The Asymmetric Leader Election Algorithm with swedish stopping: A probabilistic analysis

Guy Louchard Conrado Martinez Helmut Prodinger

March 1, 2011

Outline

Introduction

We present a probabilistic analysis, based on an urn model, of a leader election protocol that we call the *Swedish leader election protocol*. This name comes from a protocol presented by L. Bondesson , T. Nilsson, and G. Wikstrand in [1]. The goal is to select one among $n > 0$ players, by proceeding through a number of rounds. If there is only one player remaining, the protocol stops and the player is declared the leader. Otherwise, all remaining players flip a biased coin; with probability q the player survives to the next round, with probability $p = 1 - q$ the player loses (is killed) and plays no further... unless *all players lose in a given round (null round)* , so all them play again. In the classical leader election protocol, *any number of null rounds* may take place, and with probability 1 some player will ultimately be elected. In the Swedish leader election protocol there is a *maximum number τ of consecutive null rounds*, and if the threshold is attained the protocol fails without declaring a leader.

Several parameters are asymptotically analyzed, starting with n players (n large):

1. Success Probability
2. Number of rounds K_n
3. Number of null rounds T_n. We say that a round is *null* if every active player tosses tails (all killed)
4. Number W_n of players, in case of failure, that were active at the last non-null round, the so-called *left-overs*
5. Total number of coins flipped C_n

Urn model

We will proceed as in [9], with urns numbered $1, 2, \ldots$.
Let us consider the model as a sequence of n geometric iid random variables (RVs), with distribution pq^{i-1}. Each RV corresponds to a ball thrown into urn i. We have the following properties:

- We have *asymptotic independence of urns*, for all events related to an urn j containing $\mathcal{O}(1)$ balls. This is proved, by Poissonization-DePoissonization, in [8], [10] and [4] (in this paper for $p = 1/2$, but the proof is easily adapted). The error term is $\mathcal{O}(n^{-C})$ where C is a positive constant.
- We obtain *asymptotic distributions* of the interesting RVs. The number of balls in urn j containing $\mathcal{O}(1)$ balls is now Poisson-distributed with parameter $n^* q^j$. The asymptotic distibutions are related to Gumbel distributions functions or convergent series of such. The error term is $\mathcal{O}(n^{-1})$.

- We have *uniform integrability* for the moments of our RVs. To show that the limiting moments are equivalent to the moments of the limiting distributions, we need a suitable rate of convergence. This is related to a uniform integrability condition (see Loève [6, Section 11.4]). For the kind of limiting distributions we consider here, the rate of convergence is analyzed in detail in [7] and [10]. The error term is $\mathcal{O}(n^{-C})$.
- Asymptotic expressions for the moments are obtained by *Mellin transforms*. The error term is $\mathcal{O}(n^{-C})$.
- $\Gamma(s)$ decreases exponentially in the direction $i\infty$:

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}.$$

  Also, we have a *slow increase property* for all other functions we encounter. So inverting the Melin transforms is easily justified.

If we compare the approach in this paper with other ones that appeared previously, then we can notice the following. Traditionally, one would stay with *exact enumerations* as long as possible, and only at a late stage move to asymptotics. Doing this, one would, in terms of asymptotics, carry many unimportant contributions around, which makes the computations quite heavy, especially when it comes to higher moments. Here, however, *approximations* are carried out *as early as possible*, and this allows for streamlined (and often automatic) computations of the higher moments.

Notations

We will use several abbreviations for probabilities and moments in order to derive more compact expressions.

$$
\begin{aligned}
n &:= \text{number of initial players }, n \text{ large},\\
\mathcal{P}(\lambda, u) &:= e^{-\lambda}\lambda^u/u!, \ \text{(Poisson distribution)},\\
n^* &:= n\frac{p}{q},\\
Q &:= 1/q, \log := \log_Q,\\
\eta &:= j - \log n^*,\\
L &:= \ln Q,\\
\{x\} &:= \text{ fractional part of } x,\\
\tilde{\alpha} &:= \alpha/L,\\
M &:= \log p,\\
\chi_l &:= \frac{2l\pi\mathbf{i}}{L}.
\end{aligned}
$$

$$K_i := \text{ total number of rounds, starting with } i \text{ players },$$
$$T_i := \text{ total number of null rounds, starting with } i \text{ players },$$
$$W_i := \text{ total number of leftovers, starting with } i \text{ players },$$
$$C_i := \text{ total number coins flipped, starting with } i \text{ players },$$
$$R_n := \mathbb{E}(K_n), I_n := \mathbb{E}(T_n), L_n := \mathbb{E}(W_n), F_n := \mathbb{E}(C_n),$$
$$I := \text{ number of balls in the maximal non-empty urn },$$
$$J := \text{ either the position of the maximal non-empty urn, if it conta}$$
$$I > 1 \text{ balls, or the position of the last non-empty urn } \textit{before}$$
$$\text{the maximal non-empty urn, if the latter contains } I = 1 \text{ ball,}$$
$$\text{Model } 1 = M1 = \text{ we fail if we have } \tau \text{ consecutive null records },$$
$$\text{Model } 2 = M2 = \text{ we fail if we have } \tau \text{ null records, consecutive or not }.$$

Note that the maximal non-empty urn, if it contains $I \geq 2$ balls, corresponds to a null round. If the maximal non-empty urn contains $I = 1$ ball, the process is successful.

