Some combinatorial Morse theory

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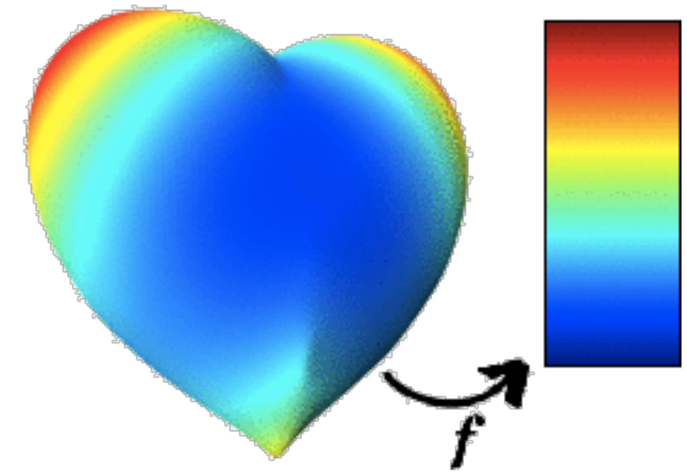
Séminaire X, 17 sep. 2012

matemática puc-rio

Acknowledgements



Luca, organizers!!

students and colleagues:

João Paixão, Renata Nascimento,
Andrei Sharf, Daniel Cohen-Or, Arik Shamir,
David Cohen-Steiner, Luca Castelli-Aleardi...

authors of the many inspirational works!

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CNPq, FAPERJ, CAPES, EDX, IMC, Matheon...

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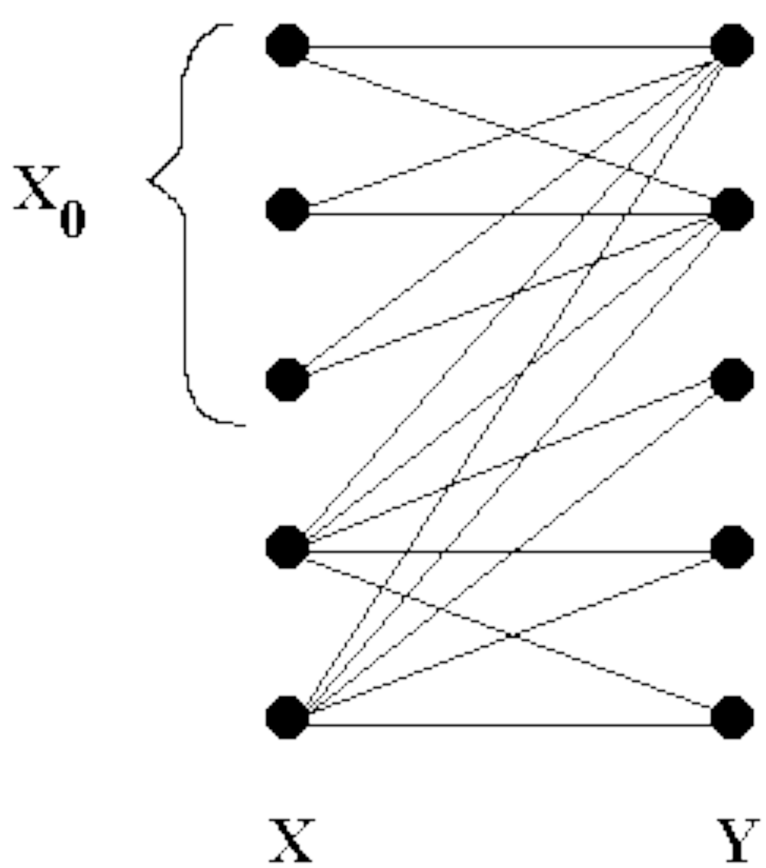
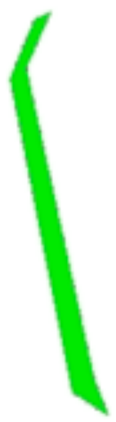
Introduction: special maximal matching

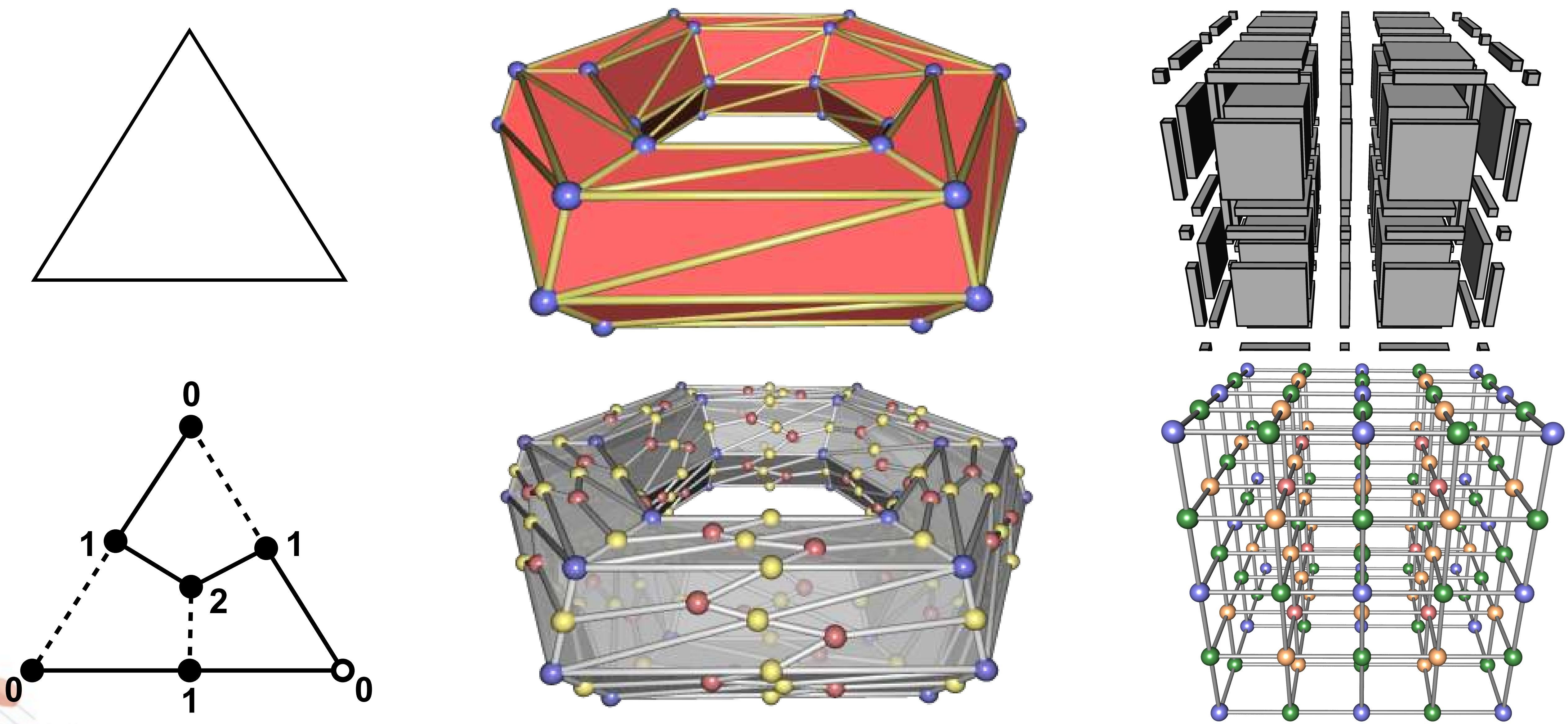


bipartite matching



topological objects

Class of graphs: Hasse diagrams



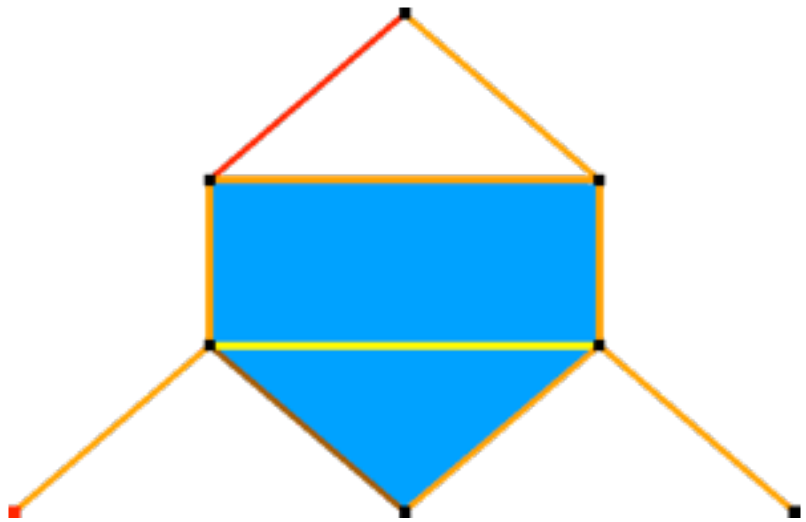
graph coding a cell complex

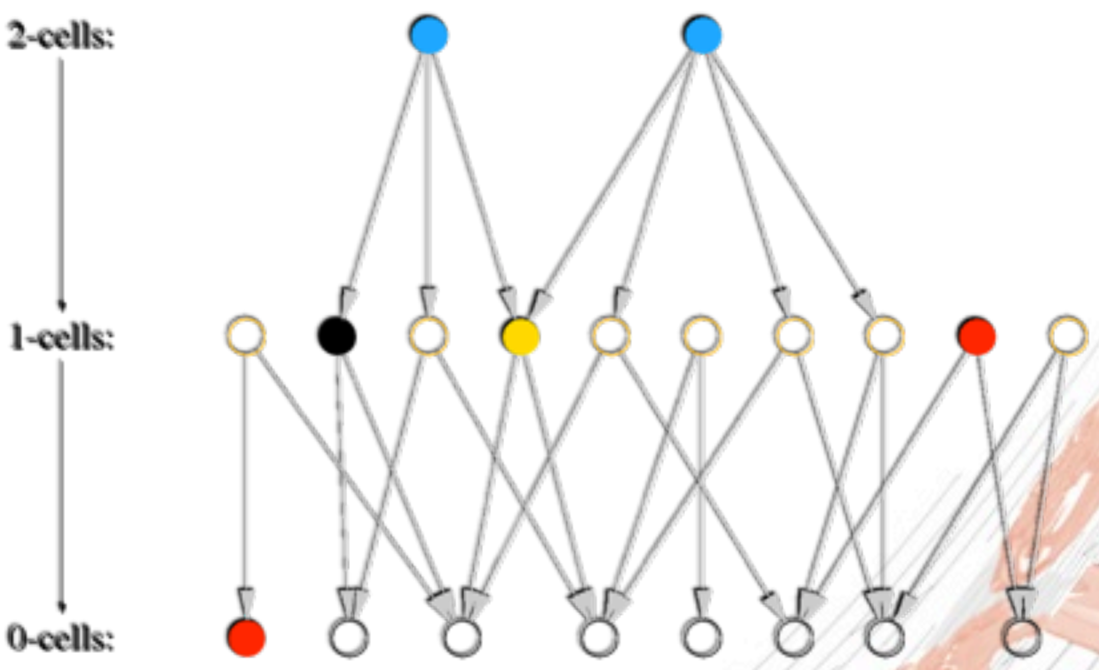
Hasse diagram

Simple oriented graph built out of K:

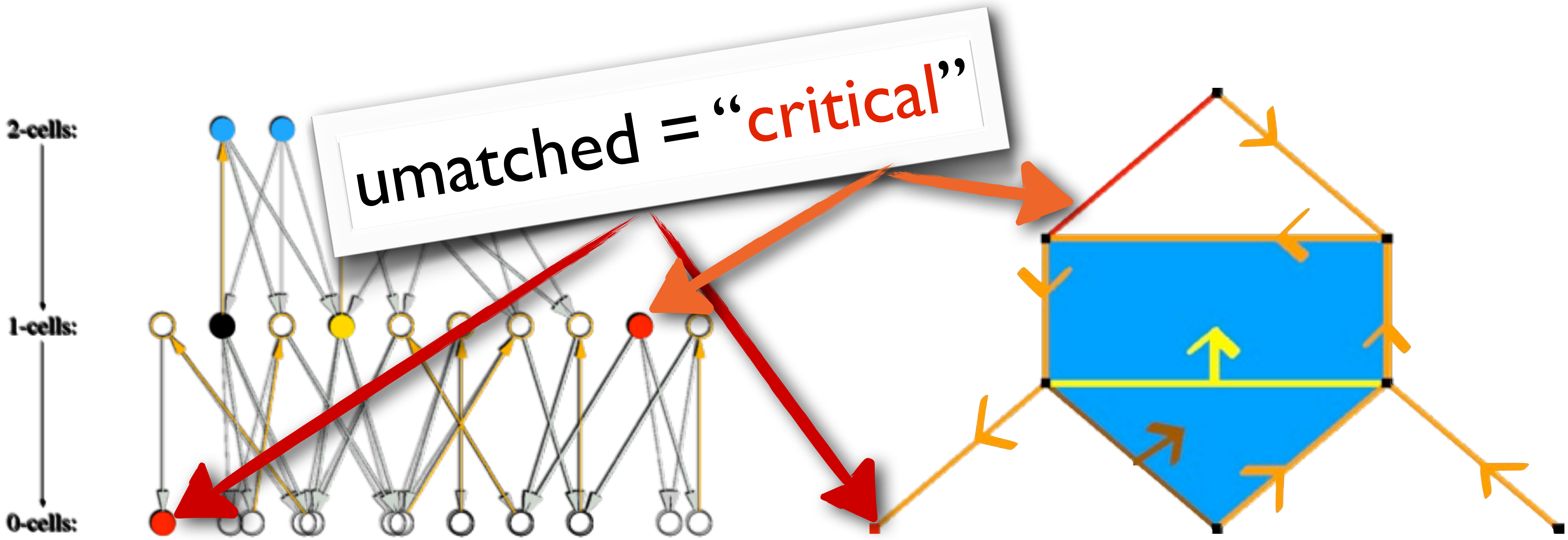
- nodes represent the cells of K

- links connect cells towards their bounding faces





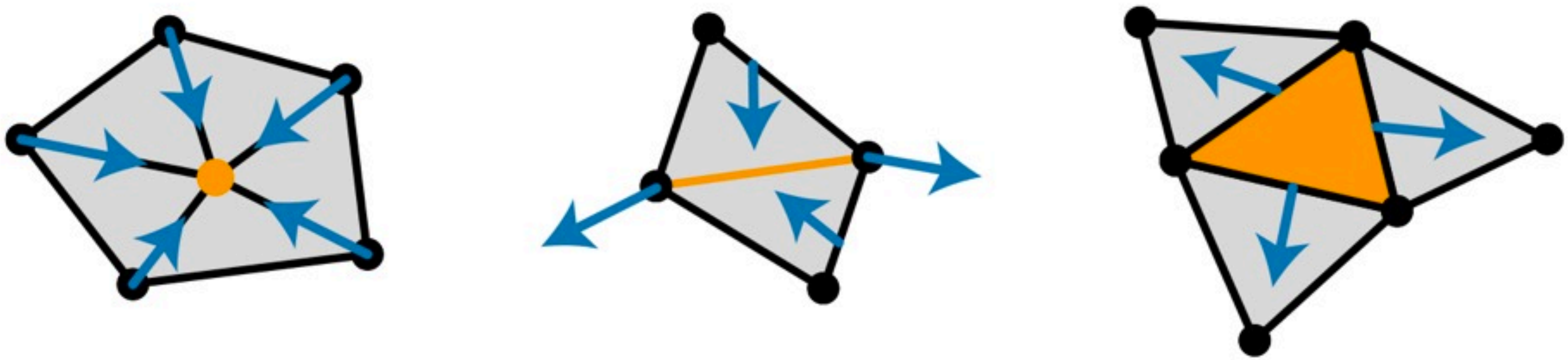
Matching in Hasse diagrams



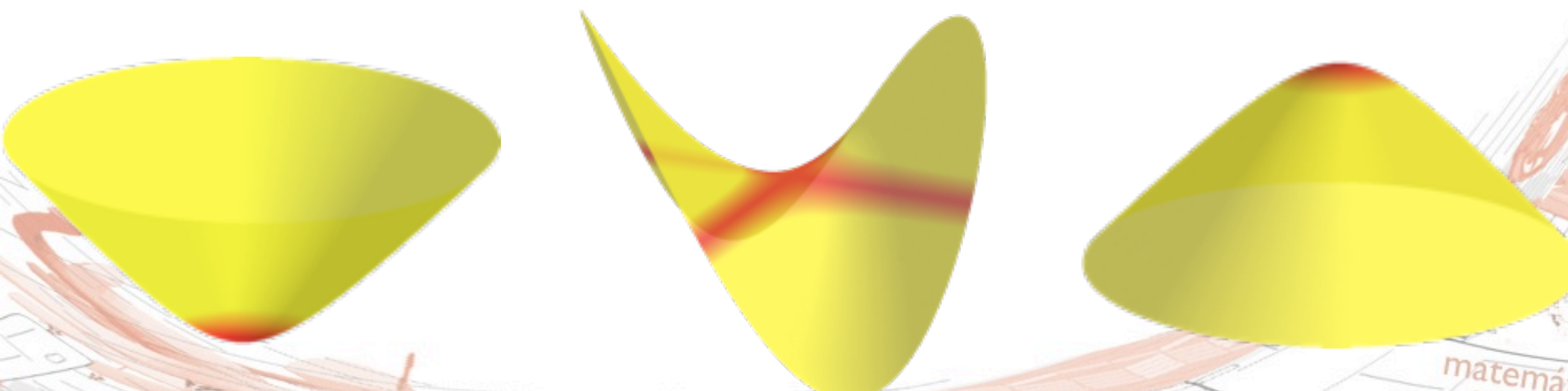
(unoriented) bipartite matching

selected links have their orientation reversed

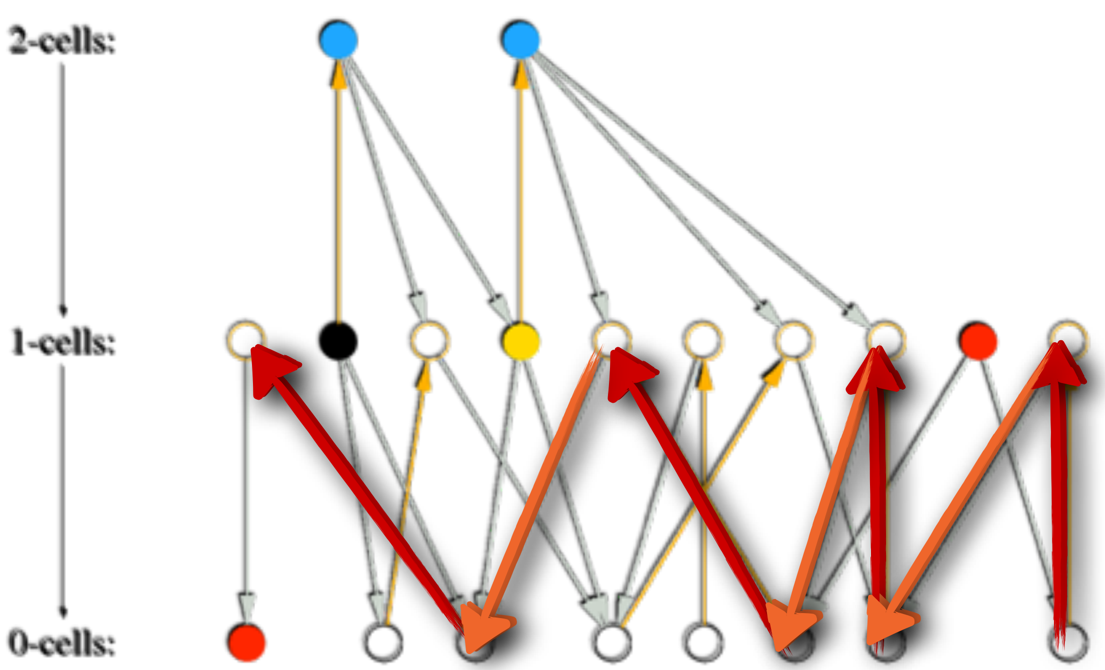
Shape of critical cells

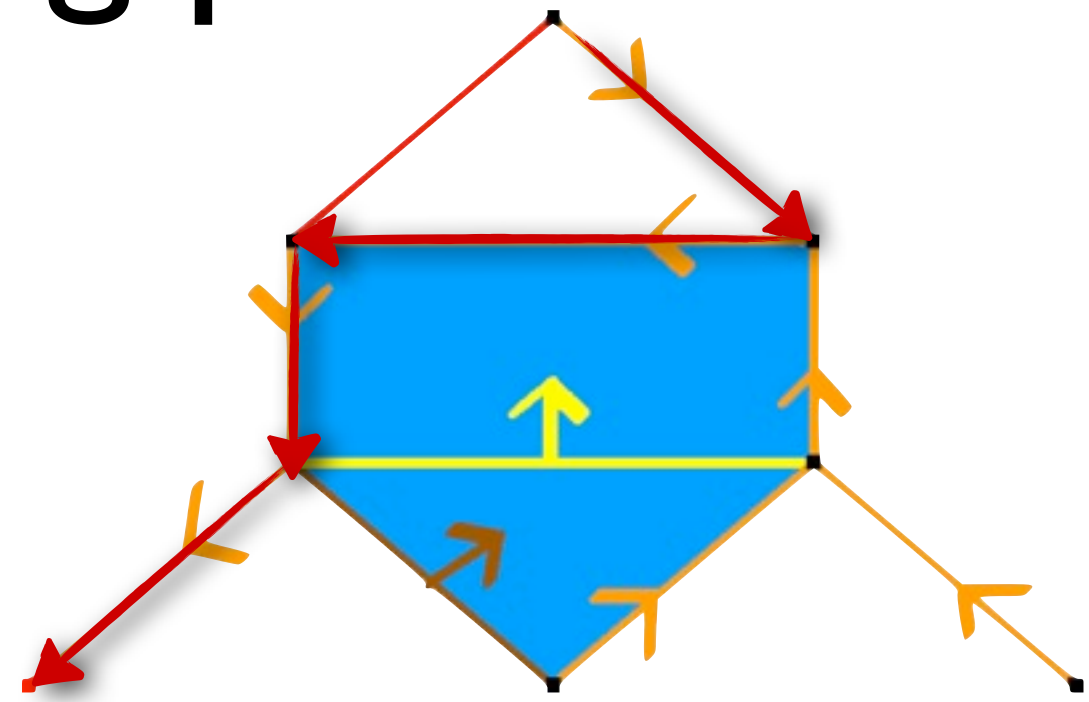


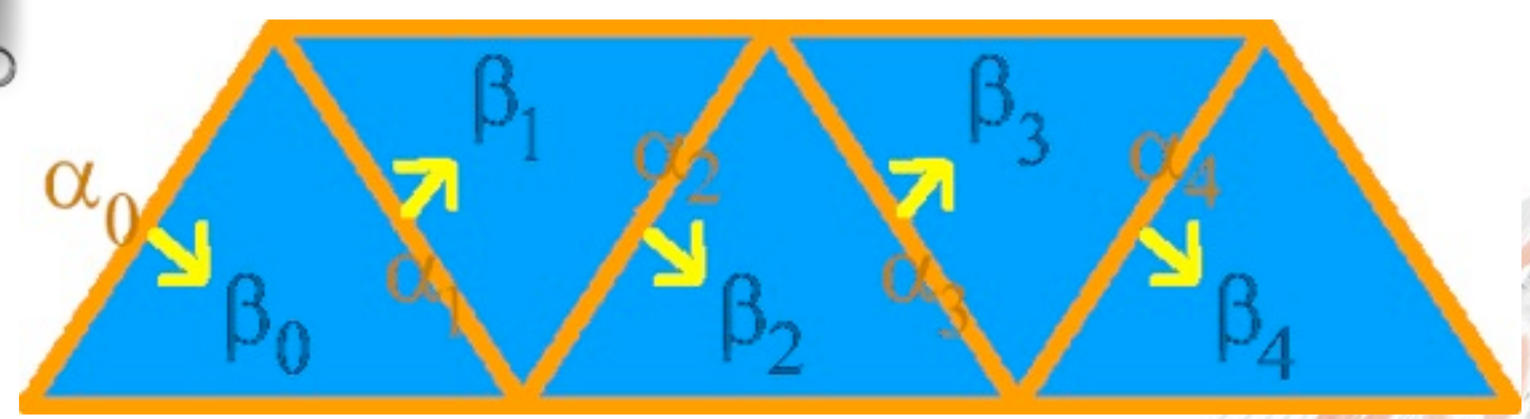
Minimum Saddle Maximum



Alternating path



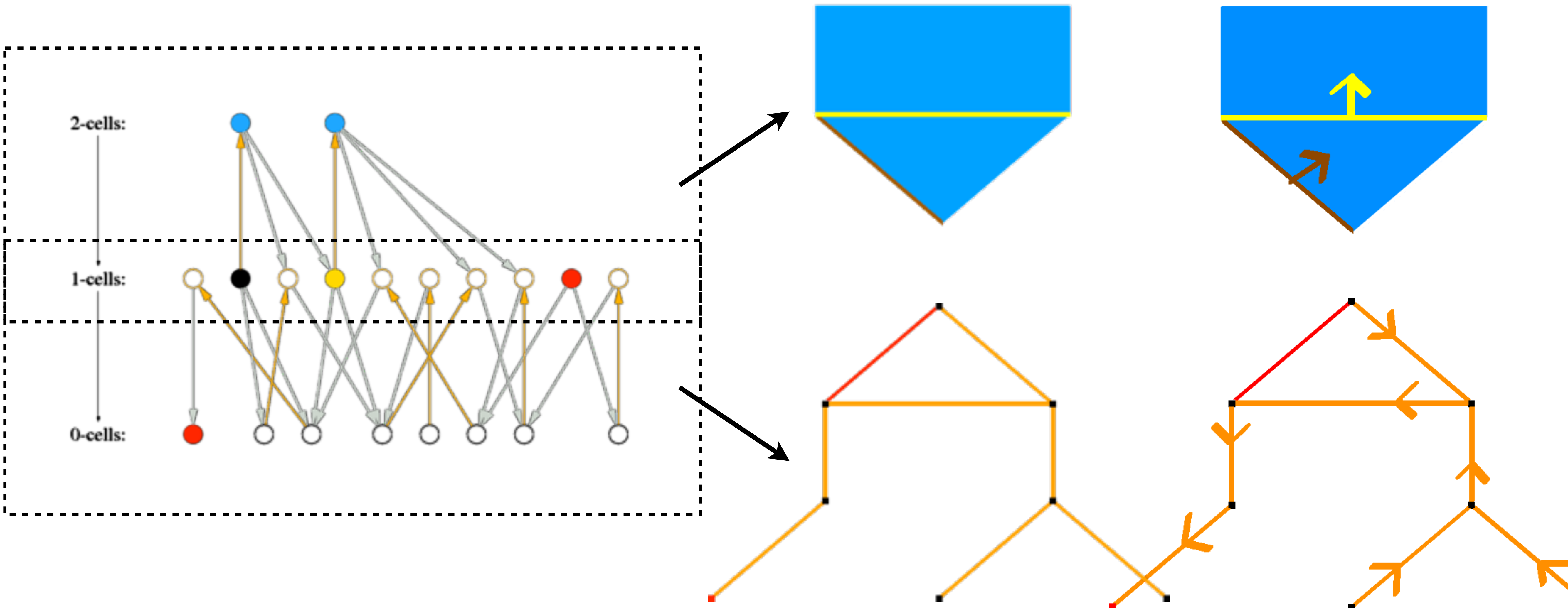




Alternating path between 2 dimensions only

"Gradient V-path"

