The growing tree distribution over Boolean functions.

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Our interest

Our aim: defining a random distribution over the set $\mathcal{F}_k$ of Boolean functions of k variables.

What's a Boolean function ?

$$\begin{aligned} f: \quad \{0,1\}^k &\longrightarrow \{0,1\} \\ (x_1,\ldots,x_k) &\longrightarrow f(x_1,\ldots,x_k) \end{aligned}$$

$1 = $ *True*
$0 = $ *False*

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1	=	*True*
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- The uniform law over the set of Boolean functions of k variables.
  [Shannon. *The synthesis of two-terminal switching circuits*]

Complexity

The complexity of f, denoted by $L(f)$, is the minimal number of connectives needed to represent f by a Boolean expression. E.g. $xXORy = (x \wedge \overline{y}) \vee (\overline{x} \wedge y)$ has complexity 3.

Shannon effect: *most* Boolean functions have exponential complexity in k - asymptotically when $k \longrightarrow \infty$.

How can we define other laws over $\mathcal{F}_k$?

Idea: consider the tree representation of a Boolean expression.

[Lefmann, Savický. *Some typical properties of large And/Or Boolean formulas*]

[Chauvin, Flajolet, Gardy, Gittenberger. *And/or trees revisited*]

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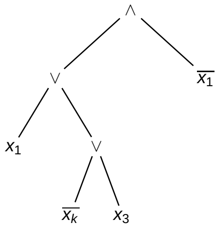
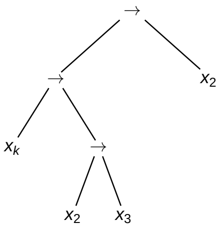
The $\wedge/\vee$ model.



connectives: $\wedge$ or $\vee$
positive and negative literals

The implication model.



unique connective: $\to$
only positive literals

Some definitions

Size of a binary tree:

The size of a binary tree is the number of its internal nodes.

Complexity of a function f:

The complexity of f is the size of the minimal trees computing f.

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a Boolean expression = a binary planar labelled tree	$\mathcal{E}_k$
$\downarrow$ "represents"	$\downarrow \Phi$
a Boolean function	$\mathcal{F}_k$

Φ is not injective.

The Catalan trees law

the set of all trees of size n labelled with k variables	uniform law $\mathbb{U}_{n,k}$
$\downarrow \Phi$	$\downarrow \Phi$
$\mathcal{F}_k$	$\mu_{n,k}$

The Catalan trees law

the set of all trees of size n labelled with k variables $\downarrow \Phi$ $\mathcal{F}_k$

uniform law $\mathbb{U}_{n,k}$ $\downarrow \Phi$ $\mu_{n,k}$

Theorem:

The sequence $(\mu_{n,k})_n$ converges to a probability law μ_k when $n \longrightarrow +\infty$.

[Lefmann, Savický] [Chauvin, Flajolet, Gardy, Gittenberger.]

Theorem:

We have $\forall f \in \mathcal{F}_k$: $\mu_k(f) = \Theta\left(\frac{1}{k^{L(f)+2}}\right)$.

[Kozik. *Subcritical pattern languages for and/or trees*]

Moreover, there is no Shannon effect. [Genitrini, Gittenberger. *No Shannon effect on probability distributions on Boolean functions induced by random expressions.*]

The Galton-Watson model

- Let us consider a critical Galton-Watson process.
- Let us label it according to one of the two labelling models, choosing for each node **uniformly** at random among the possible labels.

$\rightarrow$ a random tree labelled with k variables, random variable in $\mathcal{E}_k$ which law is denoted by Π_k.

$\rightarrow$ we denote by π_k the law induced by Π_k over $\mathcal{F}_k$ through Φ.
[Chauvin, Flajolet, Gardy, Gittenberger. *And/or trees revisited*]

Theorem [Chauvin et al.][Kozik][Genitrini, Gittenberger]:

The distribution π_k gives more weight to low complexity functions, though it weights all functions that can be represented in the chosen logical system.
There is no Shannon effect.

