Convex analysis

Master "Mathematics for data science and big data"
Preliminary version

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First two chapters in French . . . English version coming soon

Chapter 1

Introduction : Optimisation, machine learning et analyse convexe

1.1 Problèmes d'optimisation en Machine Learning

La plupart des algorithmes de Machine Learning consistent à résoudre un problème de minimisation. Autrement dit, la valeur renvoyée par l'algorithme est la solution (ou une version approchée) d'un problème de minimisation. En général, les problèmes non convexes sont difficiles, les problèmes convexes le sont moins. On va voir comment attaquer les problèmes convexes.
Donnons d'abord quelques exemples de problèmes rencontrés en apprentissage supervisé:

Example 1.1.1 (moindres carrés, régression linéaire simple ou pénalisée).

(a) Moindres carrés ordinaires:

$$\min_{x \in \mathbb{R}^p} \|Zx - Y\|^2, Z \in \mathbb{R}^{n \times p}, Y \in \mathbb{R}^n$$

(b) Lasso :

$$\min_{x \in \mathbb{R}^p} \|Zx - Y\|^2 + \lambda \|x\|_1,$$

(c) Ridge :

$$\min_{x \in \mathbb{R}^p} \|Zx - Y\|^2 + \lambda \|x\|_2^2,$$

Example 1.1.2 (Classification linéaire). On dispose d'un échantillon d'apprentissage $\mathcal{D} = \{(w_1, y_1), \dots, (w_n, y_n)\}$, $y_i \in \{-1, 1\}$, $w_i \in \mathbb{R}^p$, où les w_i sont des *features* (=régresseurs = caractéristiques) et les y_i sont des étiquettes représentant la classe de chaque observation i. On suppose que l'échantillon est obtenu par des réalisations indépendantes d'un vecteur $(W, Y) \sim P$, de loi P inconnue. Les classifieurs linéaires sont des fonctions linéaires définies sur l'espace des features, de type

$$h : w \mapsto \text{signe}(\langle x, w \rangle + x_0) \qquad (x \in \mathbb{R}^p, x_0 \in \mathbb{R})$$

Un classifieur h est donc déterminé un vecteur $\mathbf{x} = (x, x_0)$ de $\mathbb{R}^{p+1}$, x est le vecteur normal à un hyperplan séparant l'espace en une région classée "+1" et une autre classée "-1".
L'objectif est évidemment de trouver un classifieur qui, en moyenne, ne se trompe pas beaucoup, c'est à dire tel que $\mathbb{P}(h(W) = Y)$ soit aussi grande que possible.
le coût de référence est le "coût 0-1"

$$L_{01}(\mathbf{x}, w, y) = \begin{cases} 0 & \text{si } -y\,(\langle x, w \rangle + x_0) \le 0 \qquad \textit{i.e.} \text{ si } h(w) \text{ et } y \text{ sont de même signe,} \\ 1 & \text{sinon .} \end{cases}$$

L'objectif implicite des méthodes de machine learning pour la classification supervisée est le plus souvent de résoudre (au moins de manière approchée) le problème

$$\min_{\mathbf{x} \in \mathbb{R}^{p+1}} \frac{1}{n} \sum_{i=1}^{n} L_{0,1}(\mathbf{x}, w_i, y_i) \tag{1.1.1}$$

c'est à dire de minimiser le *risque empirique.*
Comme le coût L n'est pas convexe en $\mathbf{x}$, le problème (1.1.1) est *difficile.* Les méthodes classiques de Machine learning consistent à minimiser une fonction similaire à l'objectif dans (1.1.1) dans laquelle on a remplacé le coût 0-1 par un *substitut convexe*, et de rajouter un terme pénalisant la "complexité" de x, de sorte que le problème devient attaquable numériquement. Plus précisément, on veut résoudre

$$\min_{x \in \mathbb{R}^p, x_0 \in \mathbb{R}} \sum_{i=1}^{n} \varphi(-y_i(\, x^\top w_i + x_0)) + \lambda \mathcal{P}(x), \tag{1.1.2}$$

où $\mathcal{P}$ est une pénalité et φ est un substitut convexe du coût 0-1.
Différents choix de pénalités et de substituts convexes donnent lieu à des méthodes classiques de classification supervisée :

- Pour $\varphi(z) = \max(0, 1 + z)$ (Hinge loss), $\mathcal{P}(x) = \|x\|^2$, on obtient les SVM.

- Dans le cas séparable (*i.e.* lorsqu'il existe un hyperplan séparant les deux classes), on utilise l' "indicatrice infinie"

$$\mathbb{I}_A(x) = \begin{cases} 0 \text{ si } x \in A, \\ +\infty \qquad\qquad \text{si } x \in A^c, \end{cases}$$

  $(A \subset \mathcal{X})$

et on pose

$$\varphi(x) = \mathbb{I}_{\mathbb{R}^-}(x)$$

La solution du problème est l'hyperplan séparateur de marge maximale (" maximum margin hyperplane").

Le dénominateur commun des variantes de l'exemple 1.1.2 est le suivant:

- On se donne une définition d'un risque $J(x) = \mathbb{E}(L(x, D))$ on cherche x qui minimise l'espérance du coût induit par le classifieur x sur les données D.
- $D \sim \mathbb{P}$ inconnue, on cherche donc plutôt

$$\hat{x} \in \arg\min_{x \in \mathcal{X}} J_n(x) \hat{=} \frac{1}{n} \sum_{i=1}^{,} L(x, d_i)$$

- On transforme le coût $L \leftarrow L_\varphi$ de manière à obtenir une fonction $J_{n,\varphi}$ convexe en x
- on résout le problème

$$\min_{x \in \mathcal{X}} J_{n,\varphi}(x)$$

Ici, on s'attache au dernier point : comment résoudre en pratique le problème de minimisation ?

1.2 Formulation générale du problème

Dans ce cours, on considère des problèmes d'optimisation sur un espace de dimension finie $\mathcal{X} = \mathbb{R}^n$. Ces problèmes s'écrivent en toute généralité

$$\begin{gathered} \min_{x \in \mathcal{X}} f(x) \\ s.t. \text{ (sous contrainte)} \\ g_i(x) \leq 0, 1 \leq i \leq p, \quad F_i(x) = 0, 1 \leq i \leq m\,. \end{gathered} \tag{1.2.1}$$

La fonction f est la *fonction objectif* (ou *objectif*), le vecteur

$$C(x) = (g_1(x), \dots, g_p(x), F_1(x), \dots, F_m(x))$$

est le vecteur (fonctionnel) des contraintes. La région

$$K = \{x \in \mathcal{X} : \; g_i(x) \leq 0, \; 1 \leq i \leq p, \quad F_i(x) = 0, \; 1 \leq i \leq m\}$$

est l'ensemble des points *admissibles.*

- Si $K = \mathbb{R}^n$, on parle d'optimisation *sans contrainte.*

- Si $p \leq 1$ et $m = 0$, on parle d'optimisation *sous contraintes d'inégalité.*
- Si $p = 0$ et $m \geq 1$, on parle d'optimisation *sous contraintes d'égalité.*
- Si f et les contraintes sont régulières (différentiables), on parle d'optimisation *différentiable* ou *lisse.*
- Si f et les contraintes ne sont pas toutes régulières, on parle d'optimisation *non différentiable* ou *non lisse.*
- Si f et les contraintes sont convexes, on parle d'optimisation *convexe* (détails plus loin).

Résoudre le problème général (1.2.1) consiste à trouver

- un argument minimum $x^* \in \arg\min_K f$ (s'il en existe, c'est-à-dire $\arg\min_K f \neq \emptyset$),
- la *valeur* $f(x^*) = \min_{x \in K} f(x)$,

On peut réécrire tout problème sous contraintes comme un problème non contraint, grâce à la fonction indicatrice infinie $\mathbb{I}$ introduite plus haut. Notons respectivement g et F les vecteurs des contraintes d'inégalité et d'égalité. Pour $x, y \in \mathbb{R}$, on note $x \preceq y$ si $(x_1 \leq y_1, \dots, x_n \leq y_n)$ et $x \npreceq y$ sinon. Le problème (1.2.1) est équivalent à

$$\min_{x \in E} f(x) + \mathbb{I}_{g \preceq 0, F=0}(x) \tag{1.2.2}$$

Remarquons que, même si le problème initial est lisse, le nouveau problème ne l'est plus !

1.3 Algorithmes

Solutions approchées La plupart du temps, on ne peut pas résoudre (1.2.1) analytiquement. On peut en revanche proposer des algorithmes qui qui fournissent une solution approchée. Trouver une solution approchée à ϵ près (une **ϵ-solution**) consiste à trouver $\hat{x} \in K$ telle que, si le "vrai" minimum x^* existe, on ait

- $\|\hat{x} - x^*\| \leq \epsilon$,

  et/ou
- $|f(\hat{x}) - f(x^*)| \leq \epsilon$.

Modèle "boîte noire" Un cadre standard en optimisation est celui de la **boîte noire**. C'est-à-dire , on cherche à optimiser une fonction, dans une situation où

- Au départ, on n'a pas d'information sur la forme de f.
- L'algorithme n'a accès à f (et aux contraintes) que par des appels successifs à un *oracle* $\mathcal{O}(x)$. Typiquement, $\mathcal{O}(x) = f(x)$ (oracle d'ordre 0) ou $\mathcal{O}(x) = (f(x), \nabla f(x))$ (oracle d'ordre 1), ou $\mathcal{O}(x)$ peut évaluer des dérivées d'ordre supérieur de f (oracle d'ordre ≥ 2).
- A l'itération k, l'algorithme ne dispose que de l'information $\mathcal{O}(x_1), \ldots, \mathcal{O}(x_k)$ pour calculer le point suivant x_{k+1}.
- L'algorithme s'arrête au temps k si un certain critère $T_\epsilon(x_k)$ est vérifié, lui-même garantissant que x_k soit une ϵ-solution.

Performance d'un algorithme La performance est mesurée en terme de moyens de calculs nécessaires pour obtenir une solution approchée.
Elle dépend évidemment du problème considéré. Une **classe de problème** est la donnée de

- Une classe de fonctions objectifs (conditions de régularité, convexité ou autre)
- une condition sur le point de départ x_0 (par exemple, $\|x - x_0\| \leq R$)
- Un oracle.

Definition 1.3.1 (complexité d'oracle)**.** *La **complexité d'oracle** d'un algorithme $\mathcal{A}$ pour une classe de problèmes C et une précision ϵ donnée est le nombre minimal $N_\mathcal{A}(\epsilon)$ tel que, pour toute fonction objectif et point initial $(f, x_0) \in C$, en notant $N_\mathcal{A}(f, \epsilon)$ le nombre d'appels à l'oracle nécessaires pour que $\mathcal{A}$ produise une ϵ-solution, on ait*

$$N_\mathcal{A}(f, \epsilon) \leq N_\mathcal{A}(\epsilon).$$

La complexité d'oracle ainsi définie est une complexité *dans le pire des cas.*
Le temps de calcul nécessaire dépend de la complexité d'oracle mais aussi du nombre d'opérations nécessaires à chaque appel de l'oracle. Le nombre total d'opérations arithmétiques pour atteindre une ϵ-solution dans le pire des cas est appelée *complexité arithmétique.*
En pratique, c'est la complexité arithmétique qui détermine le temps de calcul, mais il est plus facile d'établir des bornes de complexité d'oracle.

1.4 Aperçu de la suite du cours

Une idée naturelle pour résoudre le problème général (1.2.1) est de partir d'un point x_0 quelconque et de proposer le point suivant x_1 dans une région où f "a des chances" d'être plus petite. Si f est différentiable, on peut, par exemple, suivre "la ligne de plus grande pente", c'est-à-dire se déplacer dans la direction de $-\nabla f$. De plus, s'il existe un minimum x^* local, on a $\nabla f(x^*) = 0$. Une idée voisine de la précédente consiste donc à annuler le gradient.
Ici, on a fait des hypothèses implicites de régularité mais un certain nombre de problèmes peuvent se poser en pratique.

- Sous quelle hypothèses la condition nécessaire $\nabla f(x) = 0$ est-elle suffisante pour que x soit un minimum local ?
- Sous quelles hypothèses un minimum local est-il global ?
- Quel cadre adopter lorsque f n'est pas différentiable ?
- Comment faire lorsque E est un espace de grande dimension ?
- Que faire si le nouveau point x_1 sort de la région admissible K ?

Le cadre adapté pour répondre aux deux premières questions est celui de l'analyse convexe. L'absence de différentiabilité peut être contournée en introduisant la notion de *sous-différentiel*. Les approches par *dualité* permettent de résoudre un problème lié à ((1.2.1)), appelé *problème dual*. On de débrouille pour que le problème dual soit plus facile à résoudre (*ex:* il s'écrit dans un espace de dimension modérée). Classiquement, Une fois la solution duale connue, le problème primal s'écrit comme un problème sans contrainte, lui aussi plus facile au résoudre que le problème initial. Par exemple, Les méthodes *proximales* peuvent être utilisées pour résoudre des problèmes contraints.

Pour aller plus loin . . .
Un panorama dans Boyd and Vandenberghe (2009), chapitre 4, ou un peu plus de rigueur (très lisible !) dans Nesterov (2004), chapitre d'introduction.

Chapter 2

Convexité et sous-differentiel

Dans tous ce cours, les fonctions d'intérêt sont définies sur une partie de $\mathcal{X} = \mathbb{R}^n$. Plus généralement, $\mathbf{E}$ désigne un espace euclidien muni (tout comme $\mathcal{X}$) d'un produit scalaire noté $\langle \cdot\, , \cdot \rangle$ et de la norme associée $\| \cdot \|$. En pratique, on aura souvent $\mathbf{E} = \mathcal{X} \times \mathbb{R}$.
notations: Si $a \leq b \in \mathbb{R} \cup \{-\infty, +\infty\}$, $(a, b]$ désigne un intervalle ouvert en a, fermé en b, avec des significations semblables pour $[a, b)$, (a, b) et $[a, b]$.

N.B Un certain nombre de propriétés immédiates sont données à démontrer en exercice. Il est plus que conseillé de le faire ! Les exercices marqués d'une * sont moins indispensables.

2.1 Convexité

Definition 2.1.1 (Ensemble convexe)**.** *Un ensemble $K \subset \mathbf{E}$ est **convexe** si*

$$\forall (x, y) \in K^2, \forall t \in [0, 1], \quad t\, x + (1 - t)\, y \in K.$$

Exercise 2.1.1.

