Navigation on a Poisson point process

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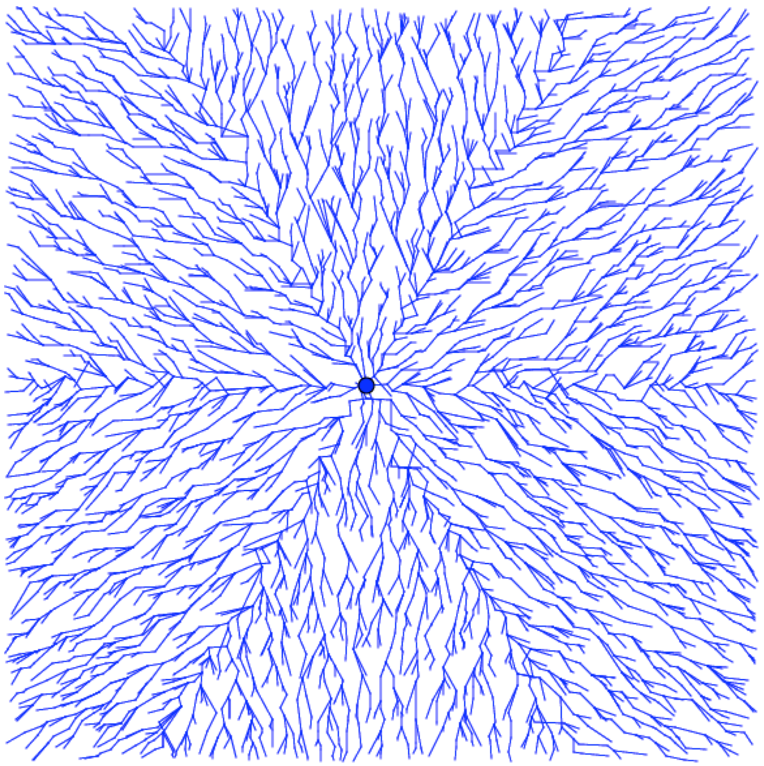
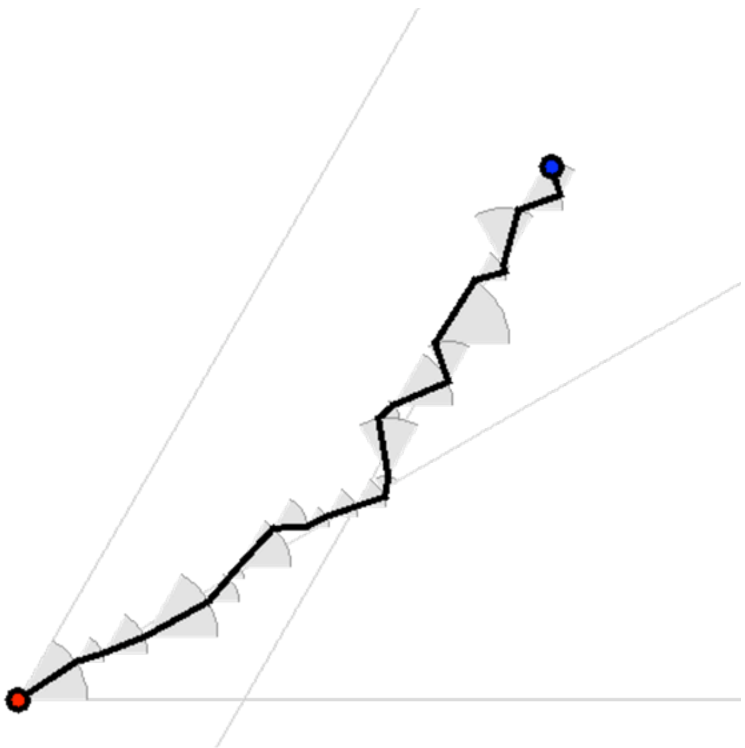




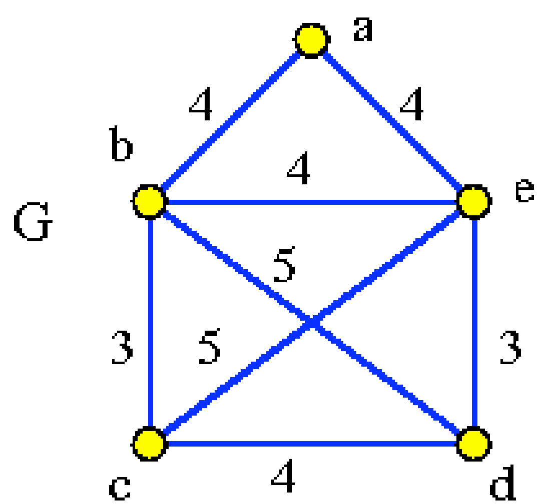
Luminy, 2011

Motivations... Notion of spanner

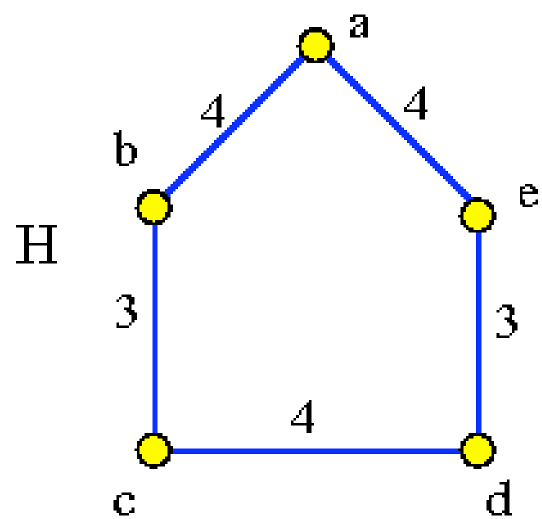
Let G be a graph with some edge lengths.
H a sub-graph of G is a s-spanner of G:

$$\forall u, v, \quad d_H(u,v) \leq s \; d_G(u,v).$$

$s =$ a stretch of the spanner



Example:



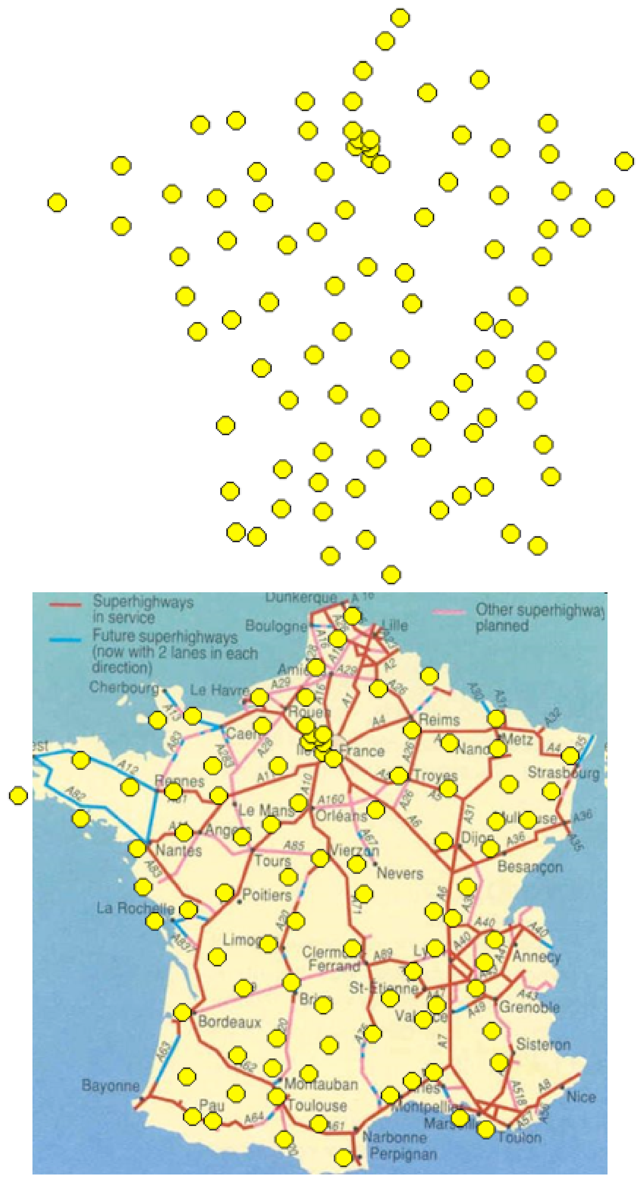
$d_G(b,e) = 4$, $d_H(b,e) = 8$.
H is a 2-spanner of G.

Motivations... Geometric graphs

(E, d_E) = plane equipped with Euclidean distance
S a subset of E
$G(S)$ the complete graph with edge length given by d_E

Goal in a lot of applications:
- construct a sub-graph with a small stretch
- that contains few edges
- small max degree
- easily routable...

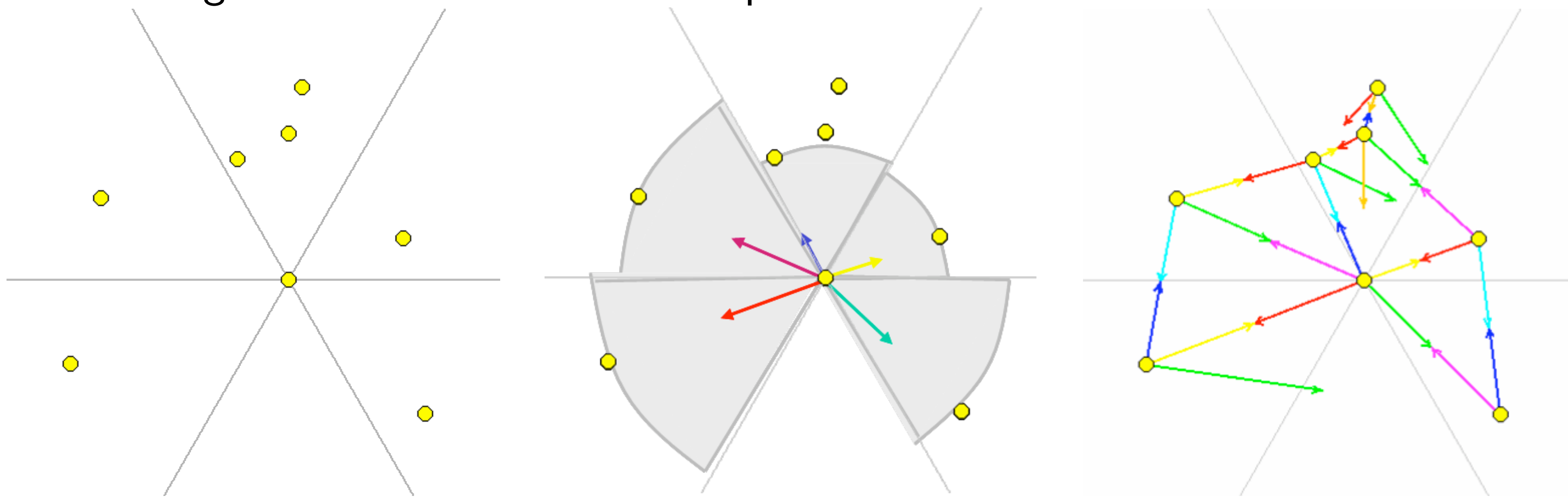
Motivations: construction/study of ad hoc network, of roads, or train lines, ...



Motivations... The construction of Yao [82] : cross navigation

Given $S \subset \mathbb{R}^2$ and a parameter $\theta = 2\pi/k$, $k \geq 6$

- The plane around each vertex s is split into k cones
- Add an edge between s and the closet point of S in each cone



The graph obtained is connected :

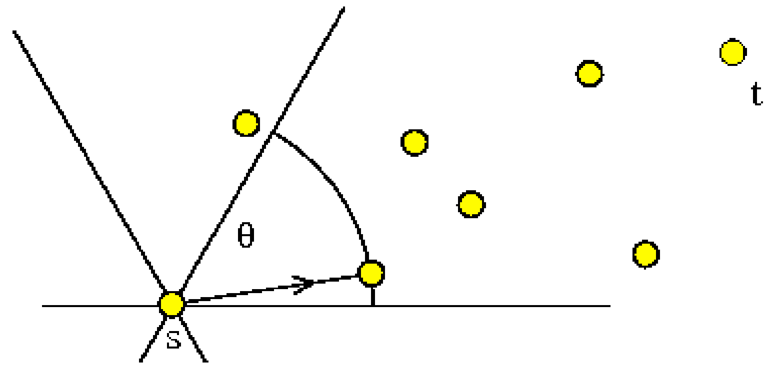
- as a at most $k\#S$ edges (linear)
- outgoing degree bounded by k
- easily routable
- Stretch for $k \geq 9$ smaller than $1/(1-2\sin(\theta/2))$