The growing tree

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A growing tree grows choosing uniformly the leaf which grows:

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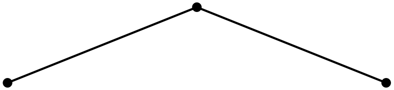

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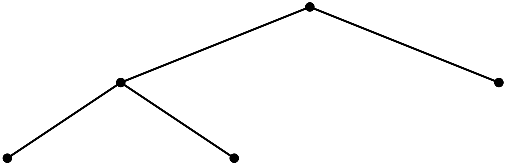

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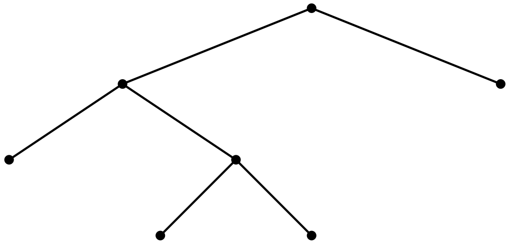

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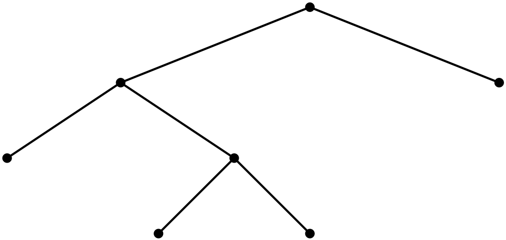

We stop after the n^{th} step.

$\longrightarrow$ non labelled random tree.
$\longrightarrow$ we label it "uniformly" at random: labelled random tree with law $\mathbb{P}_{n,k}$.
$\longrightarrow$ $p_{n,k}$ is the law induced by $\mathbb{P}_{n,k}$ over $\mathcal{F}_k$ through Φ.

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Questions

- Does $p_{n,k}$ converge when n tends to $+\infty$?
- If it does, what is the asymptotic law?
- And what is the speed of convergence?

Methods

- Analytic combinatorics.
- Probabilistic method via Yule trees.

Main results

Theorem:

In the $\wedge/\vee$ labelling model,
$p_{n,k} \longrightarrow p_k = \frac{1}{2}\delta_{True} + \frac{1}{2}\delta_{False}$ when n tends to $+\infty$, and

$$\|p_{n,k} - p_k\|_\infty = O\left(\frac{1}{\ln n}\right).$$

Theorem:

In the implication model,
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There is no Shannon effect.
The model is very simple.

Study of the growing tree law. The $\wedge/\vee$ labelling.

Analytic combinatorics approach.

Study of generating functions

$$\Phi_f(z) = \sum_{n\geq 0} p_{n,k}(f) z^n$$

Let $n \in \mathbb{N}^*$ and let $f \in \mathcal{F}_k$. We have:

$$p_{n+1,k}(f) = \frac{1}{2} \sum_{g\wedge h=f} \sum_{i=0}^{n} \frac{1}{n+1} p_{i,k}(g) p_{n-i,k}(h) + \frac{1}{2} \sum_{g\vee h=f} \sum_{i=0}^{n} \frac{1}{n+1} p_{i,k}(g) p_{n-i,k}(h).$$

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Summing over n and then deriving, we obtain a **differential** system:

$$\boxed{2\phi_f'(z) = \sum_{g\wedge h=f} \phi_g(z)\phi_h(z) + \sum_{g\vee h=f} \phi_g(z)\phi_h(z)}$$

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Simplifying the system considering equivalency classes of functions that share the same generating functions.
For example: $f \sim \bar{f}$.

Study of the differential system

$$\begin{cases} \phi_T' &= P_T(\phi_T, \phi_1, \dots, \phi_q) \\ \phi_1' &= P_1(\phi_T, \phi_1, \dots, \phi_q) \\ &\vdots \\ \phi_q' &= P_q(\phi_T, \phi_1, \dots, \phi_q) \end{cases}$$

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Remarking that the monomial Φ_T^2 only appears in the first equation, standard arguments of analysis lead to:

$$\phi_T(u) \overset{u \longrightarrow u_0}{\sim} \frac{c}{(u_0 - u)},$$

but this is only true on the real line !

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Let us apply a tauberian theorem...

Concluding the proof

$$\begin{cases} \sum_{i=1}^{n} p_{i,k}(True)u_0^i = \sum_{i=1}^{n} p_{i,k}(False)u_0^i \sim cn \\ \sum_{i=1}^{n} p_{i,k}(f)u_0^i = o(n) \text{ for all } f \notin \{True, False\}. \end{cases}$$

We easily show that $c = \frac{1}{2}$ and $u_0 = 1$.
Looking back at the induction formula over the coefficients $p_{n,k}(True)$, we get that:

$$p_{n+1,k}(True) \geq \frac{1}{n+1} \sum_{i=1}^{n} p_{n,i}(False) \xrightarrow{n\to\infty} \frac{1}{2}.$$

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Theorem:

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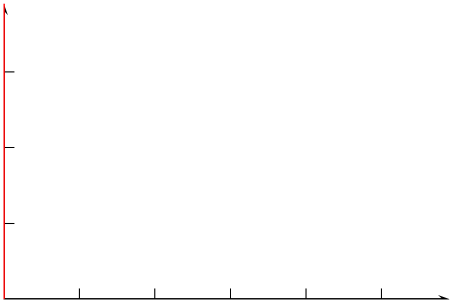
Probabilistic approach via Yule trees.

