New perspectives on the enumeration of permutation classes

Michael Albert

Department of Computer Science, University of Otago

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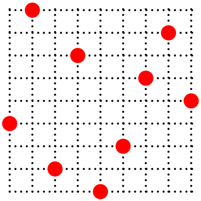
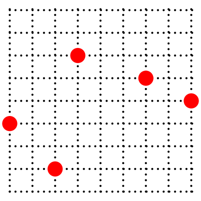
Permutation classes

Definition

A permutation class is a collection of permutations, $\mathcal{C}$, with the property that, if $\pi \in \mathcal{C}$ and we erase some points from its plot, then the permutation defined by the remaining points is also in $\mathcal{C}$.





$492713685 \in \mathcal{C}$ implies $21543 \in \mathcal{C}$

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- The objective is to try to *understand* the structure of permutation classes (or to identify when this is possible)
- Enumeration is a consequence or symptom of such understanding
- If X is a set of permutations, then $\mathrm{Av}(X)$ is the permutation class consisting of those permutations which do not dominate any permutation of X

First steps

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- All doubleton bases of lengths 4 and 3, and many of length 4 and 4 are known

Stanley-Wilf conjecture

Relative to the set of all permutations, proper permutation classes are small. Specifically:

Theorem

Let $\mathcal{C}$ be a proper permutation class. Then, the growth rate of $\mathcal{C}$,

$$\mathrm{gr}(\mathcal{C}) = \limsup |\mathcal{C} \cap \mathcal{S}_n|^{1/n}$$

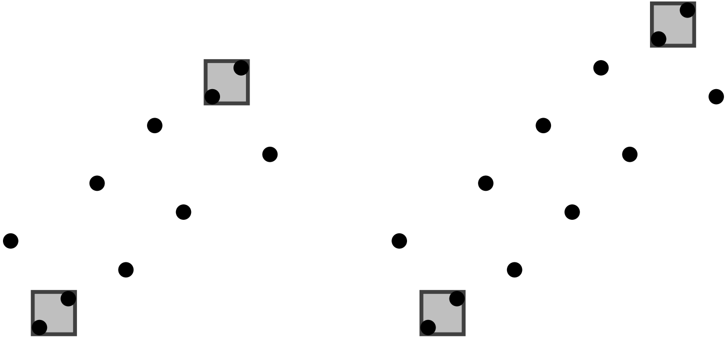
is finite.

This was known as the *Stanley-Wilf conjecture* and it was proven in 2004 by Marcus and Tardos.
The obvious next questions are:

- What growth rates can occur?
- What can be said about classes of particular growth rates?

Antichains

The subpermutation order contains infinite antichains.



Consequently, there exist $2^{\aleph_0}$ distinct enumeration sequences for permutation classes – we must be careful not to try to do too much.

Small growth rates

- Kaiser and Klazar (EJC, 2003) showed that the only possible values of $\mathrm{gr}(\mathcal{C})$ less than 2 are the greatest positive solutions of:

$$x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = 0$$

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- V (PLMS, 2011) further showed that the smallest growth rate of a class containing an infinite antichain is the unique positive solution, $\kappa \simeq 2.20557$ of

$$x^3 - 2x^2 - 1 = 0$$

  and completely characterized the set of possible growth rates below κ

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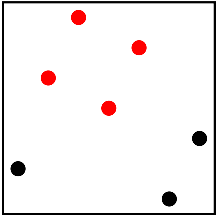

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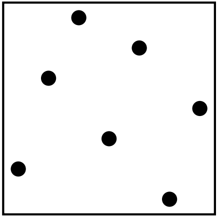

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- Simple permutations form a positive proportion of all permutations (asymptotically $1/e^2$)
- In many (conjecturally all) proper permutation classes they have density 0
- We can hope to understand a class by understanding its simples and how they *inflate*
- Specifically, this may yield functional equations of the generating function and hence computations of the enumeration and/or growth rate

Finitely many simple permutations

Theorem

If a class has only finitely many simple permutations then it has an algebraic generating function.

- A and Atkinson (2005)
- Effective 'in principle', i.e. an algorithm for computing a defining system of equations for the generating function
- Some interesting corollaries, e.g. if a class has finitely many simples and does not contain arbitrarily long decreasing permutations then it has a rational generating function
- *"The prime reason for giving this example is to show that we are not necessarily stymied if the number of simple permutations is infinite."*

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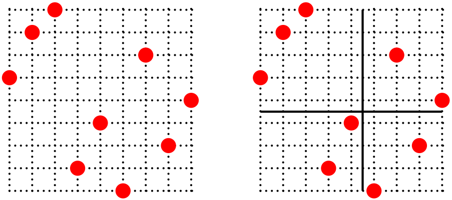
- For trees this approach goes by many names
- For words, A, At and Ruškuc (2003) give (fairly) general criteria for the encoding of a permutation class by a regular language
- Extended to a new type of encoding, the *insertion encoding* (prefigured by Viennot) by A, Linton and R (2005)