Success probability, M1.

We use the following *notations*

$$SU_i = P_2(i,S) := \text{ Probability that, starting with } i \text{ players, we succeed },$$

$$\tilde{P}_2(i,S) := \text{ Probability that, starting with } i \text{ players, we succeed,}$$
$$\text{given that the } i \text{ players were obtained in a null round },$$

$$SU_n = P_5(S) := \text{success probability, starting with } n \text{ players,}$$

$$P_6(F) := \text{failure probability, starting with } n \text{ players} = 1 - P_5(S),$$

$$\Sigma_1(i,\tau) := \sum_{v=0}^{\tau-1} (p^i)^v = \frac{1-p^{i\tau}}{1-p^i},$$

$$S_1 := \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{P}(i).$$

The tilde notation will always have the same meaning later on. Here and in the sequel, $\tilde{P}(i)$ must be replaced by $\tilde{P}_2(i,S)$ or $\tilde{P}_{10}(i,F)$ depending on the case we consider.

Recurrences.

We have the following recurrences:

$$P_2(1,S) = 1, \tilde{P}_2(1,S) = 1,$$

$$P_2(i,S) = \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_2(\ell,S)$$

$$= \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_2(\ell,S), i \geq 2,$$

$$\tilde{P}_2(i,S) = \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_2(\ell,S), i \geq 2$$

$$= \Sigma_1(i,\tau-1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_2(\ell,S) = \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)} P_2(i,S).$$

Explanation: we can have up to $\tau - 1$ or $\tau - 2$ null rounds (all killed), followed by ℓ survivors. Note that $\ell = i$ in the right-hand side leads to $P_2(i,S)$.

We also have from [9] (here and in the sequel $\sim$ always denotes $\sim_{n\to\infty}$),

$$\mathbb{P}(J=j, I=i) \sim \exp\left(-\frac{1}{p}e^{-L\eta}\right)\frac{e^{-L\eta i}}{i!}, i \geq 2, \tag{1}$$

$$\mathbb{P}(J=j, I\geq 2, S) \sim f_1(\eta),$$

$$f_1(\eta) := \sum_{i=2}^{\infty} \exp\left(-\frac{1}{p}e^{-L\eta}\right)\frac{e^{-L\eta i}}{i!}\tilde{P}_2(i,S),$$

$$\mathbb{P}(J=j, I=1) \sim f_2(\eta),$$

$$f_2(\eta) := \exp\left(-\frac{q}{p}e^{-L\eta}\right)\frac{q}{p}e^{-L\eta}\left(1-\exp\left(-e^{-L\eta}\right)\right). \tag{2}$$

Explanation:

$$\mathbb{P}(J=j, I=i) \sim \mathcal{P}\left(\frac{q}{p}e^{-L\eta}, 0\right)\mathcal{P}\left(e^{-L\eta}, i\right), i \geq 2$$

$$\mathbb{P}(J=j, I=1) \sim \mathcal{P}\left(\frac{q}{p}e^{-L\eta}, 1\right)\left[1-\mathcal{P}\left(e^{-L\eta}, 0\right)\right].$$

This gives

$$\phi_1(\alpha) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_1(\eta) d\eta = \sum_{i=2}^{\infty} \frac{(1/p)^{-i+\tilde{\alpha}}}{Li!} \Gamma(i-\tilde{\alpha}) \tilde{P}_2(i,S).$$

So the corresponding *dominant part* of $SU_n := P_5(S)$ is given by

$$\phi_1(0) = \sum_{i=2}^{\infty} \frac{p^i}{Li} \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)} P_2(i,S).$$

The corresponding *periodic part* is given by

$$\omega_{1,5} = \sum_{l\neq 0} \varphi_{1,5}(\chi_l) e^{-2l\pi \mathbf{i} \log n^*},$$

with

$$\varphi_{1,5}(\chi_l) = \phi_1(\alpha)\big|_{\alpha=-L\chi_l}.$$

We obtain

$$\varphi_{1,5}(\chi_l) = \sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!} \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)} P_2(i,S).$$

Also,

$$\phi_2(\alpha) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_2(\eta) d\eta = \frac{1}{L}\left[\left(\frac{q}{p}\right)^{\tilde{\alpha}} - q\left(\frac{1}{p}\right)^{\tilde{\alpha}}\right]\Gamma(1-\tilde{\alpha}).$$

Hence

$$\phi_2(0) = \frac{p}{L}, \text{ one ball} \tag{3}$$

$$\omega_{2,5} = \sum_{l\neq 0} \varphi_{2,5}(\chi_l) e^{-2l\pi\mathbf{i}\log n^*},$$

$$\varphi_{2,5}(\chi_l) = \frac{1}{L}\left[\left(\frac{q}{p}\right)^{-\chi_l} - q\left(\frac{1}{p}\right)^{-\chi_l}\right]\Gamma(1+\chi_l).$$

And finally,

Theorem 4.1

Success probability, M1.