Yule trees



Labelled Yule tree

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$\rightarrow$ We thus have $\mathcal{E}_t$, a continuous time process of labelled binary trees. For all $t \geq 0$, this process define a law over $\mathcal{E}_k$ which image by Φ over $\mathcal{F}_k$ by P_t.

$\rightarrow$ $P_t = p_{n(t),k}$ where $n(t)$ is the size of $\mathcal{E}_t$.

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Strategy of proof

- We have to show that only constant functions are weighted by P_t.
- Let us consider $P_t^{10} := P_t(f(a) = 1 \text{ and } f(b) = 0)$ for fixed a and $b \in \{0,1\}^k$ two assignments of the k variables.
- Let us show that P_t^{10} tends to zero when $t \longrightarrow +\infty$, independently from the choice of a and b.

Let $P_t^{\alpha\beta} = P_t(f(a) = \alpha \text{ and } f(b) = \beta)$ with fixed a and $b \in \{0,1\}^k$.

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$$P_t^{10} = \sum_{i=1}^{k} \frac{e^{-t}}{2k} \left(\mathbb{1}_{\{a_i=1 \text{ and } b_i=0\}} + \mathbb{1}_{\{a_i=0 \text{ and } b_i=1\}}\right)$$

$$+\frac{1}{2}\int_0^t \Big(P_{t-s}^{11}P_{t-s}^{10} + P_{t-s}^{10}(P_{t-s}^{11}+P_{t-s}^{01}) + P_{t-s}^{01}(P_{t-s}^{00}+P_{t-s}^{10}) + P_{t-s}^{00}P_{t-s}^{10}\Big)e^{-s}ds$$

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We can simplify this expression and deduce from it that, if $\pi(t) = \mathbb{P}_t^{10}$:

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Theorem:

$p_{n,k} \longrightarrow p_k = \tfrac{1}{2}\delta_{True} + \tfrac{1}{2}\delta_{False}$ when n tends to $+\infty$ and

$$\|p_{n,k} - p_k\|_\infty = O\left(\frac{1}{\ln n}\right)$$

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- Biasing the law over the connectives: $\mathbb{P}(\wedge) = q$ and $\mathbb{P}(\vee) = 1 - q$.

Theorem:

- If $q > \frac{1}{2}$, then $p_{n,k} \longrightarrow \delta_{False}$.
- If $q < \frac{1}{2}$, then $p_{n,k} \longrightarrow \delta_{True}$.
- If $q = \frac{1}{2}$, then $p_{n,k} \longrightarrow \frac{1}{2}(\delta_{True} + \delta_{False})$.

And the speed of convergence is of order $O\left(\frac{1}{n^{|2q-1|}}\right)$ if $q \neq \frac{1}{2}$ and of order $O\left(\frac{1}{\ln n}\right)$ otherwise.

- Biaising the law over literals: $\mathbb{P}(x_i) = \mathbb{P}(\overline{x_i}) = \nu(x_i) > 0$.
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- Biasing the law over the connectives: $\mathbb{P}(\wedge) = q$ and $\mathbb{P}(\vee) = 1 - q$.
- Biasing the law over the connective, with only positive litterals.

Theorem:

- If $q > \frac{1}{2}$, then $p_{n,k} \longrightarrow \delta_{x_1 \wedge \ldots \wedge x_k}$.
- If $q < \frac{1}{2}$, then $p_{n,k} \longrightarrow \delta_{x_1 \vee \ldots \vee x_k}$.
- If $q = \frac{1}{2}$, then $p_{n,k} \longrightarrow p_k$ where p_k only weights 2^{2^k} threshold functions.

The growing tree law has the same behaviour as the law defined from balanced trees!

[Fournier, Gardy, Genitrini. *Balanced And/Or trees and linear threshold functions*]

Conclusion

- State a *meta-theorem* which would explain the link between the saturation level of a random tree and the behaviour of the law induced over $\mathcal{F}_k$.

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Thanks for your attention!