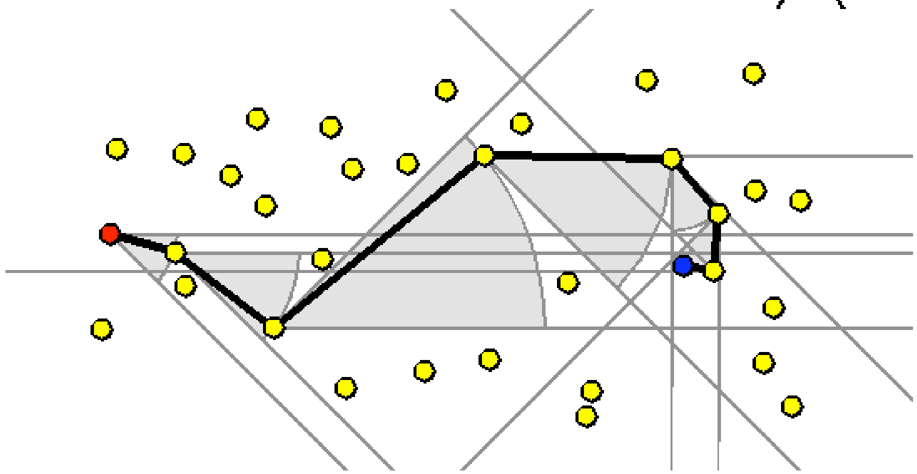
Stretch(9)≤ 3.16, Stretch(8)≤ 4.426, Stretch(7)≤ 7.56

the stretch corresponds to a worst case position of points...

Compass routing algorithm / Yao graph

Easily routable with a "compass routing algorithm" : Compute a path by selecting at each step the closest point in the sector of the target

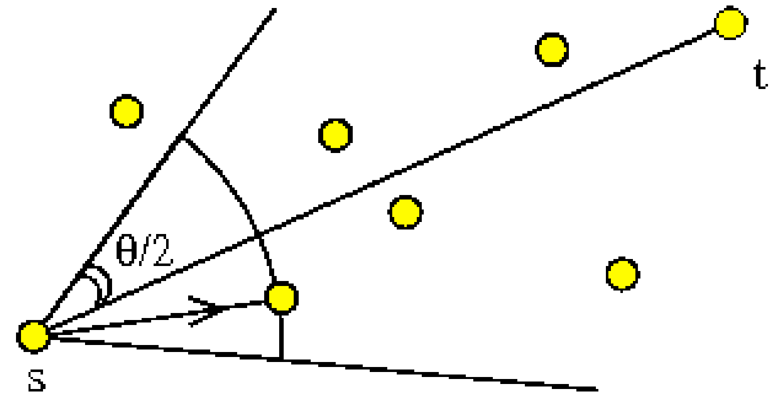


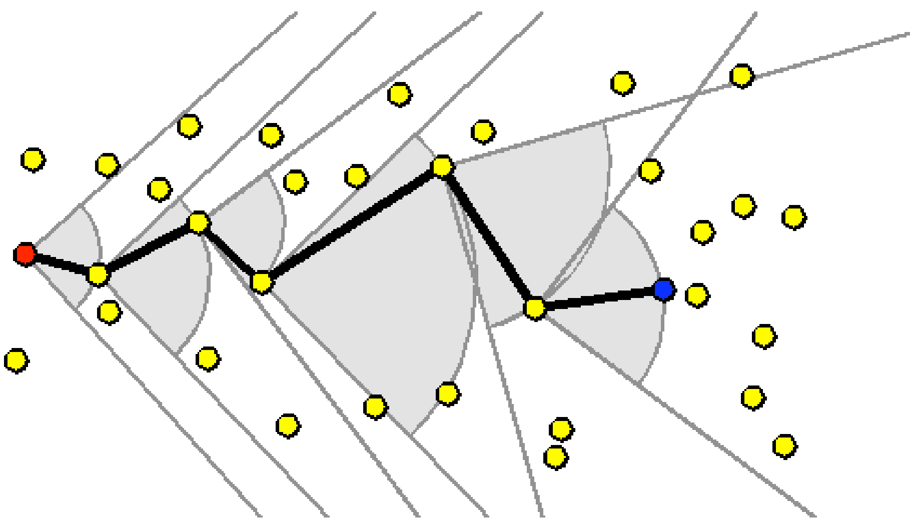


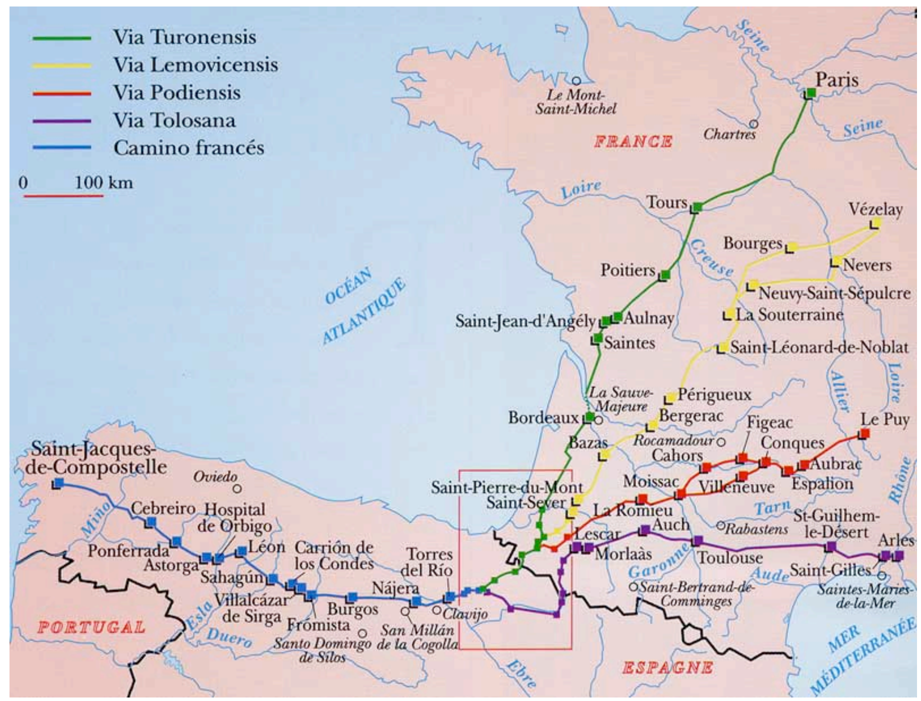
The compass routing algorithm uses as edges the edges of the Yao graph.