$$SU_n = P_5(S) \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \frac{\Sigma_1(i,\tau-1)}{\Sigma_1(i,\tau)} P_2(i,S) + \frac{p}{L}$$
$$+ \sum_{l\neq 0} \varphi_{1,5}(\chi_l) e^{-2l\pi\mathbf{i}\log n^*} + \sum_{l\neq 0} \varphi_{2,5}(\chi_l) e^{-2l\pi\mathbf{i}\log n^*}$$
$$= S_1 + \frac{p}{L} + \sum_{l\neq 0} \varphi_{1,5}(\chi_l) e^{-2l\pi\mathbf{i}\log n^*} + \sum_{l\neq 0} \varphi_{2,5}(\chi_l) e^{-2l\pi\mathbf{i}\log n^*}.$$

Of course, $P_6(F) = 1 - P_5(S)$. Note also that as $\tau \to \infty$, the dominant part gives

$$\sum_{i=2}^{\infty} \frac{p^i}{Li} + \frac{p}{L} = 1$$

as expected. For further use, we denote

$$\Pi_1 := \frac{p}{L}, \Pi_2(i) := \frac{p^i}{Li}, Pd(S) = S_1 + \frac{p}{L} \text{ (dominant part) }.$$

Moments of $K_n - \log n^*$, success case, M1.

We use the following *notations*

$P_1(i,k,S)$:= Probability that, starting with i players, we succeed after k rounds ,

$\tilde{P}_1(i,k,S)$:= Probability that, starting with i players, we succeed after k rounds , given that the i players were obtained in a null round, not preceeded by another null round ,

$x_{i,S}$:= mean number of rounds, starting with i players, with success at the end,

$\tilde{x}_{i,S}$:= mean number of rounds, starting with i players, with success at the end, given that the i players were obtained in a null round .

$$\Sigma_2(i,\tau) := \sum_{v=0}^{\tau-1} (p^i)^v (v+1) = \frac{p^{i\tau}(-1+\tau p^i - \tau)+1}{(1-p^i)^2},$$

$$\Sigma_3(i,\tau) := \sum_{v=0}^{\tau-1} (p^i)^v (v+1)^2$$
$$= \frac{p^{i\tau}(p^i + 1 - 2\tau p^i + 2\tau + \tau^2 p^{2i} - 2\tau^2 p^i + \tau^2) - p^i + 1}{(1-p^i)^3}.$$

For any of the four RVs we will use, we will denote by $\xi_i^{(2)}, \xi \in \{x, y, w, z\}$ the expectation of the square of the corresponding RVs.

In case of success, the moments of $K_n - \log n^*$ are computed as in [9], with $\tilde{x}_{i,S}, \tilde{x}_{i,S}^{(2)}$, computed as follows, instead of $x_{i,S}, x_{i,S}^{(2)}$. Here and in the sequel, we give the first two moments. All moments could be computed, only with more (algebraic and Maple) efforts. We use $f_1(\eta)$ as given by (4) of [9], with $\tilde{P}_1(i,k,S)$ instead of $P(i,k)$, ie

$$f_1(\eta) = \sum_{k=1}^{\infty}\sum_{i=2}^{\infty} \exp\left(-\frac{1}{p}e^{Lk}e^{-L\eta}\right)\frac{e^{-L\eta i}e^{Lki}}{i!}\tilde{P}_1(i,k,S).$$

$f_2(\eta)$ is given by (2).

Recurrences and first moments.

We have the following recurrences:

$$P_1(1,0,S) = 1, \tilde{P}_1(1,0,S) = 1,$$

$$P_1(1,\geq 1,S) = 0, \tilde{P}_1(1,\geq 1,S) = 0,$$

$$P_1(i,k,S) = \sum_{s=0}^{\tau-1}(p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_1(\ell, k-1-s, S), i \geq 2,$$

$$\tilde{P}_1(i,k,S) = \sum_{s=0}^{\tau-2}(p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_1(\ell, k-1-s, S), i \geq 2.$$

Explanation: we can have up to $\tau - 1$ or $\tau - 2$ null rounds (all killed), followed by ℓ survivors. This takes $s+1$ rounds already.

$$x_{i,S} = \sum_k P_1(i,k,S)k$$
$$= \sum_k \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_1(\ell, k-1-s, S)[k-1-s+s+1]$$
$$= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} x_{\ell,S}$$
$$+ \sum_k \sum_{s=0}^{\tau-1} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_1(\ell, k-1-s, S)$$
$$= \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} x_{\ell,S} + \Sigma_2(i,\tau) P_2(i,S)/\Sigma_1(i,\tau), x_{1,S} = 0,$$

$$
\begin{aligned}
x_{i,S}^{(2)} &= \sum_k P_1(i,k,S)k^2 \\
&= \sum_k \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_1(\ell, k-1-s, S)[k-1-s+s+1]^2 \\
&= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} x_{\ell,S}^{(2)} \\
&+ \sum_k \sum_{s=0}^{\tau-1} (p^i)^s (s+1)^2 \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_1(\ell, k-1-s, S) \\
&+ 2\sum_{s=0}^{\tau-1} (p^i)^s (s+1) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} x_{\ell,S}
\end{aligned}
$$

$$= \Sigma_1(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}x^{(2)}_{\ell,S}$$

$$+\Sigma_3(i,\tau)P_2(i,S)/\Sigma_1(i,\tau)+2\Sigma_2(i,\tau)\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}x_{\ell,S},x^{(2)}_{1,S}=0,$$

and similar expressions for $\tilde{x}_{i,S}, \tilde{x}^{(2)}_{i,S}$. Note that, in previous expressions, we could replace

$$\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}x_{\ell,S} \text{ by } [x_{i,S}-\Sigma_2(i,\tau)P_2(i,S)/\Sigma_1(i,\tau)]/\Sigma_1(i,\tau).$$