1. Montrer qu'une boule, un sous espace vectoriel et un sous espace affine de $\mathbb{R}^n$ sont convexes.

2. Montrer qu'une intersection quelconque de convexes est convexe.

Dans les problèmes d'optimisation sous contrainte, il sera utile de définir des fonctions de coût prenant la valeur $+\infty$ en dehors de la région admissible. Pour tout $f : \mathcal{X} \to [-\infty, +\infty]$, on note $\mathrm{dom}(f)$ est l'ensemble des points x tels que $f(x) < +\infty$. On dit que f est **propre** si $\mathrm{dom}(f) \neq \emptyset$ (c'est-à-dire $f \not\equiv +\infty$) et si f ne prend *jamais* la valeur $-\infty$.

Definition 2.1.2. *Soit $f : \mathcal{X} \to [-\infty, +\infty]$. L'**épigraphe de** f, noté* $\mathrm{epi}\, f$, *est le sous ensemble de $\mathcal{X} \times \mathbb{R}$ défini par*

$$\mathrm{epi}\, f = \{(x, t) \in \mathcal{X} \times \mathbb{R} : \quad t \geq f(x)\ \}.$$

Attention: les “ordonnées” des points de l'épigraphe sont toujours dans $(-\infty, \infty)$, par définition.

Definition 2.1.3 (Fonction convexe)**.** *$f : \mathcal{X} \to [-\infty, +\infty]$ est dite **convexe** si son épigraphe est convexe.*

Exercise 2.1.2. Montrer que

1. Si f est convexe, alors $\mathrm{dom}(f)$ est convexe.

2. si f_1, f_2 sont convexes et $a, b \in \mathbb{R}^+$, alors $af_1 + bf_2$ est convexe.

3. Si f est convexe et $x, y \in \mathrm{dom}\, f$, pour tout $t \geq 1$, $z_t = x + t(y - x)$ vérifie $f(z_t) \geq f(x) + t(f(y) - f(x)$.

4. Si f est convexe, propre, avec $\mathrm{dom}\, f = \mathcal{X}$, et si f est bornée, alors f est constante.

Proposition 2.1.1. *Une fonction $f : \mathcal{X} \to [-\infty, +\infty]$ est convexe si et seulement si*

$$\forall (x, y) \in \mathrm{dom}(f)^2,\ \forall t \in (0, 1), \quad f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

Proof. Supposons que f satisfait l'inégalité. Soient (x, u) et (y, v) deux points de l'épigraphe : $u \geq f(x)$ et $v \geq f(y)$. En particulier $(x, y) \in \mathrm{dom}(f)^2$. Soit $t \in]0, 1[$. L'inégalité implique que $f(tx + (1 - t)y) \leq tu + (1 - t)v$. Ainsi $t(x, u) + (1 - t)(y, v) \in \mathrm{epi}(f)$ ce qui prouve que $\mathrm{epi}(f)$ est convexe.
Réciproquement, supposons $\mathrm{epi}(f)$ convexe. Soient $(x, y) \in \mathrm{dom}(f)^2$. Pour (x, u) et (y, v) deux points de $\mathrm{epi}(f)$, et $t \in [0, 1]$, le point $t(x, u)+(1-t)(y, v)$ appartient à $\mathrm{epi}(f)$. Donc, $f(t(x + (1 - t)y) \leq tu + (1 - t)v$.

- Si $f(x)$ et $f(y)$ sont $> -\infty$, on peut choisir $u = f(x)$ et $v = f(y)$ ce qui démontre l'inégalité.

- Si $f(x) = -\infty$, on peut choisir u arbitrairement proche de $-\infty$. Par passage à la limite sur u, on obtient $f(t(x + (1 - t)y) = -\infty$, ce qui démontre là encore l'inégalité voulue.

□

Exercise 2.1.3. *
Soit f une fonction convexe et x, y dans $\mathrm{dom}\, f$, $t \in (0, 1)$ et $z = tx + (1-t)y$. On suppose que les trois points $(x, f(x)), (z, f(z))$ et $(y, f(y)$ sont alignés. Montrer que pour tout $u \in (0, 1)$, $f(u\, x + (1 - u)\, y) = u\, f(x) + (1 - u)\, f(y)$.

Dans la suite, la notion d'**enveloppe supérieure** d'une famille de fonctions convexes, jouera un rôle déterminant. Par définition, l'enveloppe supérieure d'une famille $(f_i)_{i\in I}$ de fonctions est la fonction $x \mapsto \sup_i f_i(x)$.

Proposition 2.1.2. *Soit $(f_i)_{i\in I}$ une famille de fonction convexe de $\mathcal{X} \to [-\infty, +\infty]$, I un ensemble d'indices* quelconque. *Alors **l'enveloppe supérieure des $(f_i)_{i\in I}$ est convexe**.*

Proof. Soit $f = \sup_{i\in I} f_i$ l'enveloppe supérieure de la famille.

(a) $epi f = \bigcap_{i\in I} \operatorname{epi} f_i$. En effet,

$$(x,t) \in \operatorname{epi} f \Leftrightarrow \forall i \in I, t \geq f_i(x) \Leftrightarrow \forall i \in I, (x,t) \in \operatorname{epi} f_i \Leftrightarrow (x,t) \in \cap_i \operatorname{epi} f_i.$$

(b) Une intersection quelconque de convexes $K = \cap_{i\in I} K_i$ est convexe (exercice 2.1.1)

(a) et (b) montrent que $\operatorname{epi} f$ est convexe, c'est-à-dire f est convexe. □

2.2 Séparation, sous-différentiel

En dimension finie, on obtient facilement des résultats de séparation, grâce à l'existence d'un "projeté orthogonal" de tout point x sur un convexe fermé. Plus précisément :

Proposition 2.2.1 (Projection)**.** *Soit $C \subset \mathbf{E}$ un convexe fermé et $x \in \mathbf{E}$.*

1. *Il existe un unique point de C, noté $P_C(x)$, tel que pour tout $y \in C$, $\|y - x\| \geq \|P_C(x) - x\|$.*
2. *$\forall y \in C$, $\langle y - P_C(x), x - P_C(x)\rangle \leq 0$.*
3. *$\forall (x,y) \in \mathbf{E}^2$, $\|P_C(y) - P_C(x)\| \leq \|y - x\|$.*

Le point $P_C(x)$ est appelé projeté (ou projection) de x sur C.

Proof.
1. Posons $d_C(x) = \inf_{y\in C} \|y - x\|$. On peut construire une suite $(y_n)_n$ de C telle que $\|y_n - x\| \to d_C(x)$. La suite étant bornée, on peut extraire une sous-suite convergeant vers y_0. Par continuité de la fonction $y \mapsto \|y - x\|$, on a bien $\|y_0 - x\| = d_C(x)$.
Montrons l'unicité. Soit $z \in C$ un point tel que $\|z - x\| = d_C(x)$. Par convexité de C, $w = (y_0 + z)/2 \in C$, donc $\|w - x\| \geq d_C(x)$. D'après l'identité du parallélogramme[1]

$$\begin{aligned} 4d_C(x)^2 &= 2\|y_0 - x\|^2 + 2\|z - x\|^2 \\ &= \|y_0 + z - 2x\|^2 + \|y_0 - z\|^2 \\ &= 4\|w - x\|^2 + \|y_0 - z\|^2 \\ &\geq 4d_C(x) + \|y_0 - z\|^2 \end{aligned}$$

[1] $2\|a\|^2 + 2\|b\|^2 = \|a + b\|^2 + \|a - b\|^2$.

Donc $\|y_0 - z\| = 0$ et $y_0 = z$.

2. Posons $p = P_C(x)$. Soit $y \in C$. Pour $\epsilon \in [0,1]$, posons $z_\epsilon = p + \epsilon(y - p)$. Par convexité, $z_\epsilon \in C$. Considérons la fonction 'distance à x au carrée':

$$\varphi(\epsilon) = \|z_\epsilon - x\|^2 \;=\; \|\epsilon(y-p) + p - x\|^2.$$

Pour $0 < \epsilon \leq 1$, $\varphi(\epsilon) \geq d_C(x)^2 = \varphi(0)$. De plus, au voisinage de $\epsilon = 0$,

$$\varphi(\epsilon) = d_C(x)^2 - 2\epsilon \langle y - p, x - p\rangle + o(\epsilon).$$

Donc $\varphi'(0) = -2 \langle y - p, x - p\rangle$. Si on avait $\varphi'(0) < 0$, alors pour ϵ proche de 0, on aurait $\varphi(\epsilon) < \varphi(0) = d_C(x)$, ce qui est impossible. D'où, $\varphi'(0) \geq 0$ et le résultat.

3. En ajoutant les inégalités

$$\begin{aligned}&\langle P_C(y) - P_C(x), x - P_C(x)\rangle \leq 0\ , \text{ et}\\ &\langle P_C(x) - P_C(y), y - P_C(y)\rangle \leq 0\ ,\end{aligned}$$

on obtient $\langle P_C(y) - P_C(x), y - x\rangle \geq \|P_C(x) - P_C(y)\|^2$. On conclut par l'inégalité de Cauchy-Schwarz. □

Une des conséquences de ce résultat de projection est la possibilité d'exhiber des "hyperplans séparateurs". Donnons d'abord deux définitions, illustrées par la figure 2.1.

Definition 2.2.1 (Séparation au sens strict et large)**.** *Soient $A, B \subset \mathbf{E}$, et H un hyperplan affine , $H = \{x \in \mathbf{E} : \langle x, w\rangle = \alpha\}$, avec $w \neq 0$. On dit que*

- *H **sépare** A **et** B **au sens large** si*

$$\begin{aligned}&\forall x \in A, \langle w, x\rangle \leq \alpha\ , \quad \textit{et}\\ &\forall x \in B, \langle w, x\rangle \geq \alpha.\end{aligned}$$

- *H **sépare** A **et** B **au sens strict** s'il existe $\delta > 0$ tel que*

$$\begin{aligned}&\forall x \in A, \langle w, x\rangle \leq \alpha - \delta\ , \quad \textit{et}\\ &\forall x \in B, \langle w, x\rangle \geq \alpha + \delta\ ,\end{aligned}$$

Le théorème suivant est un des deux résultats majeurs de cette section (avec l'existence d'un hyperplan support). Il est la conséquence directe de la proposition 2.2.1.

Theorem 2.2.1 (Séparation stricte point-convexe fermé)**.** *Soit $C \subset \mathbf{E}$ un convexe fermé et $x \notin C$. Alors il existe un hyperplan affine séparant strictement x et C.*

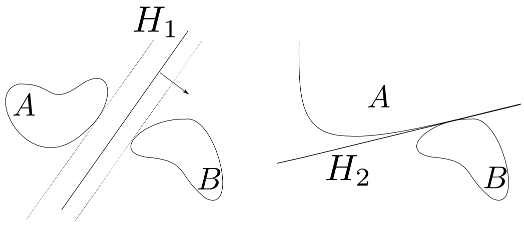

Figure 2.1: séparation stricte de A et B par H_1 , au sens faible par H_2.

Proof. Soit $p = p_C(x)$, $w = x - p$. Pour $y \in C$, d'après la proposition 2.2.1, 2., on a $\langle w, y - p\rangle \leq 0$, c'est-à-dire

$$\forall y \in C, \langle w, y\rangle \leq \langle p, w\rangle \,.$$

D'autre part, $\langle w, x - p\rangle = \|w\|^2 > 0$, donc

$$\langle w, x\rangle = \langle w, p\rangle + \|w\|^2.$$

Il suffit de poser $\delta = \|w\|^2/2 > 0$ et $\alpha = \langle w, p\rangle + \delta$ pour retrouver la définition de la séparation stricte. □

Une conséquence immédiate, très utile en pratique:

Corollary 2.2.1 (Conséquence de la séparation stricte)**.** *Soit $C \subset \mathbf{E}$ in convexe fermé et $x_0 \notin C$. Alors il existe $w \in \mathbf{E}$, tel que*

$$\forall y \in C,\ \langle w, y\rangle < \langle w, x_0\rangle$$

Dans toute la suite, on note $\mathrm{cl}(A)$ l'adhérence d'un ensemble A et $\mathrm{int}(A)$ son intérieur. On vérifie immédiatement que

Lemma 2.2.1. *Si A est convexe, alors $\mathrm{cl}(A)$ et $\mathrm{int}(A)$ sont convexes.*

Exercise 2.2.1. Démontrer le lemme 2.2.1. On pourra par exemple faire tendre des suites de A vers deux points considérés de son adhérence ; Envelopper deux points de son intérieur dans des boules.