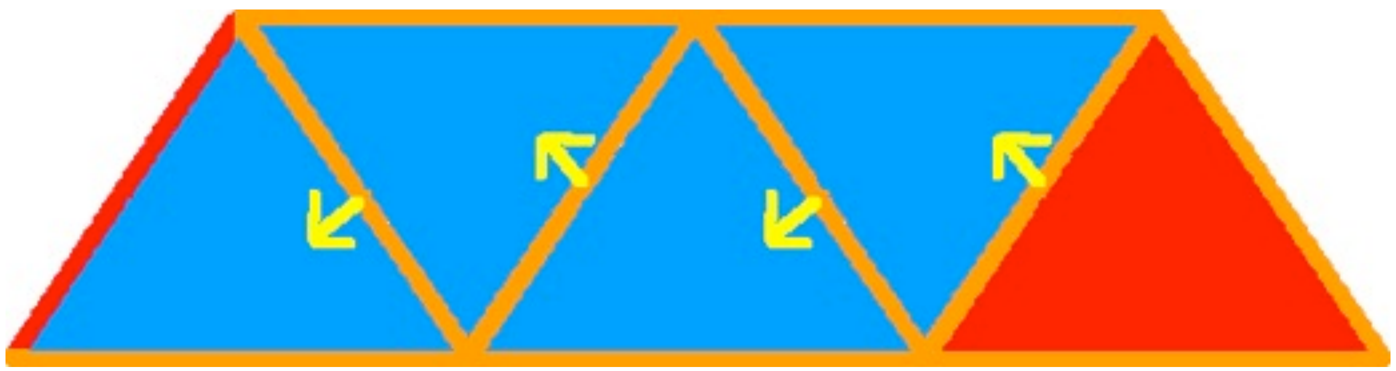
Layers of the Hasse diagram



for 2d manifolds:
alternating paths are paths in the primal / dual graph

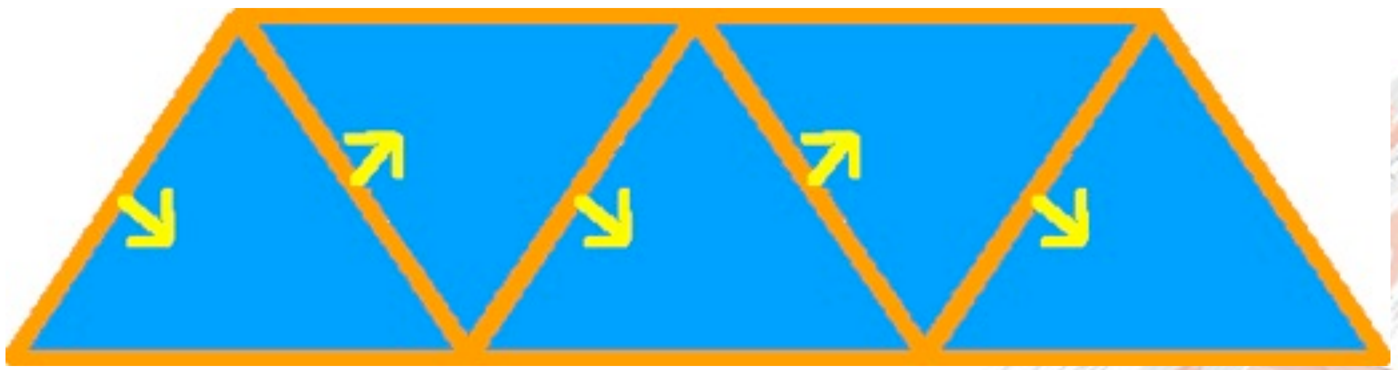
Augmenting alternating paths

improves matching

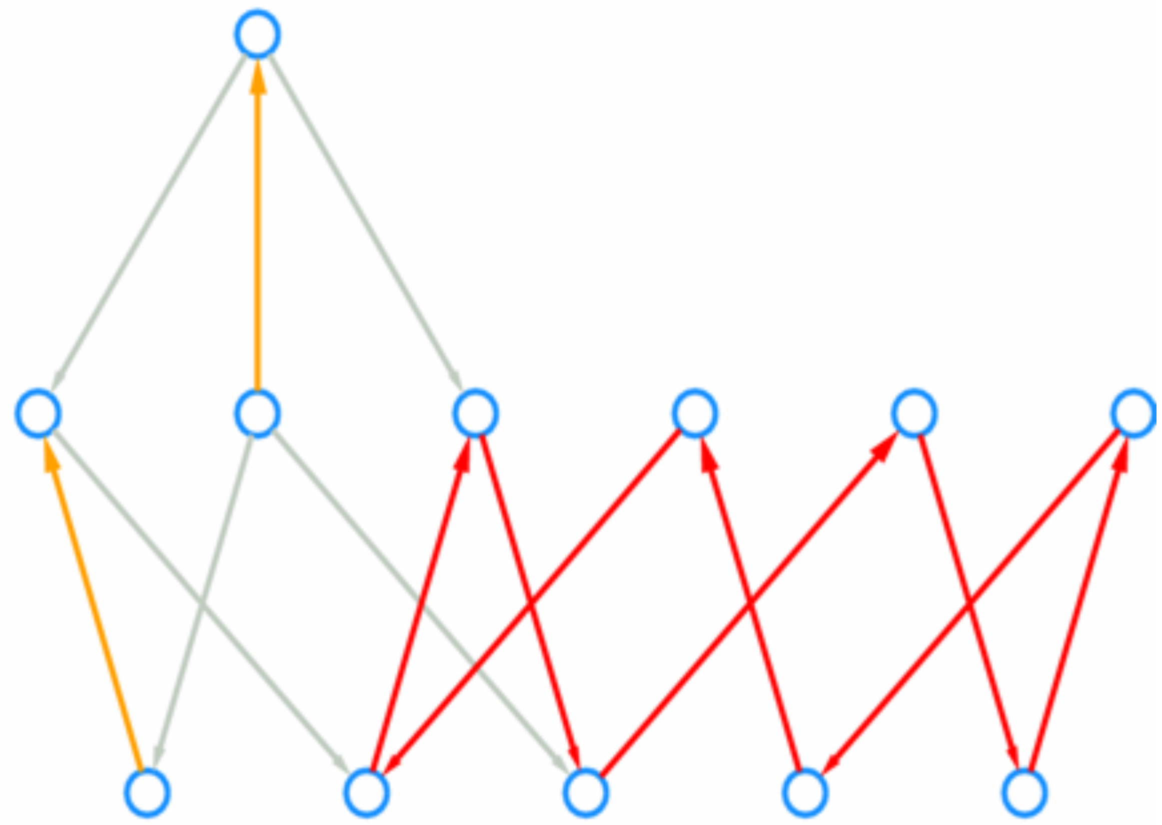


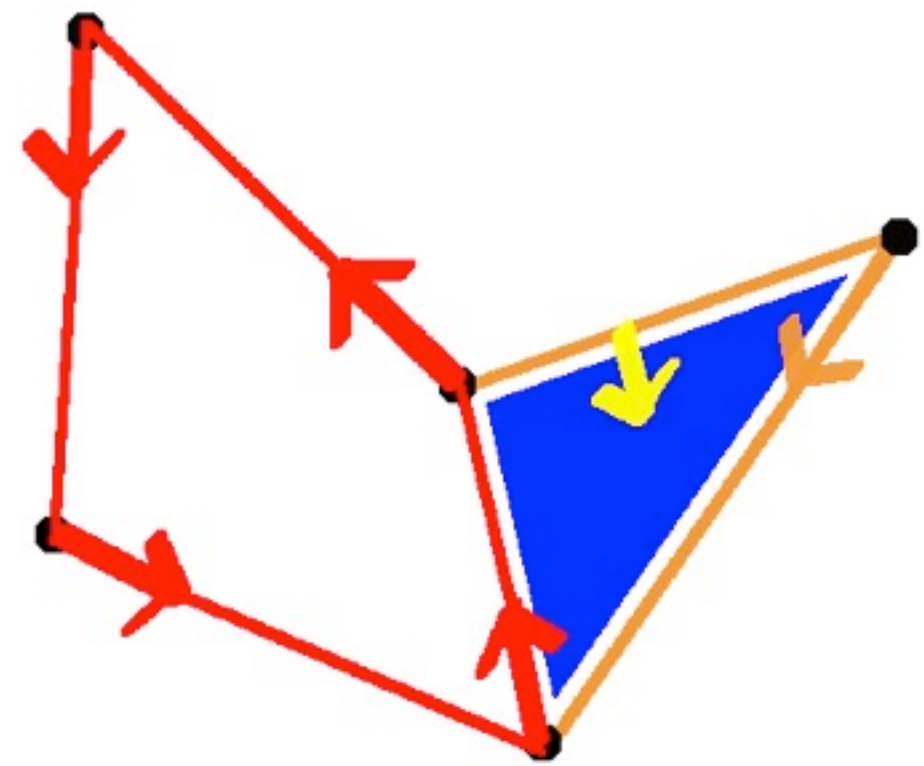
=

reduces number of critical cells



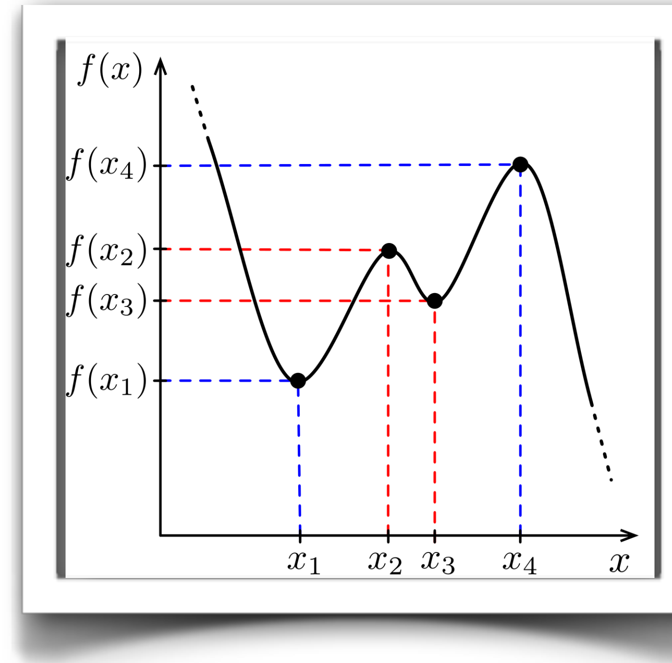
Acyclic matching





No alternating cycle

Maximal acyclic matching in Hasse diagrams

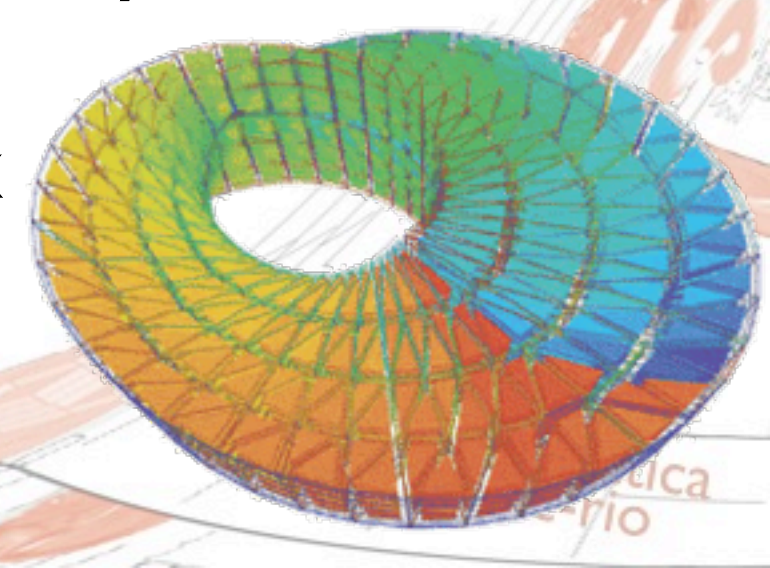


Combinatorial problem:

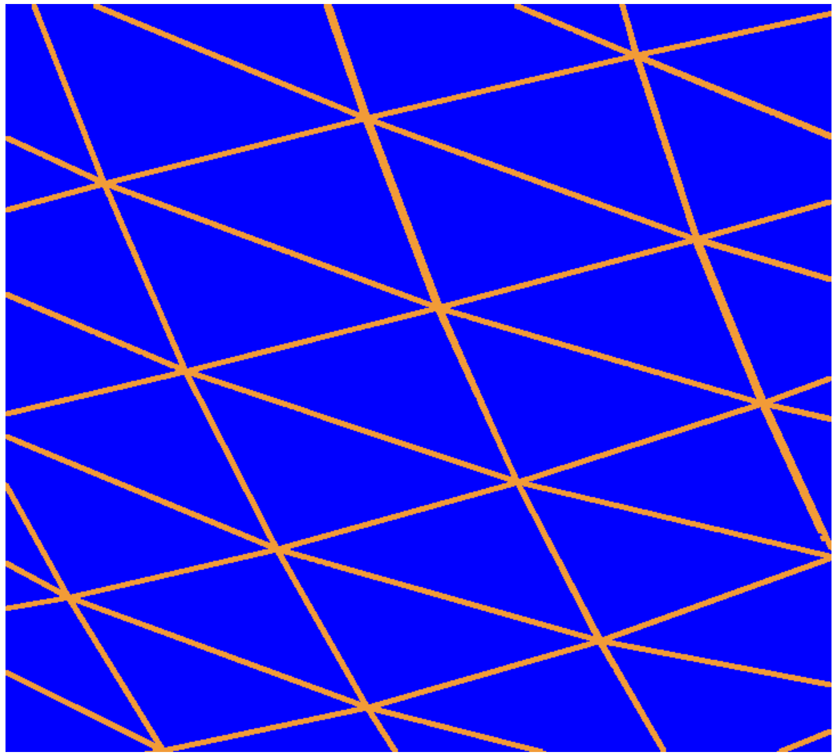
MAX SNP hard
extendable to subclass of matchings (stable?)

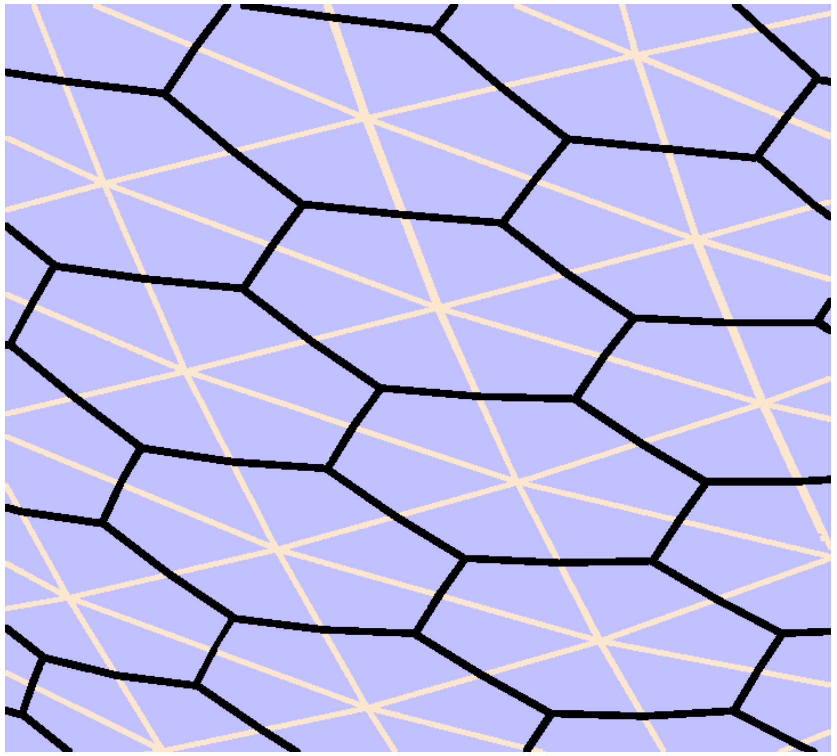
Topological problem:

critical cells are building blocks of Morse theory
lower bounds from the cell complex
characterizes homotopy of the cell complex



Simple cases: 2-manifolds





layers: primal/dual graph

acyclic matching → tree structure

maximal → spanning tree

Primal spanning tree

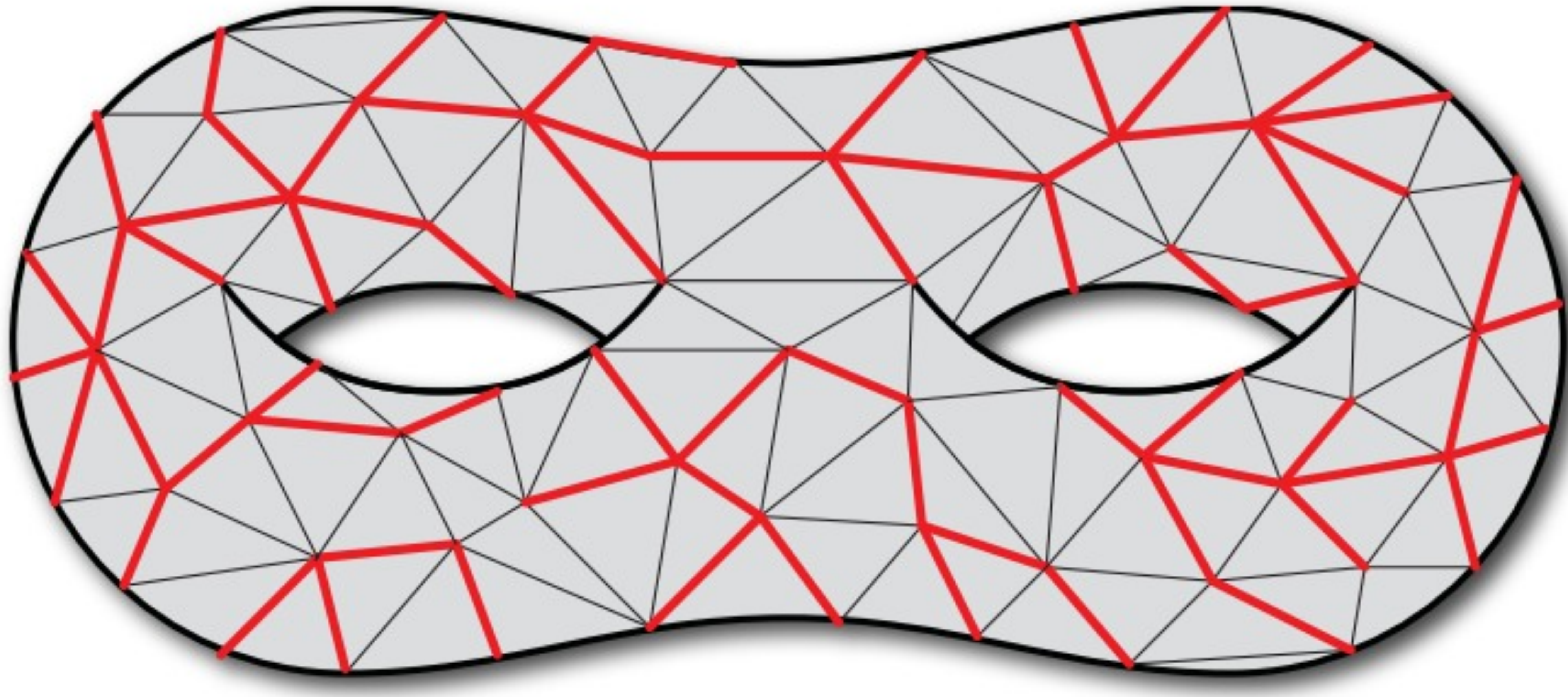
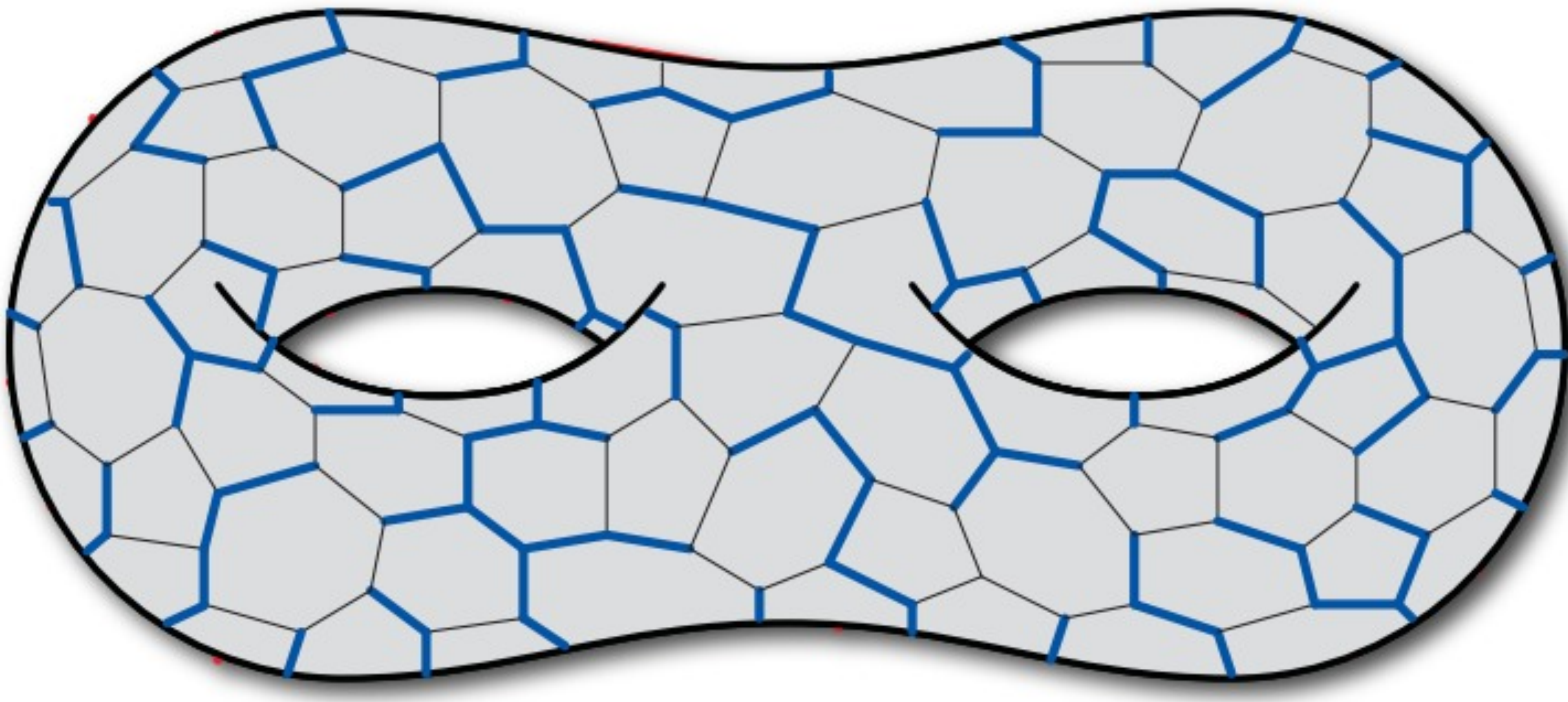


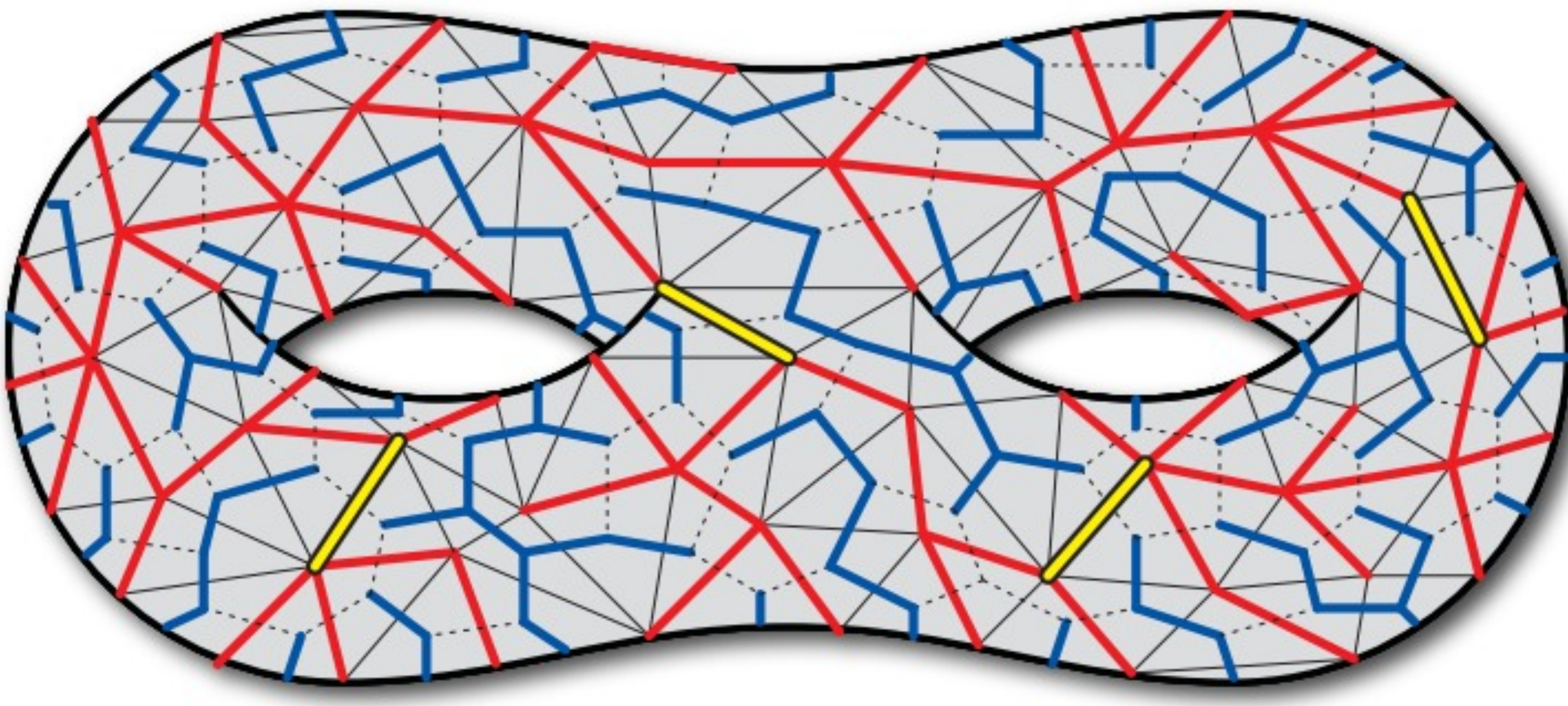
Image: Erickson 2011

Dual spanning Co-tree



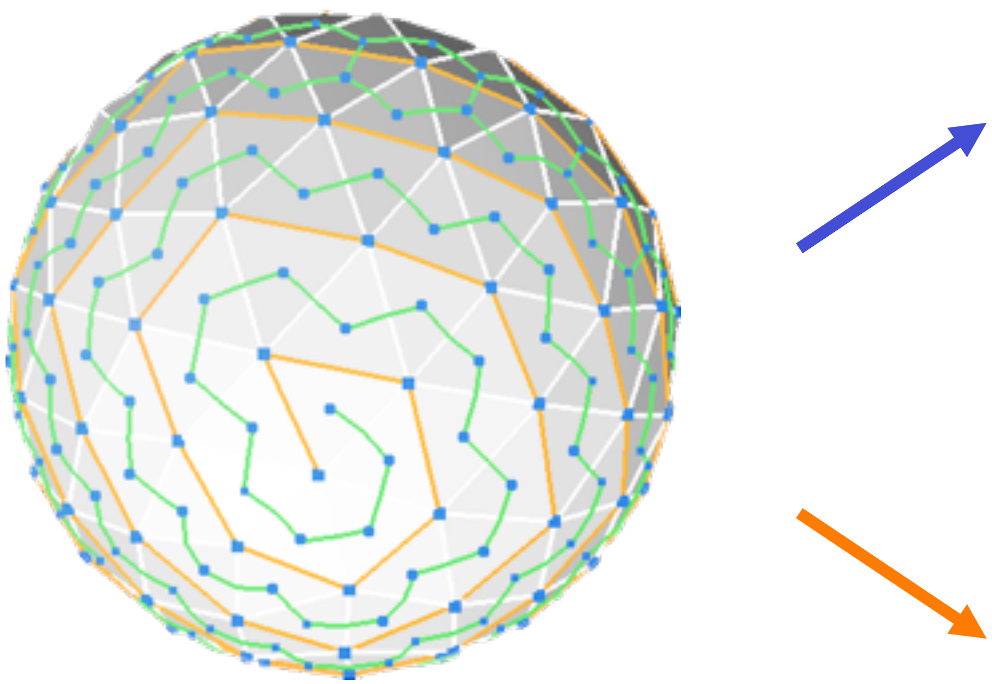
Slides from J. Erickson

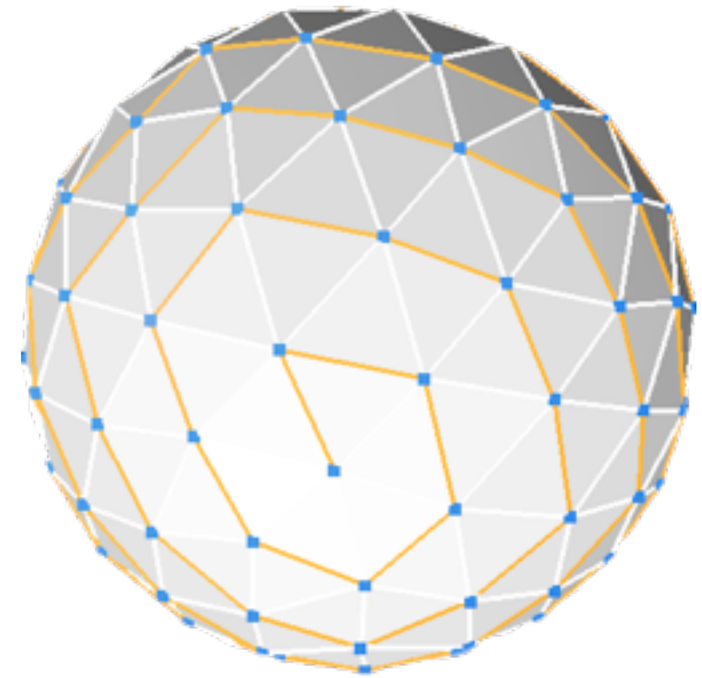
Tree-cotree decomposition

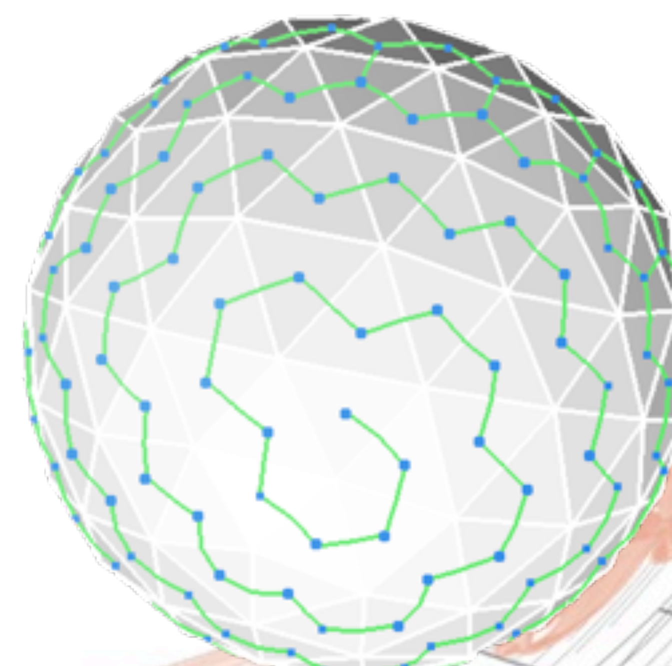


Slides from J. Erickson

Spheres: χ=2 critical







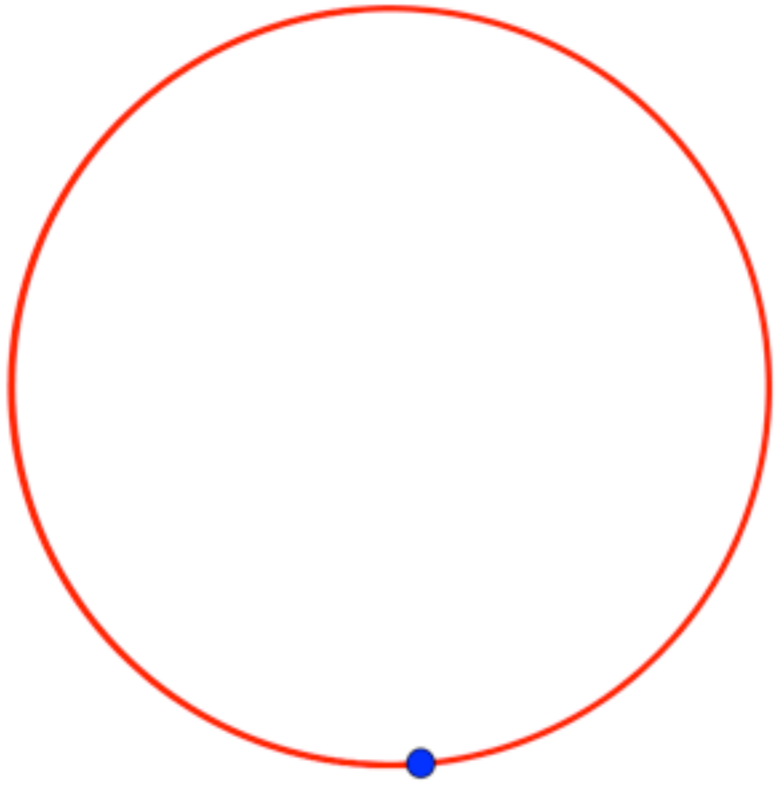
Jordan curve theorem:

dual spanning tree leaves a primal spanning tree

A stronger theorem

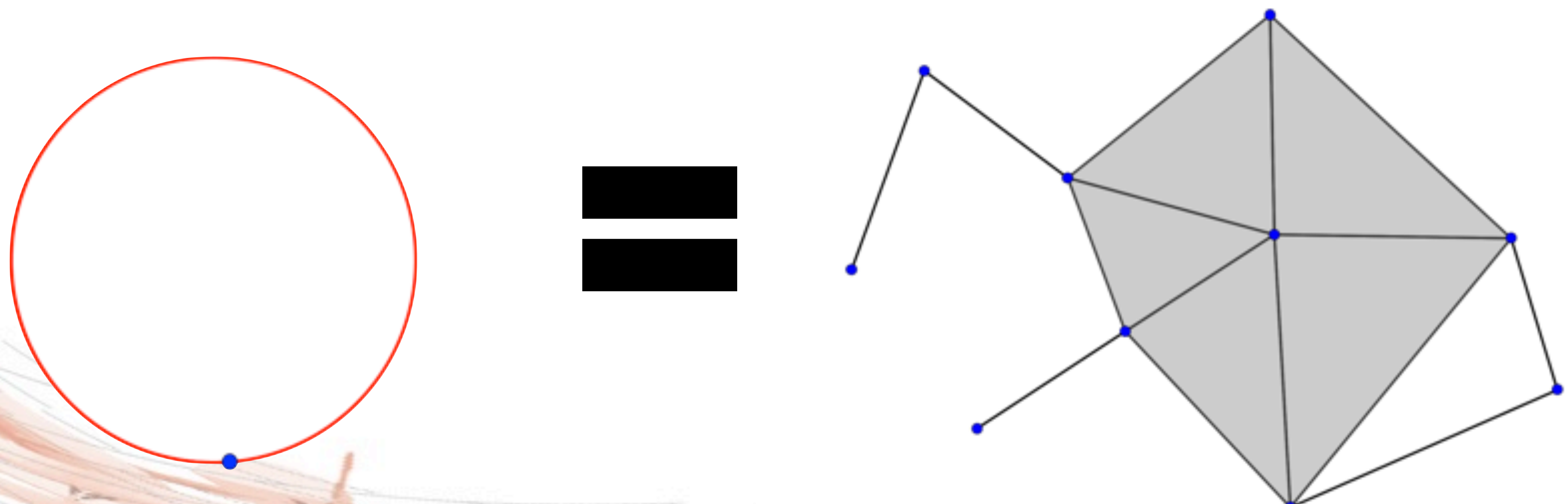
Given an acyclic matching on the Hasse diagram of a cell complex K, K is homotopy equivalent to a CW complex with only the critical cells