To obtain the moments of $K_n - \log n^*$, we plug, mutatis mutandis, $\tilde{x}_{i,S}, \tilde{x}^{(2)}_{i,S}$ into the moments given in [9]. Note that to each value $I = i \geq 2$ corresponds $\tilde{P}_2(i,S)$. Also $R_0(\chi_I)$ is no more null here and $\tilde{P}_2(1,S) = 1$ by convention. This leads to

Theorem 5.1

Moments of $K_n - \log n^*$, success case, M1.

$$R_{n,S} - \log n^* = \mathbb{E}(K_n - \log n^*) \sim U_1 - MS_1 - \frac{S_2}{L} + \frac{p\gamma}{L^2} - \frac{1+pM}{L}$$
$$+ \sum_{l\neq 0}\left[T_1(\chi_l) - MR_0(\chi_l) - \frac{R_1(\chi_l)}{L} - \frac{\Gamma(1+\chi_l)}{L}\right] e^{-2l\pi\mathbf{i}\log n},$$
$$\mathbb{E}(K_n - \log n^*)^2 \sim U_2 - 2MU_1 - 2\frac{U_4}{L} + M^2S1 + 2M\frac{S_2}{L} + \frac{S_3+S_4}{L^2}$$
$$+ \frac{p(\pi^2/6+\gamma^2)}{L^3} - \frac{2\gamma(pM+1)}{L^2} + \frac{pM^2+2M+1}{L}$$
$$+ \sum_{l\neq 0}\left\{T_2(\chi_l) - 2MT_1(\chi_l) - 2\frac{T_3(\chi_l)}{L} + M^2R_0(\chi_l)\right.$$
$$+ 2M\frac{R_1(\chi_l)}{L} + \frac{R_2(\chi_l)+R_3(\chi_l)}{L^2}$$
$$\left. +\Gamma(1+\chi_l)\left[2\frac{\psi(1+\chi_l)}{L^2} + \frac{1}{L} + 2\frac{M}{L}\right]\right\} e^{-2l\pi\mathbf{i}\log n},$$

where

$$S_2 := \sum_{i=2}^{\infty} \frac{p^i \psi(i)}{Li} \tilde{P}(i),$$

$$S_3 := \sum_{i=2}^{\infty} \frac{p^i \psi(i)^2}{Li} \tilde{P}(i),$$

$$S_4 := \sum_{i=2}^{\infty} \frac{p^i \psi(1, i)}{Li} \tilde{P}(i),$$

$$T_1(\chi_l) := \sum_{i=2}^{\infty} \frac{p^i}{Li!} \xi_i \Gamma(i + \chi_l),$$

$$T_2(\chi_l) := \sum_{i=2}^{\infty} \frac{p^i}{Li!} \xi_i^{(2)} \Gamma(i + \chi_l),$$

$$T_3(\chi_l) := \sum_{i=2}^{\infty} \frac{p^i}{Li!} \xi_i \psi(i + \chi_l) \Gamma(i + \chi_l),$$

$$U_1 := \sum_{i=2}^{\infty} \frac{p^i \xi_i}{Li},$$

$$U_2 := \sum_{i=2}^{\infty} \frac{p^i \xi_i^{(2)}}{Li},$$

$$U_4 := \sum_{i=2}^{\infty} \frac{p^i \xi_i \psi(i)}{i},$$

$$R_0(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!} \Gamma(i + \chi_l) \tilde{P}(i),$$

$$R_1(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!} \Gamma(i+\chi_l)\psi(i+\chi_l)\tilde{P}(i),$$

$$R_2(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!} \Gamma(i+\chi_l)\psi(1, i+\chi_l)\tilde{P}(i),$$

$$R_3(\chi_l) := \sum_{i=1}^{\infty} \frac{p^i}{Li!} \Gamma(i+\chi_l)\psi^2(i+\chi_l)\tilde{P}(i),$$

Note that the periodic component contains $\log n$ in the exponent (and not $\log n^*$). To obtain the moments of $K_n - \log n^*$, *given success*, we simply divide the moments given in the theorem by $P_5(S)$.

Distribution of T_n (null rounds), with success, M1.

We use the following *notations*

$$P_7(i,t,S) := \text{ Probability that, starting with } i \text{ players,}$$
$$\text{we succeed with } t \text{ null rounds },$$
$$\tilde{P}_7(i,k,S) := \text{ Probability that, starting with } i \text{ players,}$$
$$\text{we succeed with } t \text{ null rounds },$$
$$\text{given that the } i \text{ players were obtained in a null round },$$
$$P_{11}(t,S) := \text{ Probability that, starting with } n \text{ players, we succeed with } t$$
$$\Sigma_4(i,\tau) := \sum_{v=0}^{\tau-1}(p^i)^v v = \frac{p^{i\tau}(\tau p^i - \tau - p^i) + p^i}{(1-p^i)^2},$$
$$y_{i,S} := \text{ mean number of null rounds, starting with } i \text{ players,}$$
$$\text{with success at the end,}$$

and similar definitions for $\tilde{y}_{i,S}, y^{(2)}_{i,S}, \tilde{y}^{(2)}_{i,S}$.