Le deuxième résultat majeur est le suivant:

Theorem 2.2.2 (Hyperplan support)**.** *Soit $C \subset \mathbf{E}$ un convexe quelconque et soit x_0 un point de la frontière e C, $x_0 \in \partial(C) = \mathrm{cl}(C) \setminus \mathrm{int}(C)$. Alors il existe un hyperplan affine séparant x_0 et C au sens large, c'est à dire,*

$$\exists w \in \mathbf{E}, w \neq 0\ :\ \forall y \in C,\ \langle w, y\rangle \leq \langle w, x_0\rangle \,.$$

Proof. Soient C et x_0 comme dans l'énoncé.
Il existe une suite (x_n) avec $x_n \in \big(\mathrm{cl}(C)\big)^c$ et $x_n \to x_0$, sinon on aurait une boule incluse dans C contenant x_0, et x_0 serait dans $\mathrm{int}(C)$. On peut séparer strictement x_n de $\mathrm{cl}(C)$, d'après le théorème 2.2.1. Son corollaire 2.2.1 implique :

$$\forall n,\ \exists w_n \in \mathbf{E}\ :\ \forall y \in C, \langle w_n, y\rangle < \langle w_n, x_n\rangle$$

En particulier, chaque w_n est non nul. On peut donc poser $u_n = w_n/\|w_n\|$. On a encore:

$$\forall n, \forall y \in C,\ \langle u_n, y\rangle < \langle u_n, x_n\rangle \tag{2.2.1}$$

la suite (u_n) étant bornée, dans la sphère unité, on peut en extraire une suite $(u_{k_n})_n$ qui converge vers un certain $u \in \mathbf{E}$, appartenant également à la sphère unité (donc $u \neq 0$). Par passage à la limite dans (2.2.1) (à y fixé), en utilisant la continuité du produit scalaire, on a

$$\forall y \in C,\ \langle u, y\rangle \leq \langle u, x_0\rangle$$

□

Remark 2.2.1. *En dimension infinie les théorèmes 2.2.1 et 2.2.2 restent valides si l'on suppose que* $\mathbf{E}$ *est un espace de Hilbert ou même simplement un espace de Banach. C'est le "théorème d'Hahn-Banach", dont la preuve est donnée par exemple dans les toutes premières pages de Brezis (1987)*

Une des conséquences (proposition 2.2.2) du théorème 2.2.2 sera de rendre "non-vide" la définition suivante

Definition 2.2.2 (sous-différentiel)**.** *Soit* $f : \mathcal{X} \to [-\infty, +\infty]$ *et* $x \in \mathrm{dom}(f)$*. On dit que* $\phi \in \mathcal{X}$ *est un* ***sous-gradient*** *de* f *en* x *si*

$$\forall y \in \mathcal{X},\quad f(y) - f(x) \geq \langle \phi, y - x\rangle\ .$$

Le ***sous-différentiel*** *de* f *en* x*, noté* $\partial f(x)$*, est l'ensemble des sous-gradients de* f *en* x*. Par convention,* $\partial f(x) = \emptyset$ *si* $x \notin \mathrm{dom}(f)$*.*

Intérêt: les méthodes de gradient s'étendent généralement au cas non différentiable, en choisissant un sous-gradient dans le sous-différentiel.
Pour préciser dans quels cas le sous-différentiel est non vide, nous avons besoin de deux définitions de plus :

Definition 2.2.3. *Un ensemble* $A \subset \mathcal{X}$ *est appelé un* ***espace affine*** *si, pour tout* $(x, y) \in A^2$ *et pour tout* $t \in \mathbb{R}$*,* $x + t(y - x) \in A$*. L'****enveloppe affine*** $\mathcal{A}(C)$ *d'un ensemble* $C \subset \mathcal{X}$ *est* ***le plus petit espace affine*** *contenant* C*.*

Definition 2.2.4. *Soit* $C \subset \mathbf{E}$*. La* ***topologie relative à*** C *est une topologie sur* $\mathcal{A}(C)$*. Les ouverts de cette topologie sont les ensembles de type* $\{V \cap \mathcal{A}(C)\}$*, où* V *est ouvert dans* $\mathbf{E}$*.*

Definition 2.2.5. *Soit $C \subset \mathcal{X}$. L'**intérieur relatif** de C, noté* relint(C)*, est l'intérieur de C pour la topologie relative à C. Autrement dit, c'est l'ensemble des points x qui admettent un voisinage V, ouvert dans* $\mathbf{E}$*, tel que $V \cap \mathcal{A}(C) \subset C$.*

Il est clair que $\text{int}(C) \subset \text{relint}(C)$. De plus, si C est convexe, $\text{relint}(C) \neq \emptyset$. En effet :

- si C est réduit à un singleton $\{x_0\}$, alors $\text{relint}\{x_0\} = \{x_0\}$. ($\mathcal{A}(C) = \{x_0\}$ et pour un ouvert $U \subset \mathcal{X}$, tel que $x_0 \subset U$, on a bien $x_0 \in U \cap \{x_0\}$) ;
- si C contient au moins deux points x, y, alors tout point du segment ouvert $\{x + t(y - x), t \in (0, 1)\}$ est dans $\mathcal{A}(C)$.

Proposition 2.2.2. *Soit $f : \mathcal{X} \to [-\infty, +\infty]$ une fonction convexe et $x \in$* $\text{relint}(\text{dom}\, f)$*. Alors $\partial f(x)$ est non vide.*

Proof. Soit $x_0 \in \text{relint}(\text{dom}\, f)$. On suppose que $f(x_0) > -\infty$ sans quoi la preuve est triviale. On peut se restreindre au cas où $x_0 = 0$ et $f(x_0) = 0$, quitte à remplacer f par la fonction $x \mapsto f(x + x_0) - f(x_0)$. Dans ce cas, pour tout vecteur $\phi \in \mathcal{X}$,

$$\phi \in \partial f(0) \quad \Leftrightarrow \quad \forall x \in \text{dom}\, f,\ \langle \phi, x \rangle \leq f(x)\,.$$

Posons $\mathcal{A} = \mathcal{A}(\text{dom}\, f)$. $\mathcal{A}$ contient l'origine, donc c'est un espace vectoriel, euclidien.
Soit C l'adhérence de $\text{epi}\, f \cap (\mathcal{A} \times \mathbb{R})$. L'ensemble C est un convexe fermé de l'espace $\mathcal{A} \times \mathbb{R}$. On munit ce dernier du produit scalaire $\langle (x, u), (x', u') \rangle = \langle x, x' \rangle + uu'$.
Le couple $(0, 0) = (x_0, f(x_0))$ appartient à la frontière de C. On applique le théorème 2.2.2 dans l'espace $\mathcal{A} \times \mathbb{R}$: il existe $w \in \mathcal{A} \times \mathbb{R}$, $w \neq 0$, tel que

$$\forall z \in C,\ \langle w, z \rangle \leq 0$$

Notons $w = (\phi, u) \in \mathcal{A} \times \mathbb{R}$. Pour $z = (x, t) \in C$, on a donc

$$\langle \phi, x \rangle + u\, t \leq 0.$$

Soit $x \in \text{dom}(f)$. En particulier $f(x) < \infty$ et pour tout $t \geq f(x)$, $(x, t) \in C$. Ainsi,

$$\forall x \in \text{dom}(f),\ \forall t \geq f(x), \quad \langle \phi, x \rangle + u\, t \leq 0. \tag{2.2.2}$$

On faisant tendre t vers $+\infty$, on voit que $u \leq 0$.
Supposons par l'absurde que $u = 0$. On a alors $\langle \phi, x \rangle \leq 0$ pour tout $x \in \text{dom}(f)$. Comme $0 \in \text{relint}\,\text{dom}(f)$, il existe un ensemble $\tilde{V}$, ouvert dans $\mathcal{A}$, tel que $0 \in \tilde{V} \subset \text{dom}\, f$. Ainsi pour $x \in \mathcal{A}$, il existe $\epsilon > 0$ tel que $\epsilon x \in \tilde{V} \subset \text{dom}(f)$. D'après (2.2.2), $\langle \phi, \epsilon x \rangle \leq 0$, d'où $\langle \phi, x \rangle \leq 0$. De même,

$\langle \phi, -x \rangle \leq 0$. Par conséquent, $\langle \phi, x \rangle \equiv 0$ sur $\mathcal{A}$. Puisque $\phi \in \mathcal{A}$, $\phi = 0$. Finalement $w = 0$, contradiction.
On en conclut que $u < 0$. En divisant l'inégalité (2.2.2) par $-u$, et en prenant $t = f(x)$, on obtient

$$\forall x \in \mathrm{dom}(f),\ \forall t \geq f(x), \quad \big\langle \frac{-1}{u}\phi, x \big\rangle \leq f(x)\,.$$

Donc $\frac{-1}{u}\phi \in \partial f(0)$.

□

Remark 2.2.2 (la question de $-\infty$)**.**
Si $f : \mathcal{X} \to [-\infty, +\infty]$ est convexe et si relint dom f *contient un point x tel que $f(x) > -\infty$, alors f ne prend* jamais *la valeur $-\infty$, donc f est propre.*

Exercise 2.2.2. Montrez-le en utilisant la proposition 2.2.2.

Lorsque f est différentiable en $x \in \mathrm{dom}\, f$, on note $\nabla f(x)$ son gradient au point x. Le lien entre différentiation et sous-différentiel est donné par la proposition suivante:

Proposition 2.2.3. *Soit $f : \mathcal{X} \to (-\infty, \infty]$ une fonction convexe, différentiable en x. Alors $\partial f(x) = \{\nabla f(x)\}$.*

Proof. Si f est différentiable en x, le point x est nécessairement dans $\mathrm{int}\,\mathrm{dom}(f)$. Soit $\phi \in \partial f(x)$ et $t \neq 0$. Alors pour tout $y \in \mathrm{dom}(f)$, $f(y) - f(x) \geq \langle \phi, y - x \rangle$. On applique cette inégalité avec $y = x + t(\phi - \nabla f(x))$ (qui appartient bien à $\mathrm{dom}(f)$ pour t suffisamment petit). Ceci conduit à :

$$\frac{f\left(x + t(\phi - \nabla f(x))\right) - f(x)}{t} \geq \langle \phi, \phi - \nabla f(x) \rangle\ .$$

Le membre de gauche converge vers $\langle \nabla f(x), \phi - \nabla f(x) \rangle$. On obtient finalement

$$\langle \nabla f(x) - \phi, \phi - \nabla f(x) \rangle \geq 0,$$

soit $\phi = \nabla f(x)$. □

Example 2.2.1. La fonction valeur absolue $x \mapsto |x|$ définie sur $\mathbb{R} \to \mathbb{R}$ admet pour sous-différentiel l'application sign définie par

$$\mathrm{sign}(x) = \begin{cases} \{1\} & \text{si } x > 0 \\ [-1, 1] & \text{si } x = 0 \\ \{-1\} & \text{si } x < 0\,. \end{cases}$$

Exercise 2.2.3. Déterminer le sous-différentiel des fonctions suivantes aux points considérés:

1. Dans $\mathcal{X} = \mathbb{R}$, $f(x) = \mathbb{I}_{[0,1]}$, en $x = 0, x = 1$ et $0 < x < 1$.

2. Dans $\mathcal{X} = \mathbb{R}^2$, $f(x_1, x_2) = \mathbb{I}_{x_1<0}$, en x tel que $x_1 = 0$, $x_1 < 0$.

3. $\mathcal{X} = \mathbb{R}$,
$$f(x) = \begin{cases} +\infty & \text{si } x < 0 \\ -\sqrt{x} & \text{si } x \geq \end{cases}$$
en $x = 0$, et $x > 0$.

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4. $\mathcal{X} = \mathbb{R}^n$, $f(x) = \|x\|$, determine $\partial f(x)$, for any $x \in \mathbb{R}^n$.

5. $\mathcal{X} = \mathbb{R}$, $f(x) = x^3$. Show that $\partial f(x) = \emptyset$, $\forall x \in \mathbb{R}$. Explain this result.

6. $\mathcal{X} = \mathbb{R}^n$, $C = \{y : \|y\| \leq 1\}$, $f(x) = \mathbb{I}_C(x)$. Give the subdifferential of f at x such that $\|x\| < 1$ and at x such that $\|x\| = 1$.

   Hint: For $\|x\| = 1$:

   - Show that $\partial f(x) = \{\phi : \forall y \in C, \langle \phi, y - x \rangle \leq 0\}$.
   - Show that $x \in \partial f(x)$ using Cauchy-Schwarz inequality. Deduce that the cone $\mathbb{R}^+ x = \{t\, x \ : \ t \geq 0\} \subset \partial f(x)$.
   - To show the converse inclusion : Fix $\phi \in \partial f$ and pick $u \in \{x\}_\perp$ (*i.e.*, u s.t. $\langle u, x \rangle = 0$). Consider the sequence $y_n = \|x + t_n u\|^{-1}(x + t_n u)$, for some sequence $(t_n)_n, t_n > 0, t_n \to 0$. What is the limit of y_n ?

     Consider now $u_n = t_n^{-1}(y_n - x)$. What is the limit of u_n ? Conclude about the sign of $\langle \phi, u \rangle$.

     Do the same with $-u$, conclude about $\langle \phi, u \rangle$. Conclude.

7. Let $f : \mathbb{R} \to \mathbb{R}$, differentiable. Show that: f is convex, if and only if
$$\forall (x, y) \in \mathbb{R}^2, \langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0.$$

2.3 Fermat's rule, optimality conditions.

A point x is called a **minimizer** of f if $f(x) \leq f(y)$ for all $y \in \mathcal{X}$. The set of minimizers of f is denoted $\arg\min(f)$.

Proposition 2.3.1 (Fermat's rule)**.** $x \in \arg\min f \;\Leftrightarrow\; 0 \in \partial f(x)$.

Proof.
$$x \in \arg\min f \;\Leftrightarrow\; \forall y, f(y) \geq f(x) + \langle 0, y - x \rangle \Leftrightarrow 0 \in \arg\min f.$$
□

Recall that, in the differentiable, non convex case, a *necessary* condition (not a sufficient one) for $\bar{x}$ to be a local minimizer of f, is that $\nabla f(\bar{x}) = 0$. Convexity allows handling non differentiable functions, and turns the necessary condition into a sufficient one.
Besides, local minima for any function f are not necessarily global ones. In the convex case, everything works fine:

Proposition 2.3.2. *Let x be a local minimum of a convex function f. Then, x is a global minimizer.*

Proof. The local minimality assumption means that there exists an open ball $V \subset \mathcal{X}$, such that $x \in V$ and that, for all $u \in V$, $f(x) \leq f(u)$.
Let $y \in \mathcal{X}$ and t such that $u = x + t(y - x) \in V$. Then using convexity of f, $f(u) \leq tf(y) + (1-t)f(x)$. Re-organizing, we get

$$f(y) \geq t^{-1}\big(f(u) - (1-t)f(x)\big) \geq f(x).$$

□

Chapter 3

Fenchel-Legendre transformation, Fenchel Duality

We introduce now the second basic tool of convex analysis (after sub-differentials), especially useful for duality approaches: the Fenchel-Legendre transform.
One precision on notations before proceeding: If f is any mapping $\mathcal{X} \to \mathcal{Y}$, and $A \subset \mathcal{X}$, write $f(A) = \{f(x), x \in \mathcal{A}\}$.

3.1 Fenchel-Legendre Conjugate

Definition 3.1.1. *Let $f : \mathcal{X} \to [-\infty, +\infty]$. The* **Fenchel-Legendre conjugate** *of f is the function $f^* : \mathcal{X} \to [-\infty, \infty]$, defined by*

$$\begin{aligned} f^*(\phi) &= \sup_{x \in \mathcal{X}} \langle \phi, x \rangle - f(x) , \qquad \phi \in \mathcal{X}. \\ &= \sup \langle \phi, \mathcal{X} \rangle - f(\mathcal{X}) \end{aligned}$$

Notice that

$$f^*(0) = -\inf f(\mathcal{X}).$$

Figure provides a graphical representation of f^*. You should get the intuition that, in the differentiable case, if the maximum is attained in the definition of f^* at point x_0, then $\phi = \nabla f(x_0)$, and $f^*(\phi) = \langle \nabla f(x_0), x_0 \rangle - f(x_0)$. This intuition will be proved correct in proposition 3.2.3.