Idea of the proof

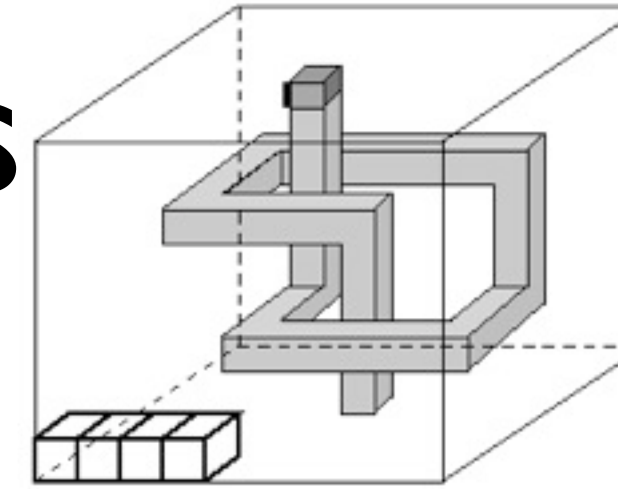


Main Theorem

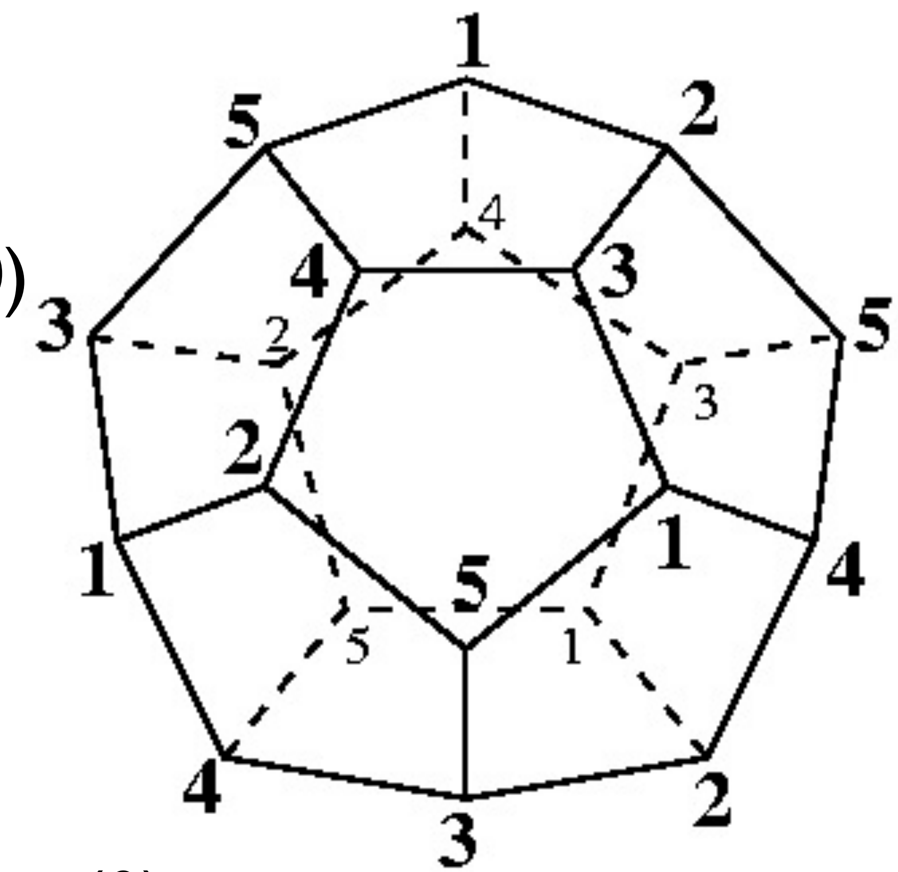
Given an acyclic matching on the Hasse diagram of a cell complex K, K is homotopy equivalent to a CW complex with only the critical cells



Topological bounds for manifolds



m(k) = # critical k-cells
β(k) = k^{th} Betti number



Euler characteristic

$\beta(n) - \beta(n-1) + \ldots \pm \beta(0) = m(n) - m(n-1) + \ldots \pm m(0)$

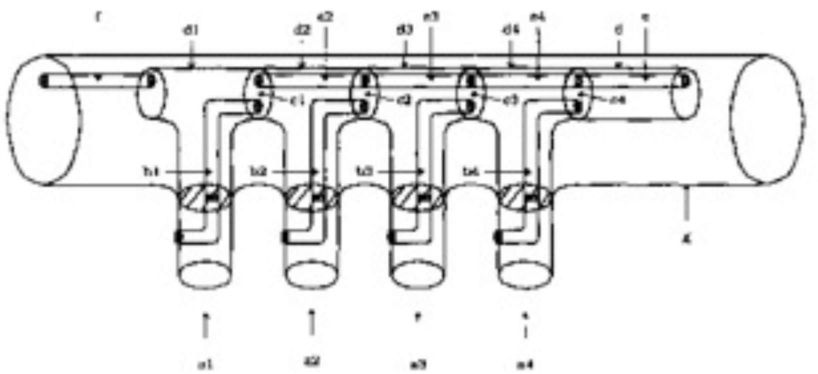
Weak Morse inequalities

$\beta(k) \leq m(k)$

Strong Morse inequalities

$\beta(k) - \beta(k-1) + \ldots \pm \beta(0) \leq m(k) - m(k-1) + \ldots \pm m(0)$

Complexity from topology



Ö. Eğecioğlu and T. F. Gonzalez

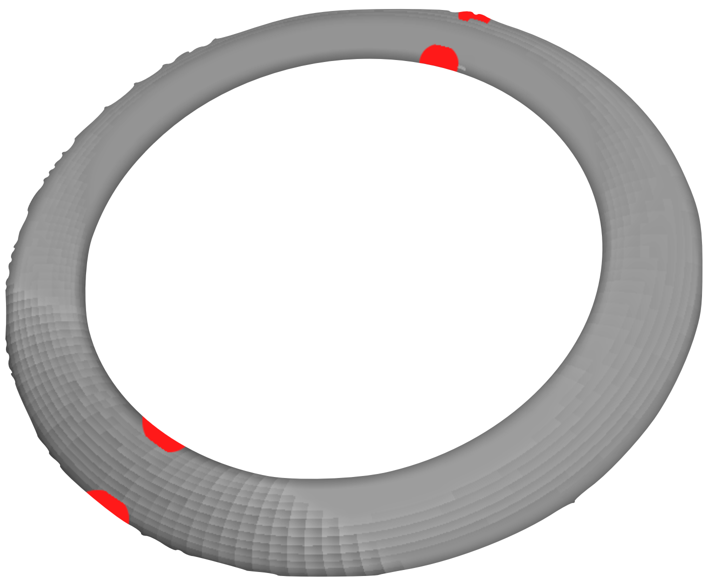
MAX SNP-hard

Reduces to collapsibility:

smaller number of simplices to remove from a 2-simplicial complex for it to collapse.

reduces to vertex cover

Quick way of computing topology!!!



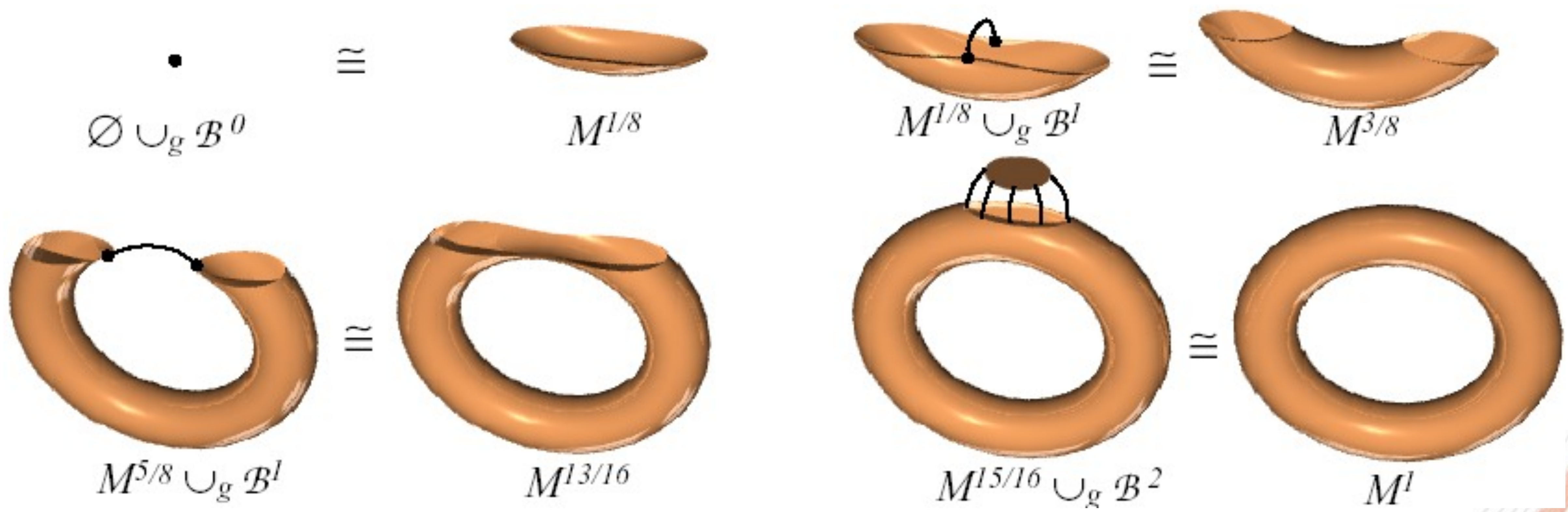
get the big picture

partially self-validated

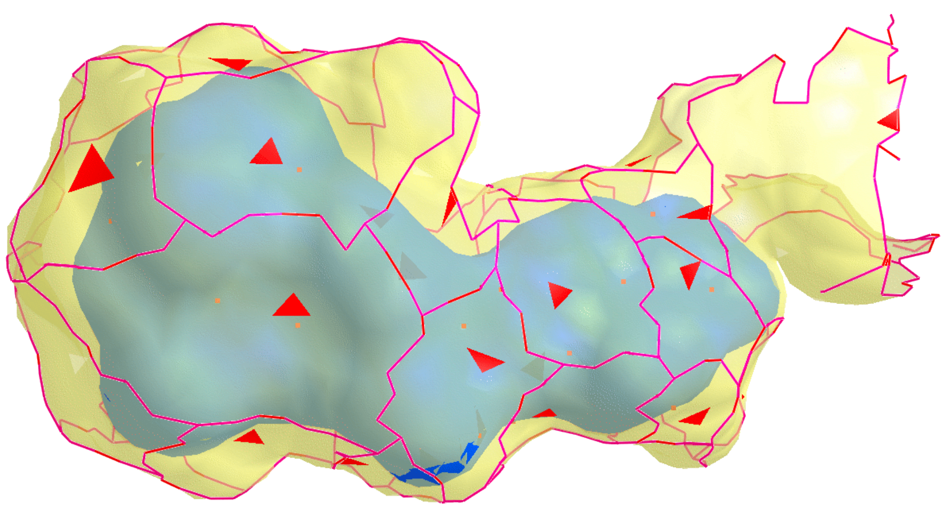
$$\chi = \sum_{i=0}^{d} (-1)^i \cdot m_i$$

global (high info) from local (low cost)

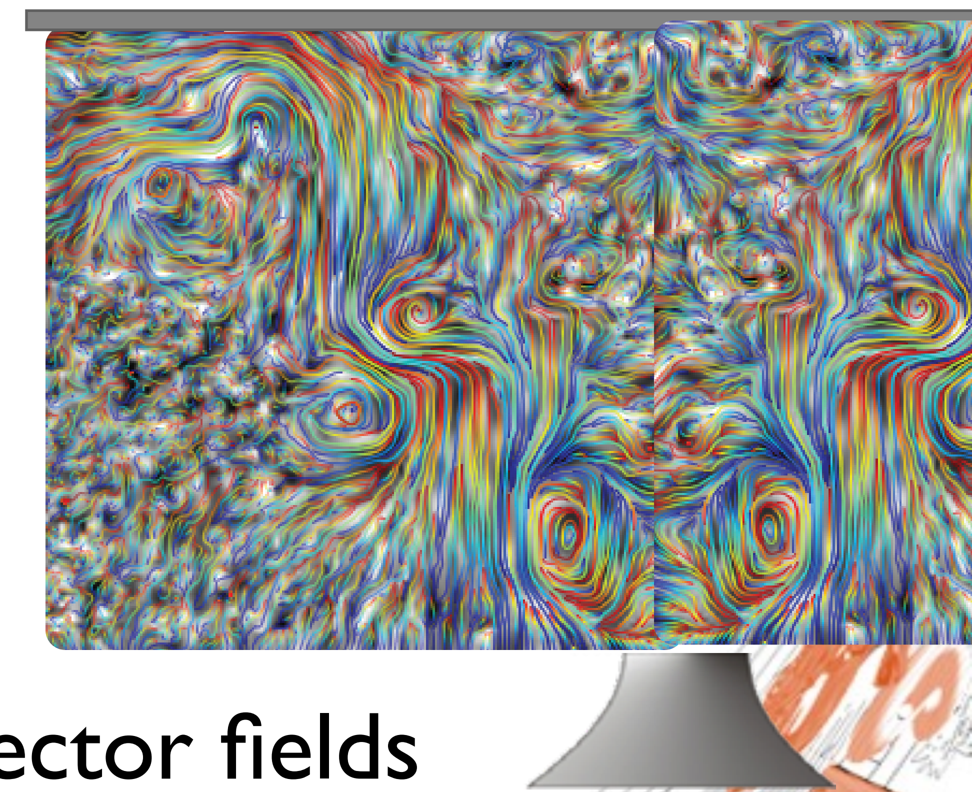
Frame by frame



Topological objects

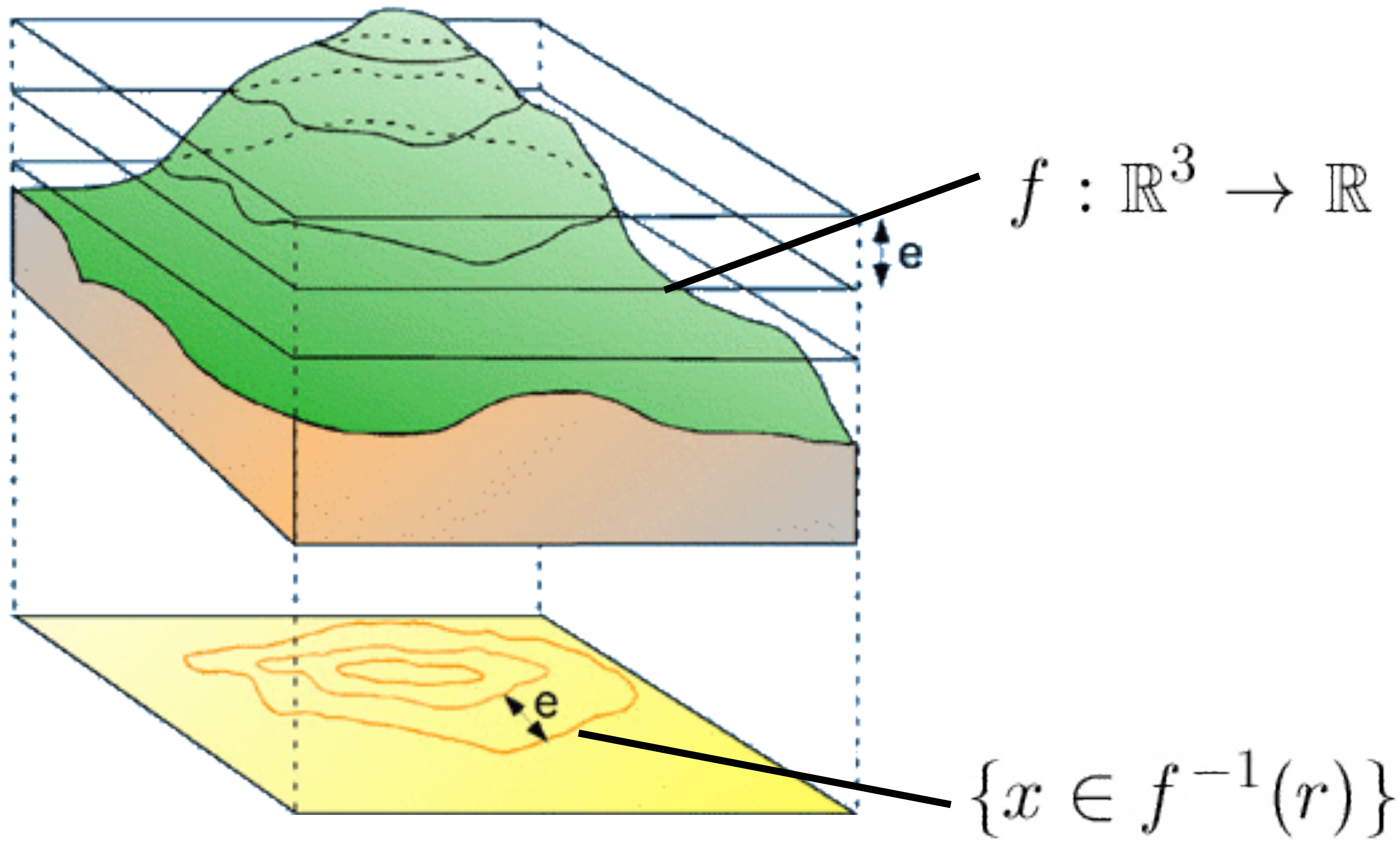


Manifolds,
Subsets of $\mathbb{R}^n$



Vector fields

Submersion intuition

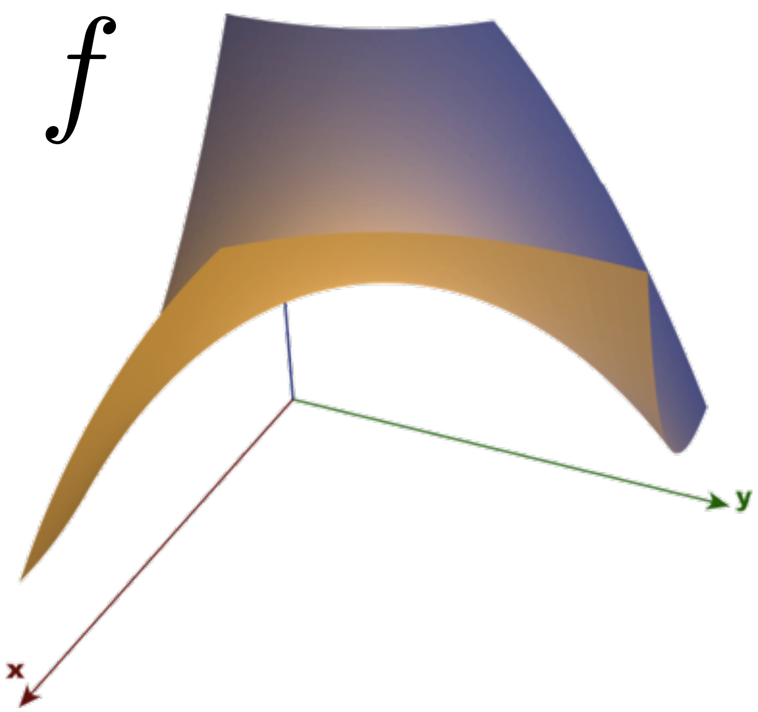


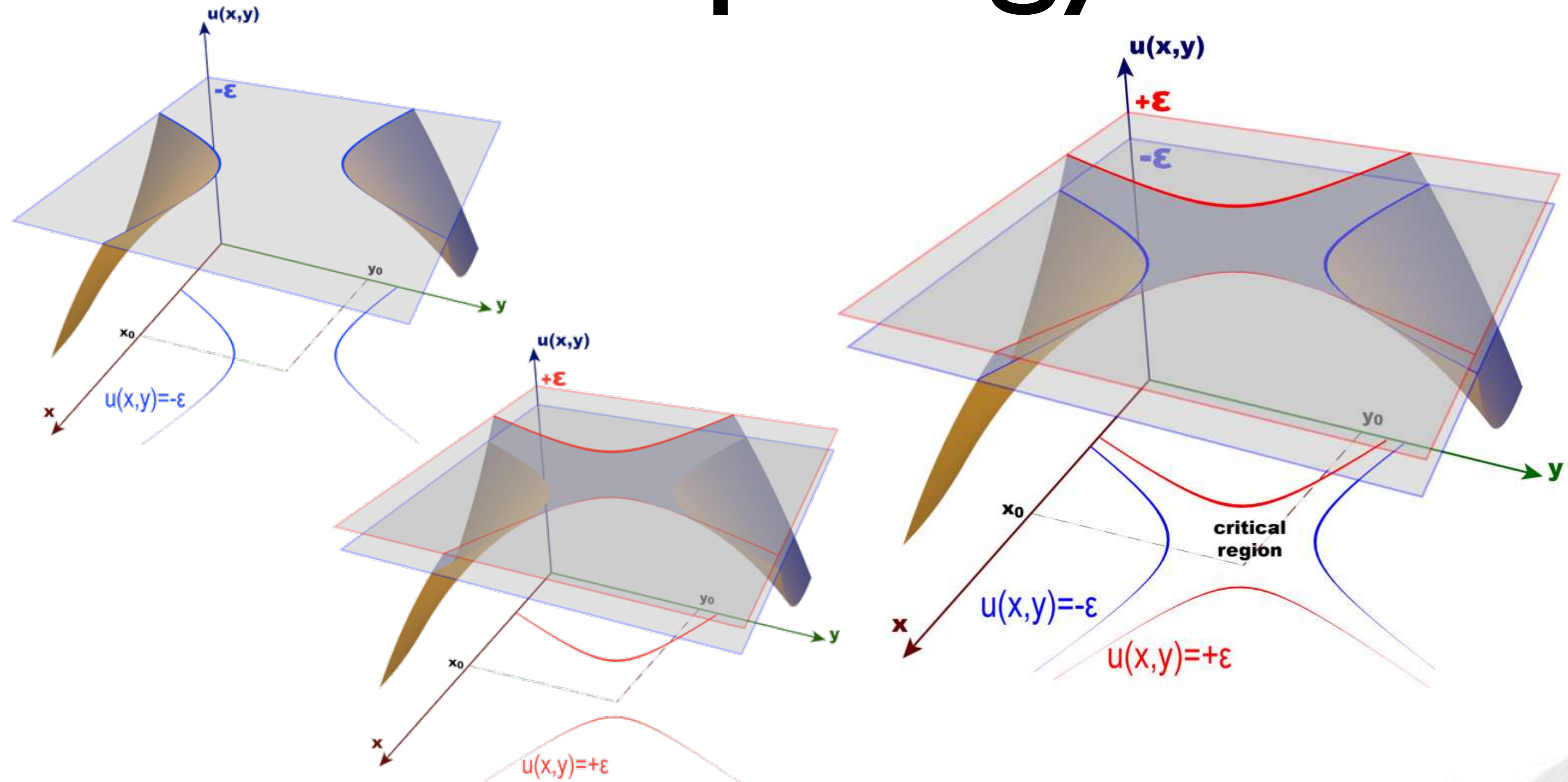
subset of $\mathbb{R}^n$ respecting a condition f

$\Rrightarrow$ closer to real data

Submersion topology

f

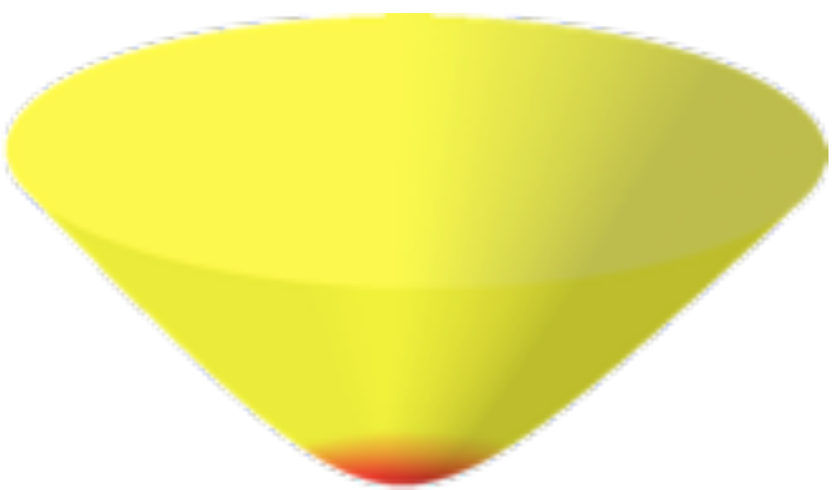




critical set of f (Morse lemma)

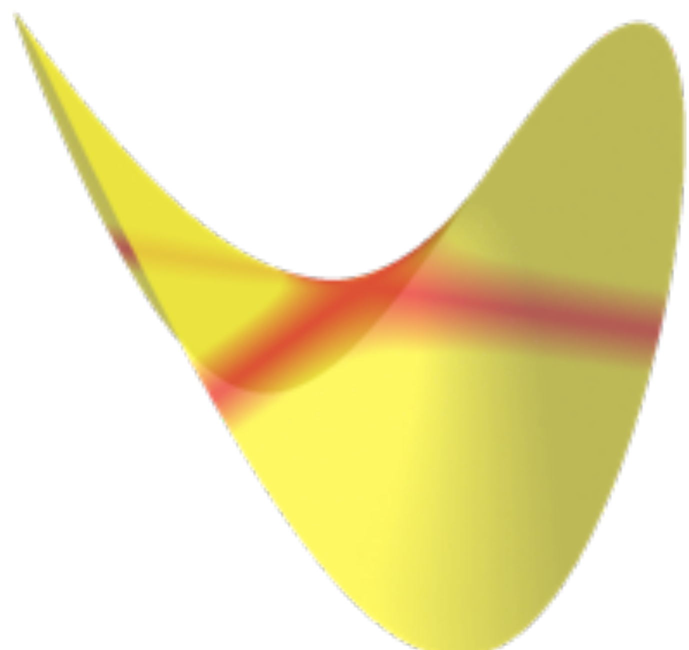
$\Rrightarrow$ global from local function analysis