Recurrences and first moments.

We have the following recurrences:

$$P_7(1,0,S)=1, \tilde{P}_7(1,0,S)=1,$$
$$P_7(1,\geq 1,S)=0, \tilde{P}_7(1,\geq 1,S)=0,$$
$$P_7(i,t,S)=\sum_{s=0}^{\tau-1}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_7(\ell,t-s,S), i\geq 2,$$
$$\tilde{P}_7(i,t,S)=\sum_{s=0}^{\tau-2}(p^i)^s\sum_{\ell=1}^{i}\binom{i}{\ell}q^\ell p^{i-\ell}P_7(\ell,t-1-s,S), i\geq 2.$$

Explanation: we can have up to $\tau-1$ or $\tau-2$ null rounds (all killed), followed by ℓ survivors. This leads to s or $s+1$ null rounds already.

$$
\begin{aligned}
y_{i,S} &= \sum_t P_7(i,t,S)t \\
&= \sum_t \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_7(\ell, t-s, S)[t-s+s], y_{1,S} = 0 \\
&= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} y_{\ell,S} \\
&+ \sum_t \sum_{s=0}^{\tau-1} (p^i)^s s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} P_7(\ell, t-s, S) \\
&= \Sigma_1(i,\tau) \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} y_{\ell,S} + \Sigma_4(i,\tau) P_2(i,S)/\Sigma_1(i,\tau),
\end{aligned}
$$

and similar expressions for $\tilde{y}_{i,S}, y^{(2)}_{i,S}, \tilde{y}^{(2)}_{i,S}$.

Again, in previous expressions, we could replace

$$\sum_{\ell=1}^{i} \binom{i}{\ell} q^{\ell} p^{i-\ell} y_{\ell,S} \text{ by } [y_{i,S} - \Sigma_4(i,\tau) P_2(i,S)/\Sigma_1(i,\tau)]/\Sigma_1(i,\tau).$$

Next, with (1),

$$\mathbb{P}(J=j, T_n=t) \sim f_7(\eta, t),$$

$$f_7(\eta,t) = \sum_{i=2}^{\infty} \exp\left(-\frac{1}{p} e^{-L\eta}\right) \frac{e^{-L\eta i}}{i!} \tilde{P}_7(i,t,S).$$

Hence

$$\phi_7(\alpha,t) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_7(\eta,t) d\eta = \sum_{i=2}^{\infty} \frac{(1/p)^{-i+\tilde{\alpha}}}{Li!} \Gamma(i-\tilde{\alpha}) \tilde{P}_7(i,t,S).$$

Note that there are no null rounds if the maximal non-empty urn contains only 1 ball.

The *dominant component* of $P_{11}(t, S)$ is given by

$$\phi_7(0,t) = \sum_{i=2}^{\infty} \frac{(1/p)^{-i}}{Li!}\Gamma(i)\tilde{P}_7(i,t,S) = \sum_{i=2}^{\infty} \frac{p^i}{Li}\tilde{P}_7(i,t,S),$$

and the *periodic component* by

$$\omega_{1,7}(t) = \sum_{l \neq 0} \varphi_7(\chi_l, t) e^{-2l\pi \mathbf{i} \log n^*},$$

with

$$\varphi_7(\chi_l, t) = \phi_7(\alpha, t)\big|_{\alpha = -L\chi_l}.$$

We obtain

$$\varphi_7(\chi_l, t) = \sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!}\tilde{P}_7(i,t,S).$$

Hence

Theorem 6.1

The asymptotic distribution of the number of null rounds, with success, M1, is given by

$$P_{11}(t,S) = \mathbb{P}(T_n = t) \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{P}_7(i,t,S) + \sum_{l\neq 0} \varphi_7(\chi_l, t) e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

The moments are given by

$$I_{n,S} = y_{n,S} \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{y}_{i,S} + \sum_{l\neq 0} \sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!} \tilde{y}_{i,S} e^{-2l\pi\mathbf{i}\{\log n^*\}},$$

$$y_{n,S}^{(2)} \sim \sum_{i=2}^{\infty} \frac{p^i}{Li} \tilde{y}_{i,S}^{(2)} + \sum_{l\neq 0} \sum_{i=2}^{\infty} \frac{p^{i+\chi_l}\Gamma(i+\chi_l)}{Li!} \tilde{y}_{i,S}^{(2)} e^{-2l\pi\mathbf{i}\{\log n^*\}}.$$

Note that $T_n = \mathcal{O}(1)$.

Other parameters.

The following parameters are asymptotically analyzed in our full report:

- Distribution of W_n (leftovers), with failure, M1
- Moments of $K_n - \log n^*$, failure case, M1
- Distribution of T_n (null rounds), with failure, M1

Distribution of C_n (coins flipped), with success, M1.

Case $l = 1$.