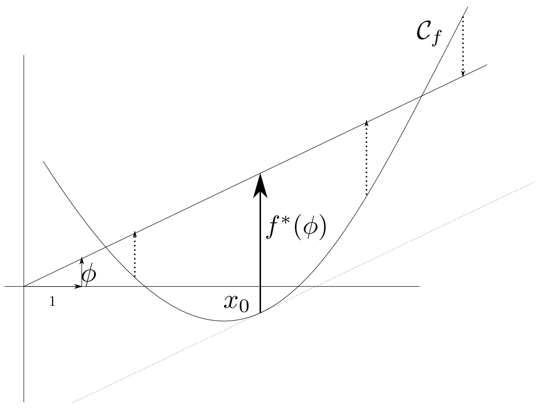

Figure 3.1: Fenchel Legendre transform of a smooth function f. The maximum positive difference between the line with slope $\tan(\phi)$ and the graph $\mathcal{C}_f$ of f is reached at x_0.

Exercise 3.1.1.
Prove the following statements.
General hint : If $h_\phi : x \mapsto \langle \phi, x \rangle - f(x)$ reaches a maximum at x^*, then $f^*(\phi) = h_\phi(x^*)$. Furthermore, h_ϕ is concave (if f is convex). If h_ϕ is differentiable, it is enough to find a zero of its gradient to get a maximum.
Indeed, $x \in \arg\min(-h_\phi) \Leftrightarrow 0 \in \partial(-h_\phi)$, and, if $-h_\phi$ is differentiable, $\partial(-h_\phi) = \{-\nabla h_\phi\}$.

1. If $\mathcal{X} = \mathbb{R}$ and f is a quadratic function (*i.e.* $f(x) = (x-a)^2 + b$), then f^* is also quadratic.

2. In $\mathbb{R}^n$, let A by a symmetric, definite positive matrix and $f(x) = \langle x, Ax \rangle$ (a quadratic function). Show that f^* is also quadratic.

3. $f : \mathcal{X} \to [-\infty, +\infty]$. Show that $f = f^* \Leftrightarrow f(x) = \frac{1}{2}\|x\|^2$.

   Hint: For the 'if' part : show first that $f(\phi) \geq \langle \phi, \phi \rangle - f(\phi)$.

   Then show that $f(\phi) \leq \sup_x \langle \phi, x \rangle - \frac{1}{2}\|x\|^2$. Conclude.

4. $\mathcal{X} = \mathbb{R}$,

$$f(x) = \begin{cases} 1/x & \text{if } x > 0; \\ +\infty & \text{otherwise .} \end{cases}$$

   then

$$f^*(\phi) = \begin{cases} -2\sqrt{-\phi} & \text{if } \phi \leq 0; \\ +\infty & \text{otherwise .} \end{cases}$$

5. $\mathcal{X} = \mathbb{R}$, $f(x) = \exp(x)$, then

$$f^*(\phi) = \begin{cases} \phi \ln(\phi) - \phi & \text{if } \phi > 0; \\ 0 & \text{if } \phi = 0; \\ +\infty & \text{if } \phi < 0. \end{cases}$$

Notice that, if $f(x) = -\infty$ for some x, then $f^* \equiv +\infty$.
Nonetheless, under 'reasonable' conditions on f, the Legendre transform enjoys nice properties, and even f can be recovered from f^* (through the equality $f = f^{**}$, see proposition 3.2.5. This is the starting point of dual approaches. To make this precise, we need a to introduce a weakened notion of continuity: semi-continuity, which allows to use separation theorems.

3.2 Lower semi-continuity

Definition 3.2.1 (Reminder : lim inf : **limit inferior**). *The **limit inferior** of a sequence $(u_n)_{n\in\mathbb{N}}$, where $u_n \in [-\infty, \infty]$, is*

$$\liminf(u_n) = \sup_{n\geq 0} \Big(\inf_{k\geq n} u_k \Big).$$

Since the sequence $V_n = \inf_{k\geq n} u_k$ is non decreasing, an equivalent definition is

$$\liminf(u_n) = \lim_{n\to\infty} \Big(\inf_{k\geq n} u_k \Big).$$

Definition 3.2.2 (Lower semicontinuous function). *A function $f : \mathcal{X} \to [-\infty, \infty]$ is called* **lower semicontinuous (l.s.c.) at** *$x \in \mathcal{X}$ if For all sequence (x_n) which converges to x,*

$$\liminf f(x_n) \geq f(x).$$

The function f is said to be **lower semicontinuous**, *if it is l.s.c. at x, for all $x \in \mathcal{X}$.*

The interest of l.s.c. functions becomes clear in the next result

Proposition 3.2.1 (epigraphical characterization). *Let $f : \mathcal{X} \to [-\infty, +\infty]$, any function*
f is l.s.c. if and only if its epigraph is closed.

Proof. If f is l.s.c., and if $(x_n, t_n) \in \operatorname{epi} f \to (\bar{x}, \bar{t})$, then, $\forall n, t_n \geq f(x_n)$. Consequently,

$$\bar{t} = \liminf t_n \geq \liminf f(x_n) \geq f(\bar{x}).$$

Thus, $(\bar{x}, \bar{t}) \in \operatorname{epi} f$, and epi f is closed.

Conversely, if f is *not* l.s.c., there exists an $x \in \mathcal{X}$, and a sequence $(x_n) \to x$, such that $f(x) > \liminf f(x_n)$, *i.e.*, there is an $\epsilon > 0$ such that $\forall n \geq 0$, $\inf_{k \geq n} f(x_k) \leq f(x) - \epsilon$. Thus, for all n, $\exists k_n \geq k_{n-1}$, $f(x_{k_n}) \leq f(x) - \epsilon$. We have built a sequence $(w_n) = (x_{k_n}, f(x) - \epsilon)$, each term of which belongs to $\operatorname{epi} f$, and which converges to a limit $\bar{w} = (f(x) - \epsilon)$ which is outside the epigraph. Consequently, $\operatorname{epi} f$ is not closed. □

There is a great variety of characterizations of l.s.c. functions, one of them is given in the following exercise.

Exercise 3.2.1. Show that a function f is l.s.c. if and only if its level sets :

$$L_{\leq \alpha} = \{x \in \mathcal{X} : f(x) \leq \alpha\}$$

are closed.
(see, *e.g.*, Rockafellar et al. (1998), theorem 1.6.)

One nice property of the family of l.s.c. functions is its stability with respect to point-wise suprema

Lemma 3.2.1. *Let $(f_i)_{i \in I}$ a family of l.s.c. functions. Then, the upper hull $f = \sup_{i \in I} f_i$ is l.s.c.*

Proof. Let C_i denote the epigraph of f_i and $C = \operatorname{epi} f$. As already shown (proof of proposition 2.1.2), $C = \cap_{i \in I} C_i$. Each C_i is closed, and any intersection of closed sets is closed, so C is closed and f is l.s.c. □

In view of proposition 3.2.1, separations theorem can be applied to the epigraph of a l.s.c. function f. The next result shows that it will also be feasible with the epigraph of f^*.

Proposition 3.2.2 (Properties of f^*)**.**
Let $f : \mathcal{X} \to [-\infty, +\infty]$ be any function.

1. *f^* is always convex, and l.s.c.*
2. *If $\operatorname{dom} f \neq \emptyset$, then $-\infty \notin f^*(\mathcal{X})$*
3. *If f is convex and proper, then f^* is convex, l.s.c., proper.*

Proof.

1. Fix $x \in \mathcal{X}$ and consider the function $h_x : \phi \mapsto \langle \phi, x \rangle - f(x)$. From the definition, $f^* = \sup_{x \in \mathcal{X}} h_x$. Each h_x is affine, whence convex. Using proposition 2.1.2, f^* is also convex. Furthermore, each h_x is continuous, whence l.s.c, so that its epigraph is closed. Lemma 3.2.1 thus shows that f^* is l.s.c.

2. From the hypothesis, there is an x_0 in $\operatorname{dom} f$. Let $\phi \in \mathcal{X}$. The result is immediate:

$$f^*(\phi) \geq h_{x_0}(\phi) = f(x_0) - \langle \phi, x_0 \rangle > -\infty.$$

3. In view of points 1. and 2., it only remains to show that $f^* \not\equiv +\infty$. Let $x_0 \in \operatorname{relint}(\operatorname{dom} f)$. According to proposition 2.2.2, there exists a subgradient ϕ_0 of f at x_0. Moreover, since f is proper, $f(x_0) < \infty$. From the definition of a subgradient,

$$\forall x \in \operatorname{dom} f, \langle \phi_0, x - x_0 \rangle \leq f(x) - f(x_0).$$

Whence, for all $x \in \mathcal{X}$,

$$\langle \phi_0, x \rangle - f(x) \leq \langle \phi_0, x_0 \rangle - f(x_0),$$

thus, $\sup_x \langle \phi_0, x \rangle - f(x) \leq \langle \phi_0, x_0 \rangle - f(x_0) < +\infty$.

Therefore, $f^*(\phi_0) < +\infty$.

□

Proposition 3.2.3 (Fenchel - Young)**.** *Let $f : \mathcal{X} \to [-\infty, \infty]$. For all $(x, \phi) \in \mathcal{X}^2$, the following inequality holds:*

$$f(x) + f^*(\phi) \geq \langle \phi, x \rangle \,,$$

With equality if and only if $\phi \in \partial f(x)$.

Proof. The inequality is an immediate consequence of the definition of f^*. The condition for equality to hold (*i.e.*, for the converse inequality to be valid), is obtained with the equivalence

$$f(x) + f^*(\phi) \leq \langle \phi, x \rangle \;\Leftrightarrow\; \forall y,\; f(x) + \langle \phi, y \rangle - f(y) \leq \langle \phi, x \rangle \;\Leftrightarrow\; \phi \in \partial f(x)\,.$$

□

An **affine minorant** of a function f is any affine function $h : \mathcal{X} \to \mathbb{R}$, such that $h \leq f$ on $\mathcal{X}$. Denote $\mathcal{AM}(f)$ the set of affine minorants of function f. One key result of dual approaches is encapsulated in the next result: under regularity conditions, if the affine minorants of f are given, then f is entirely determined !

Proposition 3.2.4 (duality, episode 0)**.** *Let $f : \mathcal{X} \to (-\infty, \infty]$ a convex, l.s.c., proper function. Then f is the upper hull of its affine minorants.*

Proof. For any function f, denote E_f the upper hull of its affine minorants, $E_f = \sup_{h \in \mathcal{AM}(f)} h$. For $\phi \in \mathcal{X}$ and $b \in \mathbb{R}$, denote $h_{\phi,b}$ the affine function $x \mapsto \langle \phi, x \rangle + b$. With these notations,

$$E_f(x) = \sup\{\langle \phi, x \rangle - b \ : \ h_{\phi,b} \in \mathcal{AM}(f)\}.$$

Clearly, $E_f \leq f$.
To show the converse inequality, we proceed in two steps. First, we assume that f is non negative. The second step consists in finding a 'change of basis' under which f is replaced with non negative function.

1. Case where f is non-negative, i.e. $f(\mathcal{X}) \subset [0, \infty])$:
Assume the existence of some $x_0 \in \mathcal{X}$, such that $t_0 = E_f(x_0) < f(x_0)$ to come up with a contradiction. The point (x_0, t_0) does not belong to the convex closed set $\operatorname{epi} f$. The strong separation theorem 2.2.1 provides a vector $\mathbf{w} = (\phi, b) \in \mathcal{X} \times \mathbb{R}$, and scalars α, b, such that

$$\forall (x, t) \in \operatorname{epi} f, \quad \langle \phi, x \rangle + bt < \alpha < \langle \phi, x_0 \rangle + bt_0. \tag{3.2.1}$$

In particular, the inequality holds for all $x \in \operatorname{dom} f$, and for all $t \geq f(x)$. Consequently, $b \leq 0$ (as in the proof of proposition 2.2.2). Here, we cannot conclude that $b < 0$: if $f(x_0) = +\infty$, the hyperplane may be 'vertical'. However, using the non-negativity of f, if $(x, t) \in \operatorname{epi} f$, then $t \geq 0$, so that, for all $\epsilon > 0$, $(b - \epsilon) t \leq b t$. Thus, (3.2.1) implies

$$\forall (x, t) \in \operatorname{epi} f, \quad \langle \phi, x \rangle + (b - \epsilon)t < \alpha.$$

Now, $b - \epsilon < 0$ and, in particular, for $x \in \operatorname{dom} f$, and $t = f(x)$,

$$f(x) > \frac{1}{b - \epsilon}(\langle -\phi, x \rangle + \alpha) := h^{\epsilon}(x).$$

Thus, the function h^{ϵ} is an affine minorant of f. Since $t_0 \geq h^{\epsilon}(x_0)$ (by definition of t_0),

$$t_0 > \frac{1}{b - \epsilon}(\langle -\phi, x_0 \rangle + \alpha),$$

i.e.

$$(b - \epsilon)t_0 \leq \langle -\phi, x_0 \rangle + \alpha$$

Letting ϵ go to zero yields

$$b\, t_0 \leq -\langle \phi, x_0 \rangle + \alpha$$

which contradicts (3.2.1)

2. General case. Since f is proper, its domain is non empty. Let $x_0 \in \operatorname{relint}(\operatorname{dom} f)$. According to proposition 2.2.2, $\partial f(x_0) \neq \emptyset$. Let $\phi_0 \in \partial f(x_0)$.