Usual critical sets



minima

new
component



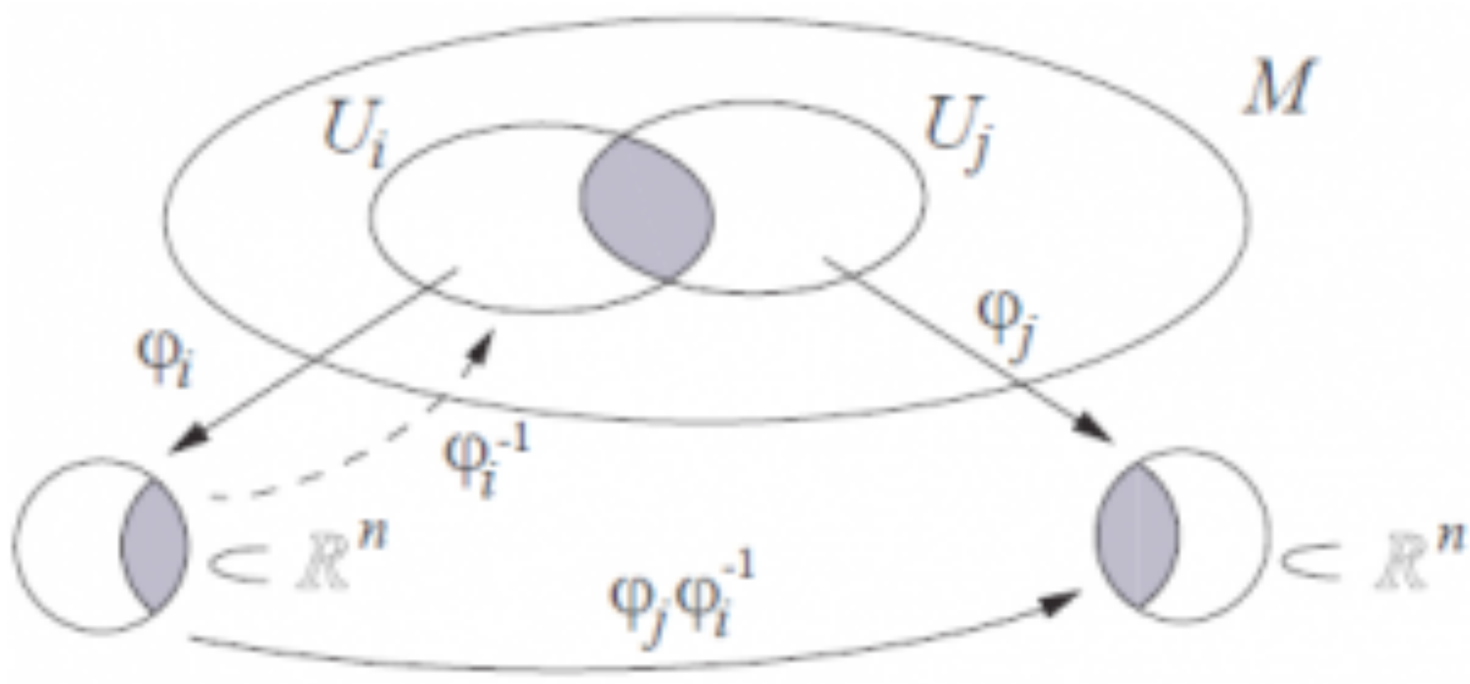
saddles

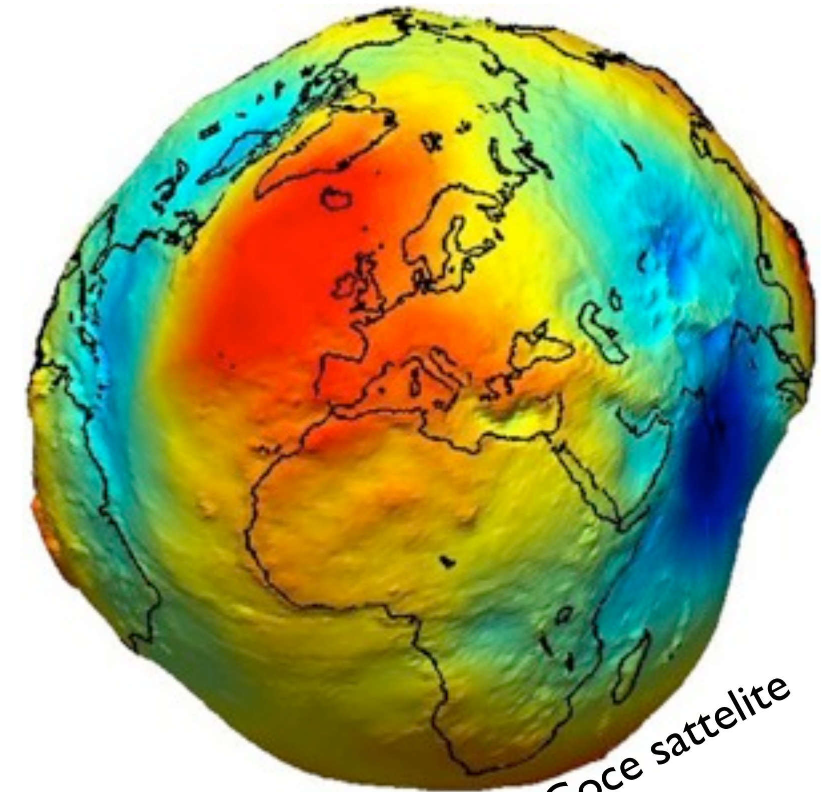
joins / splits
components

maxima

end
component

Immersion intuition

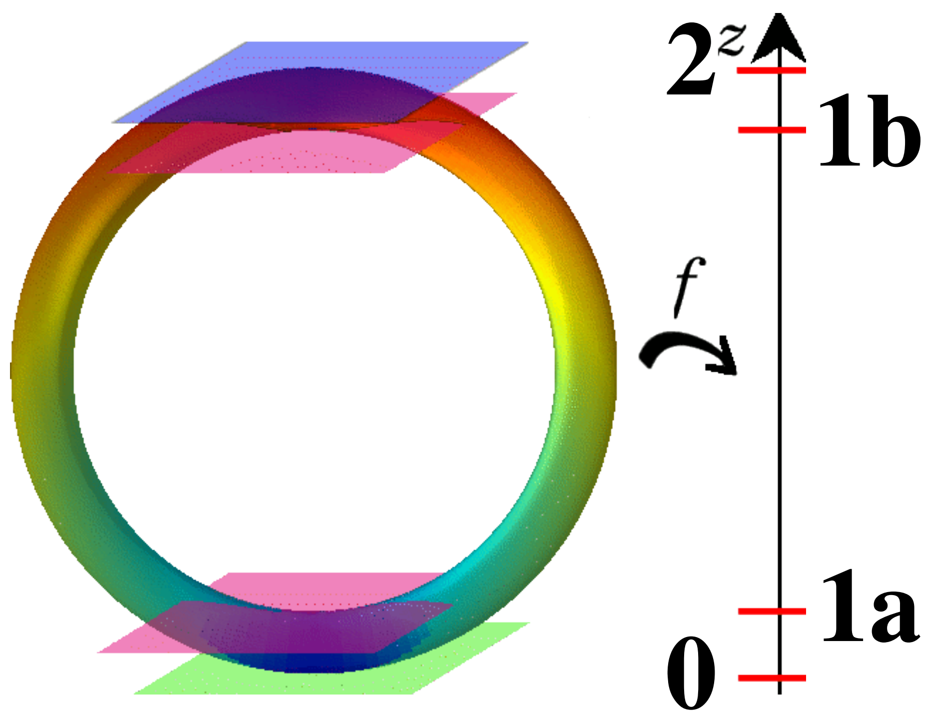


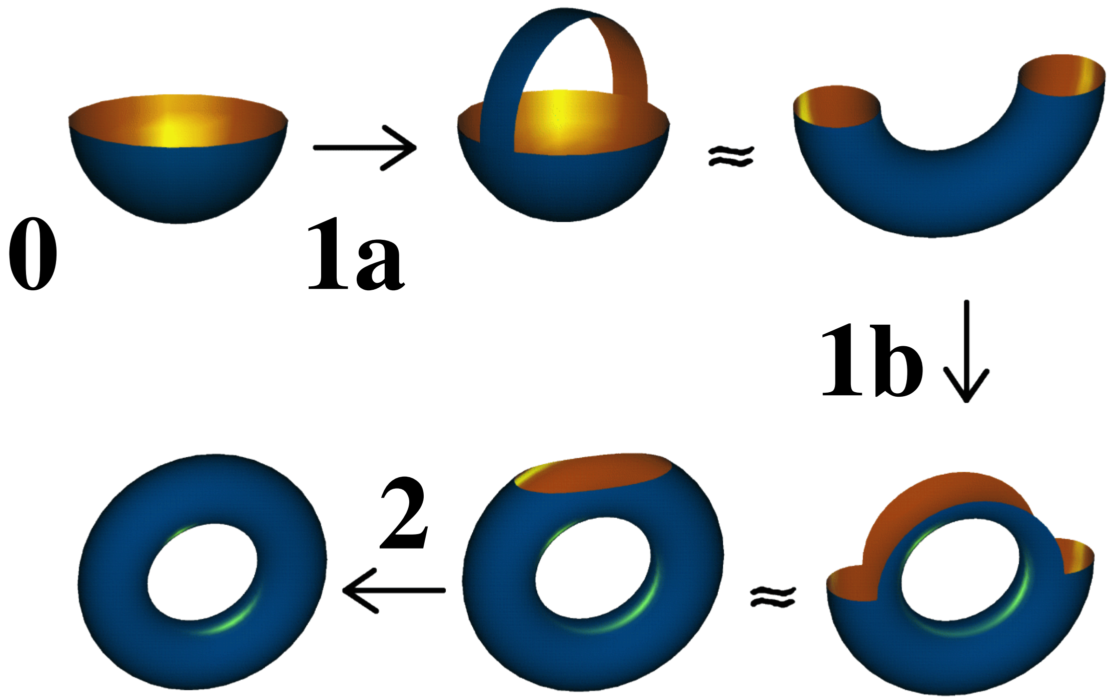


locally equivalent to $\mathbb{R}^d$

$\Rightarrow$ intuitive differential tools

Immersion Morse topology

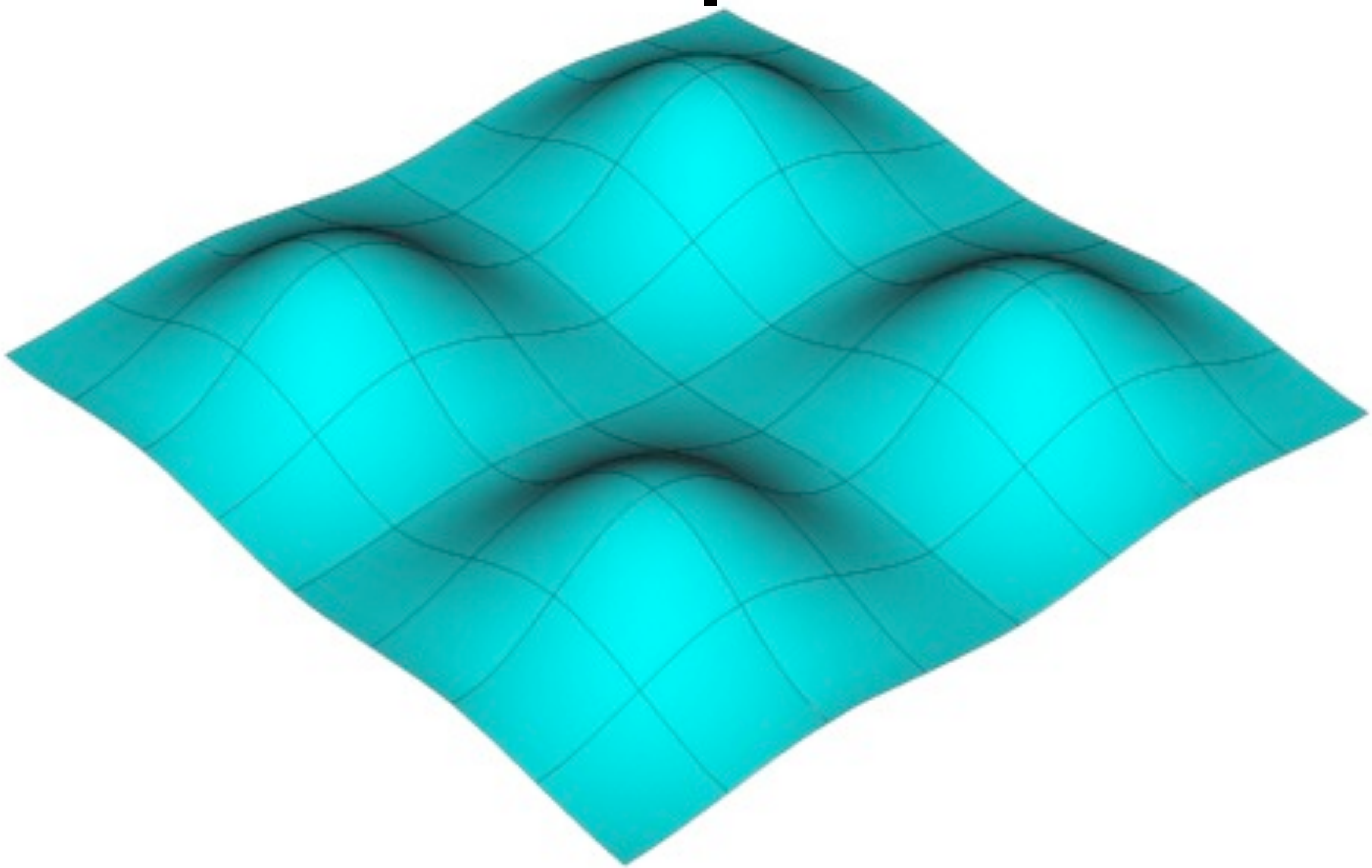




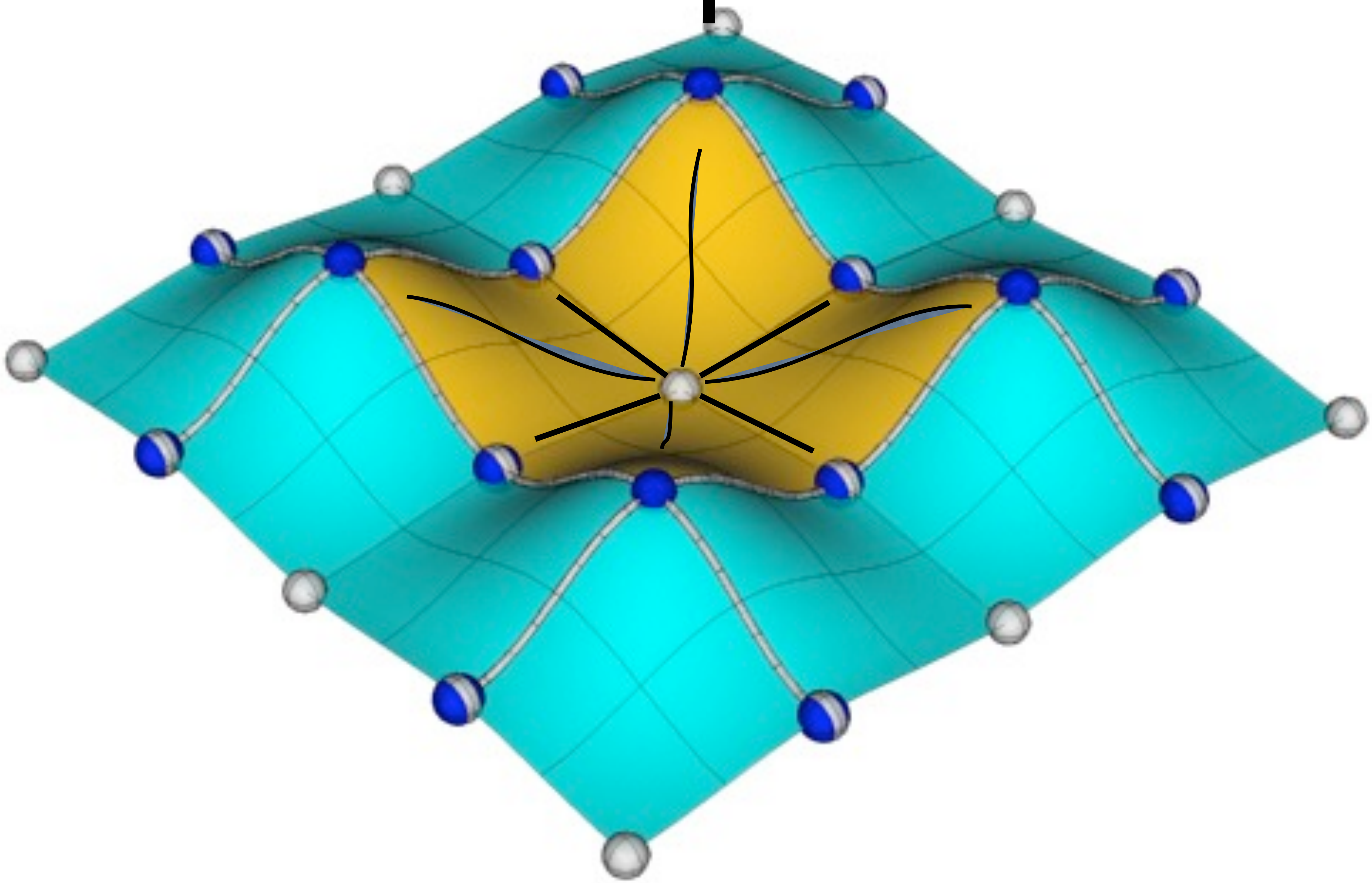
critical set of a function on the manifold