Note that this case *entails a success*. We will only deal here with the non-periodic part of our expressions. The maximal non-empty urn contains 1 ball and the position of the last non-empty urn *before* this maximal non-empty urn is denoted by J. Let us also denote by K the number of balls in urn J.

$$\mathbb{P}(J = j, K = k) \sim f_4(\eta, k), k \geq 1,$$

$$f_4(\eta, k) := \exp\left(-\frac{q}{p}e^{-L\eta}\right)\frac{q}{p}e^{-L\eta}\exp\left(-e^{-L\eta}\right)\frac{e^{-L\eta k}}{k!},$$

$$= \exp\left(-\frac{1}{p}e^{-L\eta}\right)\frac{q}{p}\frac{e^{-L\eta(k+1)}}{k!}.$$

Explanation:

$$\mathbb{P}(J = j, K = k) \sim \mathcal{P}\left(\frac{q}{p}e^{-L\eta}, 1\right)\mathcal{P}\left(e^{-L\eta}, k\right)$$

We have

$$\phi_4(\alpha, k) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_4(\eta, k) d\eta = \frac{q}{Lk!} \left(\frac{1}{p}\right)^{\tilde{\alpha}-k} \Gamma(1 - \tilde{\alpha} + k),$$

$$\Pi_4(k) := \phi_4(0, k) = \frac{q}{L} p^k.$$

Note that

$$Z_1 := \sum_{k=1}^{\infty} \Pi_4(k) = \frac{p}{L} \equiv \Pi_1 \text{ (one ball)}$$

which conforms to (3).

Another RV Δ.

Let us denote by Δ the *difference* between the maximal non-empty urn (containing 1 ball) and J. We have

$$\mathbb{P}(J=j, I=1, \Delta=\delta) \sim f_8(\eta,\delta),$$
$$f_8(\eta,\delta) := \exp\left(-\frac{q}{p}e^{-L\eta}\right) e^{-L(\eta+\delta)} \left(1-\exp\left(-e^{-L\eta}\right)\right),$$
$$= \exp(-L\delta)\left[\exp\left(-\frac{q}{p}e^{-L\eta}\right) e^{-L\eta} \left(1-\exp\left(-e^{-L\eta}\right)\right)\right],$$

which shows that Δ is asymptotically independent of J.
Explanation:

$$\mathbb{P}(J=j, I=1, \Delta=\delta) \sim \mathcal{P}\left(e^{-L(\eta+1)},0\right)\mathcal{P}\left(e^{-L(\eta+1)},0\right)\dots$$
$$\mathcal{P}\left(e^{-L(\eta+\Delta)},1\right)\mathcal{P}\left(e^{-L(\eta+\Delta+1)},0\right)\dots\left[1-\mathcal{P}\left(e^{-L\eta},0\right)\right]$$

We have

$$\phi_8(\alpha,\delta) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_8(\eta,\delta)d\eta$$
$$= \exp(-L\delta)\frac{p}{Lq}\left[\left(\frac{q}{p}\right)^{\tilde{\alpha}} - q\left(\frac{1}{p}\right)^{\tilde{\alpha}}\right]\Gamma(1-\tilde{\alpha}),$$
$$\Pi_5(\delta) := \phi_8(0,\delta) = e^{-L\delta}\frac{p^2}{Lq} = q^{\delta}\frac{p^2}{Lq}.$$

Note that

$$\sum_{\delta=1}^{\infty}\Pi_5(\delta) = \frac{p}{L} \equiv \Pi_1$$

which again conforms to (3). We have

$$\mathbb{E}(\Delta) = \frac{1}{L},$$
$$\mathbb{E}(\Delta^2) = \frac{1+q}{Lp}$$

Another RV G.

However, note carefully that the player corresponding to $I = 1$ is actually related to a *flipped coin in urn J*. So we must use a new RV $G = K + 1, G \geq 2$, with distribution

$$\Pi_6(g) = \frac{q}{L} p^{g-1}, g \geq 2$$

and

$$f_6(\eta, g) = \exp\left(-\frac{1}{p} e^{-L\eta}\right) \frac{q}{p} \frac{e^{-L\eta g}}{(g-1)!}.$$

We will also need

$$Z_5 = \mathbb{E}(G) := \sum_{g=2}^{\infty} \Pi_6(g) g = \frac{p(1+q)}{Lq}.$$

Variants.

Later on, we will use the following variants:

$$e^{-L\eta} f_6(\eta, g),$$
$$e^{-2L\eta} f_6(\eta, g),$$
$$\eta f_6(\eta, g),$$
$$e^{-L\eta} \eta f_6(\eta, g),$$
$$e^{-2L\eta} \eta f_6(\eta, g).$$

This leads respectively to $\phi_{.}(0,g)$:

$$\frac{p^g qg}{L},$$
$$\frac{qp^{g+1}g(g+1)}{L},$$
$$-\frac{p^{g-1}q[(g-1)\ln(p)+(g-1)\psi(g-1)+1]}{L^2(g-1)},$$
$$-\left[qp^g[2(g-1)+(g-1)^2\ln(p)+(g-1)^2\psi(g-1)+(g-1)\ln(p)\right.$$
$$\left.+(g-1)\psi((g-1))+1]\right]/[L^2(g-1)],$$

$\Omega_{15}(g)$ too long to display here.

This leads to $Z_7, Z_8, Z_{11}, Z_{12}, Z_{10}, Z_{13}, Z_{15},$: we *simply sum on* $g \geq 2$. Indeed, the case $l = 1$ immediately leads to a success.

