Secrets from my step set

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Journées ALEA
March 11, 2011
The story so far

Given $S = \{N, SE, W\}$ → # = ?

We have seen

- Methods for finding the generating function
  - **MBM**: Define kernel; define associated group; generate and manipulate equations; Solve using positive series extraction of a rational function
  - **KR**: Define associated boundary value problem; solve using template solution
  - **AB**: Use power of computer algebra for intelligent guessing

- Nature of OGF (Rational? Algebraic? Holonomic?) correlated to finiteness of group
What remains to be told

- Nice combinatorial conjectures
- The case of the infinite group
  - “How to” do the enumeration
  - Why it is plausible to connect infinite group with non-holonomic
- Explore the underlying combinatorics
  What is the fundamental difference between two boundaries and one?
- How does this help us understand non-holonomic series, from a combinatorial point of view?
Some nice problems for the bijective specialist
Explain this nice relation

For a step set $S$, let $W_S(t)$ be the ogf for walks counted by length, ending anywhere in the plane.

**Theorem 1.** Step set: $A = \{N, SE, W\}$

$$a(n) = [t^n]W_A(t) = M_n \Leftarrow \text{MOTZKIN numbers}$$

**Theorem 2.** Step set: $B = \{N, E, SE, S, W, NE\}$

$$b(n) = [t^n]W_B(t) = 2^n M_n$$

**Open:** Find a *combinatorial explanation* for

$$a(n) = 2^n b(n)$$
Explain the algebraic formula

Step set: \{N, NE, E, S, SW, W\}

\[ W(t) + 3tW(t)^2 + 4t^2W(t)^3 + 2t^3W(t)^4 = \frac{1}{1 - 6t} \]

This is equivalent to walks in the \(\frac{2}{3}\pi\) wedge of the triangular lattice with steps along any direction:
Much harder: Why are these algebraic?

- In each case the pumping lemma shows that you will *not* find a direct CFG, so start with a different strategy.
- *Idea:* Characterization in terms of $k$-regular sequences
The case of the infinite group
Recall the group of a walk

Consider $S = \{\text{NW, NE, SW}\}$. The kernel is

\[
K(x, y) = 1 - t(y/x + xy + x/y)
= t \left( y + \frac{1}{y} \right) x + 1 + t y \frac{1}{x} = t \left( x + \frac{1}{x} \right) y + 1 + t x \frac{1}{y}
\]

$G(S)$ is generated by $\Phi = \left( \frac{y}{x(1 + \frac{1}{y})}, y \right)$ and $\Psi = \left( x, \frac{x}{y(1 + \frac{1}{x})} \right)$
Recall the group of a walk

Consider \( S = \{\text{NW, NE, SW}\} \). The kernel is

\[
K(x, y) = 1 - t(y/x + xy + x/y)
= t\left(y + \frac{1}{y}\right)x + 1 + ty\frac{1}{x} = t\left(x + \frac{1}{x}\right)y + 1 + tx\frac{1}{y}
\]

\( G(S) \) is generated by \( \Phi = \left(\frac{y}{x(1 + \frac{1}{y})}, y\right) \) and \( \Psi = \left(x, \frac{x}{y(1 + \frac{1}{x})}\right) \)

Let us see what happens to the point \((t^2, t)\) under \( \Phi \circ \Psi \):

\[
(t^2, t) \rightarrow (t^4 + O(t^6), t^3 + O(t^5)) \rightarrow (t^6 + O(t^8), t^5 + O(t^7)) \rightarrow \ldots
\]
Not all infinite groups are created equal

The “best” walks with infinite groups are:

- Iterated Kernel approach works for all of these examples
- Distinguished from other infinite group classes in the Boundary Value Methodology: “singular”
Iterated kernel method  [JaPrRe08;MiRe09]

Kernel equation:

\[ K(x, y)Q(x, y) = xy - tx^2 Q(x, 0) - ty^2 Q(y, 0) \]

Consider one root of the kernel \( Y(x) = Y(x; t) = \frac{1-\sqrt{1-4t^2(1+x^2)}}{2t(1+x^2)} \)

\[ \Rightarrow \quad K(x, Y(x)) = 0 \]
Iterated kernel method [JaPrRe08;MiRe09]

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Rewrite equation:

\[ 0 = xY(x) - tx^2Q(x, 0) - tY(x)^2Q(Y(x), 0) \]

Iterate the root: \( (Y_0(x) := x) \)

\[ Y_n(x) := Y(Y_{n-1}(x)) = xt^n + O(xt^2) \implies K(Y_{n-1}(x), Y_n(x)) = 0 \]
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Rewrite equation in general form:

\[ 0 = Y_{n-1}(x)Y_n(x) - tY_{n-1}(x)^2Q(Y_{n-1}(x), 0) - tY_n(x)^2Q(Y_n(x), 0) \]
Take an alternating sum

\[
0 = xY_1(x) - tx^2 Q(x, 0) - tY_1(x)^2 Q(Y_1(x), 0)
\]
\[
0 = Y_1(x)Y_2(x) - tY_1^2(x)Q(Y_1(x), 0) - tY_2(x)^2 Q(Y_2(x), 0)
\]
\[
0 = Y_2(x)Y_3(x) - tY_2^2(x)Q(Y_2(x), 0) - tY_3(x)^2 Q(Y_3(x), 0)
\]
\[
0 = Y_3(x)Y_4(x) - tY_3^2(x)Q(Y_3(x), 0) - tY_4(x)^2 Q(Y_4(x), 0)
\]
\[
\vdots
\]

\[
0 = \sum (-1)^n Y_n(x)Y_{n+1}(x) - tx^2 Q(x, 0)
\]