Straight navigation in the plane:

S = finite subset of $\mathbb{R}^2$ = set of possible stops,
$\theta \in (0, 2\pi/3)$ = a parameter
A traveller wants to go from $s = s_0$ to t.
He successively stops in $(s_i, i \geq 1)$ where s_i is the first element of S in the sector of angle θ and bisecting line (s_{i-1}, t)

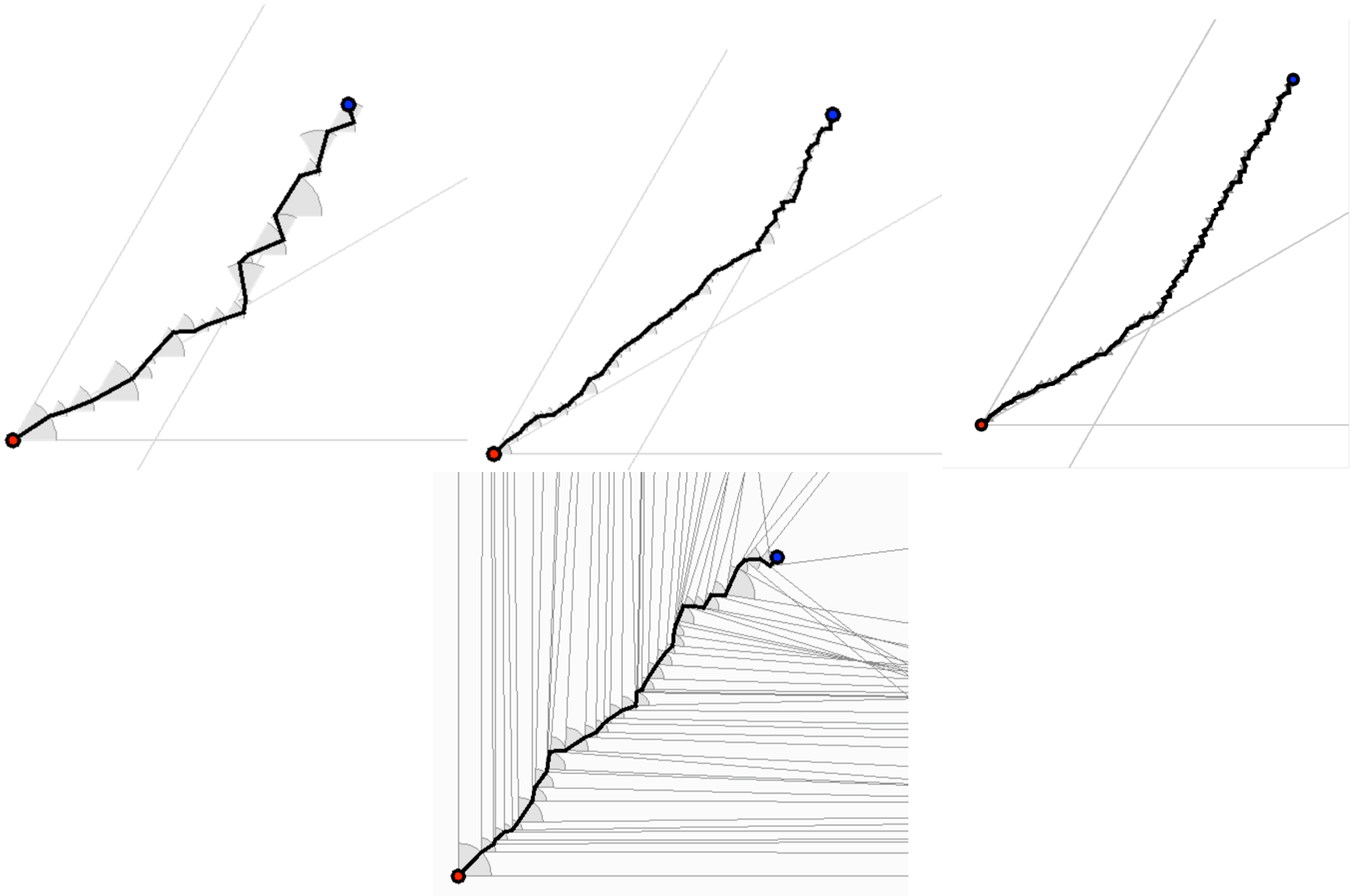






The questions...

What happens in a standard (random) situation and when the number of points goes to $+\infty$?



What is the shape of the trajectory of the traveller ? its length ?