Let us denote by $S_\Gamma(j,i)$ the sum of $(n-i)$ iid RV $\Gamma(j)$, and $\Gamma(j)$ is a truncated geometric RV $< j$. As $\Sigma_0 := \sum_1^{j-1} pq^{l-1} = 1 - q^{j-1}$, we have (we give only the terms needed in the sequel)

$$E(j) := \mathbb{E}(\Gamma(j)) = \sum_1^{j-1} pq^{l-1} l/\Sigma_0 \sim \frac{1}{p} + q^{j-1} - jq^{j-1} - jq^{2(j-1)} + \mathcal{O}\left(q^{2(j-1)}\right),$$

$$E^{(2)}(j) := \mathbb{E}(\Gamma(j)^2) = \sum_1^{j-1} pq^{l-1} l^2/\Sigma_0 \sim \frac{1+q}{p^2} + q^{j-1}\frac{1+q}{p} - j\frac{2q}{p}q^{j-1} - j^2 q^{j-1} + \mathcal{O}\left(j^2 q^{2(j-1)}\right),$$

Note that, with $j = \eta + \log n^*$,

$$q^j = e^{-L\eta}\frac{1}{n^*}$$

Moments.

This leads, by carefully taking into account the *correlation* between J and I (we expand the mean up to the $\log n^*/n^*$ term and the square mean up to the $\log n^*$ term) to

$$C_{n,1} \sim S_\Gamma(J,G) + JG, \tag{4}$$

$$\begin{aligned}
\mathbb{E}(C_{n,1}) &\sim \mathbb{E}(S_\Gamma(J,G)+JG), \\
&\sim \mathbb{E}\Bigg[(n-G)\frac{1}{p} + \frac{n}{q}\frac{e^{-L\eta}}{n^*} - \frac{n-G}{q}\frac{e^{-L\eta}}{n^*}(\log n^* + \eta) \\
&\quad -\frac{n}{q^2}\frac{e^{-2L\eta}}{n^{*2}}\log n^* + (\log n^* + \eta)G\Bigg] \\
&\sim \frac{n}{p}Z_1 - \frac{1}{p}Z_5 + \frac{n}{q}\frac{Z_7}{n^*} - \frac{n}{q}\frac{Z_7}{n^*}\log n^* + \frac{1}{q}\frac{Z_8}{n^*}\log n^* \\
&\quad - \frac{n}{q}\frac{Z_{10}}{n^*} - \frac{n}{q^2}\frac{Z_{11}}{n^{*2}}\log n^* + Z_5\log n^* + Z_{13}
\end{aligned} \tag{5}$$

$$\mathbb{E}(C_{n,1}^2) \sim \mathbb{E}((S_\Gamma(J,G) + JG)^2)$$
$$\sim \mathbb{E}[n\mathbb{E}^{(2)}(J) + (n-G)(n-G-1)(\mathbb{E}(J))^2$$
$$+ 2\mathbb{E}[(n-G)E(J)JG] + \mathbb{E}[(\log n^* + \eta)^2 G^2]. \tag{6}$$

Case $l > 1$.

We use the following *notations*

$z_{i,S} :=$ mean number of coins flipped, starting with i players, with success at the end,

and similar definitions for $\tilde{z}_{i,S}, z^{(2)}_{i,S}, \tilde{z}^{(2)}_{i,S}$. First of all, we must compute the moments of $C_{i,S}$ and $\tilde{C}_{i,S}$. This gives

$$z_{i,S} = \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} [si + i + z_{\ell,S}], z_{1,S} = 0,$$

$$z^{(2)}_{i,S} = \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} \mathbb{E}[((s+1)i + C_{\ell,S})^2]$$

$$= \sum_{s=0}^{\tau-1} (p^i)^s \sum_{\ell=1}^{i} \binom{i}{\ell} q^\ell p^{i-\ell} [((s+1)i)^2 + 2(s+1)iz_{\ell,S} + z^{(2)}_{\ell,S}],$$

and similar expressions for $\tilde{z}_{i,S}, \tilde{z}^{(2)}_{i,S}$.

Next, with (1),

$$\mathbb{P}(J=j, I=i) \sim f_5(\eta, i),$$

$$f_5(\eta, i) := \exp\left(-\frac{1}{p}e^{-L\eta}\right)\frac{e^{-L\eta i}}{i!},$$

$$\phi_5(\alpha, i) = \int_{-\infty}^{\infty} e^{\alpha\eta} f_5(\eta, i) d\eta = \frac{(1/p)^{-i+\tilde{\alpha}}}{Li!}\Gamma(i-\tilde{\alpha}),$$

$$\Pi_2(i) := \phi_5(0, i) = \frac{p^i}{Li},$$

$$P_0 := \sum_2^{\infty} \Pi_2(i) = 1 - p/L$$

Variants.

