} & \quad \tilde{B}_{n+1} \\
\quad & \quad & \\
& \quad & \\
\end{align*}
\]
Other affine types

There are 3 classical types

\[ \tilde{A}_{n-1}, \tilde{C}_n, \tilde{B}_{n+1}, \tilde{D}_{n+2}, \tilde{E}_6, \tilde{E}_7, \tilde{G}_2, \tilde{F}_4, \tilde{E}_8 \]

**Theorem [BJN '12]**
For each irreducible affine group \( W \), the sequence of coefficients of \( W^{FC}(q) \) is ultimately periodic, with period recorded in the following table.

<table>
<thead>
<tr>
<th>Affine Type</th>
<th>( \tilde{A}_{n-1} )</th>
<th>( \tilde{C}_n )</th>
<th>( \tilde{B}_{n+1} )</th>
<th>( \tilde{D}_{n+2} )</th>
<th>( \tilde{E}_6 )</th>
<th>( \tilde{E}_7 )</th>
<th>( \tilde{G}_2 )</th>
<th>( \tilde{F}_4, \tilde{E}_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Periodicity</td>
<td>( n )</td>
<td>( n + 1 )</td>
<td>( (n + 1)(2n + 1) )</td>
<td>( n + 1 )</td>
<td>4</td>
<td>9</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>
Type $\tilde{C}$

$$
\tilde{C}_4^{FC}(q) = 1 + 5q + 14q^2 + 29q^3 + 47q^4 + 64q^5 + 76q^6 + 81q^7 + 80q^8 + 75q^9 + 68q^{10} + 63q^{11} + 61q^{12} + 59q^{13} + 59q^{14} + 60q^{15} + 59q^{16} + 59q^{17} + 59q^{18} + 59q^{19} + 60q^{20} + 59q^{21} + 59q^{22} + 59q^{23} + 59q^{24} + 60q^{25} + 59q^{26} + 59q^{27} + \cdots
$$

We obtain here also certain heaps corresponding to paths, but there are in addition infinitely many exceptional FC heaps, certain “zigzag heaps”.
Type $\tilde{C}$

Two families of paths survive for large enough length:

1. Finite factors of

2. $t \ s_1 \ s_2 \ s_3 \ u$

Path $L$
Type $\tilde{D}$

$\tilde{D}_4(q) = 1 + 5q + 14q^2 + 28q^3 + 39q^4 + 44q^5 + 45q^6 + 34q^7 + 30q^8 + 36q^9 + 30q^{10} + 30q^{11} + 36q^{12} + 30q^{13} + 30q^{14} + 36q^{15} + 30q^{16} + 30q^{17} + 36q^{18} + 30q^{19} + 30q^{20} + 36q^{21} + 30q^{22} + 30q^{23} + 36q^{24} + 30q^{25} + 30q^{26} + 36q^{27} + 30q^{28} + 30q^{29} + 36q^{30} + 30q^{31} + 30q^{32} + 36q^{33} + 30q^{34} + 30q^{35} + 36q^{36} + 30q^{37} + 30q^{38} + \cdots$.

Here the minimal period is 3, while the period predicted by the theorem is 6.
Type $\tilde{B}$

$$\tilde{B}_3^{FC}(q) = 1 + 4q + 9q^2 + 15q^3 + 19q^4 + 21q^5 + 21q^6 + 18q^7 + 17q^8 + 19q^9 + 18q^{10} + 17q^{11} + 19q^{12} + 17q^{13} + 17q^{14} + 20q^{15} + 17q^{16} + 17q^{17} + 19q^{18} + 17q^{19} + 18q^{20} + 19q^{21} + 17q^{22} + 17q^{23} + 19q^{24} + 18q^{25} + 17q^{26} + 19q^{27} + 17q^{28} + 17q^{29} + 20q^{30} + 17q^{31} + 17q^{32} + 19q^{33} + 17q^{34} + 18q^{35} + 19q^{36} + 17q^{37} + 17q^{38} + 19q^{39} + 18q^{40} + 17q^{41} + 19q^{42} + 17q^{43} + 17q^{44} + 20q^{45} + 17q^{46} + 17q^{47} + 19q^{48} + 17q^{49} + 18q^{50} + 19q^{51} + 17q^{52} + 17q^{53} + 19q^{54} + 18q^{55} + 17q^{56} + 19q^{57} + 17q^{58} + 17q^{59} + 20q^{60} + 17q^{61} + 17q^{62} + 19q^{63} + 17q^{64} + 18q^{65} + 19q^{66} + 17q^{67} + 17q^{68} + 19q^{69} + 18q^{70} + 17q^{71} + 19q^{72} + 17q^{73} + 17q^{74} + 20q^{75} + 17q^{76} + \cdots$$
The period is 15 in this case, corresponding to \((n + 1)(2n + 1)\) for \(n = 2\).
Further questions

- All of this work can be easily restricted to deal with FC involutions.

- Other statistics to consider, e.g. descent numbers.

- Formulas for our generating functions? (and not just functional equations/recurrences).

- (Affine case) Repartition of the alcoves corresponding to FC elements.

- Classification: for which Coxeter groups $W$ is it true that $W^{FC}(q)$ has periodic coefficients?
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THANK YOU
Type $\tilde{A}_2$
Type $\tilde{A}_2$
Type $\tilde{A}_2$
Type $\tilde{B}_2$
Type $\tilde{B}_2$
Type $\tilde{G}_2$
Type $\tilde{G}_2$
Type $\tilde{G}_2$