Using Fenchel-Young inequality, for all $x \in \mathcal{X}$, $\varphi(x) := f(x) + f^*(\phi_0) - \langle \phi_0, x\rangle \geq 0$. The function φ is non negative, convex, l.s.c., proper (because equality in Fenchel-Young ensures that $f^*(\phi_0) \in \mathbb{R}$). Part *1.* applies :

$$\forall x \in \mathcal{X}, \varphi(x) = \sup_{(\phi,b):h_{\phi,b}\in\mathcal{AM}(\varphi)} \langle \phi, x\rangle + b\,. \tag{3.2.2}$$

Now, for $(\phi, b) \in \mathcal{X} \times \mathbb{R}$,

$$\begin{aligned} h_{\phi,b} \in \mathcal{AM}(\varphi) &\Leftrightarrow \forall x \in \mathcal{X}, \langle \phi, x\rangle + b \leq f(x) + f^*(x_0) - \langle \phi_0, x\rangle \\ &\Leftrightarrow \forall x \in \mathcal{X}, \langle \phi + \phi_0, x\rangle + b - f^*(x_0) \leq f(x) \\ &\Leftrightarrow h_{\phi+\phi_0, b-f^*(x_0)} \in \mathcal{AM}(f). \end{aligned}$$

Thus, (3.2.2) writes as

$$\forall x \in \mathcal{X}, f(x) + f^*(\phi_0) - \langle \phi_0, x\rangle = \sup_{(\phi,b)\in\Theta(f)} \langle \phi - \phi_0, x\rangle + b + f^*(x_0)\,.$$

In other words, $x \in \mathcal{X}, f(x) = E_f(x)$. □

The announced result comes next:

Definition 3.2.3 (Fenchel Legendre biconjugate). *Let $f : \mathcal{X} \to [-\infty, \infty]$, any function. The biconjugate of f (under Fenchel-Legendre conjugation), is*

$$\begin{aligned} f^{**}: \ & \mathcal{X} \to [-\infty, \infty] \\ & x \mapsto f^*(f^*(x)) = \sup_{\phi\in\mathcal{X}} \langle \phi, x\rangle - f^*(\phi). \end{aligned}$$

Proposition 3.2.5 (Involution property, Fenchel-Moreau). *If f is convex, l.s.c., proper, then $f = f^{**}$.*

Proof. Using proposition 3.2.4, it is enough to show that $f^{**}(x) = E_f(x)$

1. From Fenchel-Young, inequality, for all $\phi \in \mathcal{X}$, the function $x \mapsto h_\phi(x) = \langle \phi, x\rangle - f^*(\phi)$ belongs to $\mathcal{AM}(f)$. Thus,

   $$\mathcal{AM}^* = \{h_\phi, \phi \in \mathcal{X}\} \subset \mathcal{AM}(f),$$

   so that

   $$f^{**}(x) = \sup_{h\in\mathcal{AM}^*} h(x) \leq \sup_{h\in\mathcal{AM}(f)} h(x) = E_f(x).$$

2. Conversely, let $h_{\phi,b} \in \mathcal{AM}(f)$. Then, $\forall x, \langle \phi, x\rangle - f(x) \leq -b$, so

   $$f^*(\phi) = \sup_x \langle \phi, x\rangle - f(x) \leq -b.$$

   Thus,

   $$\forall x, \quad \langle \phi, x\rangle - f^*(\phi) \geq \langle \phi, x\rangle + b = h(x).$$

   In particular, $f^{**}(x) \geq h(x)$. Since this holds for all $h \in \mathcal{AM}(f)$, we obtain

   $$f^{**}(x) \geq \sup_{h\in\mathcal{AM}(f)} h(x) = E_f(x).$$

$\square$

One local condition to have $f(x) = f^{**}(x)$ at some point x is the following.

Proposition 3.2.6. *Let $f : \mathcal{X} \to [-\infty, \infty]$ a convex function, and let $x \in \operatorname{dom} f$.*

$$\textit{If} \;\; \partial f(x) \neq \emptyset, \;\; \textit{then} \;\; f(x) = f^{**}(x).$$

Proof. Let $\lambda \in \partial f(x)$. This is the condition for equality in Fenchel-Young inequality (proposition 3.2.3), *i.e.*

$$f(x) + f^*(\lambda) - \langle \lambda, x \rangle = 0 \tag{3.2.3}$$

Consider the function $h_x(\phi) = f^*(\phi) - \langle \phi, x \rangle$. Equation (3.2.3) writes as

$$h_x(\lambda) = -f(x).$$

The general case in Fenchel Young writes

$$\forall \phi \in \mathcal{X}, \;\; h_x(\phi) \geq -f(x) = h_x(\lambda).$$

Thus, λ is a minimizer of h_x,

$$\lambda \in \operatorname*{arg\,min}_{\phi \in \mathcal{X}} h_x(\phi) = \operatorname*{arg\,max}_{\phi \in \mathcal{X}} (-h_x(\phi))$$

In other words,

$$f(x) = -h_x(\lambda) = \sup_{\phi} -h_x(\phi) = \sup_{\phi} \langle \phi, x \rangle - f^*(\phi) = f^{**}(x).$$

$\square$

Exercise 3.2.2. Let $f : \mathcal{X} \to (-\infty, +\infty]$ a proper, convex, l.s.c. function. Show that

$$\partial(f^*) = (\partial f)^{-1}$$

where, for $\phi \in \mathcal{X}$, $(\partial f)^{-1}(\phi) = \{x \in \mathcal{X} : \;\; \phi \in \partial f(x)\}$.
Hint : Use Fenchel-Young inequality to show one inclusion, and the property $f = f^{**}$ for the other one.

3.3 Fenchel duality**

This section may be skipped at first reading.
Dual approaches use the fact that, under 'qualification assumptions', the optimal value of a *primal* problem is also that of a *dual problem*. In the following definition, think of f as the objective function, whereas g summarizes the constraints, *e.g.* $g = \mathbb{I}_{\tilde{g}(x) \leq 0}$. For applications in optimization, it is convenient to consider a linear transformation $M : \mathcal{X} \to \mathcal{Y}$ and let g be defined on $\mathcal{Y}$.

Definition 3.3.1 (Fenchel duality : primal and dual problems). *Let* $f : \mathbf{E} \to [-\infty, \infty]$, $g : \mathcal{Y} :\to [-\infty, \infty]$ *two convex functions.*
Let $M : \mathcal{X} \to \mathcal{Y}$ *a linear operator and denote* M^* *its adjoint,* i.e. $\langle y, M\,x \rangle = \langle M^* y, x \rangle, \forall (x, y) \in \mathcal{X} \times \mathcal{Y}$.
The primal value *associated to* f *and* g *is*

$$p = \inf_{x \in \mathcal{X}} f(x) + g(M\,x).$$

A point x *is called* primal optimal *if* $x \in \arg\min_{\mathcal{X}}(f + g)$.
The dual value *of the problem is*

$$\begin{aligned} d &= \sup_{\phi \in \mathcal{X}} \left(-f^*(M^*\phi) - g^*(-\phi) \right). \\ &= -\inf_{\phi \in \mathcal{X}} \left(f^*(M^*\phi) + g^*(-\phi) \right). \end{aligned}$$

The dual gap *is the difference*

$$\Delta = p - d$$

Proposition 3.3.1 (Dual gap). *In the setting of definition 3.3.1, the dual gap is always non negative,*

$$p \geq d$$

Proof. From Fenchel-Young inequality, for all $x \in \mathcal{X}$ and $\phi \in \mathcal{Y}$,

$$\begin{aligned} \forall (x, \phi) \in \mathcal{X} \times \mathcal{Y}, \quad & f(x) + f^*(M^*\phi) \geq \langle x, M^*\phi \rangle \ ; \\ & g(Mx) + g^*(-\phi) \geq -\langle Mx, \phi \rangle \ . \end{aligned}$$

Adding the two yields $f(x) + g(Mx) \geq -f^*(M^*\phi) - g^*(-\phi)$; taking the infimum in the left-hand side and the supremum in the right-hand side gives the result. □

The interesting case is the *zero-duality gap* situation, when $p = d$, allowing two solve the (hopefully easier) dual problem as an intermediate step to find a primal solution.
Before proceeding, we need to define operations on ensembles

Definition 3.3.2 (addition and transformations of ensembles). *Let* $A, B \subset \mathcal{X}$. *The Minkowski sum and difference of* A *and* B *are the sets*

$$\begin{aligned} A + B &= \{x \in \mathcal{X} : \ \exists a \in A, \exists b \in B, \ x = a + b\} \\ A - B &= \{x \in \mathcal{X} : \ \exists a \in A, \exists b \in B, \ x = a + b\} \end{aligned}$$

Let $\mathcal{Y}$ *another space and* M *any mapping from* $\mathcal{X}$ *to* $\mathcal{Y}$. *Then* MA *is the image of* A *by* M,

$$MA = \{y \in \mathcal{Y} : \ \exists a \in A, \ y = Ma\}.$$

Now, we can give a condition ensuring a zero-duality gap:

Theorem 3.3.1 (Fenchel-Rockafellar). *In the setting of definition 3.3.1, if*

$$0 \in \operatorname{relint}(\operatorname{dom} g - M \operatorname{dom} f)\,, \tag{3.3.1}$$

then $p = d$*,* i.e.

$$\inf_{x \in \mathbf{E}} \Big(f(x) + g(M\,x) \Big) = -\inf_{\phi \in \mathcal{Y}} \Big(f^*(M^*\phi) + g^*(-\phi) \Big)\,. \tag{3.3.2}$$

Besides, the dual value is attained as soon as it is finite.

Proof. Let p and d the primal and dual values. In view of proposition 3.3.1, we only need to prove that $p \leq d$.
Introduce the **value function**

$$\vartheta(y) = \inf_{x \in \mathcal{X}} \left(f(x) + g(Mx + y) \right)\,. \tag{3.3.3}$$

Notice that $p = \vartheta(0)$. Furthermore, for $\phi \in \mathcal{X}$,

$$\begin{aligned}
\vartheta^*(-\phi) &= \sup_{u \in \mathcal{X}} \langle -\phi, u \rangle - \vartheta(u) \\
&= \sup_{u \in \mathcal{X}} \langle -\phi, u \rangle - \inf_{x \in \mathcal{X}} f(x) + g(Mx + u) \\
&= \sup_{u \in \mathcal{X}} \sup_{x \in \mathcal{X}} \langle -\phi, u \rangle - f(x) - g(Mx + u) \\
&= \sup_{x \in \mathcal{X}} \big(\sup_{u \in \mathcal{X}} \langle -\phi, Mx + u \rangle - g(Mx + u) \big) + \langle \phi, Mx \rangle - f(x) \\
&= \sup_{x \in \mathcal{X}} \big(\sup_{\tilde{u} \in \mathcal{X}} \langle -\phi, \tilde{u} \rangle - g(\tilde{u}) \big) + \langle \phi, Mx \rangle - f(x) \\
&= g^*(-\phi) + f^*(M^*\phi)
\end{aligned} \tag{3.3.4}$$

We shall show later on, that ϑ is convex and that its domain is $\operatorname{dom}(g) - M \operatorname{dom}(f)$. Admit it temporarily. The qualification hypothesis (3.3.1), together with proposition 2.2.2, thus imply that $\partial\vartheta(0)$ is non empty. Let $\lambda \in \partial\vartheta(0)$. Equality in Fenchel-Young writes : $\vartheta(0) + \vartheta(\lambda) = \langle \lambda, 0 \rangle = 0$. Thus, we have

$$\begin{aligned}
p = \vartheta(0) &= -\vartheta^*(\lambda) \\
&= -g^*(\lambda) - f^*(-M^*\lambda) \qquad \text{from}(3.3.4) \\
&\leq \sup_{\phi} -g^*(-\phi) - f^*(M^*\phi) = d
\end{aligned}$$

whence, $p \leq d$ and the proof is complete.

convexity of ϑ: Let $u, v \in \mathrm{dom}(\vartheta)$ and $t \in (0,1)$. We need to check that $\vartheta(tu + (1-t)v) \leq t\vartheta(u) + (1-t)\vartheta(v)$. For any $\bar{x} \in \mathcal{X}$, we have $\vartheta(tu+(1-t)v) \leq f(\bar{x}) + g(M\bar{x} + tu + (1-t)v)$. Pick $(x, y) \in \mathcal{X}^2$ and fix $\bar{x} = tx + (1-t)y$. The latter inequality becomes

$$\begin{aligned}\vartheta(tu+(1-t)v) &\leq f(tx+(1-t)y) + g\big(t(Mx+u) + (1-t)(My+v)\big)\\ &\leq t\big(f(x) + g(Mx+u)\big) + (1-t)\big(f(y) + g(My+v)\big)\,.\end{aligned}$$

Taking the infimum of the right hand side with respect to x and y concludes the proof.

domain of ϑ There remains to check that $\mathrm{dom}(\vartheta) = \mathrm{dom}(g) - M\,\mathrm{dom}(f)$. It is enough to notice that

$$y \in \mathrm{dom}(\vartheta) \Leftrightarrow \exists x \in \mathrm{dom}\, f \ : \ g(Mx+y) < +\infty,$$

so that

$$\begin{aligned}y \in \mathrm{dom}(\vartheta) &\Leftrightarrow \exists x \in \mathrm{dom}\, f \ : \ Mx + y \in \mathrm{dom}\, g\\ &\Leftrightarrow \exists x \in \mathrm{dom}\, f, \exists u \in \mathrm{dom}\, g \ : \ u = Mx + y\\ &\Leftrightarrow \exists x \in \mathrm{dom}\, f, \exists u \in \mathrm{dom}\, g \ : \ u - Mx = y\\ &\Leftrightarrow y \in \mathrm{dom}\, g - M\,\mathrm{dom}\, f\end{aligned}$$

□

3.4 Operations on subdifferentials

Until now, we have seen example of subdifferential computations on basic functions, but we haven't mentioned how to derive the subdifferentials of more complex functions, such as sums or linear transforms of basic ones. A basic fact from differential calculus is that, when all the terms are differentiable, $\nabla(f+g) = \nabla f + \nabla g$. Also, if M is a linear operator, $\nabla(g \circ M)(x) = M^*\nabla g(Mx)$. Under qualification assumptions, these properties are still valid in the convex case, up to replacing the gradient by the subdifferential and point-wise operations by set operations.

Proposition 3.4.1. *Let $f : \mathcal{X} \to (-\infty, +\infty]$, $g : \mathcal{Y} \to (-\infty, \infty]$ two convex functions and let $M : \mathcal{X} \to \mathcal{Y}$ a linear operator. If the qualification condition* (3.3.1) *holds, then*

$$\forall x \in \mathcal{X}, \partial(f + g \circ M)(x) = \partial f(x) + M^* \partial g(Mx)$$

Proof. Let us show first that $\partial f(\,\cdot\,) + M^*\partial g(M\,\cdot\,) \subset \partial(f + g \circ M)(\,\cdot\,)$. Let $x \in \mathcal{X}$ and $\phi \in \partial f(x) + M^*\partial g \circ M(x)$, which means that $\phi = u + M^*v$ where $u \in \partial f(x)$ and $v \in \partial g \circ M(x)$. In particular, none of the latter subdifferentials is empty, which implies that $x \in \operatorname{dom} f$ and $x \in \operatorname{dom}(g \circ M)$. By definition of u and v, for $y \in \mathcal{X}$,

$$\begin{cases} f(y) - f(x) \geq \langle u, y - x \rangle \\ g(My) - g(Mx) \geq \langle v, M(y-x) \rangle = \langle M^* v, y - x \rangle . \end{cases}$$

Adding the two inequalities,

$$(f + g \circ M)(y) - (f + g \circ M)(x) \geq \langle \phi, y - x \rangle .$$

Thus, $\phi \in \partial(f + g \circ M)(x)$ and $\partial f(x) + M^* \partial g(Mx) \subset \partial(f + g \circ M)(x)$.