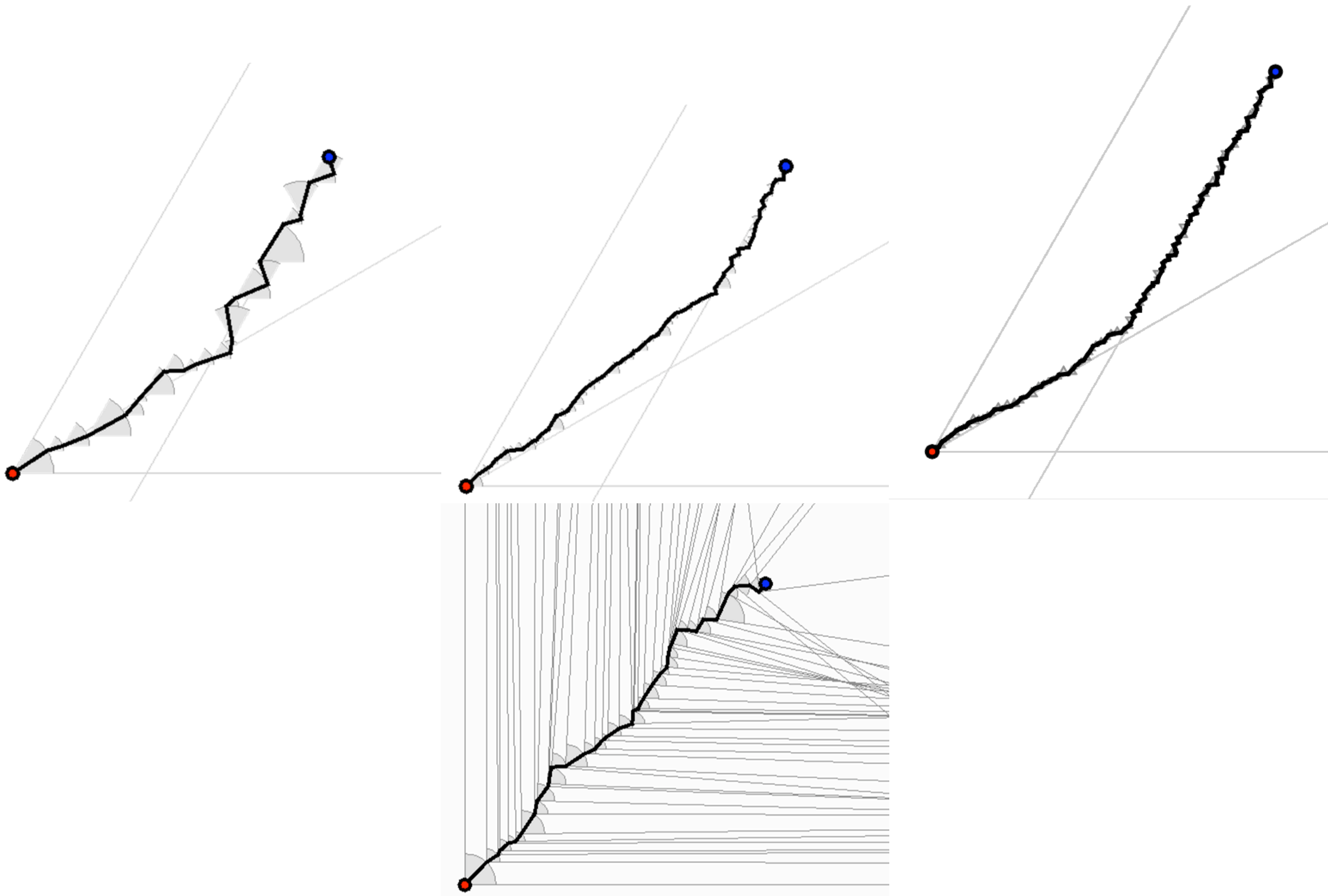
What is the corresponding stretch $= \sup \dfrac{\text{distance done}}{\text{Euclidean distance}}$

Model of random points

Domain fixed : $\mathcal{D}$ open, bounded, simply connected.

set of stopping place S : Poisson point process with intensity nf, and $n \to +\infty$,

$$\text{where } f = \text{ a non zero Lipschitz function on } \mathcal{D}$$



The results

– Description of the limiting paths between two points (and of various cost functions)

– Description of the traveller position according to the time between two points

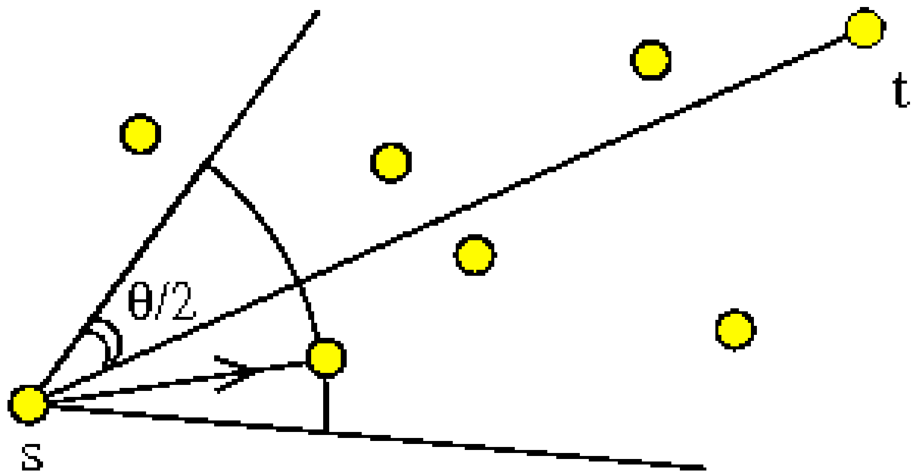
– Computation of the limiting "stretch", $\sup \dfrac{\text{distance done}}{\text{Euclidean distance}}$
(global result)

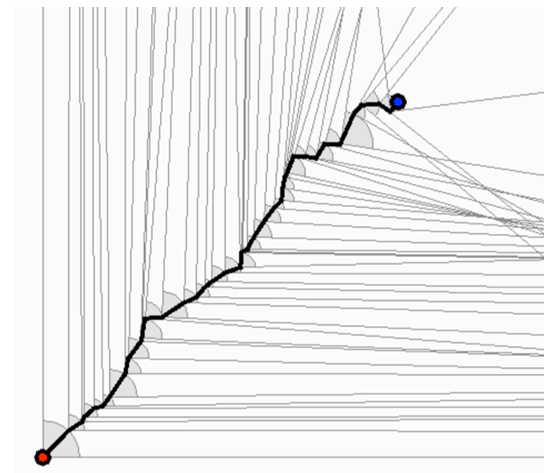
Description of the limiting path in the straight case

It is a segment !