⇛ global from local function analysis

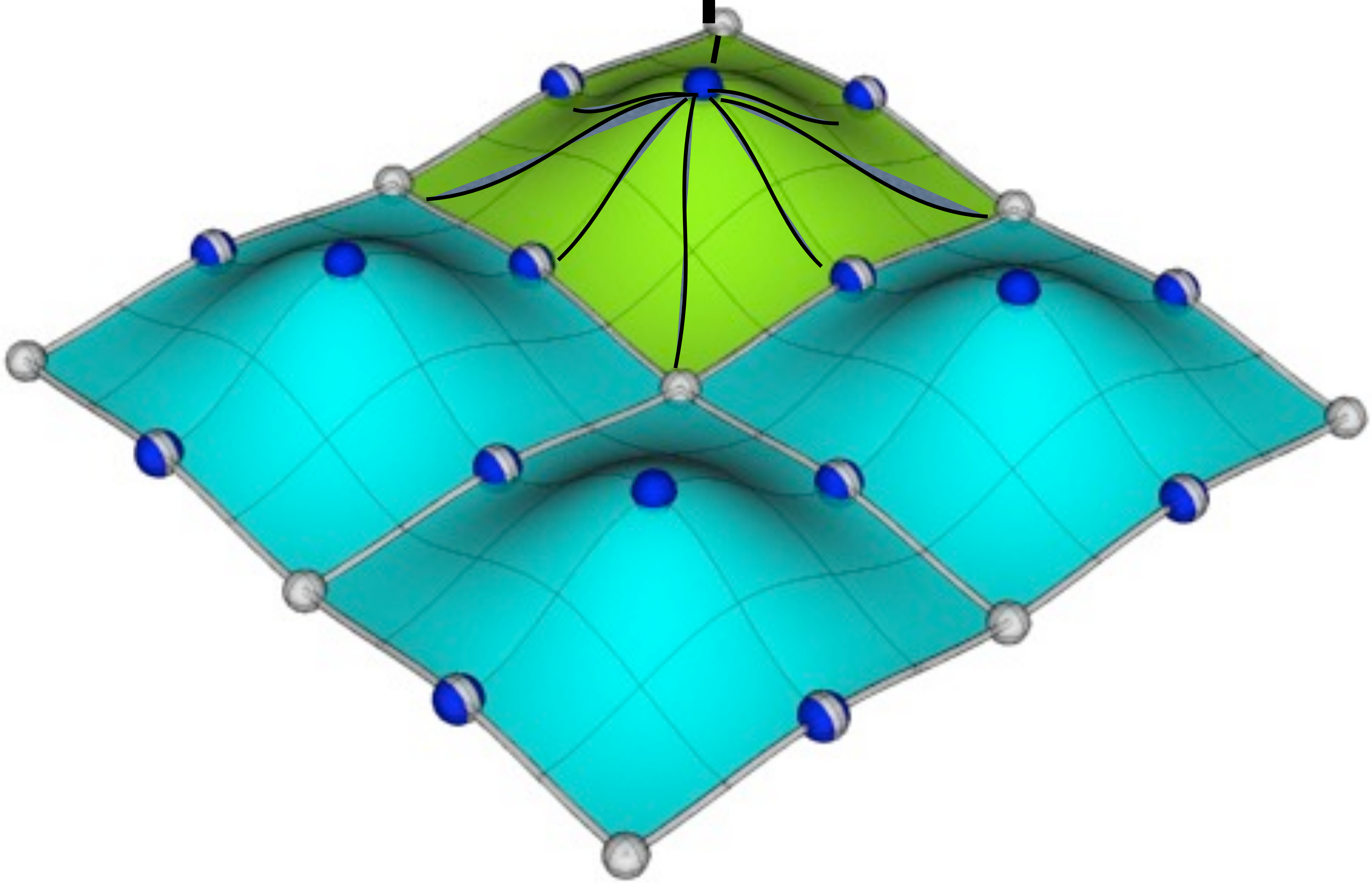
Morse-Smale Decomposition



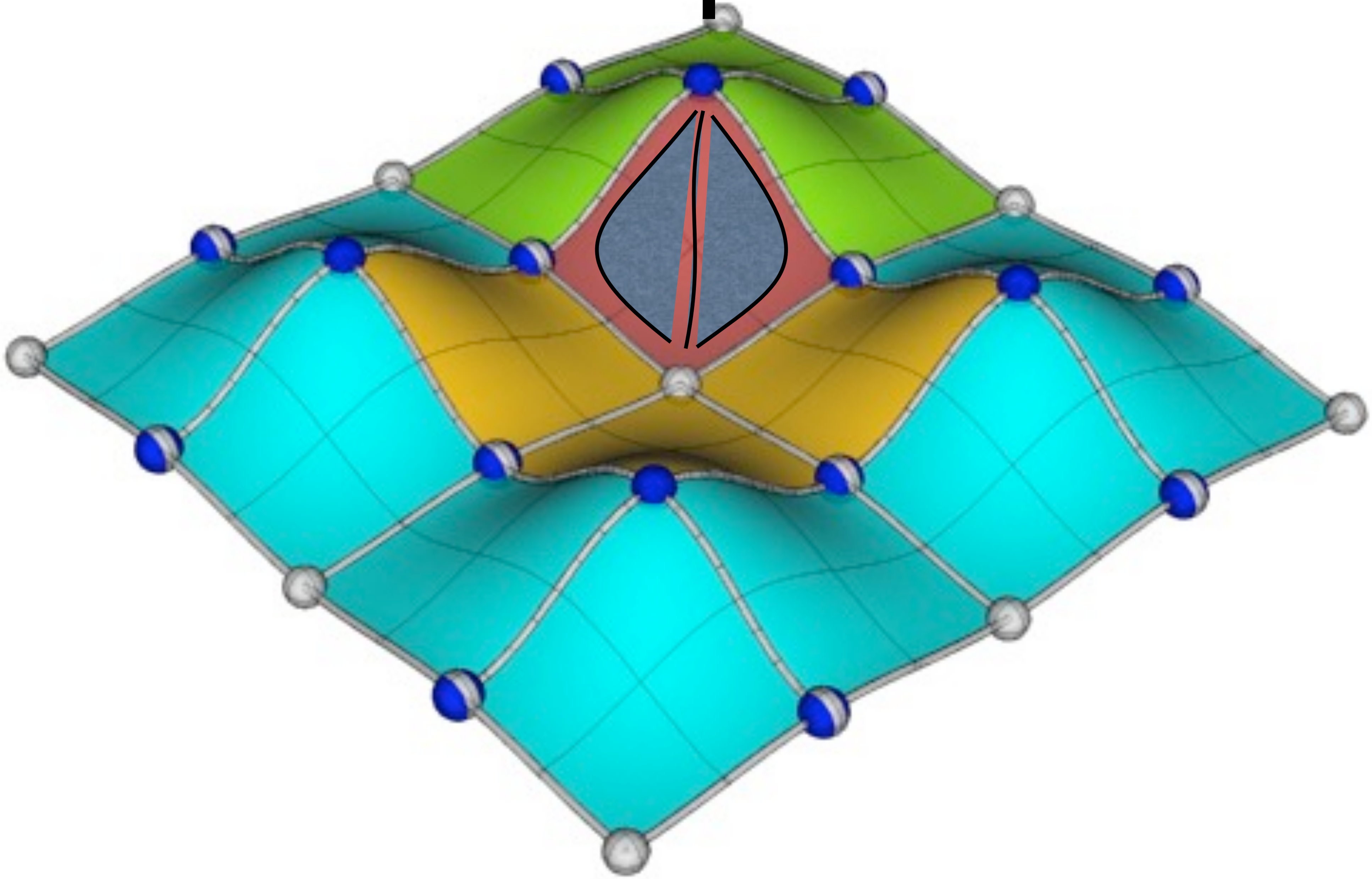
Morse-Smale Decomposition



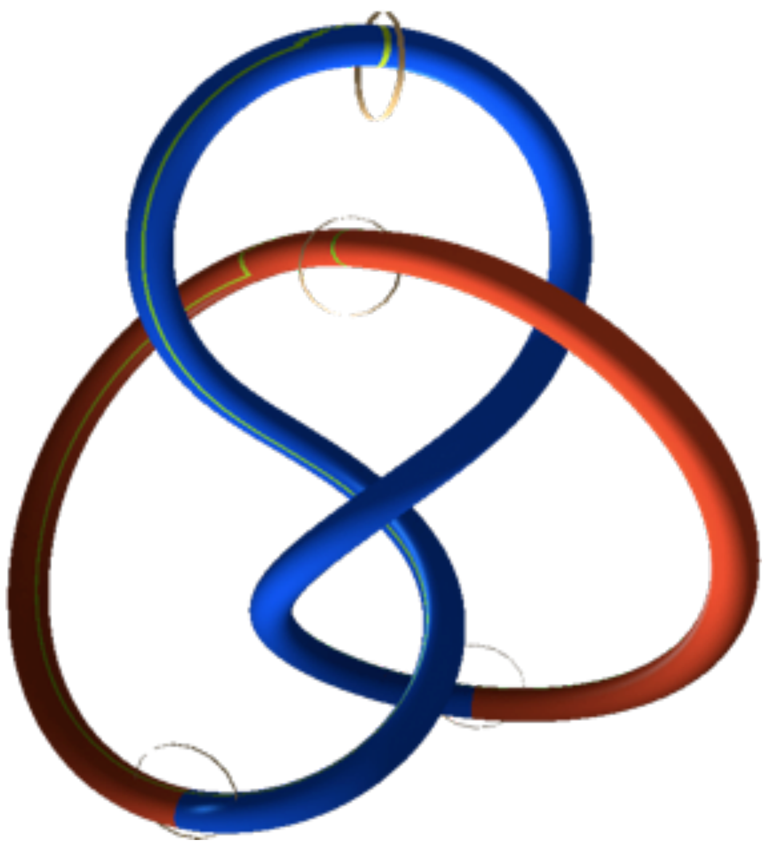
Morse-Smale Decomposition

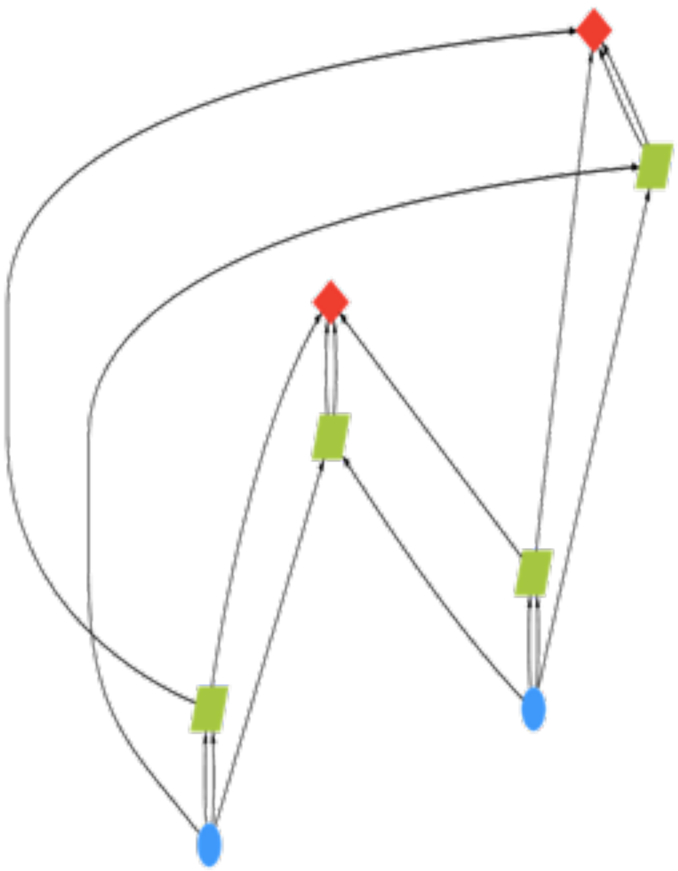


Morse-Smale Decomposition



Morse-Smale complex

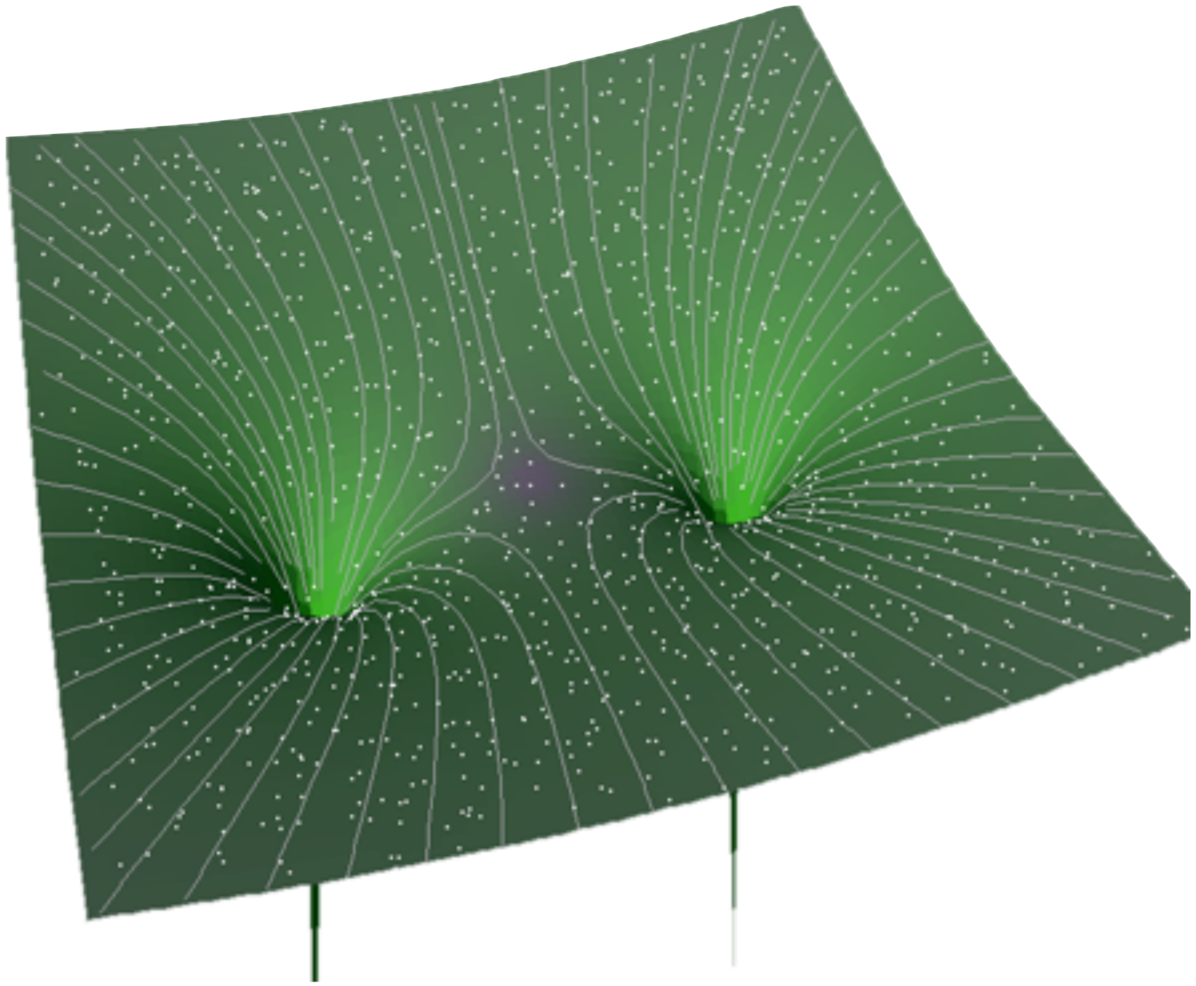


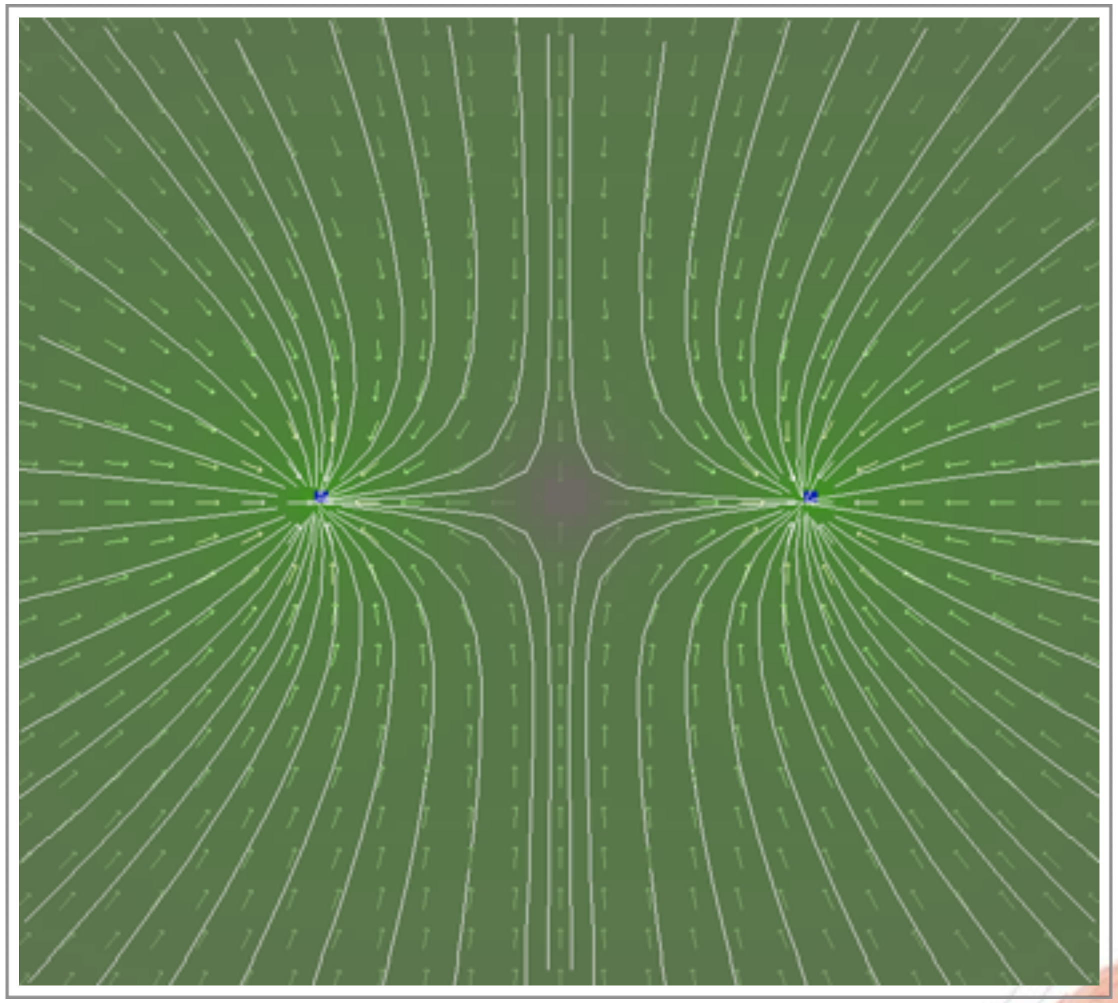


relation between critical points

⇛ local function analysis + graph

Vector field



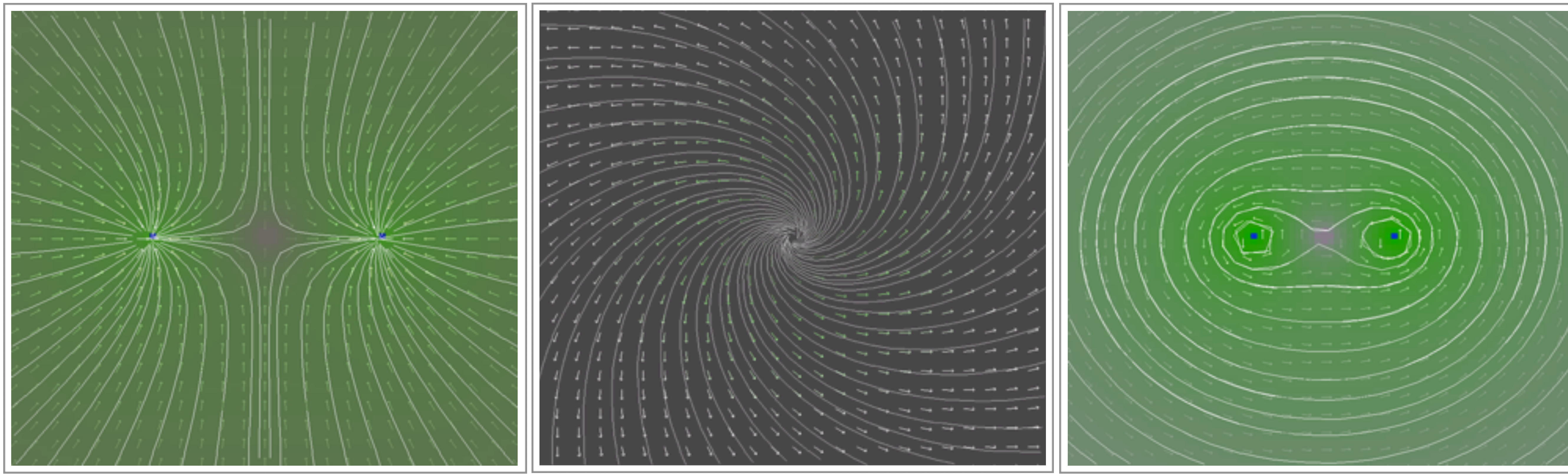


© http://www.falstad.com/vector/

essentially tubular flow

sparse invariant sets

Vector field topology

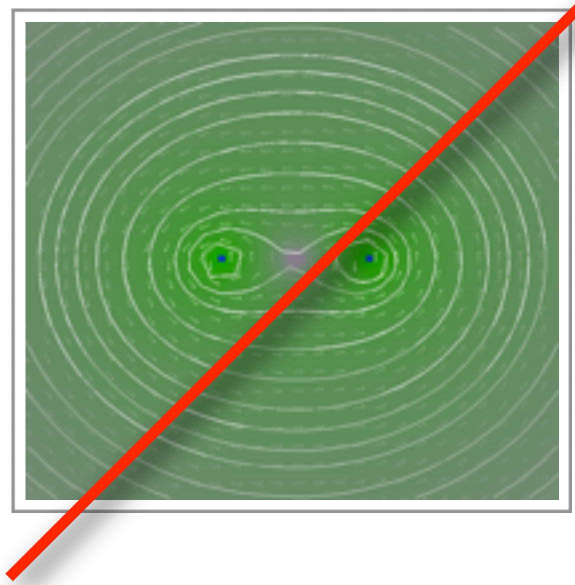


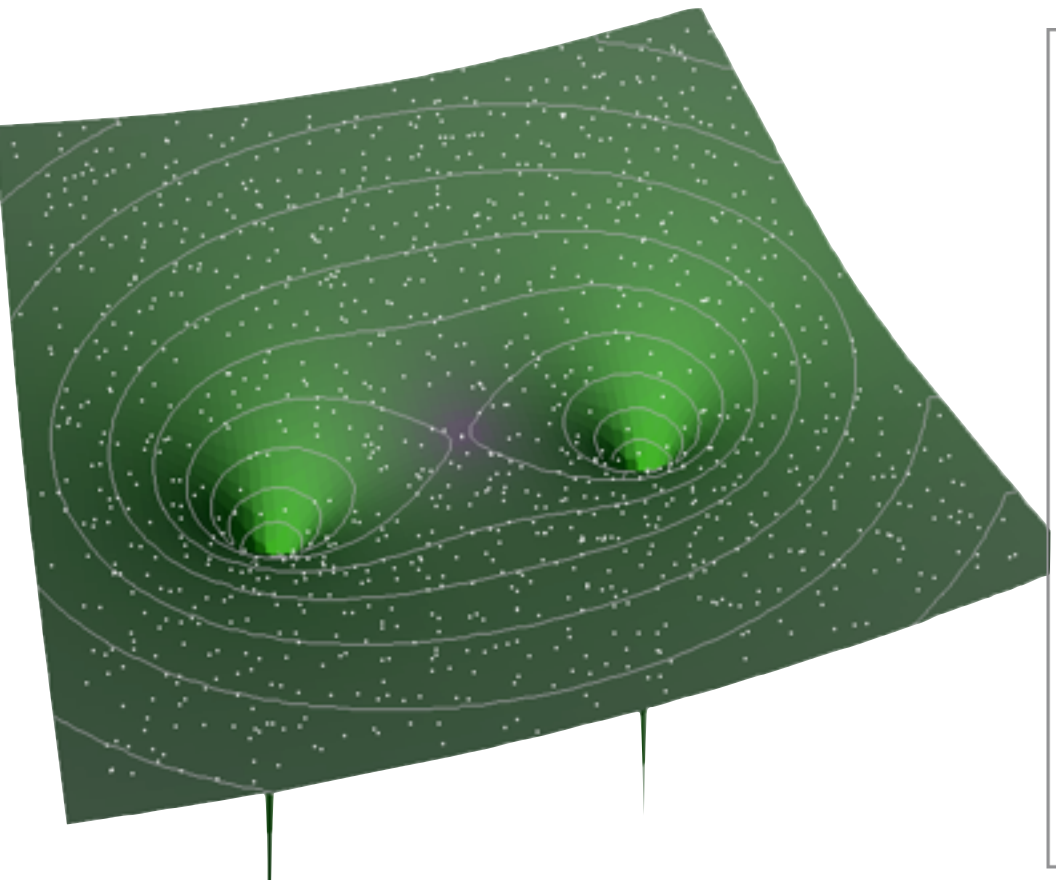
isolated singularities behavior

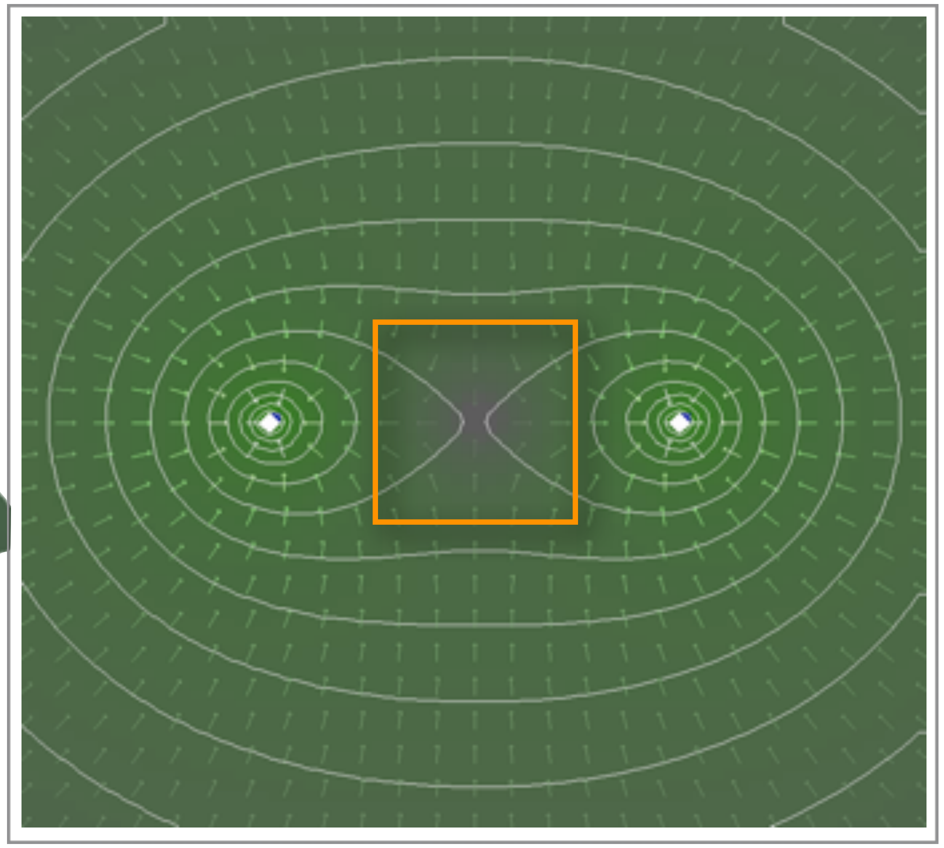
⇛ local analysis (Hartman Grobman)+graph

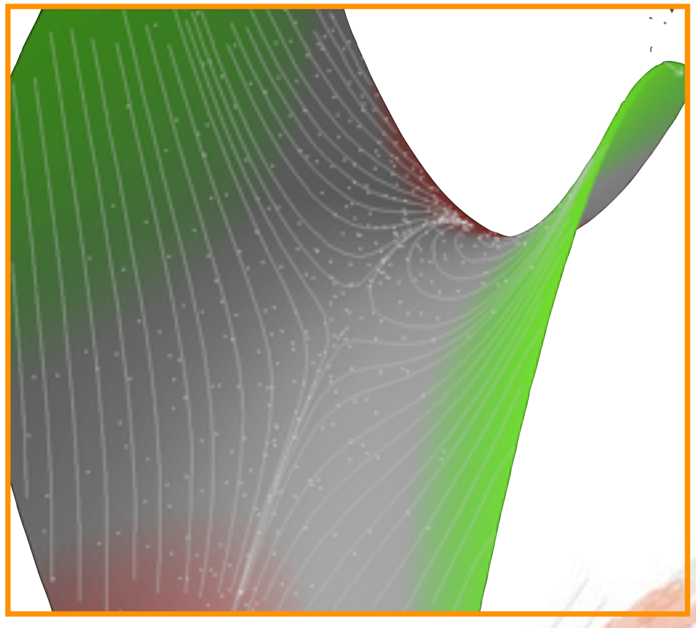
+ closed orbits + non-generic

Gradient vector field





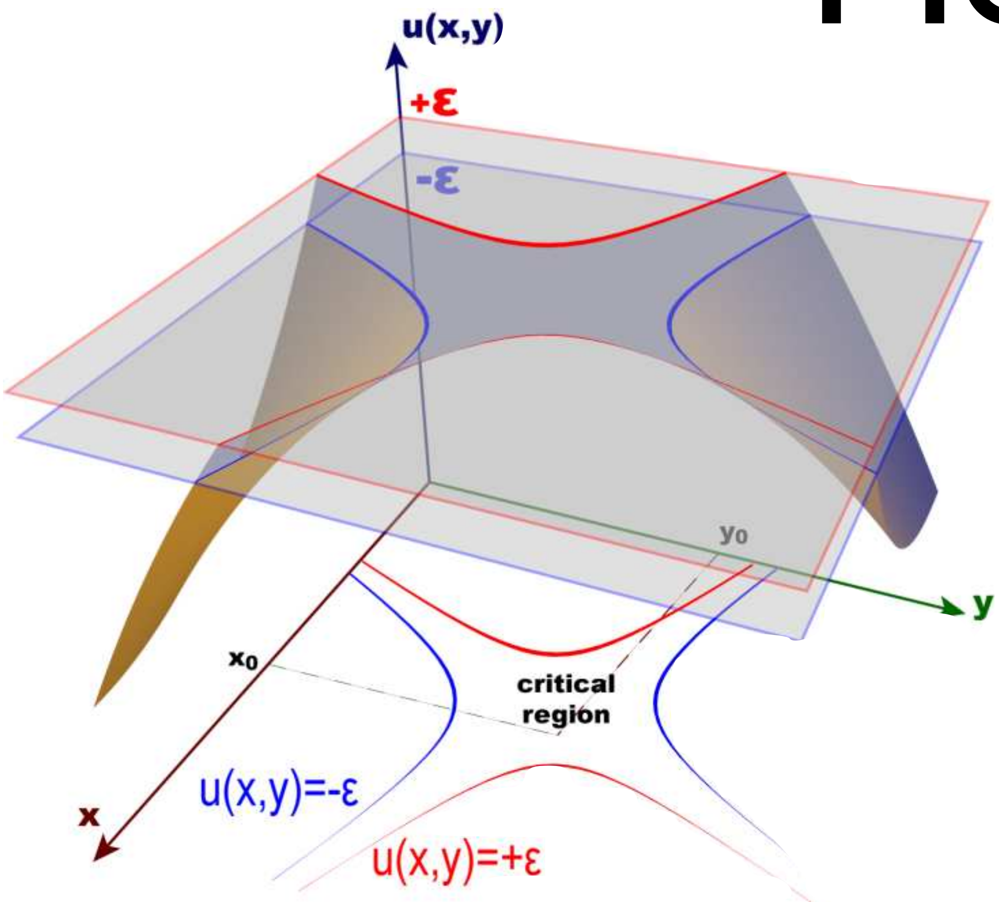




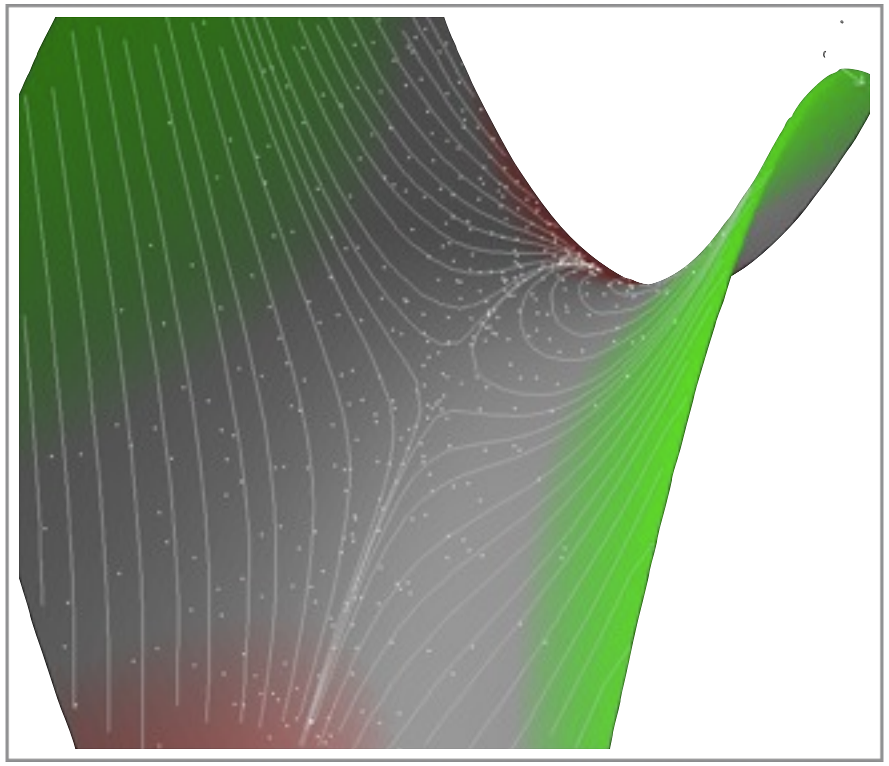
generic gradient

$\Rrightarrow$ Morse-Smale structure

Morse theory

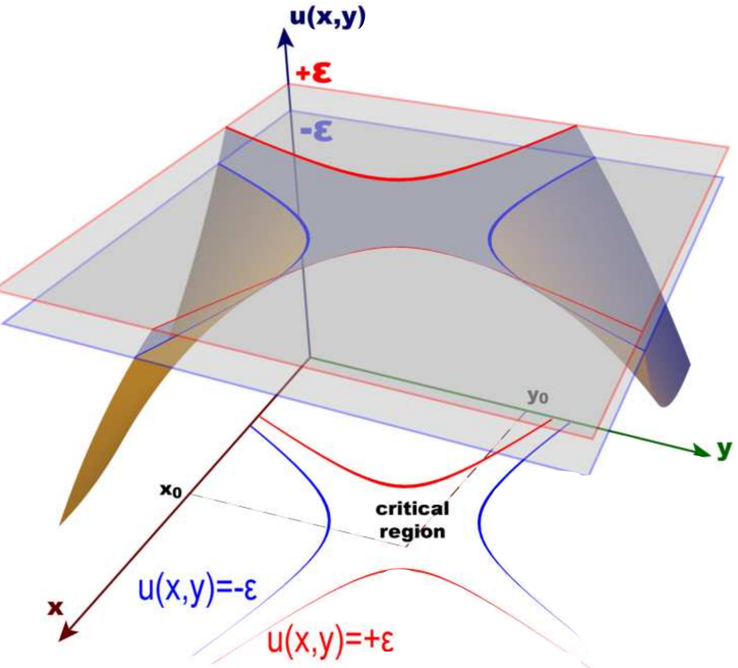


$$\chi = \sum_{i=0}^{d} (-1)^i \cdot m_i$$



topology from local function analysis

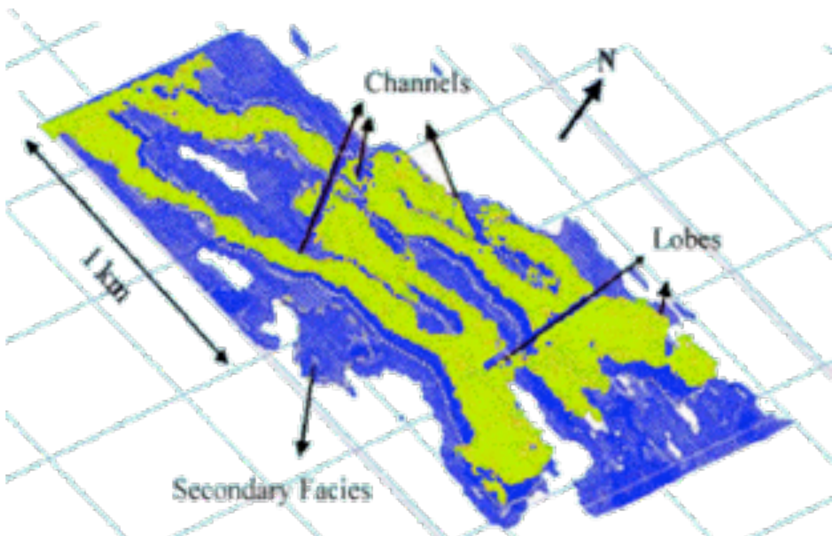
+ Smale complex / topological graph



Morse theory

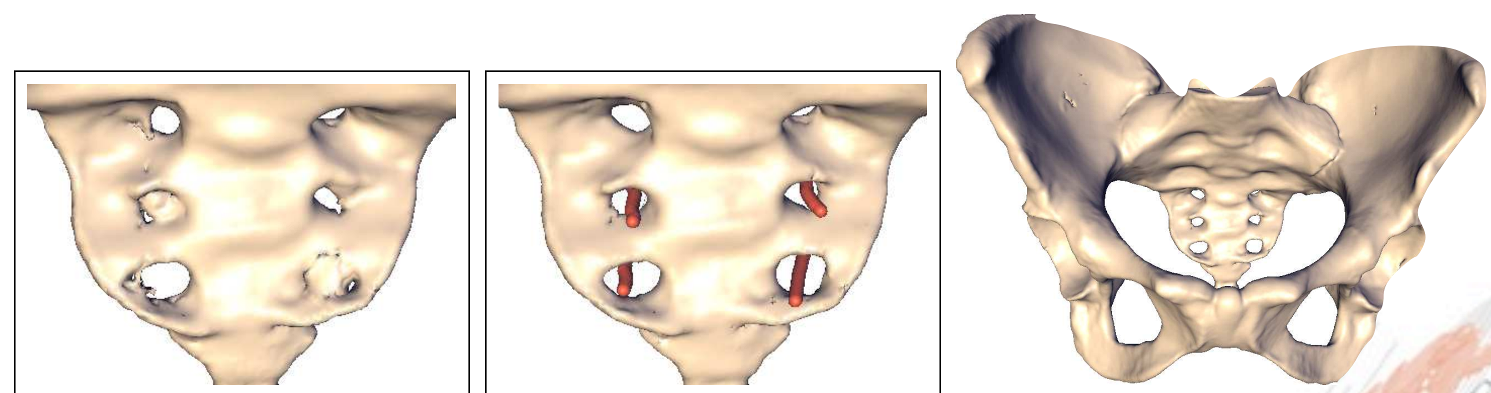
Manifold	$\mathcal{M} \subset \mathbb{R}^n$
Function	$f : \mathcal{M} \to \mathbb{R}$
Critical point	$\mathbf{x} \in \mathcal{M}, \partial f(\mathbf{x}) = 0$
Index	$\#\{\lambda \in_d Eig(\partial^2 f), \lambda < 0\}$
Topology	$\chi = \sum_{i=0} (-1)^i \cdot m_i \quad \ldots$