The proof of the converse inclusion requires to use Fenchel-Rockafellar theorem 3.3.1, and **may be skipped at first reading.**
Notice first that $\operatorname{dom}(f + g \circ M) = \{x \in \operatorname{dom} f \ : \ Mx \in \operatorname{dom} g\}$. The latter set is non empty: to see this, use assumption (3.3.1) : $0 \in \operatorname{dom} g - M \operatorname{dom} f$, so that $\exists\ (y, x) \in \operatorname{dom} g \times \operatorname{dom} f \ : \ 0 = y - Mx$.
Thus, let $x \in \operatorname{dom}(f + g \circ M)$. Then $x \in \operatorname{dom} f$ and $Mx \in \operatorname{dom} g$.
Assume $\phi \in \partial(f + g \circ M)(x)$. For $y \in \mathcal{X}$,

$$f(y) + g(My) - (f(x) + g(Mx)) \geq \langle \phi, y - x \rangle ,$$

thus, x is a minimizer of the function $\varphi : y \mapsto f(y) - \langle \phi, y \rangle + g(My)$, which is convex. Using Fenchel-Rockafellar theorem 3.3.1, where $f - \langle \phi, \cdot \rangle$ replaces f, the dual value is attained : there exists $\psi \in \mathcal{Y}$, such that

$$f(x) - \langle \phi, x \rangle + g(Mx) = -(f - \langle \phi, . \rangle)^*(-M^* \psi) - g^*(\psi) .$$

It is easily verified that $(f - \langle \phi, \cdot \rangle)^* = f^*(\cdot + \phi)$. Thus,

$$f(x) - \langle \phi, x \rangle + g(Mx) = -f^*(-M^* \psi + \phi) - g^*(\psi) .$$

In other words,

$$f(x) + f(-M^* \psi + \phi) - \langle \phi, x \rangle + g(Mx) + g^*(\psi) = 0 ,$$

so that

$$[f(x) + f(-M^* \psi + \phi) - \langle -M^* \psi + \phi, x \rangle] + [g(Mx) + g^*(\psi) - \langle \psi, Mx \rangle] = 0 .$$

Each of the terms within brackets is non negative (from Fenchel-Young inequality). Thus, both are null. Equality in Fenchel-Young implies that $\psi \in \partial g(Mx)$ and $-M^* \psi + \phi \in \partial f(x)$. This means that $\phi \in \partial f(x) + M^* \partial g(Mx)$, which concludes the proof. □

Chapter 4

Lagrangian duality

4.1 Lagrangian function, Lagrangian duality

In this chapter, we consider the convex optimization problem

$$\text{minimize over } \mathbb{R}^n : \quad f(x) + \mathbb{I}_{g(x) \preceq 0} \ . \tag{4.1.1}$$

(*i.e.* minimize $f(x)$ over $\mathbb{R}^n$, under the constraint $g(x) \preceq 0$),
where $f : \mathbb{R}^n \to (-\infty, \infty]$ is a convex, proper function; $g(x) = (g_1(x), \dots, g_p(x))$, and each $g_i : \mathbb{R}^n \to (-\infty, +\infty)$ is a convex function ($1 \leq i \leq p$). Here, we do not allow $g(x) \in \{+\infty, -\infty\}$, for the sake of simplicity. This condition may be replaced by a weaker one :

$$0 \in \operatorname{relint}(\operatorname{dom} f - \cap_{i=1}^{p} \operatorname{dom} g_i,)$$

without altering the argument.

It is easily verified that under these conditions, the function $x \mapsto f(x) + \mathbb{I}_{g(x) \preceq 0}$ is convex.

Definition 4.1.1 (primal value, primal optimal point)**.** *The* ***primal value*** *associated to (4.1.1) is the infimum*

$$p = \inf_{x \in \mathbb{R}^n} f(x) + \mathbb{I}_{g(x) \preceq 0}.$$

A point $x^* \in \mathbb{R}^n$ is called **primal optimal** *if*

$$p = f(x^*) + \mathbb{I}_{g(x^*) \preceq 0}.$$

Notice that, under our assumption, $p \in [-\infty, \infty]$. Also, there is no guarantee about the existence of a primal optimal point, *i.e.* that the primal value be attained.

Since (4.1.2) may be difficult to solve, it is useful to see this as an 'inf sup' problem, and solve a 'sup inf' problem instead (see definition 4.1.3 below). To make this precise, we introduce the Lagrangian function.

Definition 4.1.2. *The* **Lagrangian function** *associated to problem (4.1.1) is the function*

$$L : \mathbb{R}^n \times \mathbb{R}^{+p} \longrightarrow [-\infty, +\infty]$$
$$(x, \phi) \mapsto f(x) + \langle \phi, g(x) \rangle$$

(where $\mathbb{R}^{+p} = \{\phi \in \mathbb{R}^p, \phi \succeq 0\}$).

The link with the initial problem comes next:

Lemma 4.1.1 (constrained objective as a supremum)**.** *The constrained objective is the supremum (over ϕ) of the Lagrangian function,*

$$\forall x \in \mathbb{R}^n,\ \ f(x) + \mathbb{I}_{g(x) \preceq 0} = \sup_{\phi \in \mathbb{R}^{+p}} L(x, \phi)$$

Proof. Distinguish the case $g(x) \preceq 0$ and $g(x) \npreceq 0$.
(a) If $g(x) \npreceq 0, \exists i \in \{1, \ldots, p\} : g_i(x) > 0$. Choosing $\phi_t = te_i$ (where $\mathbf{e} = (e_1, \ldots, e_p)$ is the canonical basis of $\mathbb{R}^p$)), $t \geq 0$, then $\lim_{t \to \infty} L(x, \phi_t) = +\infty$, whence $\sup_{\phi \in \mathbb{R}^{+p}} L(x, \phi) = +\infty$. On the other hand, in such a case, $\mathbb{I}_{g(x) \preceq 0} = +\infty$, whence the result.

(b) If $g(x) \preceq 0$, then $\forall \phi \in \mathbb{R}^{+p}, \langle \phi, g(x) \rangle \leq 0$, and the supremum is attained at $\phi = 0$. Whence, $\sup_{\phi \succeq 0} L(x, \phi) = f(x)$.
On the other hand, $\mathbb{I}_{g(x) \leq 0} = 0$, so $f(x) + \mathbb{I}_{g(x) \preceq 0} = f(x)$. The result follows. $\square$

Equipped with lemma 4.1.1, the primal value associated to problem (4.1.1) writes

$$p = \inf_{x \in \mathbb{R}^n} \sup_{\phi \in \mathbb{R}^{+p}} L(x, \phi). \tag{4.1.2}$$

One natural idea is to exchange the order of inf and sup in the above problem. Before proceeding, the following simple lemma allows to understand the consequence of such an exchange.

Lemma 4.1.2. *Let $F : A \times B \to [-\infty, \infty]$ any function. Then,*

$$\sup_{y \in B} \inf_{x \in A} F(x, y) \leq \inf_{x \in A} \sup_{y \in B} F(x, y).$$

Proof. $\forall (\bar{x}, \bar{y}) \in A \times B$,

$$\inf_{x \in A} F(x, \bar{y}) \leq F(\bar{x}, \bar{y}) \leq \sup_{y \in B} F(\bar{x}, y).$$

Taking the supremum over $\bar{y}$ in the left-hand side we still have

$$\sup_{\bar{y} \in B} \inf_{x \in A} F(x, \bar{y}) \leq \sup_{y \in B} F(\bar{x}, y).$$

Now, taking the infimum over $\bar{x}$ in the right-hand side yields

$$\sup_{\bar{y}\in B}\inf_{x\in A} F(x,\bar{y}) \le \inf_{\bar{x}\in A}\sup_{y\in B} F(\bar{x},y).$$

up to to a simple change of notation, this is

$$\sup_{y\in B}\inf_{x\in A} F(x,y) \le \inf_{x\in A}\sup_{y\in B} F(x,y).$$

□

Definition 4.1.3 (Dual problem, dual function, dual value)**.**
The **dual value** *associated to (4.1.2) is*

$$d = \sup_{\phi\in\mathbb{R}^{+p}}\inf_{x\in\mathbb{R}^n} L(x,\phi).$$

The function

$$\mathcal{D}(\phi) = \inf_{x\in\mathbb{R}^n} L(x,\phi)$$

is called the **Lagrangian dual function***. Thus, the* **dual problem** *associated to the primal problem (4.1.1) is*

$$\text{maximize over } \mathbb{R}^{+p}: \quad \mathcal{D}(\phi).$$

A vector $\lambda \in \mathbb{R}^{+p}$ is called **dual optimal** *if*

$$d = \mathcal{D}(\lambda).$$

Without any further assumption, there is no reason for the two values (primal and dual) to coincide. However, as a direct consequence of lemma 4.1.2, we have :

Lemma 4.1.3. *Let p and d denote respectively the primal and dual value for problem (4.1.1). Then,*

$$d \le p\ .$$

Proof. Apply lemma 4.1.2. □

One interesting property of the dual function, for optimization purposes, is :

Lemma 4.1.4. *The dual function $\mathcal{D}$ is concave.*

Proof. For each fixed $x \in \mathbb{R}^n$, the function

$$h_x : \phi \mapsto L(x,\phi) = f(x) + \langle \phi, g(x)\rangle$$

is affine, whence concave on $\mathbb{R}^{+p}$. In other words, the negated function $-h_x$ is convex. Thus, its upper hull $h = \sup_x(-h_x)$ is convex. There remains to notice that

$$\mathcal{D} = \inf_x h_x = -\sup_x(-h_x) = -h,$$

so that $\mathcal{D}$ is concave, as required. □

4.2 Zero duality gap

The inequality $d \leq p$ (lemma 4.1.3) leads us to the last definition

Definition 4.2.1. *The* **dual gap** *associated to problem (4.1.1) is the non-negative difference*

$$\Delta = p - d \ .$$

The remaining of this section is devoted to finding conditions under which the primal and dual values do coincide, also called **zero duality gap** conditions. Notice, that, under such conditions, it is legitimate to solve the dual problem instead of the primal one. The course of ideas is very similar to the proof of Fenchel-Rockafellar theorem 3.3.1.

Introduce the **Lagrangian value function**

$$\mathcal{V}(b) = \inf_{x \in \mathbb{R}^n} f(x) + \mathbb{I}_{g(x) \preceq b} \ , \quad b \in \mathbb{R}^p. \tag{4.2.1}$$

Thus, $\mathcal{V}(b)$ is the infimum of a perturbated version of problem (4.1.1), where the constraints have been shifted by a constant b. Notice that

$$p = \mathcal{V}(0).$$

The remaining of the argument relies on manipulating the Fenchel conjugate and biconjugate of $\mathcal{V}$. The following result is key to provide zero duality gap conditions and allows to understand why we have introduced $\mathcal{V}$.

Proposition 4.2.1 (conjugate and biconjugate of the value function)**.**
The Fenchel conjugate of the Lagrangian value function satisfies, for $\phi \in \mathbb{R}^p$,

$$\mathcal{V}^*(-\phi) = \begin{cases} -\mathcal{D}(\phi) & \text{if } \phi \succeq 0 \\ +\infty & \text{otherwise,} \end{cases} \tag{4.2.2}$$

and the dual value d is related to $\mathcal{V}$ via

$$\mathcal{V}^{**}(0) = d \tag{4.2.3}$$

To prove proposition 4.2.1, the following technical lemma is needed (the proof of which may be skipped at first reading).

Lemma 4.2.1. *The Lagrangian value function $\mathcal{V}$ is convex.*

Proof. We need to show that, for $a, b \in \text{dom}(\mathcal{V})$, and $\alpha \in (0,1)$,

$$\mathcal{V}(\alpha a + (1-\alpha)b) \leq \alpha \mathcal{V}(a) + (1-\alpha)\mathcal{V}(b).$$

Let a, b and α as above. For $x, y \in \text{dom}(f)$, let $u_{x,y} = \alpha x + (1-\alpha)y$ and $\gamma = \alpha a + (1-\alpha)b$. Since g is component-wise convex, we have $g(u_{xy}) \preceq \alpha g(x) + (1-\alpha)g(y)$; whence

$$
\begin{aligned}
\mathbb{I}_{g(u_{xy})\preceq\gamma} &\leq \mathbb{I}_{\alpha g(x)+(1-\alpha)g(y)\preceq\gamma} \\
&= \mathbb{I}_{\alpha g(x)+(1-\alpha)g(y)\preceq\alpha a+(1-\alpha)b} \\
&\leq \alpha\mathbb{I}_{g(x)\preceq a} + (1-\alpha)\mathbb{I}_{g(y)\preceq b},
\end{aligned}
\tag{4.2.4}
$$

where the last inequality follows from

$$
\{g(x) \preceq a\ , g(y) \preceq b\} \Rightarrow \alpha g(x) + (1-\alpha g(y) \preceq \alpha a + (1-\alpha)b,
$$

and the fact that, for any $t > 0$, $t\,\mathbb{I} = \mathbb{I}$.
Using (4.2.4) and the convexity of f, we get

$$
f(u_{xy}) + \mathbb{I}_{g(u_{xy})} \leq \alpha\left(f(x) + \mathbb{I}_{g(x)\preceq a}\right) + (1-\alpha)\left(f(y) + \mathbb{I}_{g(y)\preceq b}\right). \tag{4.2.5}
$$

Taking the infimum in (4.2.5) with respect to x and y yields

$$
\begin{aligned}
\inf_{(x,y)\in\operatorname{dom} f} f(u_{xy}) + \mathbb{I}_{g(u_{xy})\leq\gamma} \leq & \inf_{(x,y)\in\operatorname{dom} f}\Big[\alpha\left(f(x) + \mathbb{I}_{g(x)\preceq a}\right) + \dots \\
& \dots(1-\alpha)\left(f(y) + \mathbb{I}_{g(y)\preceq b}\right)\Big] \\
&= \inf_{x\in\mathbb{R}^n}\left[\alpha\left(f(x) + \mathbb{I}_{g(x)\preceq a}\right)\right] + \dots \\
&\dots \inf_{y\in\mathbb{R}^n}\left[(1-\alpha)\left(f(y) + \mathbb{I}_{g(y)\leq b}\right)\right] \\
&= \alpha\ \mathcal{V}(a) + (1-\alpha)\ \mathcal{V}(b).
\end{aligned}
$$