For any $\alpha < 1/8$, there exists $d > 0$ such that for n large enough (unif. in s, t)

$$\mathbb{P}_{nf}(d_H(\mathsf{Path}(s,t),[s,t]) > n^{-\alpha}) \leq \exp(-n^d)$$



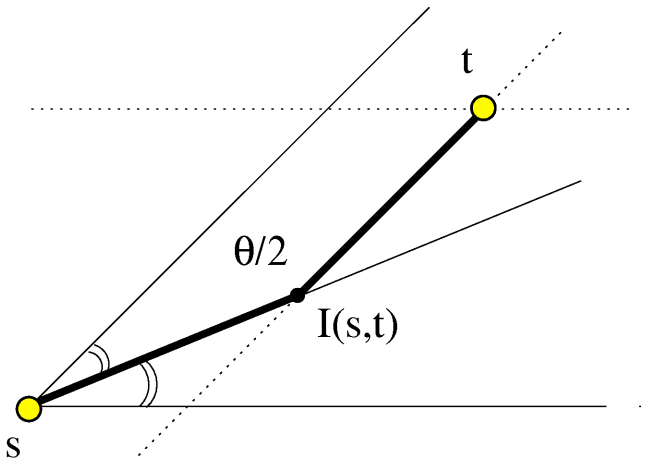


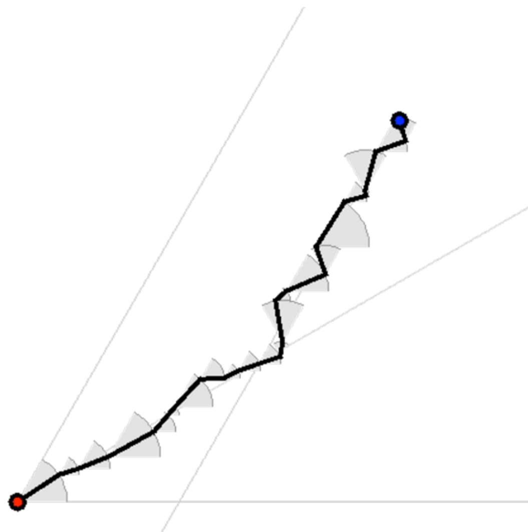
– The traveller can turn around the target, but very close of it !

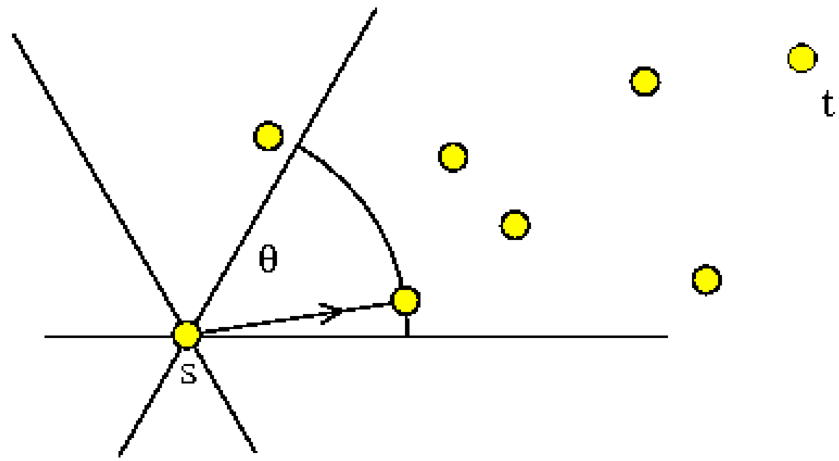
– f has no influence on the limiting trajectory