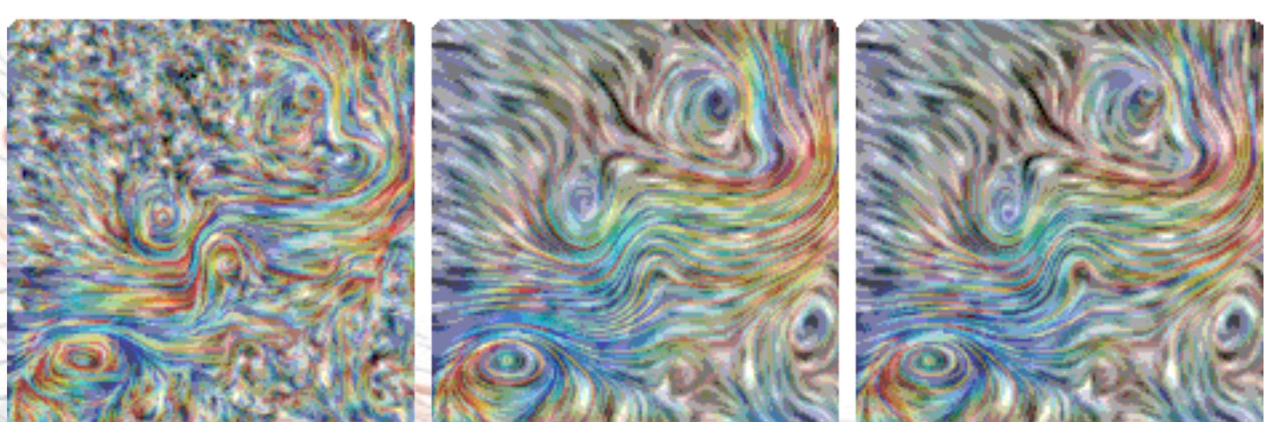
Applications that motivated me



reservoir characterization
from huge seismic data

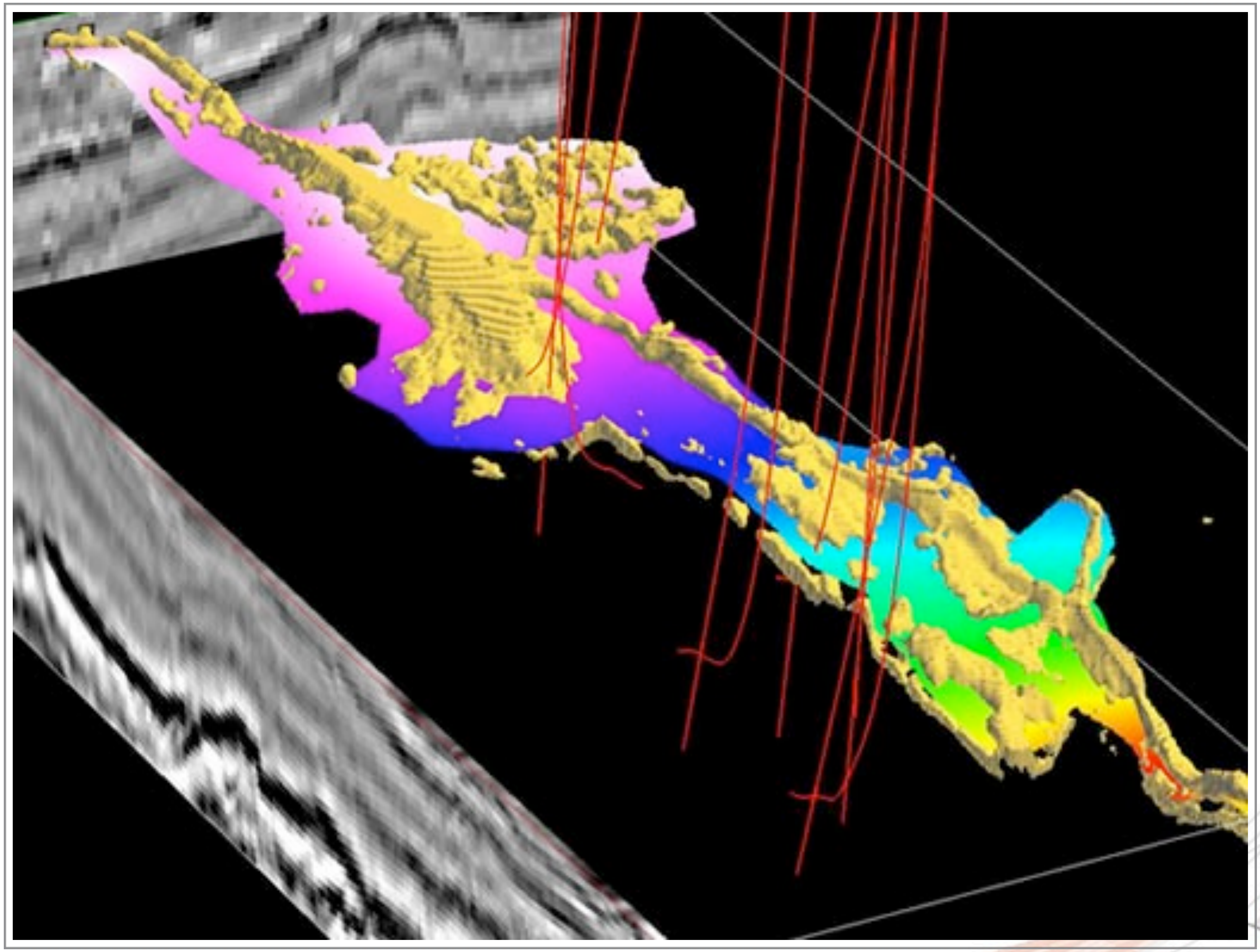
surface extraction
and reconstruction





vector field de-noising

Isosurface extraction



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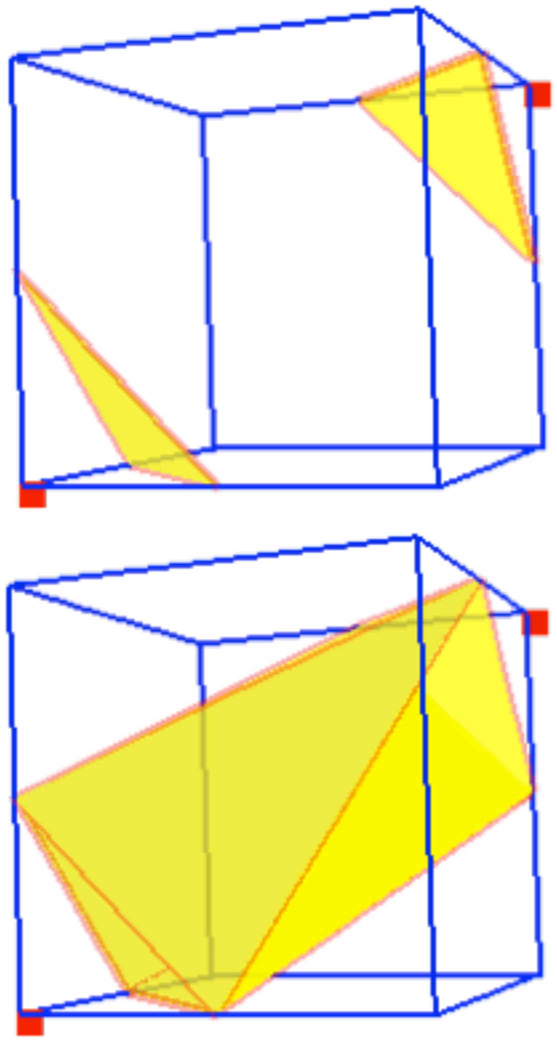
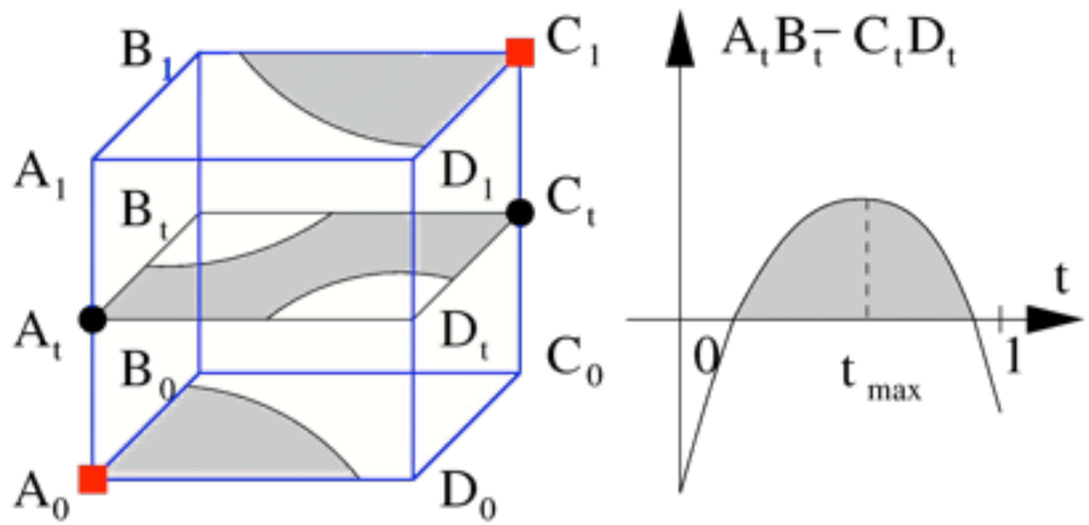
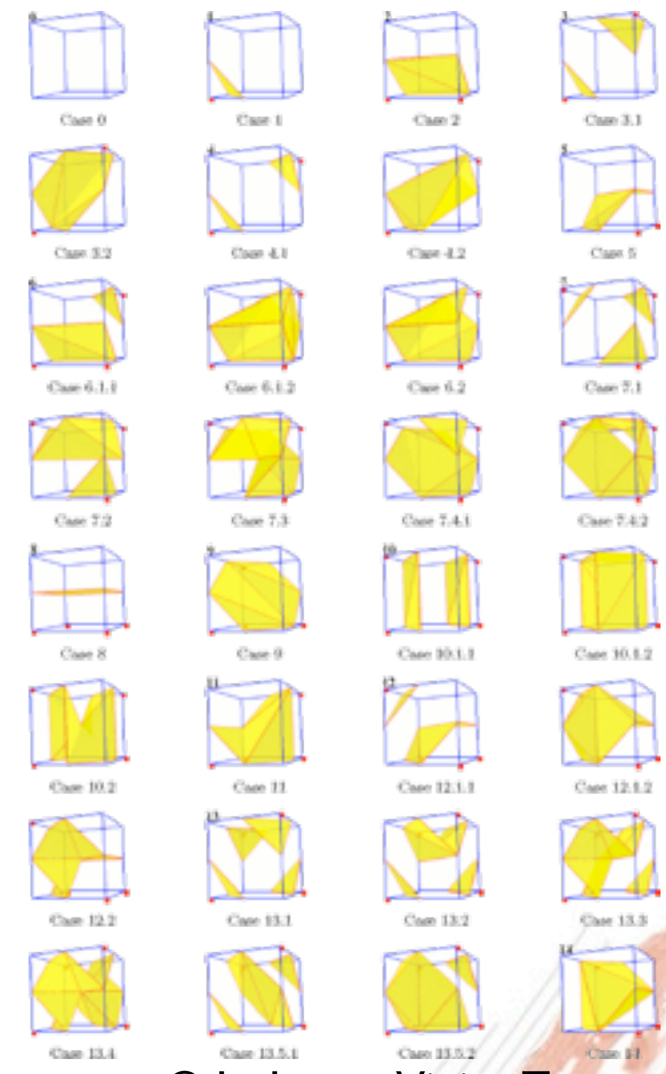
Isosurface extraction







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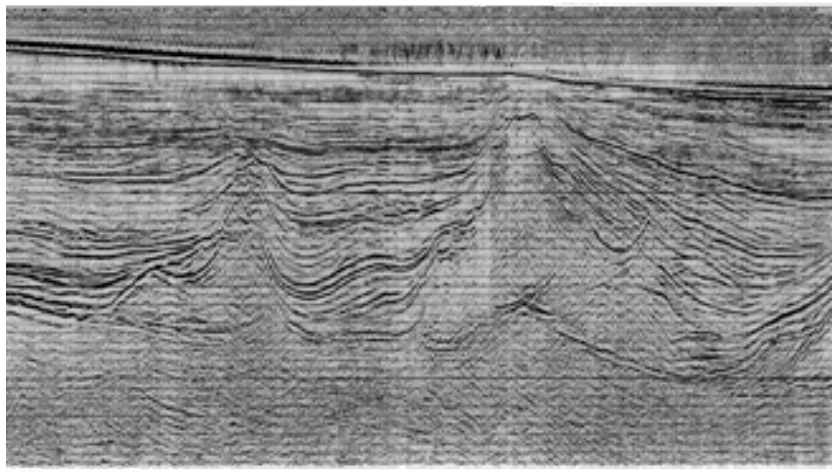
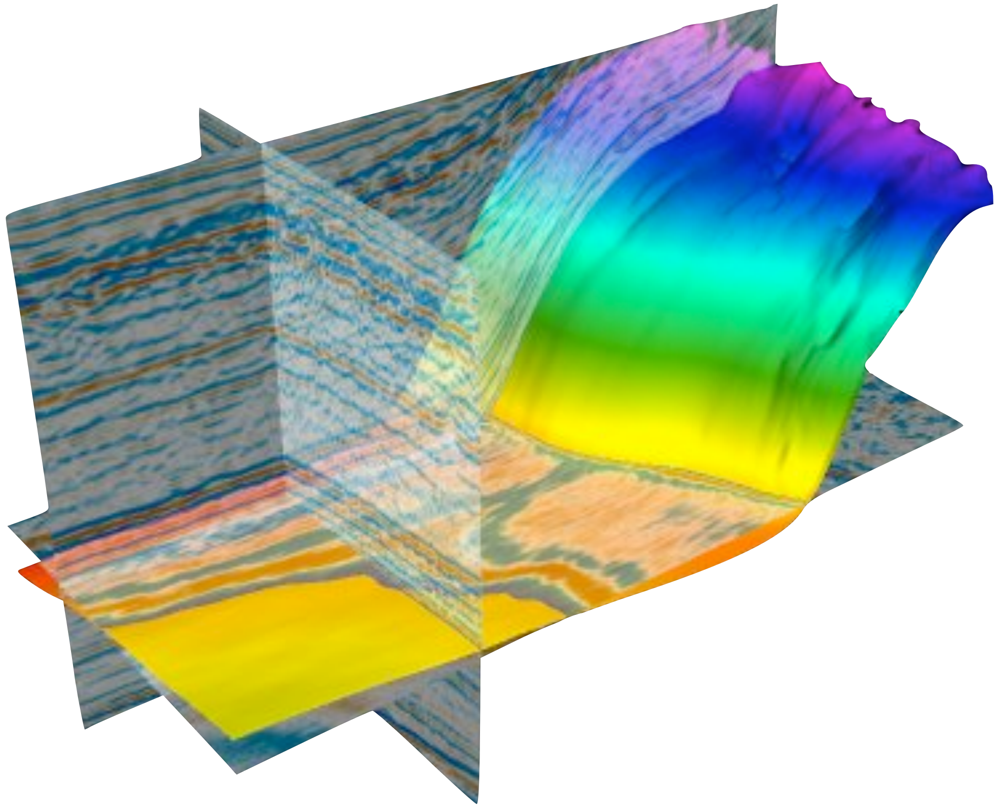
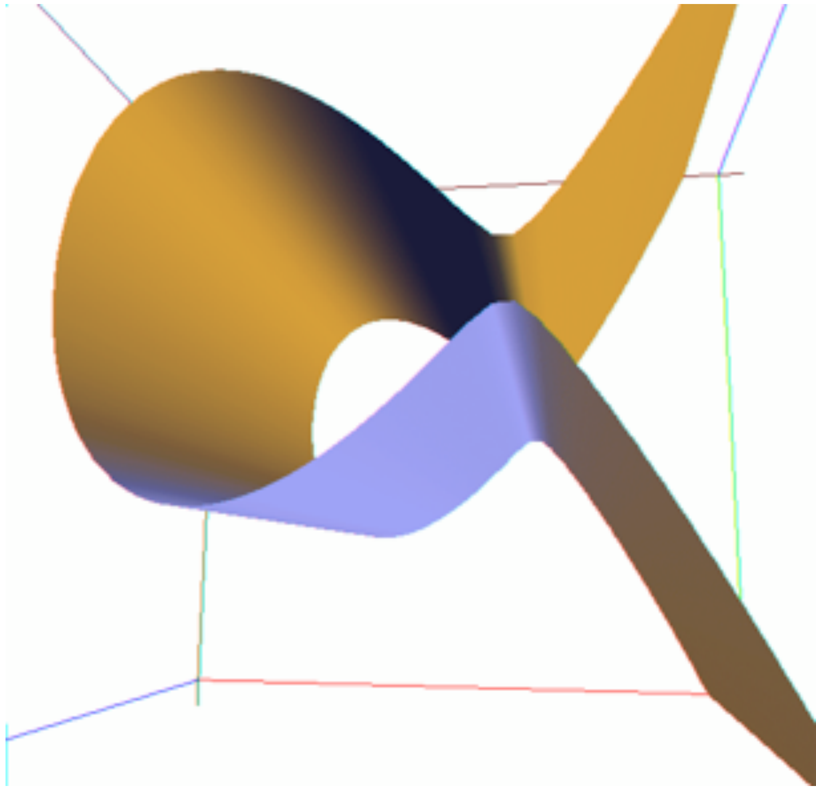
Topological cases of Marching Cubes

⇛ differentiable function analysis

Large isosurface topology





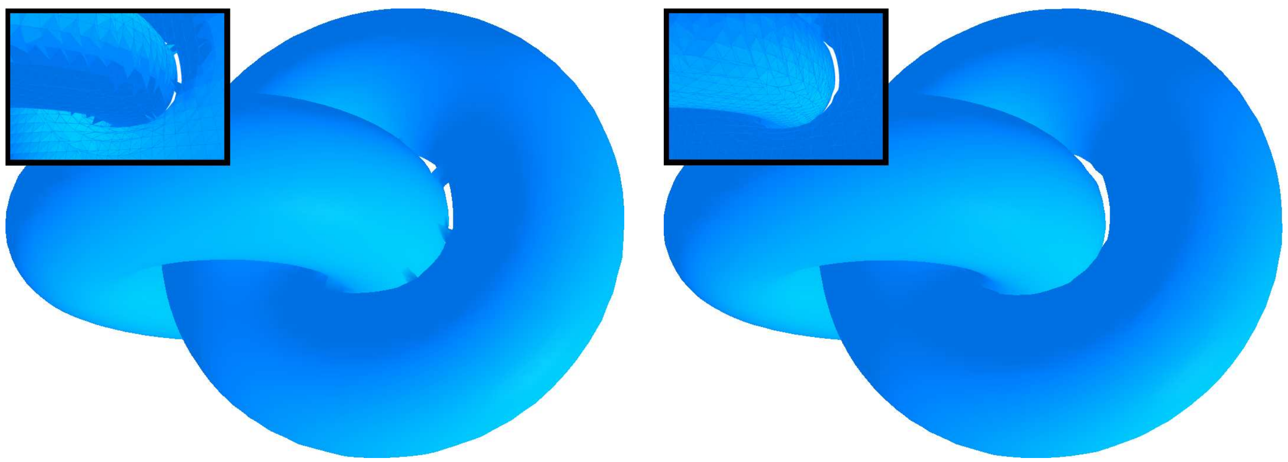


Topology without the isosurface

Mid-scale control and filtering

global + efficiency ⇛ Forman's line

Some Isosurfaces' Topology



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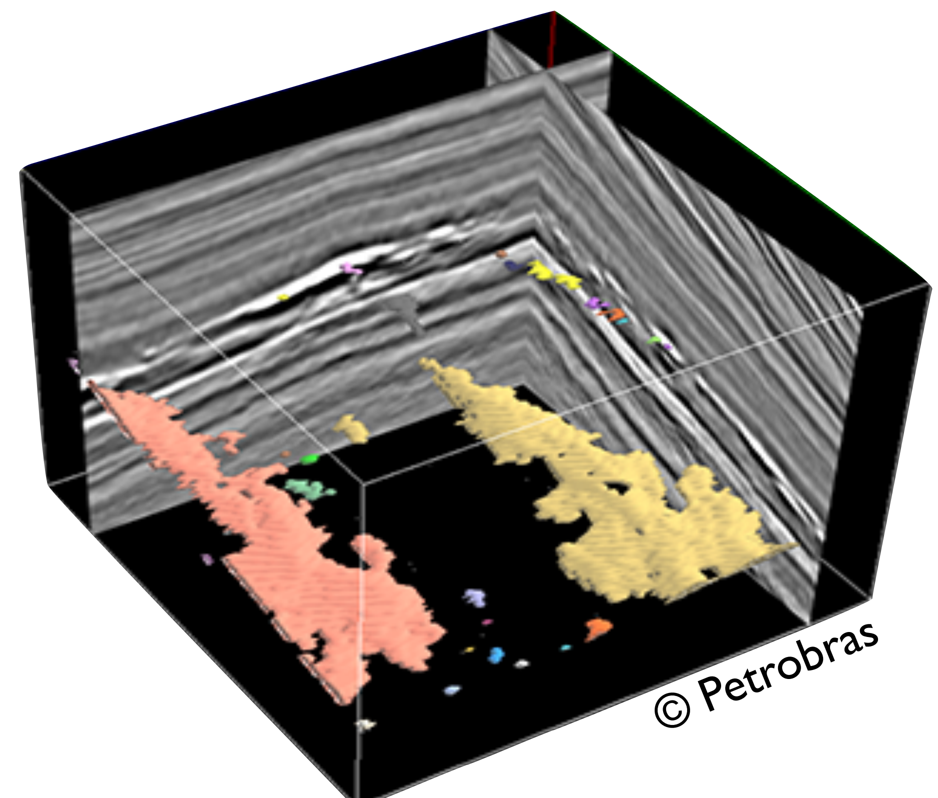
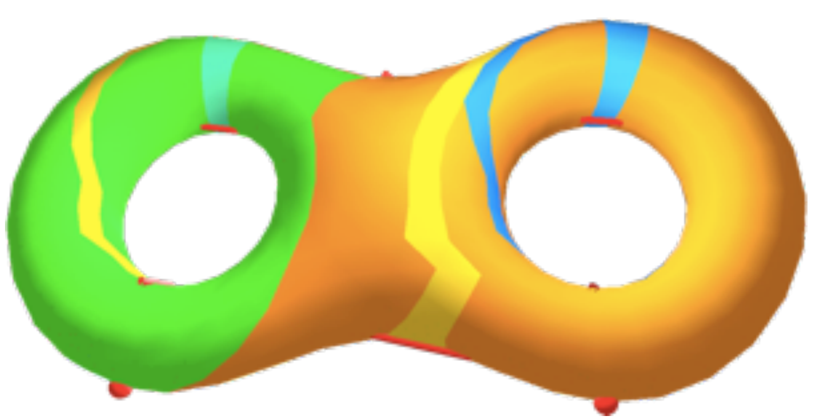
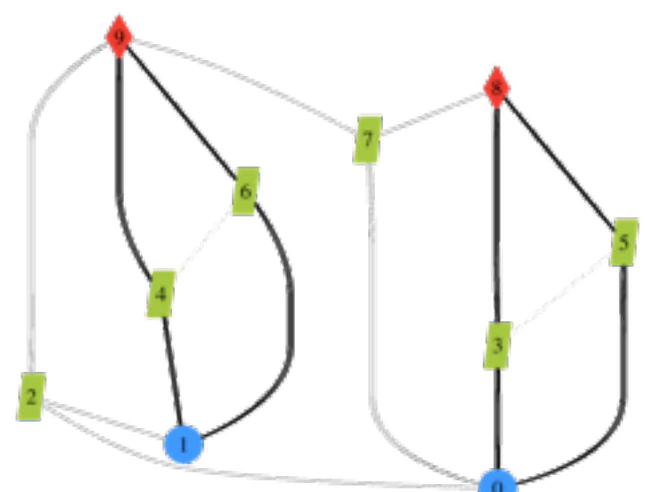
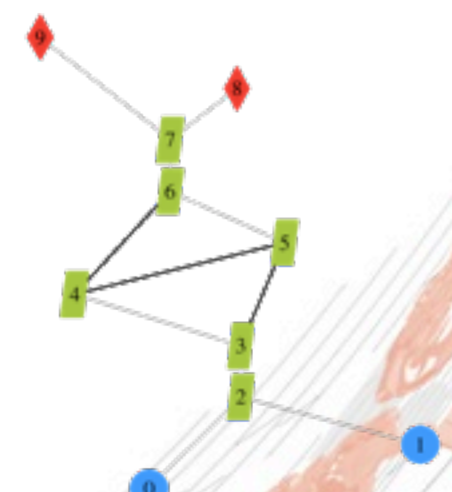




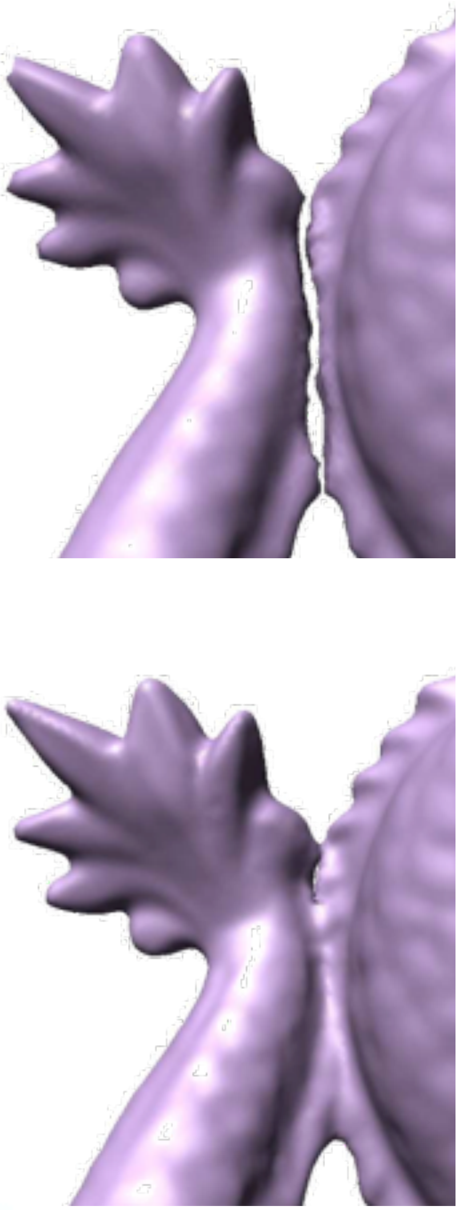


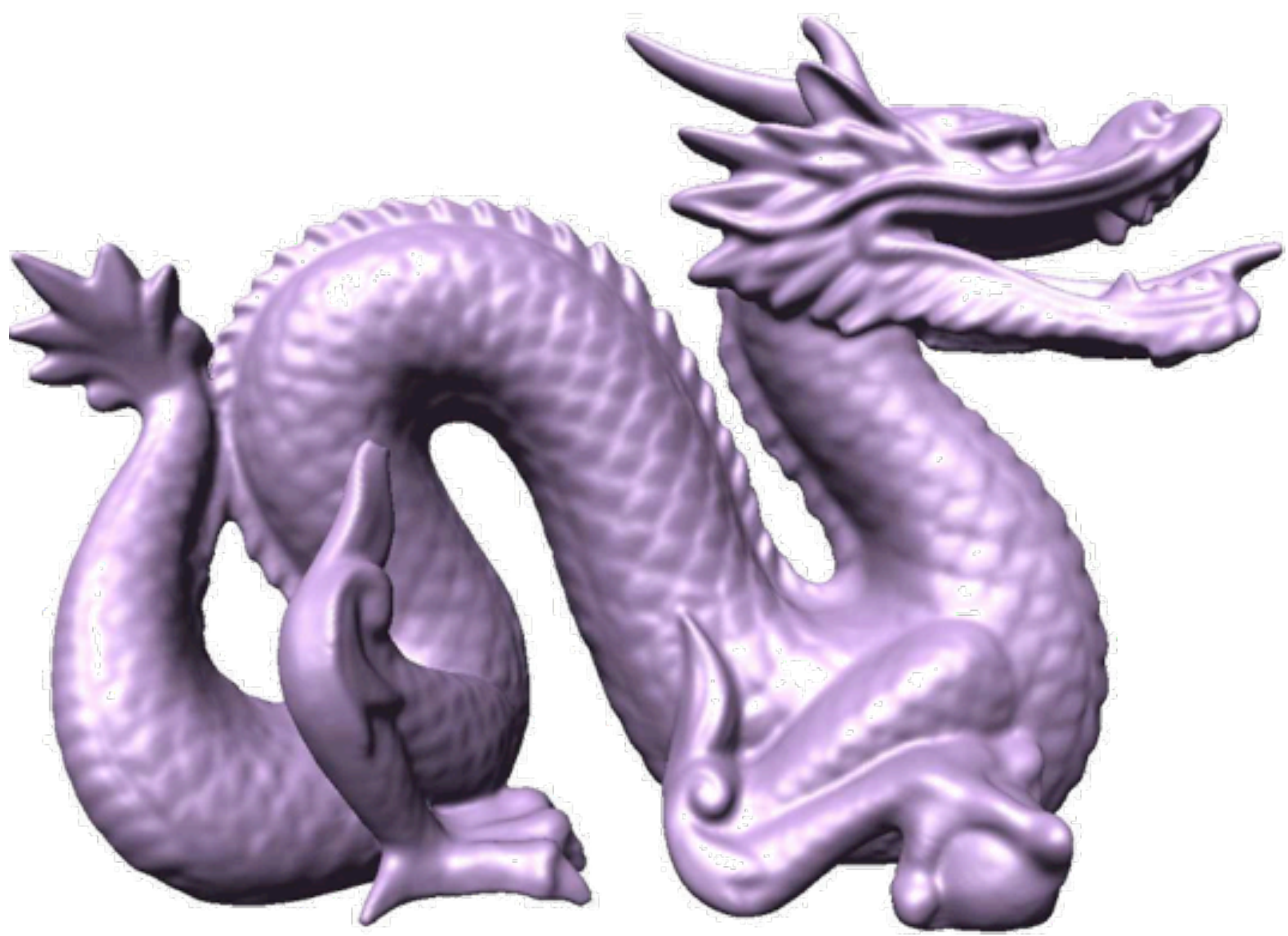
Smale complex



Reeb graph

Surface reconstruction





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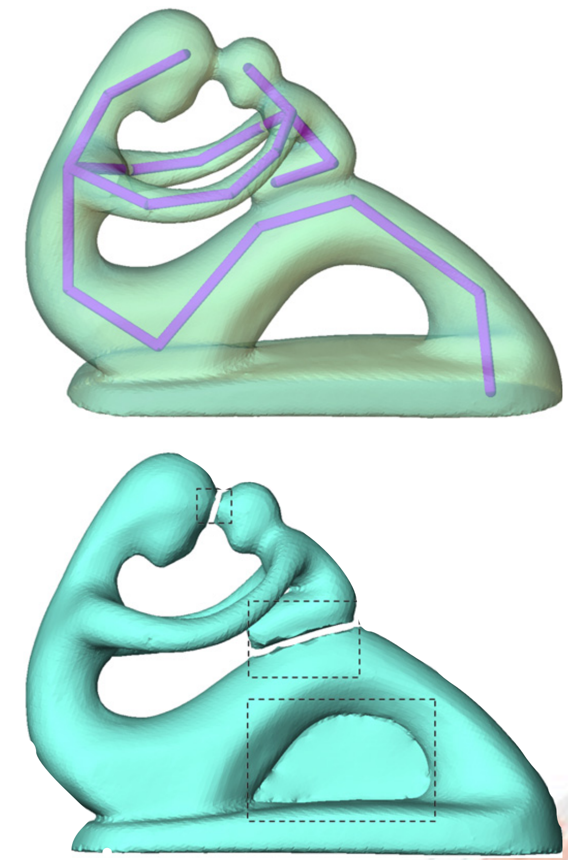
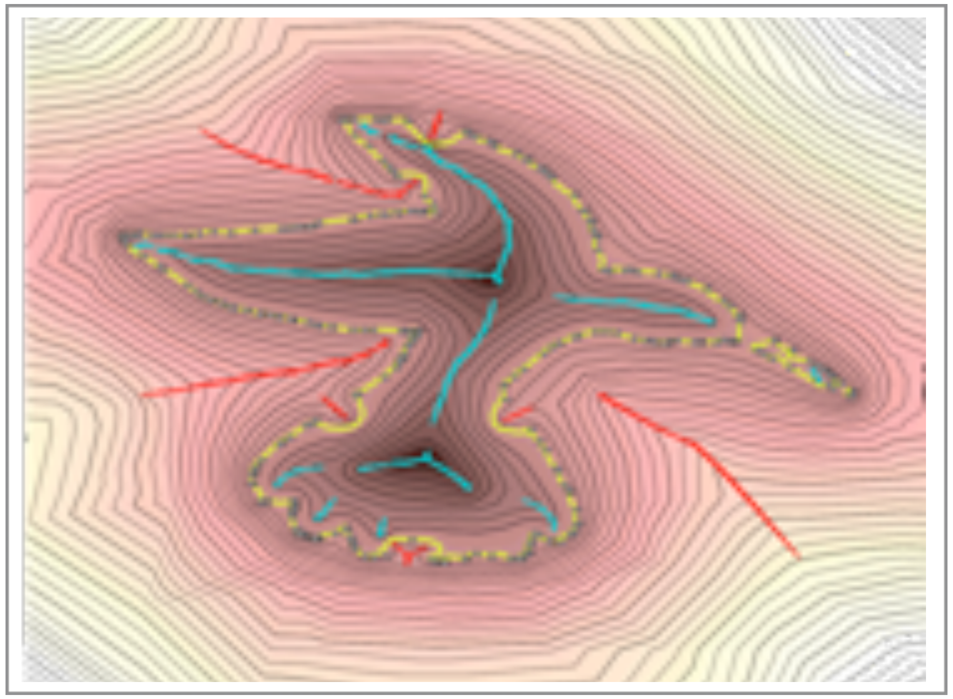


© Ju, Zhou, Hu

noisy, sparse point set

$\Rrightarrow$ correct topology?