For the second line, we used the fact that the infimum in the definition of $\mathcal{V}$ may be taken over $\operatorname{dom} f$, since on the complementary set of the latter, $f(x) + \mathbb{I}_{g(x)\preceq c} = +\infty$, for all $c \in \mathbb{R}^p$.
Finally, notice that if $A \subset B$, $\inf_A(\dots) \geq \inf_B(\dots)$, thus the left-hand side in the above inequalities is greater than, or equal to $\mathcal{V}(\gamma)$. The result follows. □

proof of proposition 4.2.1.
We first prove (4.2.2). For $\phi \in \mathbb{R}^p$, by definition of the Fenchel conjugate,

$$
\begin{aligned}
\mathcal{V}^*(-\phi) &= \sup_{y\in\mathbb{R}^p} \langle -\phi, y\rangle - \mathcal{V}(y) \\
&= \sup_{y\in\mathbb{R}^p} \langle -\phi, y\rangle - \inf_{x\in\mathbb{R}^n}\left[f(x) + \mathbb{I}_{g(x)\preceq y}\right] \\
&= \sup_{y\in\mathbb{R}^p} \langle -\phi, y\rangle + \sup_{x\in\mathbb{R}^n}\left[-f(x) - \mathbb{I}_{g(x)\preceq y}\right] \\
&= \sup_{y\in\mathbb{R}^p}\sup_{x\in\mathbb{R}^n} \langle -\phi, y\rangle - f(x) - \mathbb{I}_{g(x)\preceq y} \\
&= \sup_{x\in\mathbb{R}^n}\Big[\sup_{y\in\mathbb{R}^p} \underbrace{\langle -\phi, y\rangle - \mathbb{I}_{g(x)\preceq y}}_{\varphi_x(y)}\Big] - f(x).
\end{aligned}
\tag{4.2.6}
$$

For fixed $x \in \operatorname{dom} f$, consider the function $\varphi_x : y \mapsto \langle -\phi, y\rangle + \mathbb{I}_{g(x)\preceq y}$. Distinguish the cases $\phi \succeq 0$ and $\phi \not\succeq 0$.
a) If $\phi \not\succeq 0$: Let $i \in \{1, \dots, p\}$ such that $\phi_i < 0$, and let $x \in \operatorname{dom} f$. Choose $y = g(x)$ and $\tilde{y}_t = y + t\ e_i$, so that $g(x) \preceq y_t, \forall t \geq 0$. Equation (4.2.6) implies:

$$\forall x, \forall y, \quad \mathcal{V}^*(-\phi) \geq \varphi_x(y) - f(x),$$

in particular

$$\begin{aligned}
\forall t \geq 0, \quad \mathcal{V}^*(-\phi) &\geq \langle -\phi, y_t\rangle - \mathbb{I}_{g(x)\preceq y_t} - f(x) \\
&= \langle -\phi, y_t\rangle - f(x) \\
&= \langle -\phi, y\rangle + t\phi_i - f(x) \\
&\xrightarrow[t\to+\infty]{} +\infty
\end{aligned}$$

Thus, $\mathcal{V}^*(-\phi) = +\infty$.
b) If $\phi \succeq 0$: Fix $x \in \operatorname{dom} f$. The function φ_x is componentwise non-increasing in y on the feasible set $\{y : y \succeq g(x)\}$, and $\varphi_x(y) = -\infty$ if $y \not\succeq g(x)$, so that

$$\sup_{y\in\mathbb{R}^p} \varphi_x(y) = \varphi(g(x)) = \langle -\phi, g(x)\rangle .$$

Thus, (4.2.6) becomes

$$\begin{aligned}
\mathcal{V}^*(-\phi) &= \sup_{x\in\mathbb{R}^n} \langle -\phi, g(x)\rangle - f(x), \\
&= -\inf_{x\in\mathbb{R}^n} \underbrace{f(x) + \langle \phi, g(x)\rangle}_{L(x,\phi)} \\
&= -\mathcal{D}(\phi)
\end{aligned}$$

This is exactly (4.2.2).

The identity $d = \mathcal{V}^{**}(0)$ is now easily obtained : by definition of the biconjugate,

$$\begin{aligned}
\mathcal{V}^{**}(0) &= \sup_{\phi\in\mathbb{R}^p} -\mathcal{V}^*(\phi) = \sup_{\phi\in\mathbb{R}^p} -\mathcal{V}^*(-\phi) \quad \text{(by symmetry of } \mathbb{R}^p) \\
&= \sup_{\phi\in\mathbb{R}^p, \phi\succeq 0} \mathcal{D}(\phi) \qquad \text{(using (4.2.2))} \\
&= d \qquad \text{(by definition of } d),
\end{aligned}$$

and the proof is complete. □

We shall see next that under a condition of **constraint qualification**, the primal and dual values coincide and that the dual optimal λ brings some knowledge about the primal optimal x^*. Roughly speaking, we say that the constraints are qualified if the problem is satisfiable (there exists some point in $\operatorname{dom} f$, such that $g(x) \preceq 0$, *i.e.*, $\mathcal{V}(0) < \infty$), and if, moreover, the constraints can even be strengthened without altering the satisfiability of the problem: we ask (again, roughly speaking), that, for $b \preceq 0$, close to 0, $\mathcal{V}(b) < +\infty$.

Exercise 4.2.1. Show that, without any further assumption, if $\mathcal{V}(0) < +\infty$, then, $\forall b \succeq 0$, $\mathcal{V}(b) < +\infty$.

Definition 4.2.2 (constraint qualification)**.** *The constraints $g(x) \preceq 0$ in the convex problem (4.1.1) are called* **qualified** *if*

$$0 \in \operatorname{relint} \operatorname{dom} \mathcal{V}. \tag{4.2.7}$$

Now comes the main result of this section

Proposition 4.2.2 (Zero duality gap condition)**.**
If the constraints are qualified for the convex problem (4.1.1), then

1. $p < +\infty$
2. $p = d$ *(*i.e.*, the duality gap is zero).*
3. *(Dual attainment at some $\lambda \succeq 0$):*

$$\exists \lambda \in \mathbb{R}^p, \lambda \succeq 0, \textit{such that} \;\; d = \mathcal{D}(\lambda).$$

Proof.

1. The condition of the statement implies that $0 \in \operatorname{dom} \mathcal{V}$. Thus, $p = \mathcal{V}(0) < +\infty$.
2. Proposition 2.2.2 implies $\partial\mathcal{V}(0) \neq \emptyset$. Thus, proposition 3.2.6 shows that $\mathcal{V}(0) = \mathcal{V}^{**}(0)$. Using proposition 4.2.1 ($d = \mathcal{V}^{**}(0)$), we obtain

$$d = \mathcal{V}(0) = p.$$

3. Using proposition 2.2.2, pick $\phi_0 \in \partial\mathcal{V}(0)$. Notice that the value function is (componentwise) non increasing (the effect of a non negative b in $\mathcal{V}(b)$ is to relax the constraint), so that

$$\begin{aligned} \forall b \succeq 0, \; \langle b, \phi_0 \rangle &\leq \mathcal{V}(b) - \mathcal{V}(0) \qquad \text{(subgradient)} \\ &\leq 0. \end{aligned}$$

This implies $\phi_0 \preceq 0$. Fenchel-Young equality yields $\mathcal{V}(0) + \mathcal{V}^*(\phi_0) = 0$. Thus, proposition 4.2.1 shows that

$$\begin{aligned} \mathcal{D}(-\phi_0) &= -\mathcal{V}^*(\phi_0) = \mathcal{V}(0) \\ &= p \\ &= d \\ &= \sup_{\phi \in \mathbb{R}^{+p}} \mathcal{D}(\phi), \end{aligned}$$

whence, $\lambda = -\phi_0$ is a maximizer of $\mathcal{D}$ and satisfies $\lambda \preceq 0$, as required.

□

Before proceeding to the celebrated KKT (Karush,Kuhn,Tucker) theorem, let us mention one classical condition under which the constraint qualification condition (4.2.7) holds

Proposition 4.2.3 (Slater conditions)**.** *Consider the convex optimization problem (4.1.1). Assume that*

$$\exists \bar{x} \in \operatorname{dom} f : \ \forall i \in \{1, \ldots, p\}, g_i(\bar{x}) < 0.$$

Then, the constraints are qualified, in the sense of (4.2.7) ($0 \in \operatorname{relint} \operatorname{dom} \mathcal{V}$*).*

Exercise 4.2.2. Prove proposition 4.2.3.

4.3 Saddle points and KKT theorem

Introductory remark (Reminder: KKT theorem in smooth convex optimization)**.** *You may have already encountered the KKT theorem, in the smooth convex case: If f and the g_i's ($1 \le i \le p$) are convex, differentiable, and if the constraints are qualified in some sense (*e.g.*, Slater) it is a well known fact that, x^* is primal optimal if and only if, there exists a Lagrange multiplier vector $\lambda \in \mathbb{R}^p$, such that*

$$\lambda \succeq 0, \quad \langle \lambda, g(x^*) \rangle = 0, \quad \nabla f(x^*) = -\sum_{i \in I} \lambda_i \nabla g_i(x^*).$$

(where I is the set of active constraints, i.e. *the i's such that $g_i(x) = 0$.)*
The last condition of the statement means that, if only one g_i is involved, and if there is no minimizer of f within the region $g_i < 0$, the gradient of the objective and that of the constraint are colinear, in opposite directions.
The objective of this section is to obtain a parallel statement in the convex, non-smooth case, with subdifferentials instead of gradients.

First, we shall prove that, under the constraint qualification condition (4.2.7), the solutions for problem (4.1.1) correspond to saddle points of the Lagrangian function.

Definition 4.3.1 (Saddle point)**.** *Let $F : A \times B \to [-\infty, \infty]$ any function, and A, B two sets. The point $(x^*, y^*) \in A \times B$ is called a* **saddle point** *of F if, for all $(x, y) \in A \times B$,*

$$F(x^*, y) \leq F(x^*, y^*) \leq F(x, y^*).$$

Proposition 4.3.1 (primal attainment and saddle point)**.**
Consider the convex optimization problem (4.1.1) and assume that the constraint qualification condition (4.2.7) holds. The following statements are equivalent:

(i) The point x^* is primal-optimal,

(ii) $\exists \lambda \in \mathbb{R}^{+p}$, such that the pair (x^*, λ) is a saddle point of the Lagrangian function L.

Furthermore, if (i) or (ii) holds, then

$$p = d = L(x^*, \lambda).$$

Proof. From proposition 4.2.2, under the condition $0 \in \operatorname{relint} \operatorname{dom} \mathcal{V}$, we know that the dual value is attained at some $\lambda \in \mathbb{R}^{+p}$. We thus have, for such a λ,

$$d = \mathcal{D}(\lambda) = \inf_{x \in \mathbb{R}^n} L(x, \lambda) \tag{4.3.1}$$

(the second equality is just the definition of $\mathcal{D}$).
Assume that *(i)* holds. Using the Lagrangian formulation of the constrained objective (lemma 4.1.1), $f(x) + \mathbb{I}_{g(x) \leq 0} = \sup_{\phi \succeq 0} L(x, \phi)$, saying that x^* is primal optimal means

$$p = \sup_{\phi \succeq 0} L(x^*, \phi) \tag{4.3.2}$$

In view of (4.3.1) and (4.3.2),

$$d = \inf_x L(x, \lambda) \leq L(x^*, \lambda) \leq \sup_{\phi \succeq 0} L(x^*, \phi) = p \tag{4.3.3}$$

Since $p = d$ (proposition 4.2.2), all the above inequalities are equalities, thus

$$L(x^*, \lambda) = \sup_{\phi} L(x, \phi),$$

which is the first inequality in the definition of a saddle point.
Furthermore, using equality in (4.3.3) again,

$$L(x^*, \lambda) = \inf_x L(x, \lambda),$$

which is the second inequality in the definition of a saddle point. We thus have, for $(x, \phi) \in \mathbb{R}^n \times \mathbb{R}^{+p}$,

$$L(x^*, \phi) \leq L(x^*, \lambda) \leq L(x, \lambda)$$

which is *(ii)*
Conversely, assume *(ii)*. The second inequality from the definition of a saddle point writes

$$L(x^*, \lambda) = \inf_x L(x^*, \lambda) = \mathcal{D}(\lambda). \tag{4.3.4}$$

The second inequality is

$$L(x^*, \lambda) = \sup_{\phi \succeq 0} L(x^*, \phi). \tag{4.3.5}$$

Thus

$$\begin{aligned} p = \inf_x \sup_{\phi \succeq 0} L(x, \phi) &\leq \sup_{\phi \succeq 0} L(x^*, \phi) \\ &= L(x^*, \lambda) \qquad (\text{ from } \ (4.3.5)) \\ &= \mathcal{D}(\lambda) \qquad (\text{ from } \ (4.3.4)) \\ &\leq \sup_{\phi \succeq 0} \mathcal{D}(\phi) \\ &= d. \end{aligned}$$

Since we know (lemma 4.1.2) that $d \leq p$, all the above inequalities are equalities, so that λ is a maximizer of $\mathcal{D}$, $p = d = L(x^*, \lambda)$. Finally,

$$\inf_x \sup_\phi L(x, \phi) = \sup_\phi L(x^*, \phi),$$

which means that x^* is a minimizer of

$$x \mapsto \sup_{\phi \succeq 0} L(x^*, \phi) = f(x) + \mathbb{I}_{g(x) \preceq \lambda}$$

(lemma 4.1.1. In other words, x^* is primal optimal. □

The last ingredient of KKT theorem is the complementary slackness properties of λ. If (x^*, λ) is a saddle point of the Lagrangian, and if the constraints are qualified, then $g(x^*) \preceq 0$. Call $I = \{i_1, \dots, i_k\}$, $k \leq p$, the set of **active constraints** at x^*, *i.e.*,

$$I = \Big\{ i \in \{1, \dots, p\} \ : \ g_i(x^*) = 0 \Big\}.$$

the indices i such that $g_i(x^*) = 0$.

Proposition 4.3.2. *Consider the convex problem (4.1.1) and assume that the constraints satisfiability condition* $0 \in \operatorname{relint} \operatorname{dom} \mathcal{V}$ *is satisfied.*
The pair (x^*, λ) *is a saddle point of the Lagrangian, if an only if*

$$\begin{cases} g(x^*) \preceq 0 & & \textit{(admissibility)} \\ \lambda \succeq 0, \quad \langle \lambda, g(x^*) \rangle = 0, & (i) & \textit{(complementary slackness)} \\ 0 \in \partial f + \sum_{i \in I} \lambda_i \partial g_i(x^*). & (ii) & \end{cases} \tag{4.3.6}$$

Remark 4.3.1. *The condition (4.3.6) (ii) may seem complicated at first view. However, notice that, in the differentiable case, this the usual 'colinearity of gradients' condition in the KKT theorem :*

$$\nabla f(x^*) = -\sum_{i \in I} \lambda_i \nabla g_i(x^*).$$

Proof. Assume that (x^*, λ) is a saddle point of L. By definition of the Lagrangian function, $\lambda \succeq 0$. The first inequality in the saddle point property implies $\forall \phi \in \mathbb{R}^{+p}, L(x^*, \phi) \leq L(x^*, \lambda)$, which means

$$f(x^*) + \langle \phi, g(x^*) \rangle \leq f(x^*) + \langle \lambda, g(x^*) \rangle,$$

i.e.

$$\forall \phi \in \mathbb{R}^{+p}, \quad \langle \phi - \lambda, g(x^*) \rangle \leq 0.$$

Since x^* is primal optimal, and the constraints are qualified, $g(x^*) \preceq 0$. For $i \in \{1, \ldots, p\}$,

- If $g_i(x) < 0$, then choosing $\phi = \begin{cases} \lambda_j & (j \neq i) \\ 0 & (j = i) \end{cases}$ yields $-\lambda_i g_i(x^*) \leq 0$, whence $\lambda_i \leq 0$, and finally $\lambda_i = 0$. Thus, $\lambda_i g_i(x^*) = 0$.
- If $g_i(x) = 0$, then $\lambda_i g_i(x^*) = 0$ as well.