This works because \( Y_n(x) = xt^n + O(xt^2) \), hence the sum converges as a formal power series.
Singularities spring eternal

Theorem

\[ W(t) = (1 - 3t)^{-1}(1 - 2 \sum_{n \geq 0}(-1)^n Y_n(1)Y_{n+1}(1)). \]

The set \( \bigcup_n \text{poles}(Y_n(1)) \) is infinite, and is a subset of \( \text{poles}(W(t)) \). Consequently, \( W(t) \) is not holonomic.
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Proof

\[ Y_n(x) \bigg|_{t=\frac{q}{1+q^2}} = q^n + \ldots; \]
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- \( Y_n(x)|_{t=\frac{q}{1+q^2}} = q^n + \ldots; \)
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- \( Y_n(x)|_{t=\frac{q}{1+q^2}} = q^n + \ldots \);
- Valid power series in \( q \);
- Show \( \sum_{n \geq 0}(-1)^n Y_n(1)Y_{n+1}(1) \) convergent, except: when denominator is zero and along the branch cut of \( Y_1 \).
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Proof

- \( Y_n(x)|_{t=\frac{q}{1+q^2}} = q^n + \ldots \);
- Valid power series in \( q \);
- Show \( \sum_{n \geq 0}(-1)^n Y_n(1)Y_{n+1}(1) \) convergent, except: when denominator is zero and along the branch cut of \( Y_1 \).
- Show singularities don’t cancel.
IKM: Selectively applicable

The next models to analyze are non-singular. A good one to check is \{N, NE, E, SW\}

\[ Y(Y(x)) = x \]
The combinatorics of restricted lattice paths
One dimensional case

**Meanders:** Walks with steps of the form \((1, k), k \in \mathbb{Z}\) that start at \((0,0)\), end above or on the axis and never go below the axis.

![Graph showing meanders](image)

Combinatorics well understood \(+\) asymptotic formulas \([\text{BaFl01}]\)

**Theorem 4.** Consider a simple aperiodic walk. The number of paths of length \(n\), \([z^n]W(z,1)\), is \(P(1)^n\) exactly. Set

\[
\bar{Y}_1(z) := \prod_{j=2}^{c} (1 - u_j(z)).
\]

The asymptotic number of meanders depends on the sign of the drift \(\delta = P'(1)\) as follows:

\[\delta = 0: \quad [z^n]F(z,1) \sim \nu_0 \frac{P(1)^n}{\sqrt{\pi n}} \left(1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots\right)\]

\[
\nu_0 := \sqrt{2 \frac{P(1)}{P''(1)} \psi_1(\rho)}, \quad \rho = P(\tau)^{-1} = P(1)^{-1};
\]

\[\delta < 0: \quad [z^n]F(z,1) \sim \nu_0^\pm \frac{P(\tau)^n}{2\sqrt{\pi n^3}} \left(1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots\right)\]

\[
\nu_0^\pm := -\sqrt{2 \frac{P(\tau)^3}{P''(\tau)P(\tau) - P(1)}} \psi_1(\rho), \quad \rho = P(\tau)^{-1};
\]

\[\delta > 0: \quad [z^n]F(z,1) \sim \xi_0 P(1)^n + \nu_0^\pm \frac{P(\tau)^n}{2\sqrt{\pi n^3}} \left(1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots\right)\]

\[
\xi_0 := (1 - u_1(\rho_1))\psi_1(\rho_1), \quad \rho_1 := P(1)^{-1}.
\]
Asymptotics: Finite group cases

\[ w_n = [z^n] W_S(t) \quad w_n \sim \kappa n^\alpha \rho^{-n} \]

drift = vector sum of elements in the step set

<table>
<thead>
<tr>
<th>Description</th>
<th>Example</th>
<th># steps</th>
<th>( \rho )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 no drift; symmetric</td>
<td>{N, E, S, W}</td>
<td># steps</td>
<td>-1</td>
<td>((- \frac{1}{2}))</td>
</tr>
<tr>
<td>2 up drift; symmetric</td>
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<td>(-\frac{1}{2})</td>
<td>(-1)</td>
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<td>(P(\tau))</td>
<td>-2 ((-\frac{3}{2}))</td>
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<td># steps</td>
<td>(-\frac{3}{2})</td>
<td></td>
</tr>
<tr>
<td>5 no drift; Krewer erases</td>
<td>{NE, S, W}</td>
<td># steps</td>
<td>(-\frac{3}{2})</td>
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</tr>
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<td># steps</td>
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</tr>
<tr>
<td>7 no drift; G-B</td>
<td>{NW, W, SE, E}</td>
<td># steps</td>
<td>-2</td>
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</tr>
</tbody>
</table>

These results (guessed numerically [BoKa09]) are predicted by the meander arising as a horizontal projection:

e.g. \{N, SW, S, SE\} \rightarrow \{\uparrow, \downarrow, \downarrow, \downarrow\}

\[ P(u) = \# \text{up steps } u + \# \text{side steps } + \# \text{down steps } u^{-1}; \tau : P'(\tau) = 0. \]
Asymptotics: Infinite group case

There are no walks with an infinite group and drift=0. Infinite group case is similar for positive drift, and negative drift along an axis. Numerical studies are inconclusive in the case of negative drift in two directions.
The big picture
A combinatorial understanding of holonomy

GOAL

A theory of holonomic functions akin to the Chomsky-Schützenberger understanding of algebraic functions.

Holonomic functions in the combinatorial context only pop out when there is substantial structure. What is it?

Example: Lattice paths

- symmetry across $y$-axis is sufficient.
- symmetry across line $x = y$ is insufficient
- zero drift/rotational/reversal symmetry sufficient in 2D but maybe not in 3D
- Which symmetries affect the Galois group of the kernel?
Merci beaucoup.