Surface reconstruction



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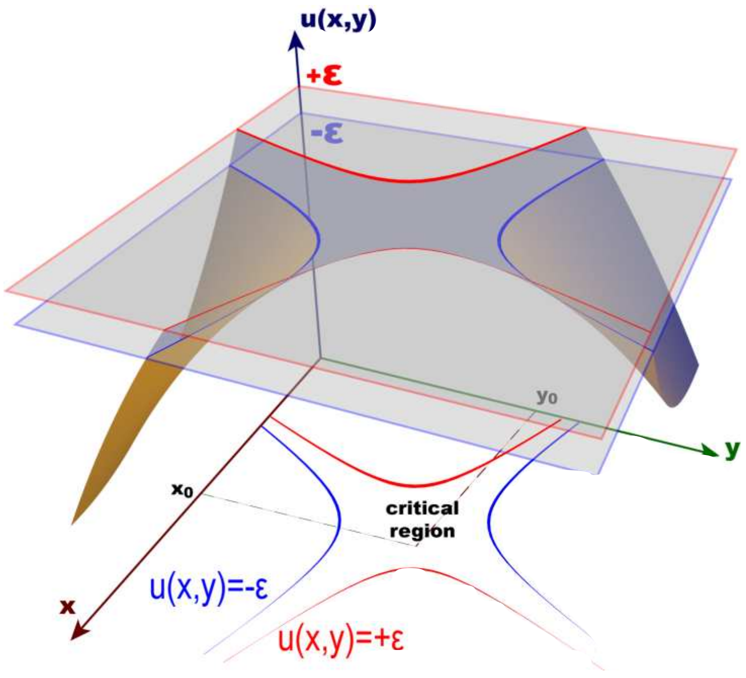
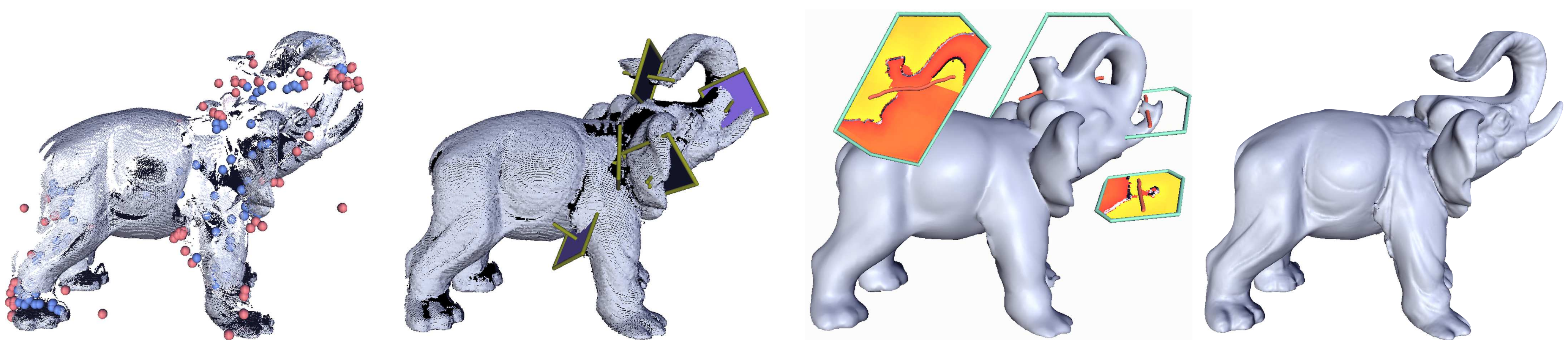


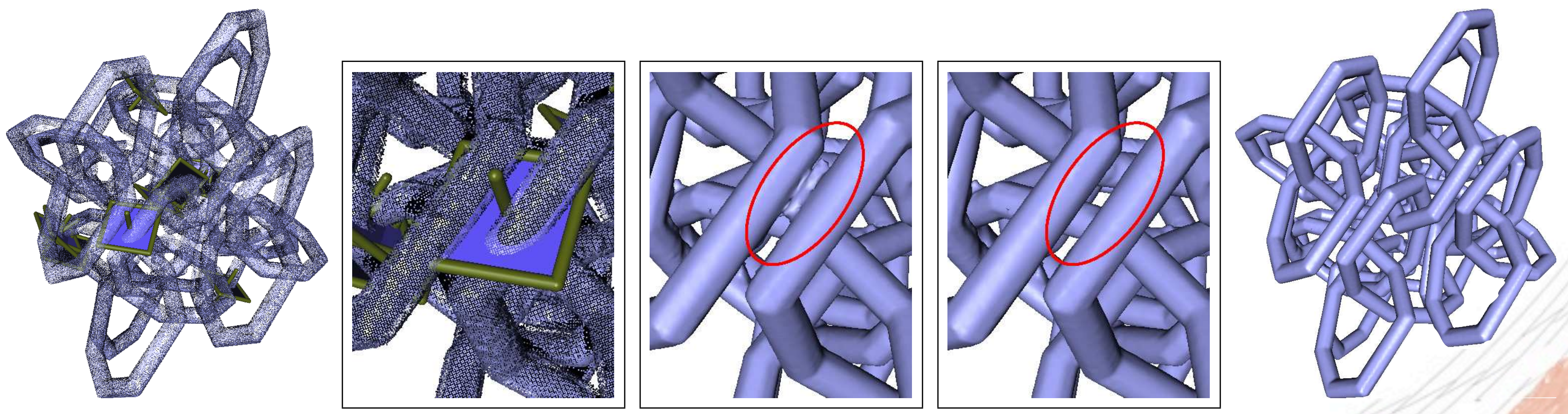
interactive topology edition

local critical regions

⇛ Banchoff's line

Topology-aware reconstruction





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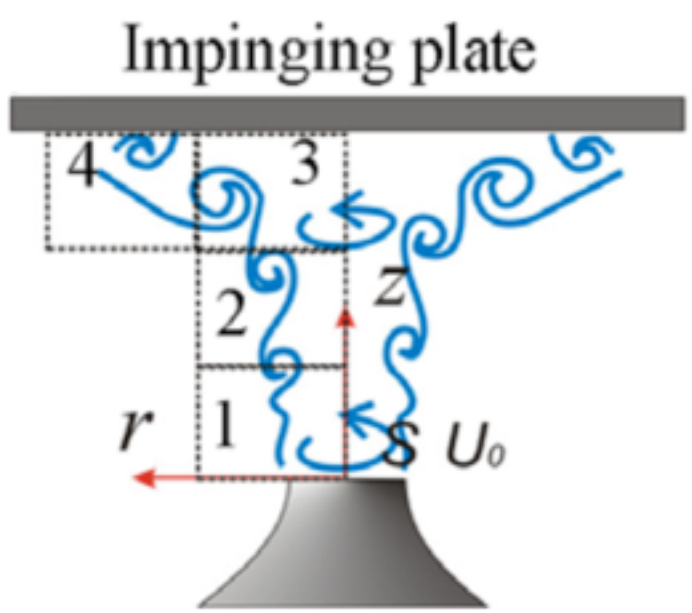
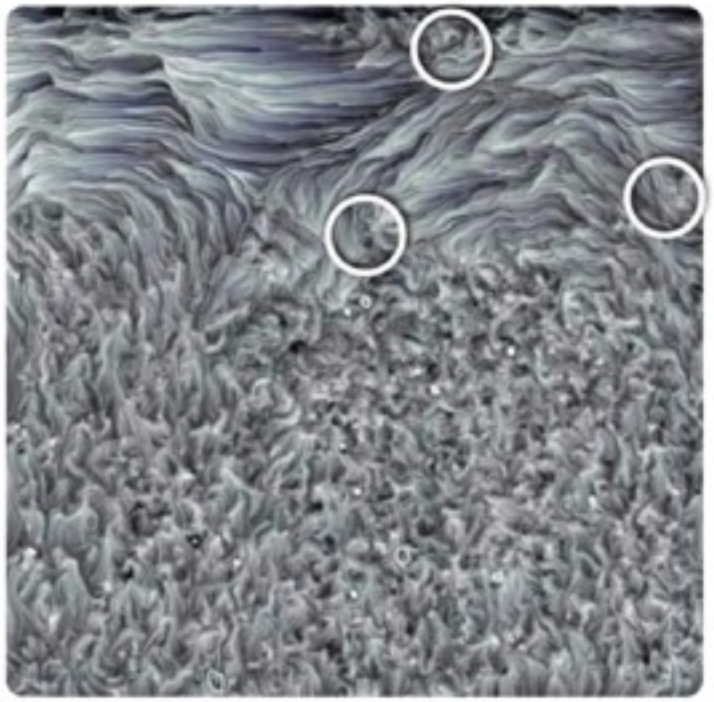
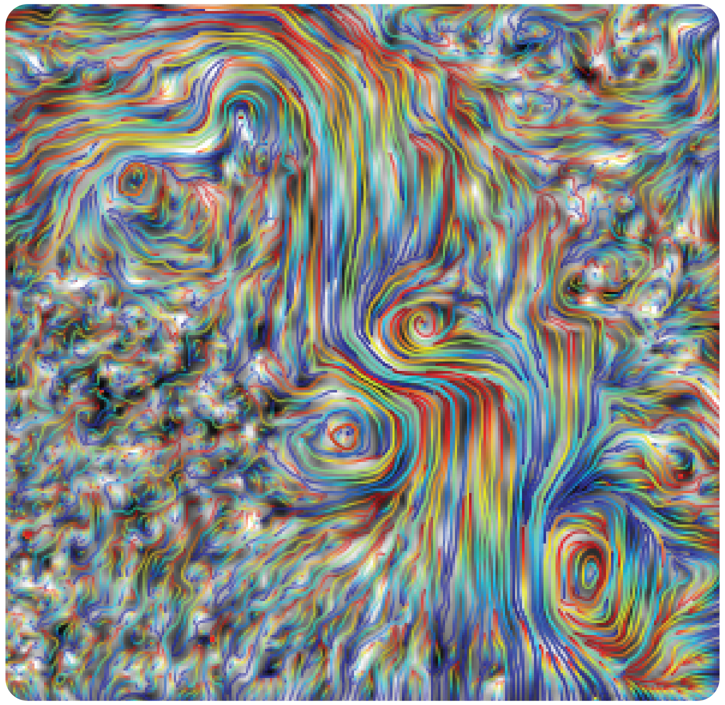
Vector field de-noising



Mechanical Dept, PUC-Rio



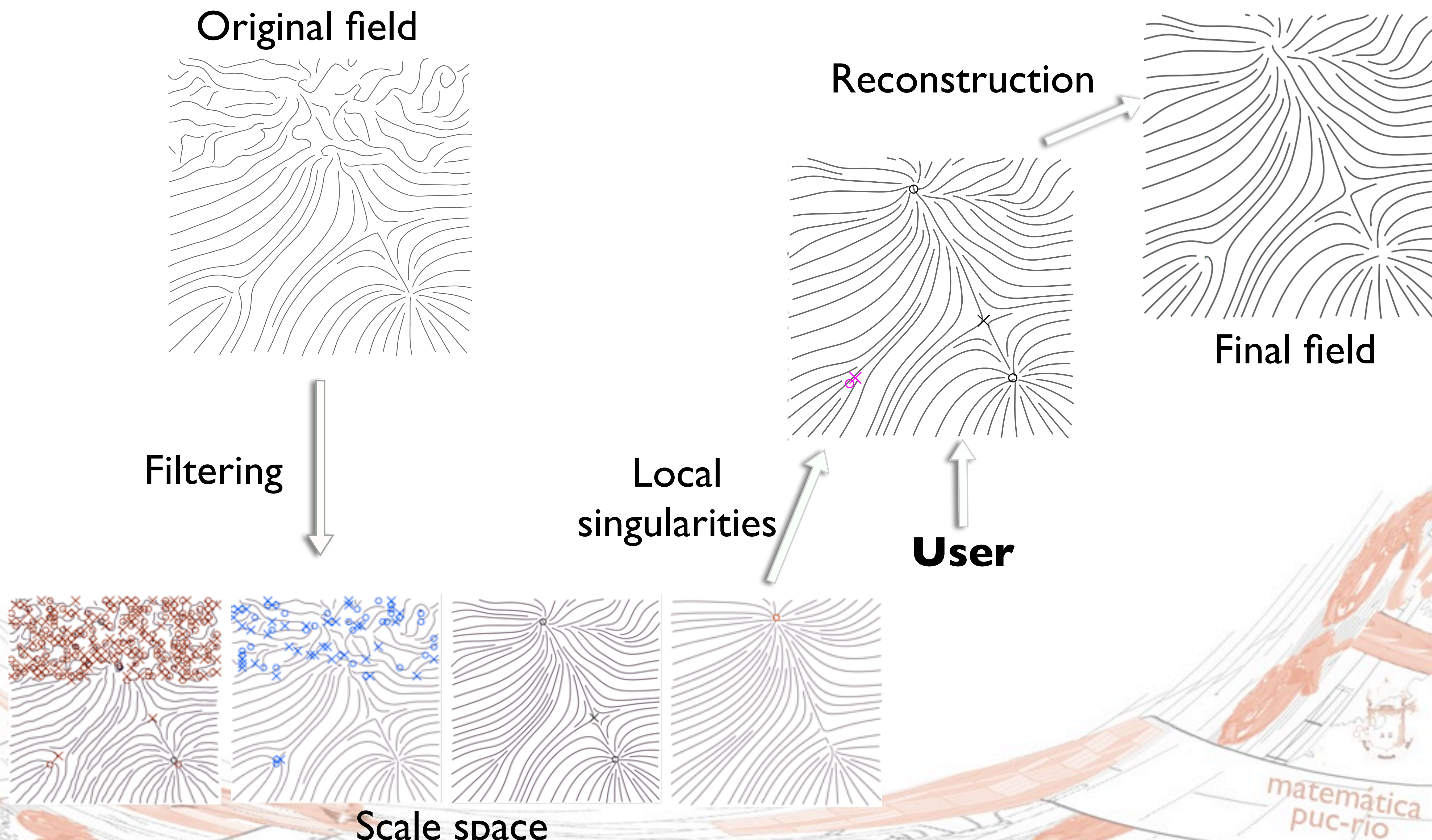


noise at the scale of the data

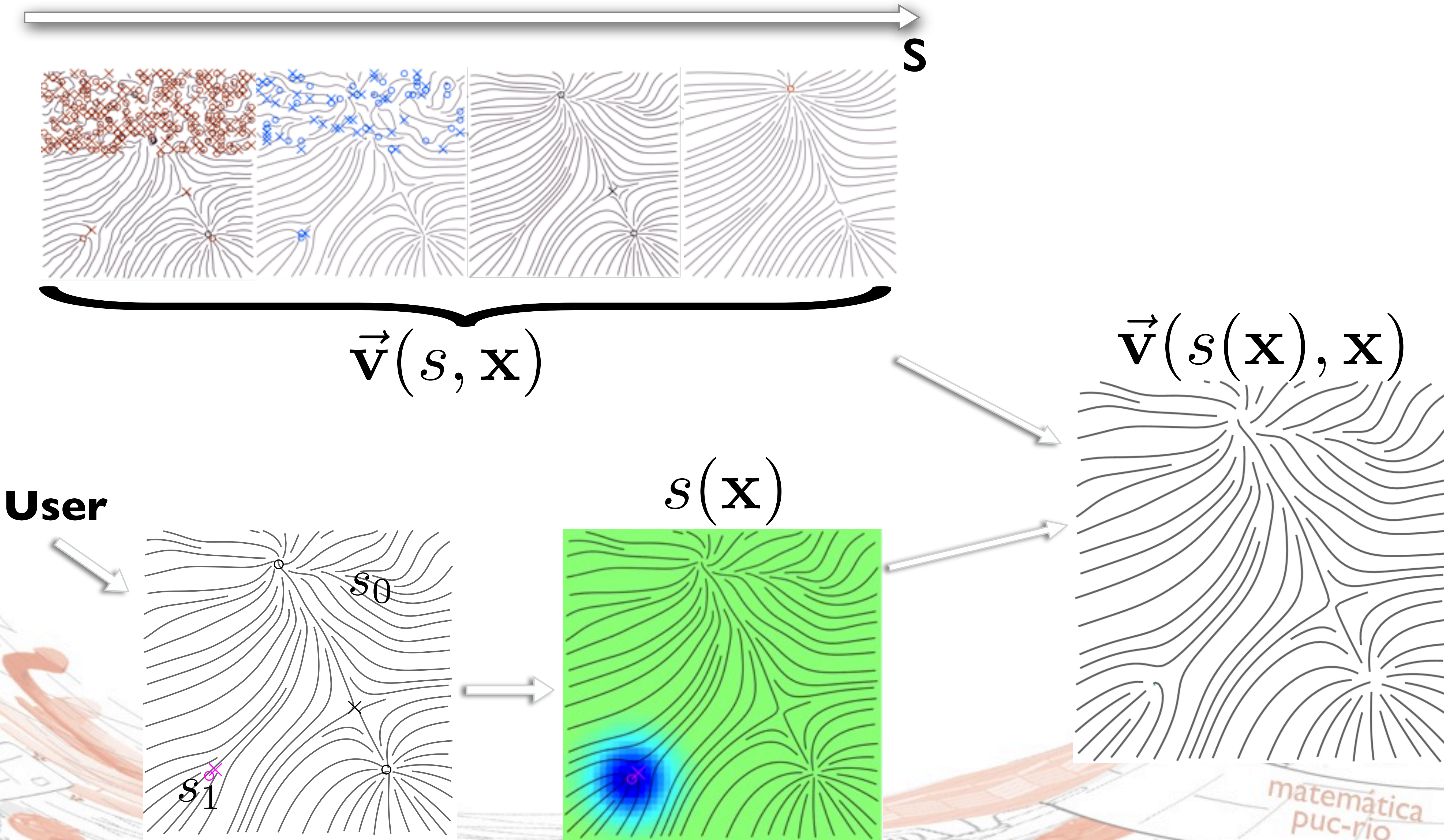
clean data + "important" vortices

local interpolation analysis

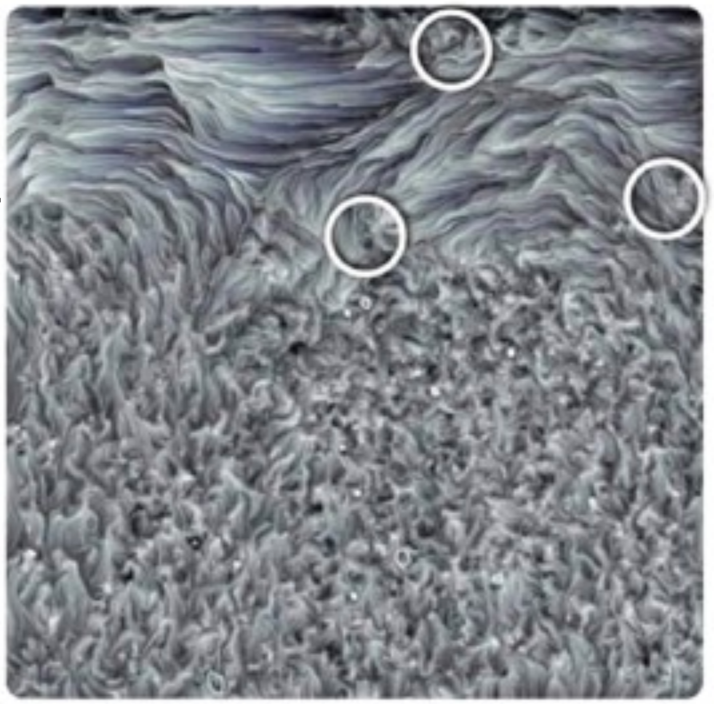
Interactive de-noising



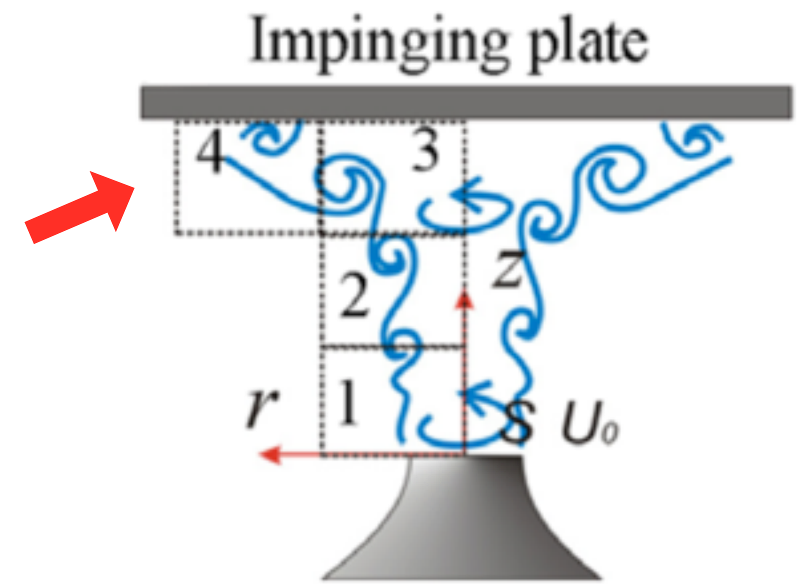
Scale-dependent singularity



Topology-aware de-noising



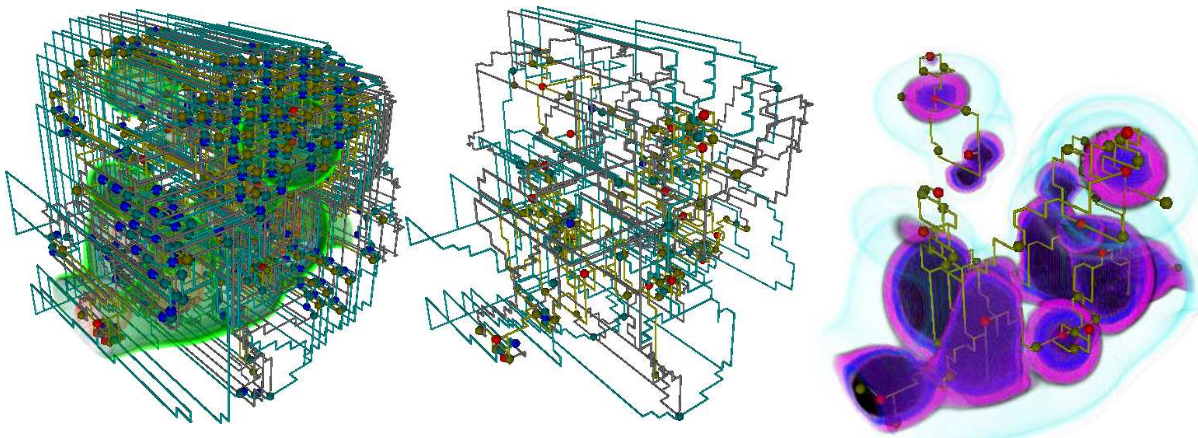
Original



Smoothed

Reconstructed

Some common points



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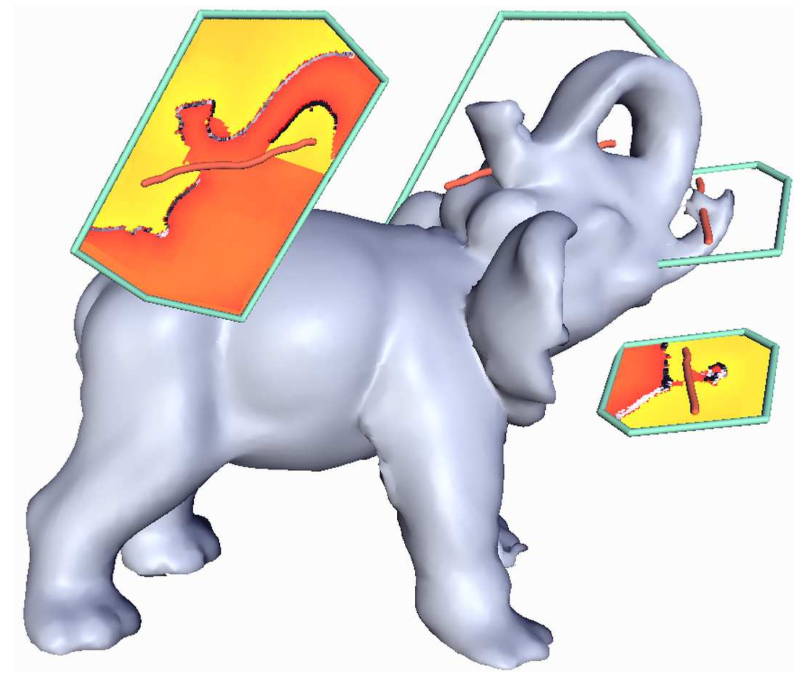
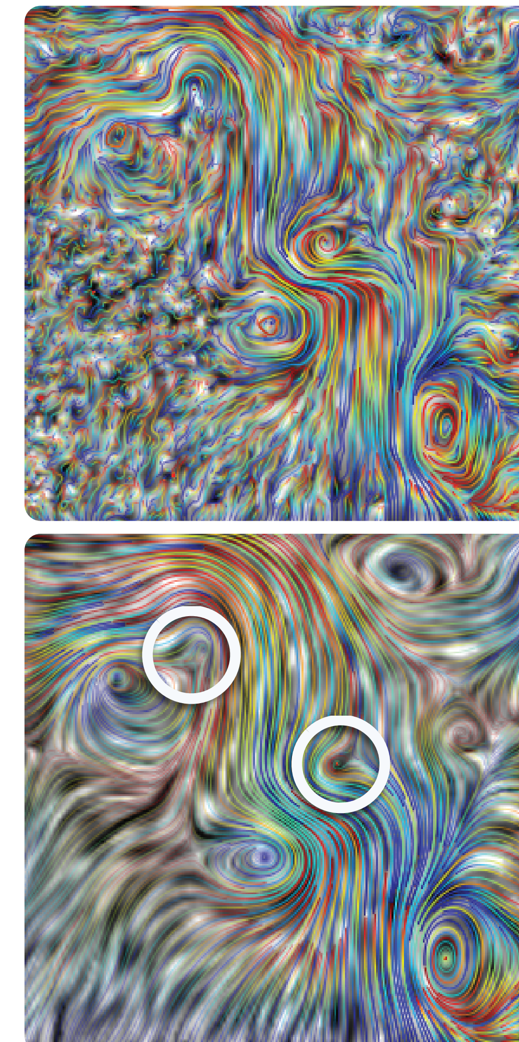




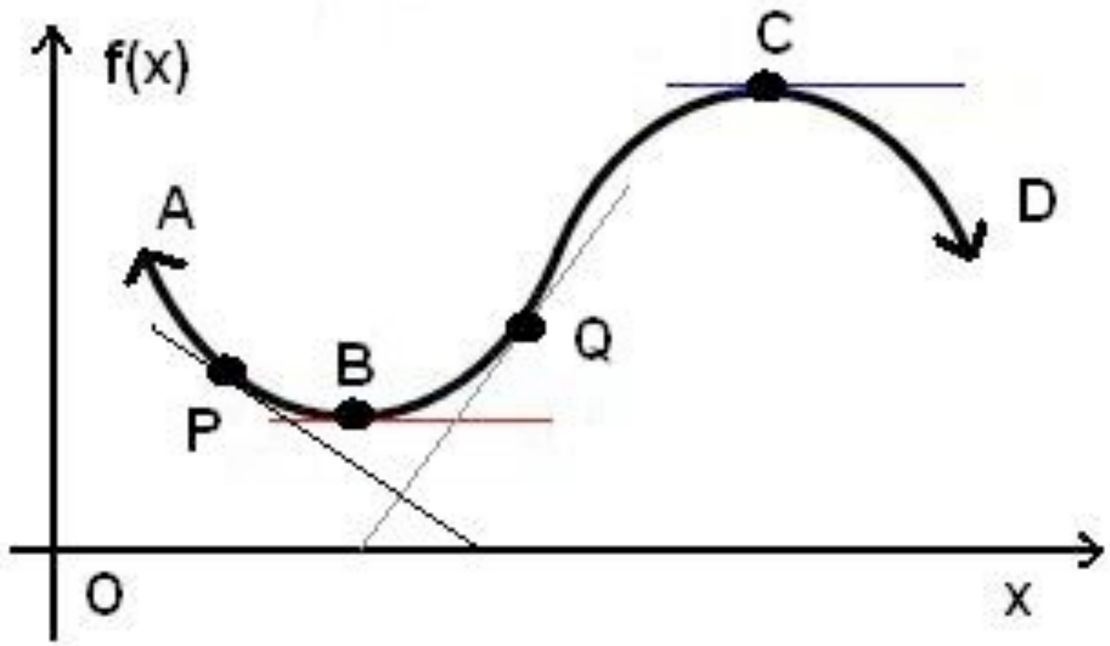
singular points only

$\Rrightarrow$ several applications

topology: intuitive interfaces

noise / scale problems

Analysis is hard to compute



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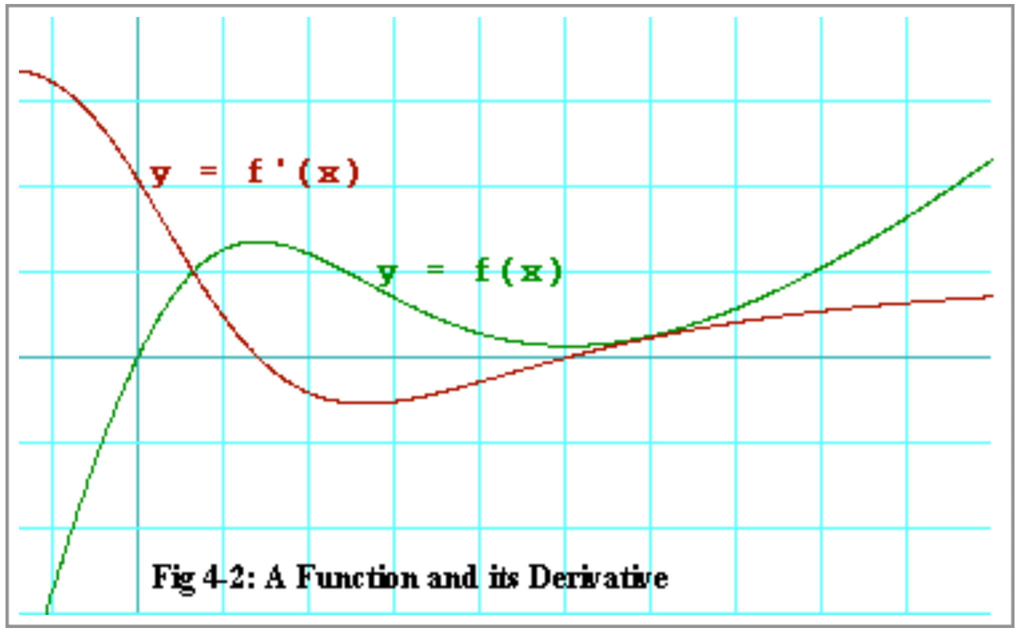

Fig 4-2: A Function and its Derivative

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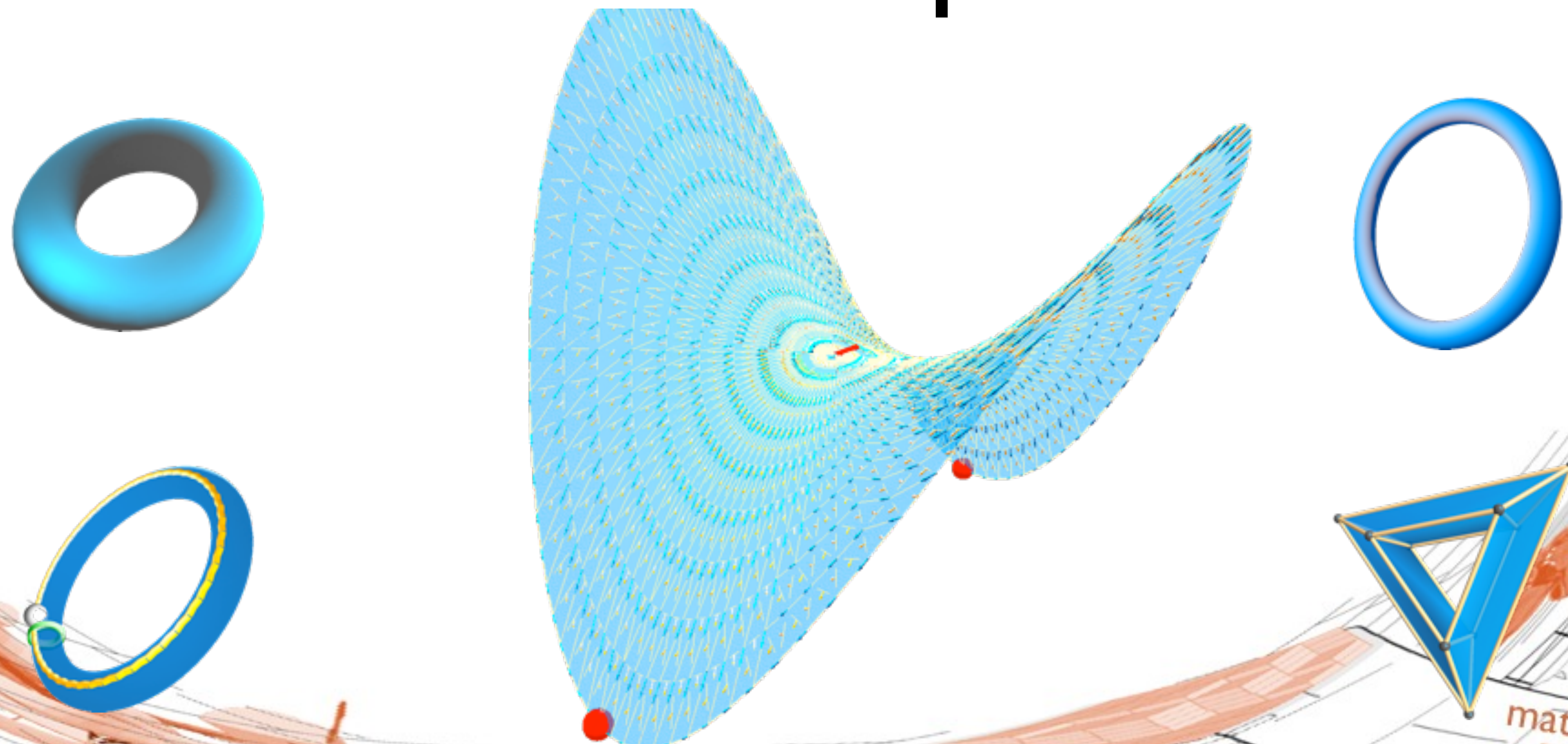
But intuitive:

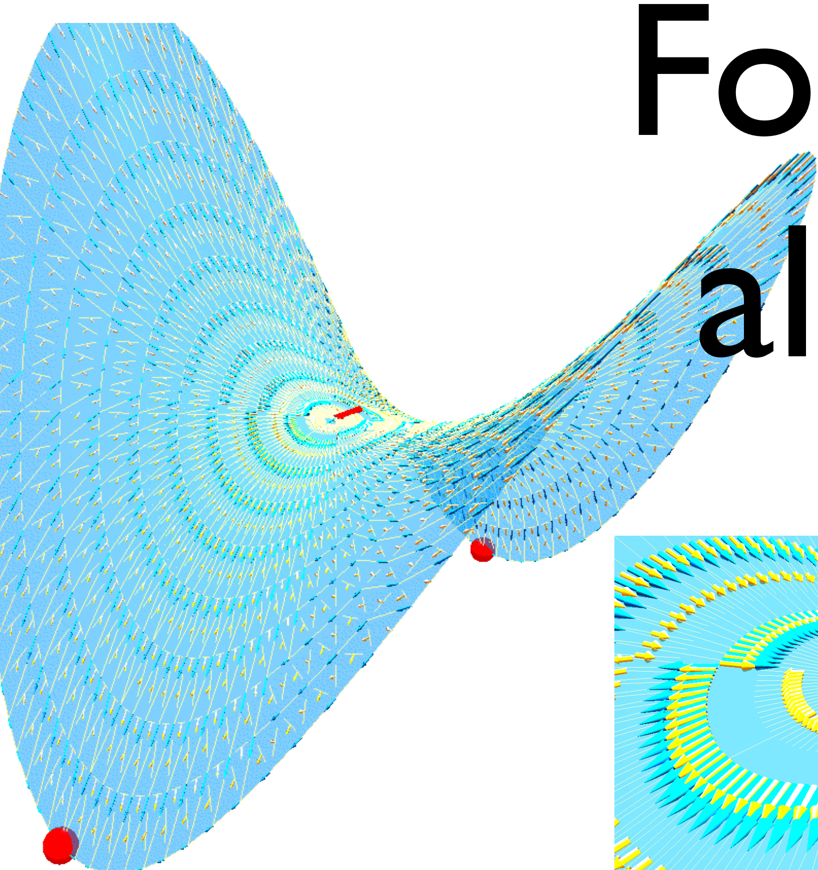
quick and ready insight

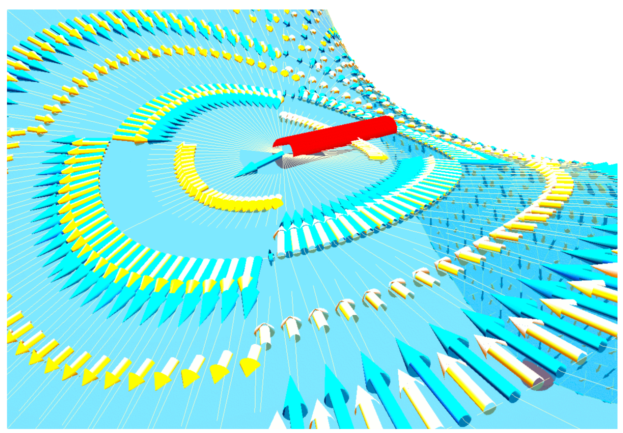
immediate apprehension or cognition

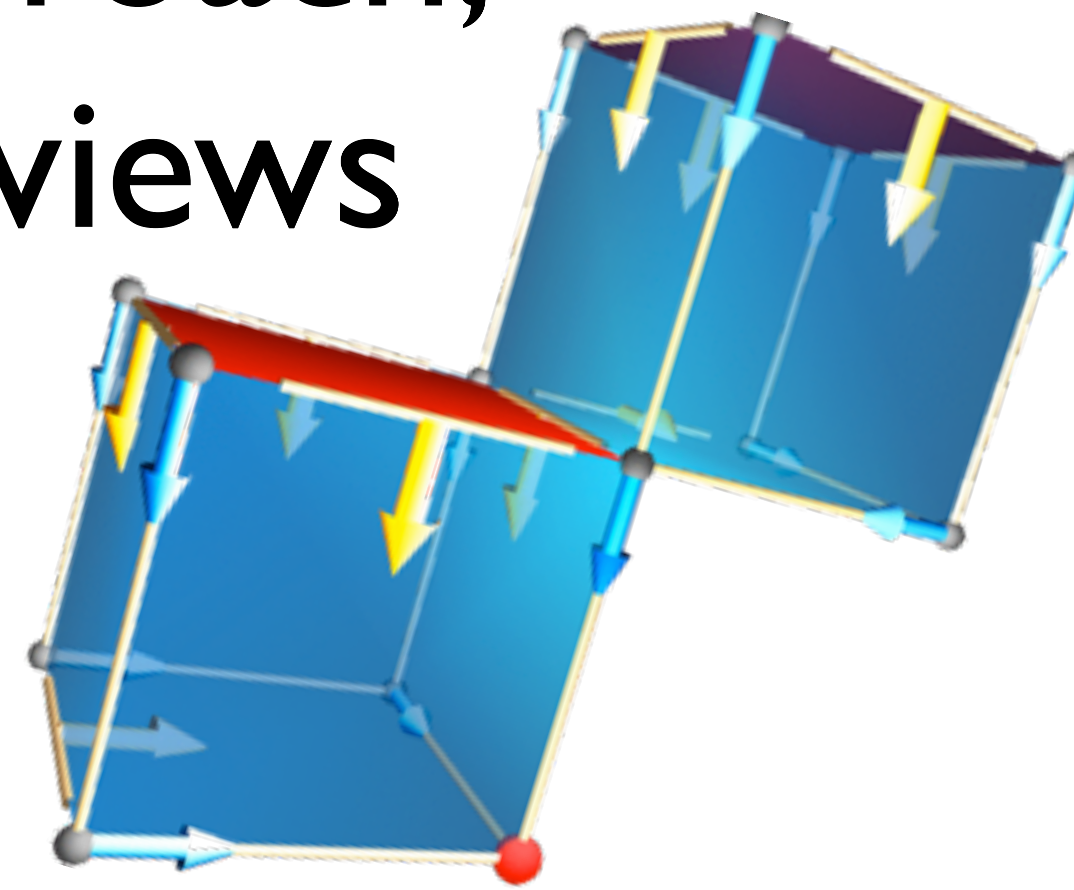
Combinatorial optimizations



Forman’s approach, algorithmic views







matching

combinatorial field

⇛ matching along the flow

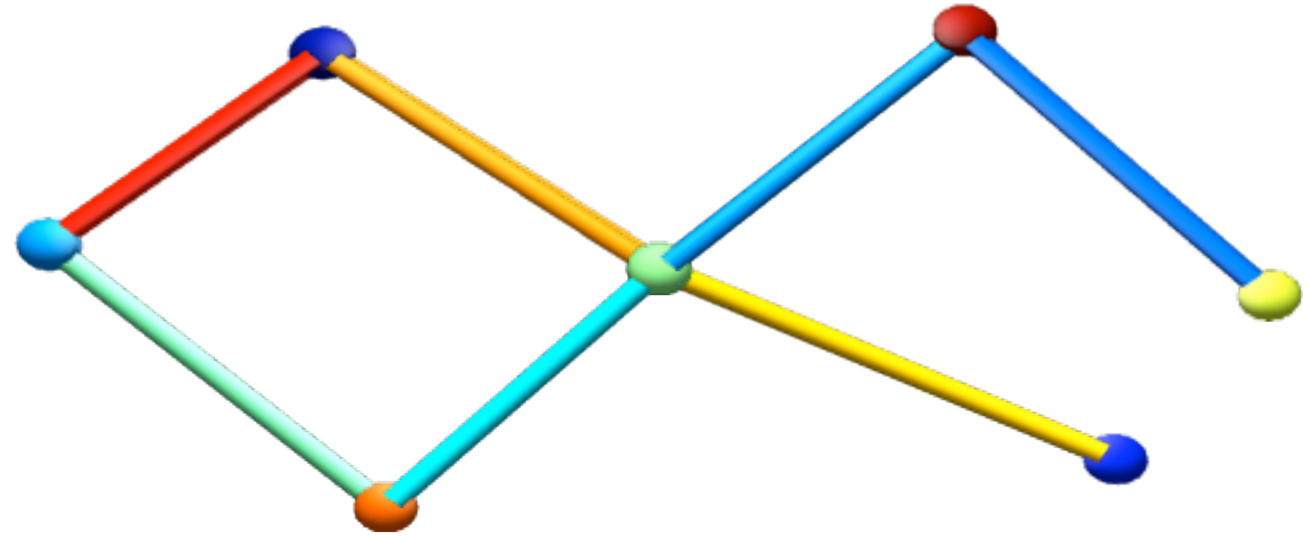
⇛ critical = unmatched

tree

gradient field

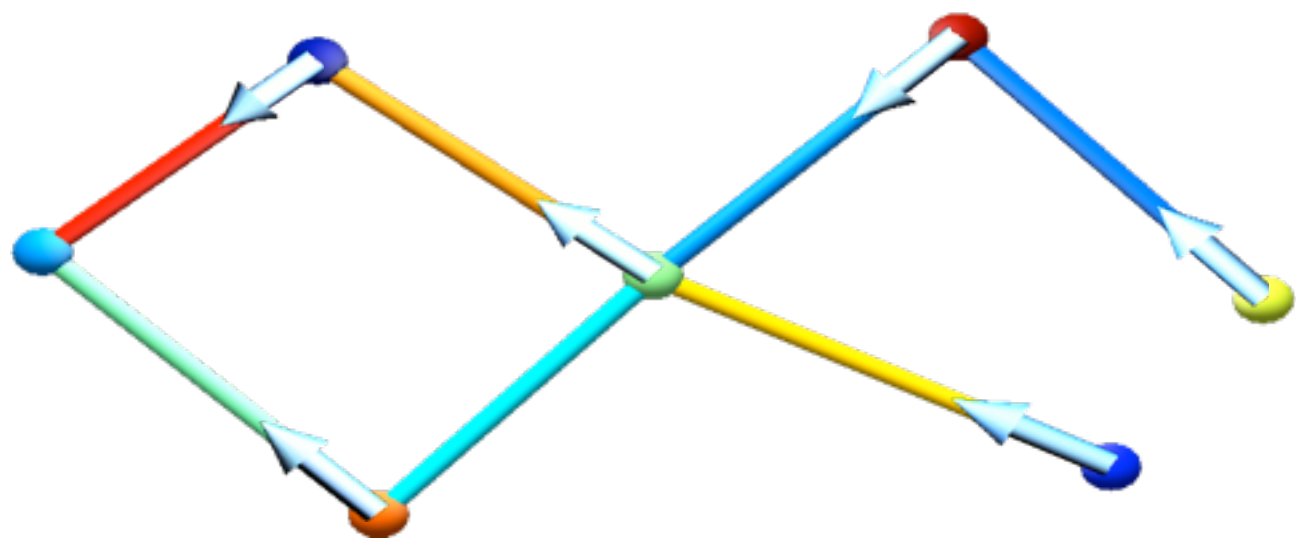
no closed gradient path ⇛ acyclic

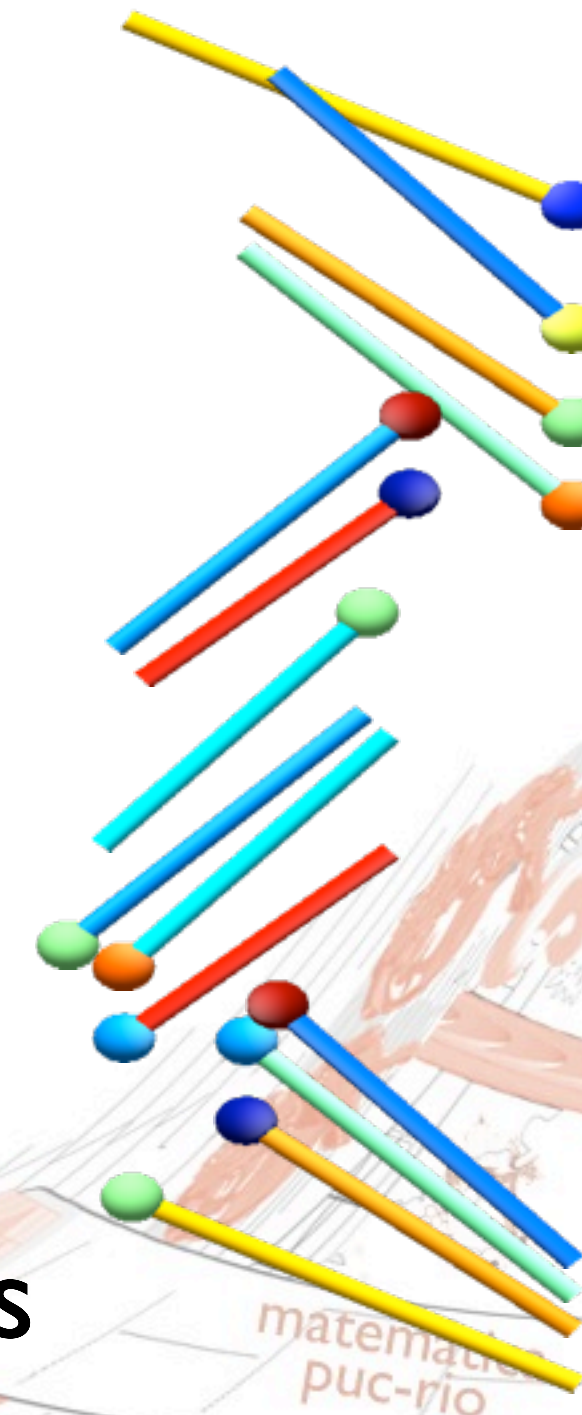
Geometric Morse function



- Smooth Morse function = greedy order
  orient first

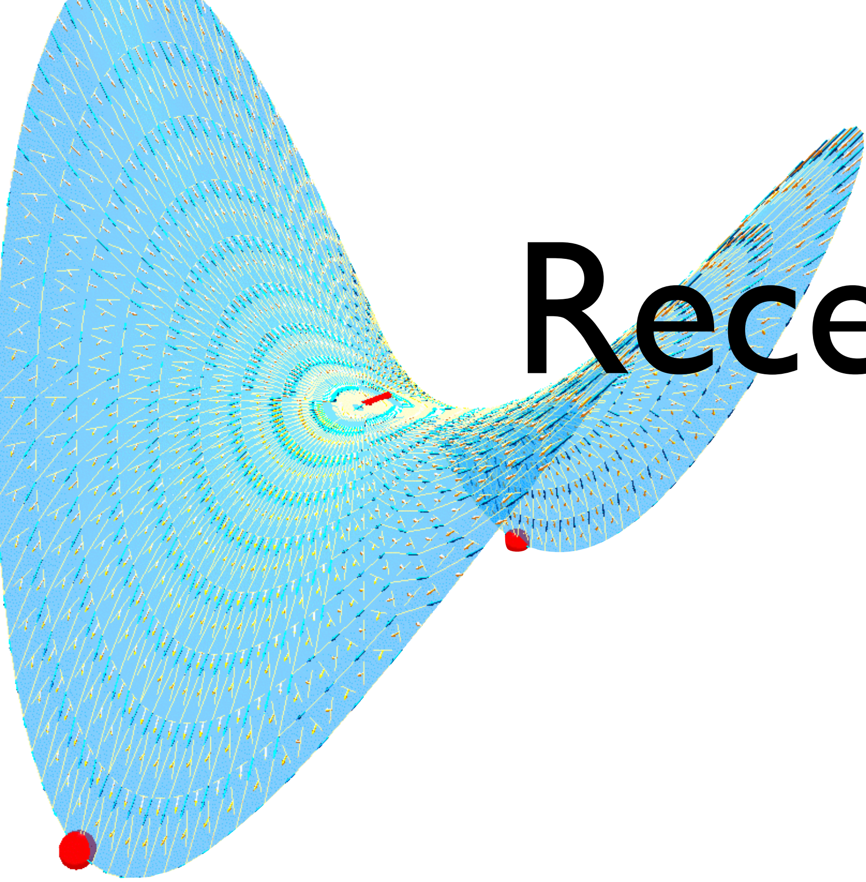
Elementary optimization

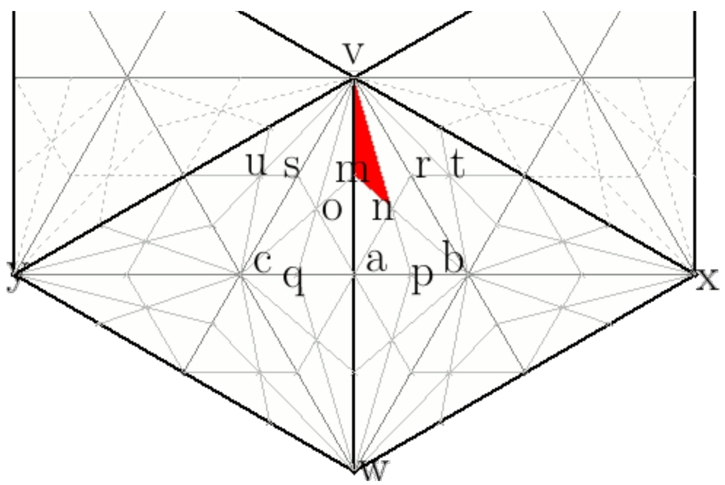


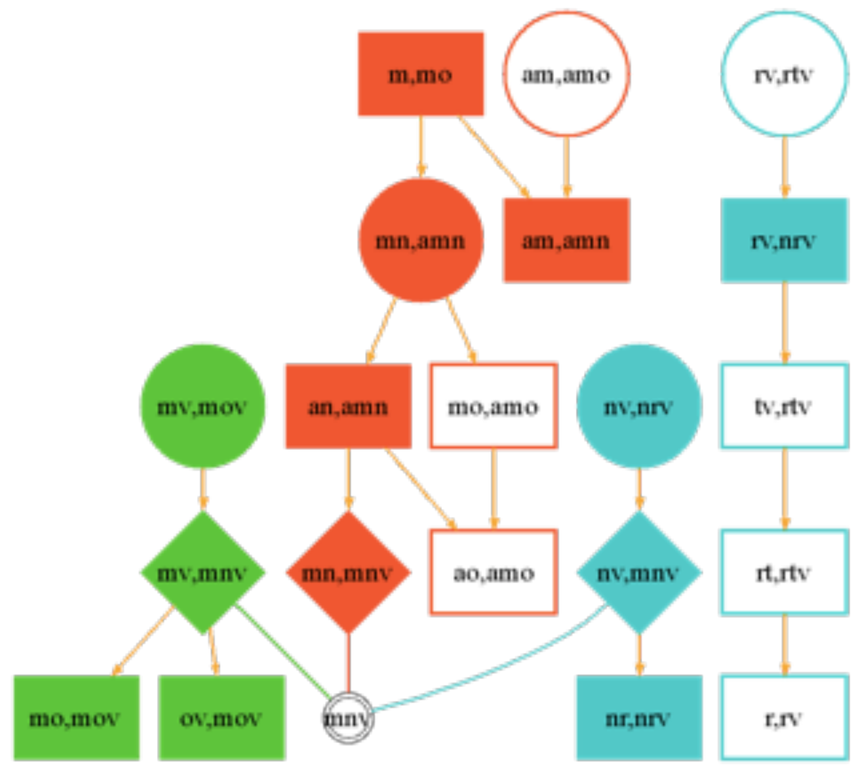


can also enforce critical cells

Recent improvements



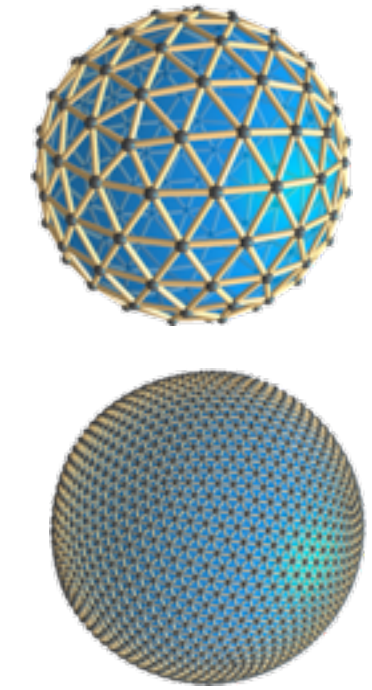


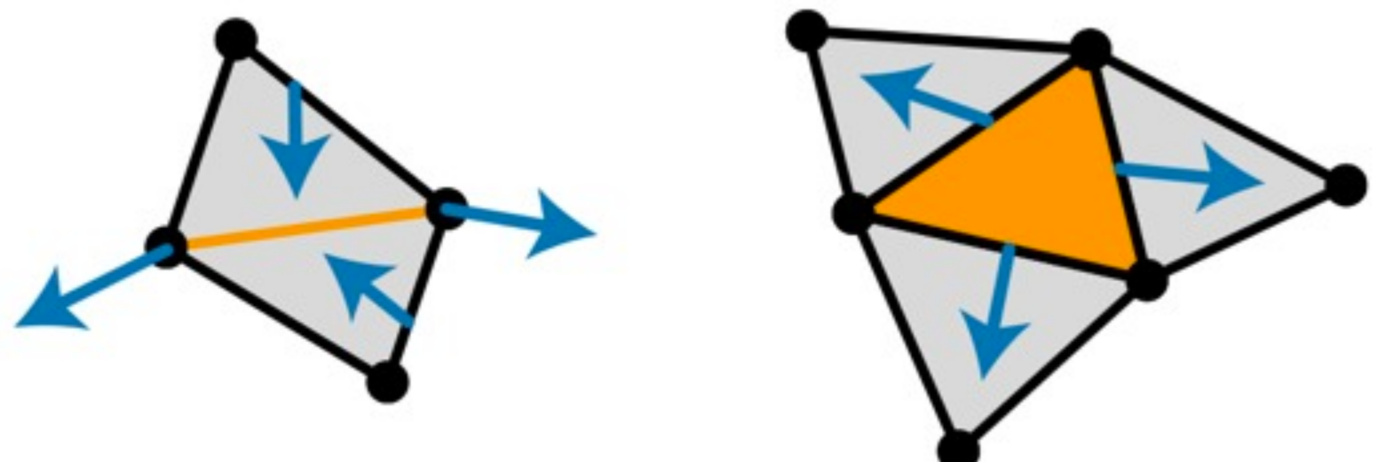


© L.

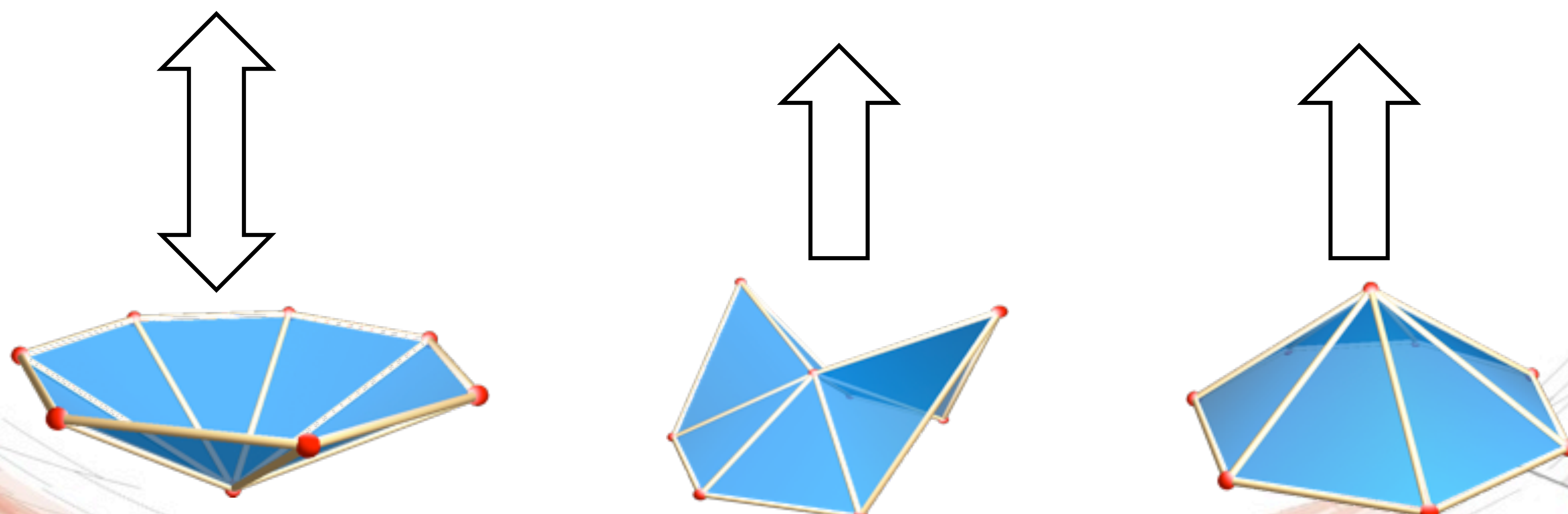
On triangulated surface, greedy construction of Forman's vector field keeps Banchoff's critical set for slowly varying function $f : \mathcal{K}_0 \hookrightarrow \mathbb{R}$

Critical cells in the right place

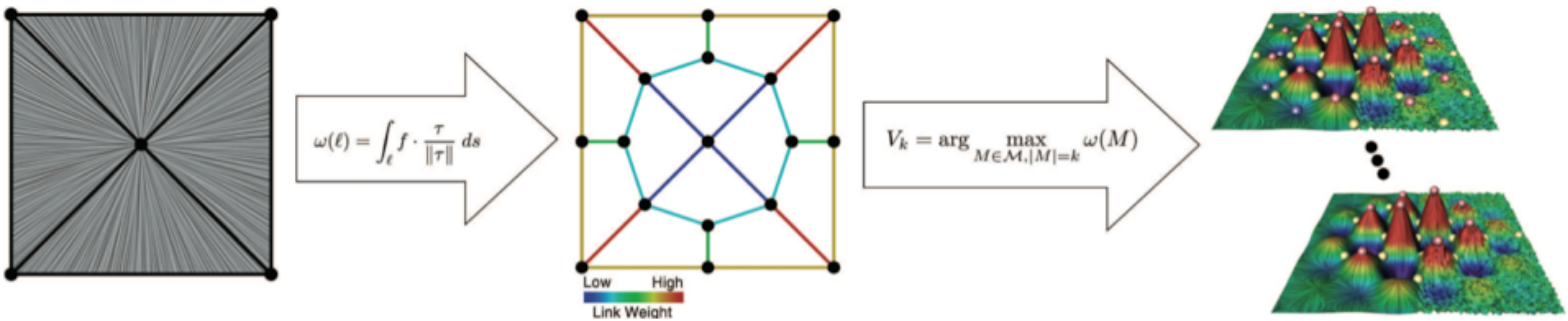




Minimum Saddle Maximum



Maximal weight matching



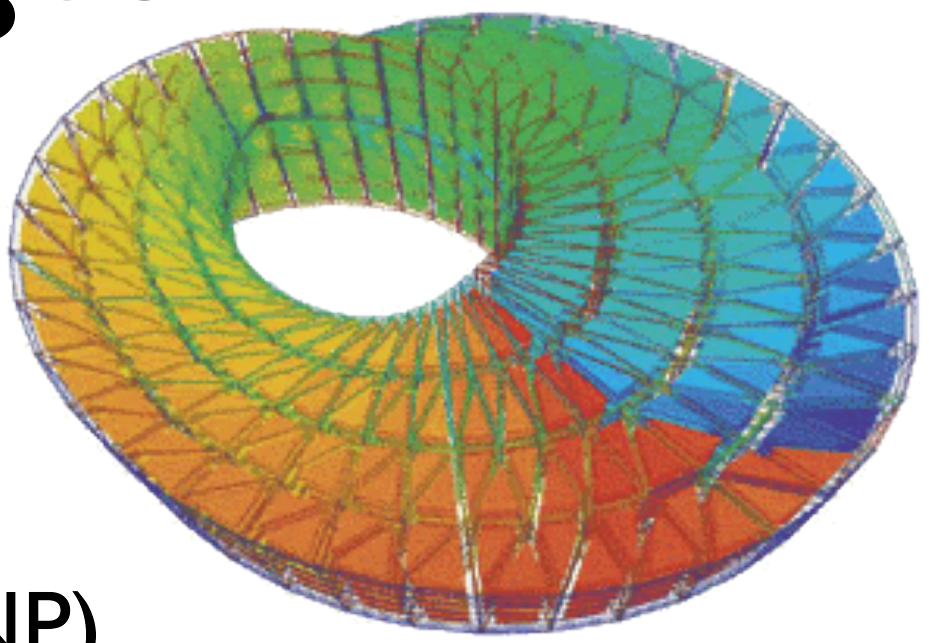
Forman's critical set results from global construction:

number of critical cells

quality of the field approximation

Scale-dependent critical set

Next challenges



Higher dimension (besides NP)

L., Lopes, Tavares, Joswig, Pfetsch...

More general cases (infinite complexes)

Ayala, Vilches...

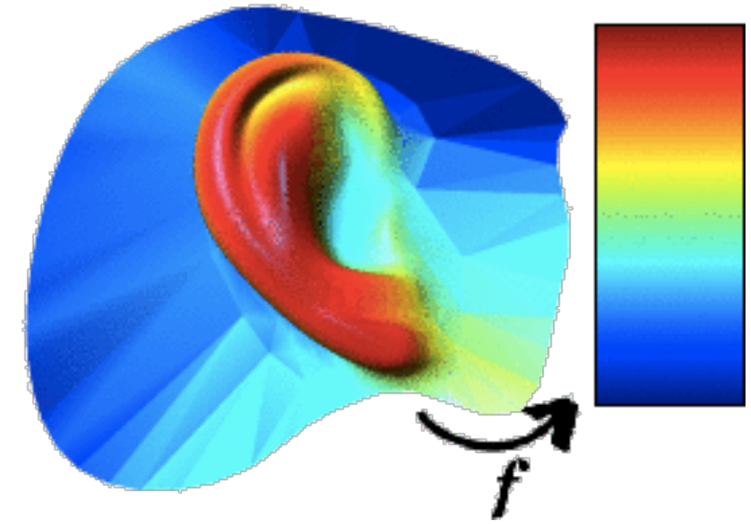
More complex objects (tensors, $\{f_i\}$)

Forman, Tricoche, Tong, Desbrun...

More theoretical guarantees

L., Zhang, Mischaikow...

More matchings



Thank you for your attention!

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