Description of the limiting path in the cross case

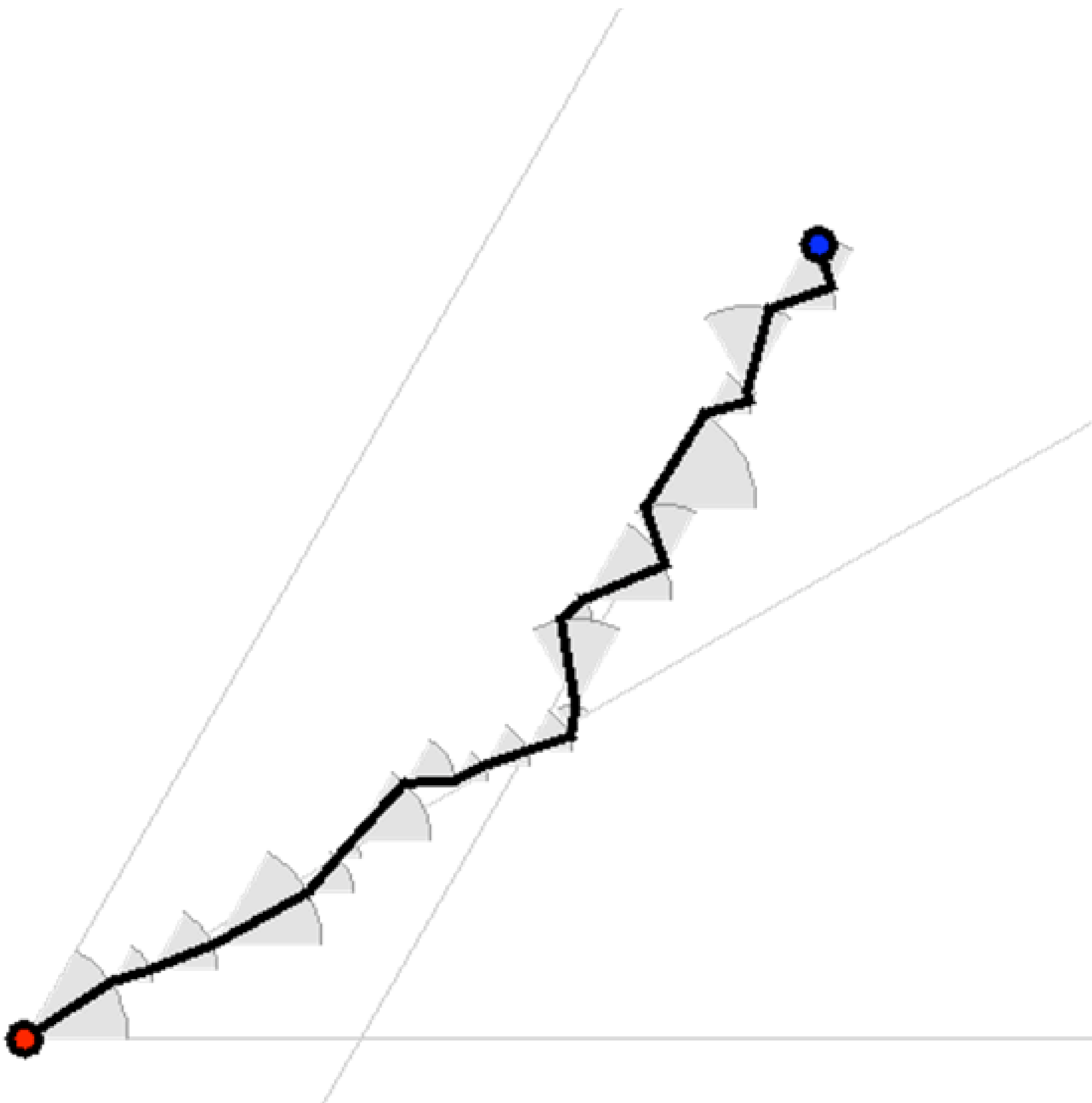
It is the union of two segments







$$\mathbb{P}_{nf}(d_H(\mathsf{Path}(s,t), [s, I(s,t)] \cup [I(s,t), t]) > n^{-\alpha}) \leq \exp(-n^d)$$



Straight case : Computation of the limiting stretch

For $\theta < 2\pi/3$, any $\varepsilon > 0$ and $c \geq 0$,

$$\mathbb{P}_{nf}\left(\sup_{(s,t)\in\mathcal{D},\ [s,t]\subset\mathcal{D}[a],|t-s|\geq n^{c-1/2}}\left|\frac{|\mathsf{Path}(s,t)|}{|s-t|}-Q_1\right|\geq\varepsilon\right)\xrightarrow[n]{}0.$$

Where

$$Q_1=\frac{\theta/2}{\sin(\theta/2)}$$

$\boxed{\Rightarrow \text{ this is close to 1, much smaller than the theoretical bounds.}}$

the limiting distance done by the traveller does not depend on f

Cross case : computation of the limiting stretch

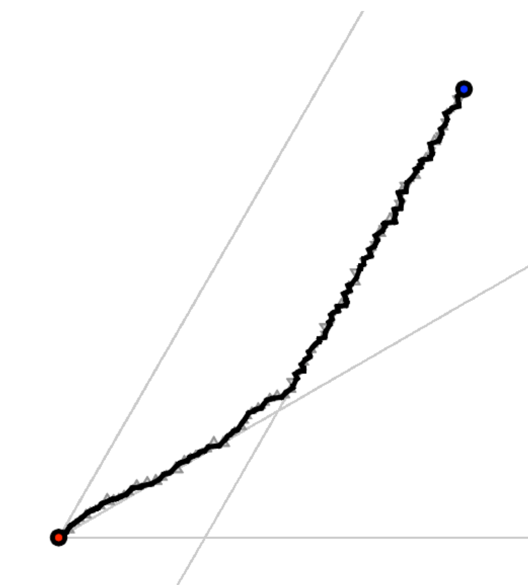
The maximum stretch obtained when t is on cross(s)
For $\theta \leq \pi/3$,

$$\mathbb{P}_{nf}\left(\left|\sup_{(s,t)\in\mathcal{D},\ \mathsf{Path}^{\infty}[s,t]\subset\mathcal{D}[a],|t-s|\geq n^{c-1/2}} \frac{|\mathsf{Path}(s,t)|}{|s-t|} - Q_2\right| \geq \varepsilon\right) \underset{n}{\longrightarrow} 0.$$

where

$$Q_2 = \frac{1}{2}\left(\frac{1}{\cos^2(\theta/2)} + \frac{\mathsf{arcsinh}(\tan(\theta/2))}{\sin(\theta/2)}\right).$$

$\boxed{\Rightarrow \text{ this is close to 1, much smaller than the theoretical bounds}}$



Towards the proofs

The results is obtained first for one path

To get this uniformity, three steps:

– deviations of order n^{-d} arise with proba. exponentially small for one path,

– this is then extended to a thin grid having a polynomial number of points

– then it is extended to all starting points and targets, by checking that any path can be split in at most 12 parts (with huge proba.), corresponding to paths between points of the grid.

Straight case : position according to the time

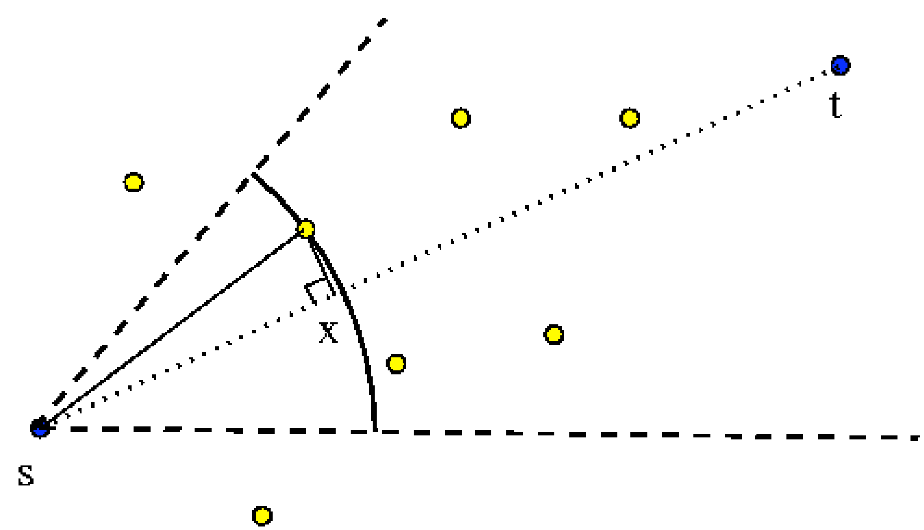
$\Delta_c =$ size of a stage under a homogeneous Poisson PP $\mathbb{P}_c$ then

$$\Delta_c \overset{(d)}{=} \Delta_1/\sqrt{c}.$$

Under $\mathbb{P}_{nf}$, at position s, the value of a stage is $\sim \frac{\Delta_1}{\sqrt{nf(s)}}$.