As a consequence, $\lambda_j g_j(x^*) = 0$ for all j, and (4.3.6 (i)) follows.
Furthermore, the saddle point condition implies that x^* is a minimizer of the function $L_\lambda : x \mapsto L(x, \lambda) = f(x) + \sum_{i \in I} \lambda_i g_i(x)$. (the sum is restricted to the active set of constraint, due to (i)). From Fermat's rule,

$$0 \in \partial \left[f + \sum \lambda_i g_i \right]$$

Since $\operatorname{dom} g_i = \mathbb{R}^n$, the condition for the subdifferential calculus rule 3.4.1 is met and an easy recursion yield $0 \in \partial f(x^*) + \sum_{i \in I} \partial(\lambda_i g_i(x^*))$. As easily verified, $\partial \lambda_i g_i = \lambda_i \partial g_i$, and (4.3.6 (ii)) follows.

Conversely, assume that λ satisfies (4.3.6) By Fermat's rule, and the subdifferential calculus rule 3.4.1, condition (4.3.6) (ii) means that x^* is a minimizer of the function $h_\lambda : x \mapsto f(x) + \sum_{i \in I} \lambda_i g_i(x)$. using the complementary slackness condition ($\lambda_i = 0$ for $i \notin I$), $h_\lambda(x) = L(x, \lambda)$, so that the second inequality in the definition of a saddle point holds:

$$\forall x, L(x^*, \lambda) \leq L(x, \lambda).$$

Furthermore, for any $\phi \succeq 0 \in \mathbb{R}^p$,

$$L(x^*, \phi) - L(x^*, \lambda) = \langle \phi, g(x^*) \rangle - \langle \lambda, g(x^*) \rangle = \langle \phi, g(x^*) \rangle \leq 0,$$

since $g(x^*) \preceq 0$. This is the second inequality in the saddle point condition, and the proof is complete.

□

Definition 4.3.2. *Any vector $\lambda \in \mathbb{R}^p$ which satisfies (4.3.6) is called a* **Lagrange multiplier** *at x^* for problem (4.1.1).*

The following theorem summarizes the arguments developed in this section

Theorem 4.3.1 (KKT (Karush, Kuhn,Tucker) conditions for optimality)**.** *Assume that the constraint qualification condition (4.2.7) is satisfied for the convex problem (4.1.1). Let $x^* \in \mathbb{R}^n$. The following assertions are equivalent:*

(i) x^* is primal optimal.

(ii) There exists $\lambda \in \mathbb{R}^{+p}$, such that (x^*, λ) is a saddle point of the Lagrangian function.

(iii) There exists a Lagrange multiplier vector λ at x^*, i.e. *a vector $\lambda \in \mathbb{R}^p$, such that the* **KKT conditions***:*

$$\begin{cases} g(x^*) \preceq 0 & \textit{(admissibility)} \\ \lambda \succeq 0, \quad \langle \lambda, g(x^*) \rangle = 0, & \textit{(complementary slackness)} \\ 0 \in \partial f + \sum_{i \in I} \lambda_i \partial g_i(x^*). & \textit{('colinearity of subgradients')} \end{cases}$$

are satisfied.

Proof. The equivalence between (ii) and (iii) is proposition 4.3.2; the one between (i) and (ii) is proposition 4.3.1. □

4.4 Examples, Exercises and problems

In addition to the following exercises, a large number of feasible and instructive exercises can be found in Boyd and Vandenberghe (2009), chapter 5, pp 273-287.

Exercise 4.4.1 (Examples of duals, Borwein and Lewis (2006), chap.4)**.** Compute the dual of the following problems. In other words, calculate the dual function $\mathcal{D}$ and write the problem of maximizing the latter as a convex minimization problem.

1. Linear program

$$\inf_{x \in \mathbb{R}^n} \langle c, x \rangle$$

$$\text{under constraint } Gx \preceq b$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^p$ and $G \in \mathbb{R}^{p \times n}$.

Hint : you should find that the dual problem is again a linear program, with equality constraints.

2. Linear program on the non negative orthant

$$\inf_{x\in\mathbb{R}^n} \langle c, x\rangle + \mathbb{I}_{x\succeq 0}$$
$$\text{under constraint } Gx \preceq b$$

Hint : you should obtain a linear program with inequality constraints again.

3. Quadratic program

$$\inf_{x\in\mathbb{R}^n} \frac{1}{2}\langle x, Cx\rangle$$
$$\text{under constraint } Gx \preceq b$$

where C is symmetric, positive, definite.

Hint : you should obtain an unconstrained quadratic problem.

- Assume in addition that the constraints are linearly independent, *i.e.* $\operatorname{rang}(G) = p$, *i.e.* $G = \begin{pmatrix} w_1^\top \\ \vdots \\ w_p^\top \end{pmatrix}$, where $(w_1, \dots, w_p)$ are linearly independent. Compute then the dual value.

Exercise 4.4.2 (dual gap)**.** Consider the three examples in exercise 4.4.1, and assume, as in example 3., that the constraints are linearly independent. Show the duality gap is zero under the respective following conditions:

1. Show that there is zero duality gap in examples 1 and 3 (linear and quadratic programs).

   Hint : Slater.

2. For example 2, Assume that $\exists \bar{x} > 0 : G\bar{x} = b$. Show again that the duality gap is zero.

   Hint (spoiler) : Show that $0 \in \operatorname{int}\operatorname{dom}\mathcal{V}$. In other words, show that for all $y \in \mathbb{R}^p$ close enough to 0, there is some small $\bar{u} \in \mathbb{R}^n$, such that $x = \hat{x} + \bar{u}$ is admissible, and $G\,x \leq b + y$.
   To do so, exhibit some $u \in \mathbb{R}^n$ such that $Gu = -\mathbf{1}_p$ (why does it exist ?) Pick t such that $\hat{x} + tu > 0$. Finally, consider the 'threshold' $Y = -t\mathbf{1}_p \prec 0$ and show that, if $y \succ Y$, then $\mathcal{V}(y) < \infty$. Conclude.

Exercise 4.4.3 (Gaussian Channel, Water filling.)**.** In signal processing, a *Gaussian channel* refers to a transmitter-receiver framework with Gaussian noise: the transmitter sends an information X (real valued), the receiver observes $Y = X + \epsilon$, where ϵ is a noise.

A Channel is defined by the joint distribution of (X, Y). If it is Gaussian, the channel is called *Gaussian*. In other words, if X and ϵ are Gaussian, we have a Gaussian channel.

Say the transmitter wants to send a word of size p to the receiver. He does so by encoding each possible word w of size p by a a certain vector of size n, $\mathbf{x}_n^w = (x_1^w, \dots, x_n^w)$. To stick with the Gaussian channel setting, we assume that the x_i^w's are chosen as $i.i.d.$ replicates of a Gaussian, centered random variable, with variance x.
The receiver knows the code (the dictionary of all 2^p possible $\mathbf{x}_n^w$'s) and he observes $\mathbf{y}_n = \mathbf{x}_n^w + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_n))$. We wants to recover w.
The *capacity* of the channel, in information theory, is (roughly speaking) the maximum ratio $C = n/p$, such that it is possible (when n and p tend to ∞ while $n/p \equiv C$), to recover a word w of size p using a code $\mathbf{x}_n^w$ of length n.

For a Gaussian Channel , $C = \log(1 + x/\sigma^2)$. (x/σ^2 is the ratio signal/noise). For n Gaussian channels in parallel, with $\alpha_i = 1/\sigma_i^2$, then

$$C = \sum_{i=1}^{n} \log(1 + \alpha_i x_i).$$

The variance x_i represents a *power* affected to channel i. The aim of the transmitter is to maximize C under a *total power constraint* : $\sum_{i=1}^n x_i \le P$. In other words, the problem is

$$\max_{x \in \mathbb{R}^n} \sum_{i=1}^{n} \log(1 + \alpha_i x_i) \quad \text{under constraints : } \forall i, x_i \ge 0, \quad \sum_{i=1}^{n} x_i \le P. \tag{4.4.1}$$

1. Write problem (4.4.1) as a minimization problem under constraint $g(x) \preceq 0$. Show that this is a convex problem (objective and constraints both convex).

2. Show that the constraints are qualified. (hint: Slater).

3. Write the Lagrangian function

4. Using the KKT theorem, show that a primal optimal x^* exists and satisfies :

   - $\exists K > 0$ such that $x_i = \max(0, K - 1/\alpha_i)$.
   - K is given by
   $$\sum_{i=1}^{n} \max(K - 1/\alpha_i, 0) = P$$

5. Justify the expression *water filling*

Exercise 4.4.4 (Max-entropy)**.** Let $p = (p_1, \dots, p_n)$, $p_i > 0$, $\sum_i p_i = 1$ a probability distribution over a finite set. If $x = (x_1, \dots, x_n)$ is another probability distribution ($x_i \geq 0$), an if we use the convention $0 \log 0 = 0$, the entropy of x with respect to p is

$$H_p(x) = -\sum_{i=1}^{n} x_i \log \frac{x_i}{p_i}.$$

To deal with the case $x_i < 0$, introduce the function $\psi : \mathbb{R} \to (-\infty, \infty]$:

$$\psi(u) = \begin{cases} u \log(u) & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ +\infty & \text{otherwise .} \end{cases}$$

If $g : \mathbb{R}^n \to \mathbb{R}^p$, the general formulation of the max-entropy problem under constraint $g(x) \preceq 0$ is

$$\begin{aligned} \texttt{maximize}_{\text{ over } \mathbb{R}^n} \quad & \sum_i \big(-\psi(x_i) + x_i \log(p_i) \big) \\ \text{under constraints } & \sum x_i = 1; g(x) \preceq 0. \end{aligned}$$

In terms of minimization, the problem writes

$$\inf_{x \in \mathbb{R}^n} \sum_{i=1}^{n} \psi(x_i) - \langle x, c \rangle + \mathbb{I}_{\langle \mathbf{1}_n, x \rangle = 1} + \mathbb{I}_{g(x) \preceq 0}. \tag{4.4.2}$$

with $c = \log(p) = (\log(p_1), \dots, \log(p_n))$ and $\mathbf{1}_n = (1, \dots, 1)$ (the vector of size n which coordinates are equal to 1).

A : preliminary questions

1. Show that

$$\partial \mathbb{I}_{\langle \mathbf{1}_n, x \rangle} = \begin{cases} \{\lambda_0 \mathbf{1}_n \ : \ \lambda_0 \in \mathbb{R}\} := \mathbb{R}\mathbf{1}_n & \text{if } \sum_i x_i = 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

2. Show that ψ is convex
   hint : compute first the Fenchel conjugate of the function exp, then use proposition 3.2.2.

   Compute $\partial\psi(u)$ for $u \in \mathbb{R}$.

3. Show that

$$\partial\Big(\sum_i \psi(x_i)\Big) = \begin{cases} \sum_i (\log(x_i) + 1)\, \mathbf{e}_i & \text{if } x \succ 0 \\ \emptyset & \text{otherwise,} \end{cases}$$

   where $(\mathbf{e}_i, \dots, \mathbf{e}_n)$ is the canonical basis of $\mathbb{R}^n$.

4. Check that, for any set A, $A + \emptyset = \emptyset$.

5. Consider the unconstrained optimization problem, (4.4.2) where the term $\mathbb{I}_{g(x) \preceq 0}$ has been removed. Show that there exists a unique primal optimal solution, which is $x^* = p$.

   Hint: Do not use Lagrange duality, apply Fermat's rule (section 2.3) instead. Then, check that the conditions for subdifferential calculus rules (proposition 3.4.1) apply.

B : Linear inequality constraints In the sequel, we assume that the constraints are linear, independent, and independent from $\mathbf{1}_n$, *i.e.*: $g(x) = Gx - b$, where $b \in \mathbb{R}^p$, and G is a $p \times n$ matrix,

$$G = \begin{pmatrix} (\mathbf{w}^1)^\top \\ \vdots \\ (\mathbf{w}^p)^\top \end{pmatrix},$$

where $\mathbf{w}^j \in \mathbb{R}^n$, and the vectors $(\mathbf{w}^1, \dots, \mathbf{w}^p, \mathbf{1}_n)$ are linearly independent. We also assume the existence of some point $\hat{x} \in \mathbb{R}^n$, such that

$$\forall i, \hat{x}_i > 0, \ \sum_i \hat{x}_i = 1, \ G\hat{x} = b. \tag{4.4.3}$$

1. Show that the constraints are qualified, in the Lagrangian sense (4.2.7).

   Hint (spoiler) : proceed as in exercise 4.4.2, (2). This time, you need to introduce a vector $u \in \mathbb{R}^n$, such that $Gu = -\mathbf{1}_p$ and $\sum u_i = 0$ (again, why does it exist ?). The remaining of the argument is similar to that of exercise 4.4.2, (2).

2. Using the KKT conditions, show that any primal optimal point x^* must satisfy :
   $\exists Z > 0, \exists \lambda \in \mathbb{R}^{+p}$:

   $$x_i^* = \frac{1}{Z}\, p_i \ \exp\big[-\sum_{j=1}^p \lambda_j \mathbf{w}_i^j \big] \quad (i \in \{1, \dots, n\})$$

   (this is a Gibbs-type distribution).

List of symbols

$\liminf$ Limit inferior, page 17

$\partial f(x)$ Subdifferential of function f at x, page 13

E_f Upper hull of affine minorants of function f., page 19

f^* Fenchel-Legendre transform of function f, page 16

$\mathcal{AM}(f)$ Set of affine minorants of function f, page 19

l.s.c. Lower semicontinuous, page 17

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