Grid classes

- The notion of *griddable* class was central to V's characterization of small permutation classes
- Loosely, a griddable class is associated with a matrix whose entries are (simpler) permutation classes
- All permutations in the class can be chopped apart into sections that correspond to the matrix entries



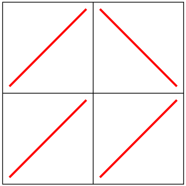
Geometric monotone grid classes

In a *geometric* grid class, the permutations need to be drawn from the points of a particular representation in $\mathbb{R}^2$

Theorem (A, At, Bouvel, R and V (to appear TAMS))

Every geometrically griddable class:

- *is partially well ordered;*
- *is finitely based;*
- *is in bijection with a regular language and thus has a rational generating function.*



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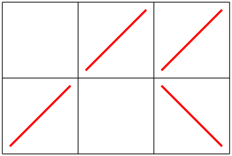
- If $\mathcal{C}$ is geometrically griddable, then every subclass of $\langle \mathcal{C} \rangle$ has an algebraic generating function
- If $\mathcal{C}$ is geometrically griddable and $\mathcal{U}$ is strongly rational, then $\mathcal{C}[\mathcal{U}]$ is also strongly rational
- Every small permutation class has a rational generating function

Quasi-applications

- These ideas, together with a certain amount of number eight wire or duct tape, i.e. “un peu de rafistolage” can be used to compute enumerations for some (arguably) interesting classes
- Examples from the basic environment of geometric grid classes are considered in A, At and Brignall: *The Enumeration of Three Pattern Classes using Monotone Grid Classes* (EJC 19.3 (2012) P20)
- Examples for inflations of geometric grid classes are considered in A, At and V: *Inflations of Geometric Grid Classes: Three Case Studies* (arxiv.org/abs/1209.0425)

$\mathrm{Av}(4312, 3142)$

- Every simple permutation in this class lies in the geometric grid class:

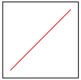


- This yields a regular language for the simple permutations
- The allowed inflations of these permutations are easily described, yielding a recursive description of the class
- This leads to an equation for its generating function:

$$\begin{aligned}(x^3 - 2x^2 + x)f^4 \quad &+ \quad (4x^3 - 9x^2 + 6x - 1)f^3 \\ &+ \quad (6x^3 - 12x^2 + 7x - 1)f^2 \\ &+ \quad (4x^3 - 5x^2 + x)f \\ &+ \quad x^3 \qquad\qquad\qquad\qquad = \quad 0\end{aligned}$$

Av(321)

This class is in some sense a limit of geometric grid classes:



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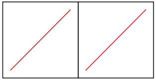

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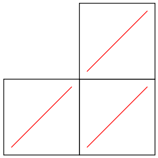

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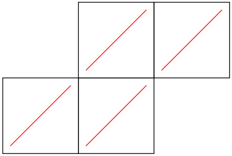

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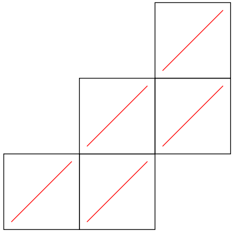

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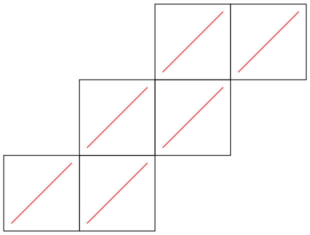

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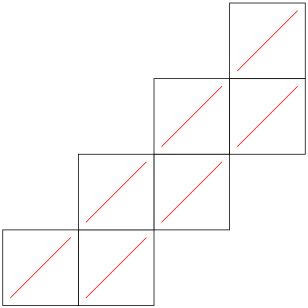

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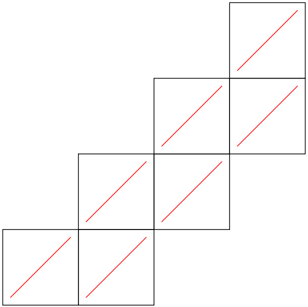
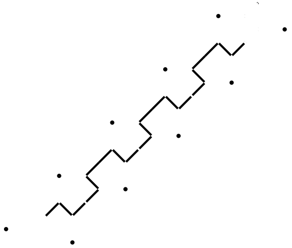


Conjecture

Every finitely based proper subclass of Av(321) *has a rational generating function*

Av(4231, 35142, 42513, 351624)

- Enumerating this class was mentioned as a challenge problem by Alexander Woo at *Permutation Patterns 2012*
- These permutations index "Schubert varieties defined by inclusions"
- The simple permutations look like



- Somehow, this leads to an enumeration of the class (a complicated rational function in $\sqrt{1-4x}$)

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- *Permutation Patterns 2013*: July 1-5, Paris