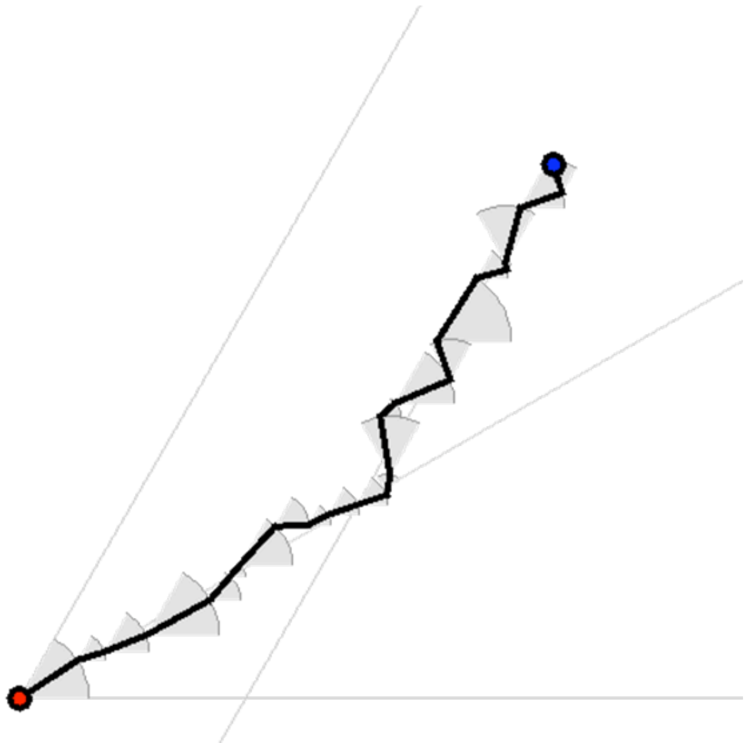
The speed at position s is of order $c/\sqrt{nf(s)}$ (speed = length per stage).
The asymptotic position function $\mathrm{Pos}_{s,t}^{(n)}(.\sqrt{n})$ is given by the solution of an ODE

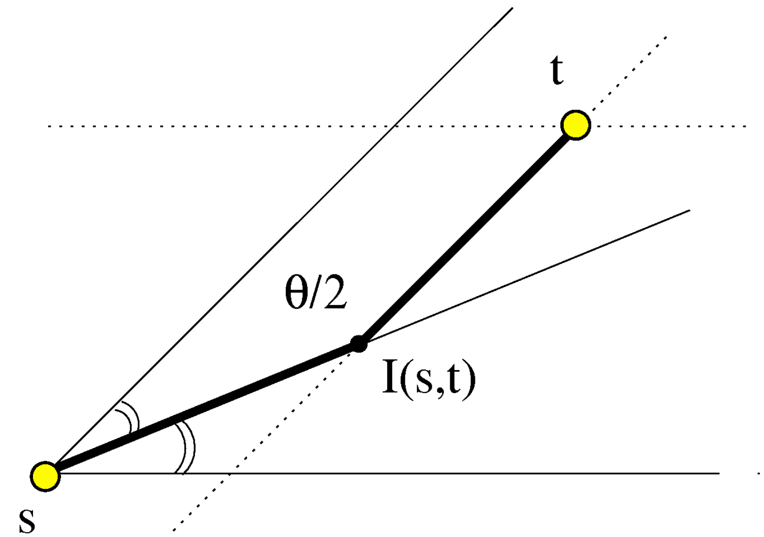
$$\begin{cases} \mathrm{Pos}_{s,t}(0) = s \\ \dfrac{\partial\,\mathrm{Pos}_{s,t}(x)}{\partial x} = \dfrac{\mathbb{E}(x_1)e^{i\arg(t-s)}}{\sqrt{f(\mathrm{Pos}_{s,t}(x))}} \end{cases}$$



– The number of stops and the places of the stops depend on f

Cross case : position according to the time

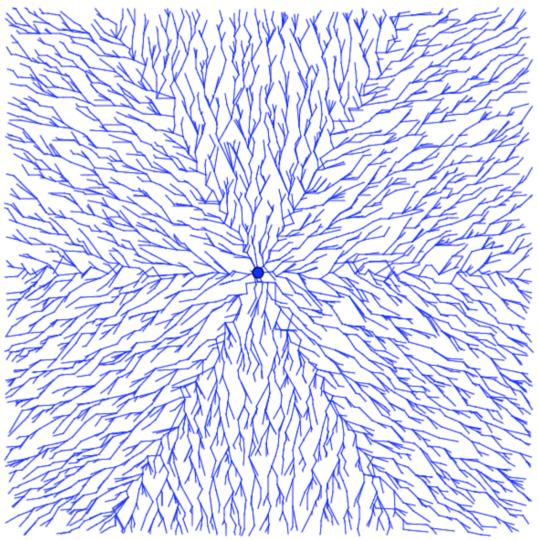


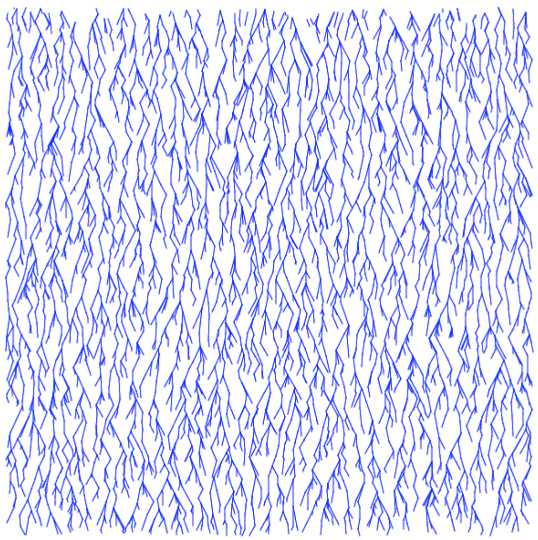


The asymptotic limiting position function is the concatenation of the solution of two different ODEs, with different speeds

To end... some references

Bordenave and Baccelli *Radial spanning tree of a Poisson point process*
Ferrari, Fontes and Wu : *Two-dimensional Poisson trees converge to the Brownian web*





Aldous & coauthors: *sequence of papers concerning short routes in road networks.*
Devroye & coauthors: *max degree in Yao graphs*

Navigation graph for a variant of Yao's graph