Later on, we will use the following variants:

$$e^{-L\eta} f_5(\eta, i),$$
$$e^{-2L\eta} f_5(\eta, i),$$
$$\eta f_5(\eta, i),$$
$$e^{-L\eta} \eta f_5(\eta, i),$$
$$e^{-2L\eta} \eta f_5(\eta, i)$$

This leads respectively to $\phi_{.}(0,i)$:

$$\frac{p^i p}{L},$$
$$\frac{p^i p^2(i+1)}{L},$$
$$-\frac{p^i[\ln(p)+\psi(i)]}{L^2 i},$$
$$-\frac{p^i p[i\ln(p)+i\psi(i)+1]}{L^2 i},$$
$$-\frac{p^i p^2[i^2\ln(p)+i^2\psi(i)+2i+i\ln(p)+i\psi(i)+1]}{L^2 i}$$

Multiplying by $\tilde{P}_2(i,S)$ and summing on $i \geq 2$, this leads to $S_7, S_5, S_8, S_{11}, S_{12}, S_{10}, S_{13}, S_{15}$. Indeed, the case $I > 1$ does *not* immediately leads to a success. Again we expand the mean up to the $\log n^*/n^*$ term and the square mean up to the $\log n^*$ term.

Moments.

We have

$$
\begin{aligned}
C_{n,2} &\sim S_\Gamma S(J,I) + JI + \tilde{C}_{I,S}, \qquad (7)\\
\mathbb{E}(C_{n,2}) &\sim \mathbb{E}(S_\Gamma(J,I) + JI + \tilde{z}_{I,S}),\\
&\sim \mathbb{E}(S_\Gamma(J,I) + JI) + U_1,\\
&\sim \mathbb{E}\left[(n-I)\frac{1}{p} + \frac{n}{q}\frac{e^{-L\eta}}{n^*} - \frac{n-I}{q}\frac{e^{-L\eta}}{n^*}(\log n^* + \eta)\right.\\
&\left. -\frac{n}{q^2}\frac{e^{-2L\eta}}{n^{*2}}\log n^* + (\log n^* + \eta)I\right] + U_1\\
&\sim \frac{n}{p}S_1 - \frac{1}{p}S_5 + \frac{n}{q}\frac{S_7}{n^*} - \frac{n}{q}\frac{S_7}{n^*}\log n^* + \frac{1}{q}\frac{S_8}{n^*}\log n^*\\
&- \frac{n}{q}\frac{S_{10}}{n^*} - \frac{n}{q^2}\frac{S_{11}}{n^{*2}}\log n^* + S_5\log n^* + S_{13} + U_1 \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(C_{n,2}^2) &\sim \mathbb{E}((S_\Gamma(J,I)+JI)^2) + 2\mathbb{E}[(S_\Gamma(J,I)+JI)\tilde{C}_{I,S}] \\
&\sim \mathbb{E}((S_\Gamma(J,I)+JI)^2) + 2\mathbb{E}\left[\left[\frac{n}{p} - \frac{n}{q}\frac{e^{-L\eta}}{n^*}\log n^* + I\log n^*\right]\tilde{z}_{I,S}\right] \\
&\sim \mathbb{E}((S_\Gamma(J,I)+JI)^2) + 2\left[\frac{n}{p}U_1 - \frac{n}{q}\frac{U_3}{n^*}\log n^* + U_5\log n^*\right] \\
&\sim \mathbb{E}[n\mathbb{E}^{(2)}(J) + (n-I)(n-I-1)(\mathbb{E}(J))^2 + 2\mathbb{E}[(n-I)E(J)JI] \\
&+ \mathbb{E}[(\log n^* + \eta)^2 I^2] + 2\left[\frac{n}{p}U_1 - \frac{n}{q}\frac{U_3}{n^*}\log n^* + U_5\log n^*\right]
\end{aligned}
\tag{9}
$$

General case.

The *total mean* is given by (we provide here only two terms)

$$\mathbb{E}(C_n) \sim \mathbb{E}(C_{n,1}) + \mathbb{E}(C_{n,2}) \sim n\left(\frac{p}{L} + S_1\right)\frac{1}{p} + \left(-\frac{Z_7}{p} + Z_5 - \frac{S_7}{p} + S_5\right)\log n^*$$

But the first term amounts to the mean of a sum of n GEOM pq^{l-1} RVs. Indeed, the GEOM pq^{l-1} RV has mean $\frac{1}{p}$, second moment $\frac{1+q}{p^2}$ and variance $\frac{q}{p^2}$. This is easy to explain: from (4) and (7), the correction $\tilde{C}_{l,S}$ is asymptotically $\mathcal{O}(1)$ and the correction $-\Delta$ is also asymptotically $\mathcal{O}(1)$. Similarly

$$\mathbb{E}(C_n^2) \sim \mathbb{E}(C_{n,1}^2) + \mathbb{E}(C_{n,2}^2) \sim n^2\left(\frac{p}{L} + S_1\right)\frac{1}{p^2} + n\left(-\frac{2(Z_7 - pZ_5)}{p^2} - \frac{2(S_7 - pS_5)}{p^2}\right)\log n^*$$

and the *variance* is finally given by (we must adequatey condition on the dominant success probability $Pd(S) := \frac{p}{L} + S_1$)

$$\mathbb{V}(C_n) \sim Pd(S)\left[\frac{\mathbb{E}(C_n^2)}{Pd(S)} - \left(\frac{\mathbb{E}(C_n)}{Pd(S)}\right)^2\right]$$
$$\sim Pd(S)n\frac{q}{p^2}$$

So we obtain

Theorem 8.1

The moments of C_n in case of success, M1, are given by (Maple knows more terms, of course, in particular the $\log^2 n^*$ and $\log n^*$ terms of the variance)

$$\mathbb{E}(C_n) \sim n\left(\frac{p}{L} + S_1\right)\frac{1}{p} + \left(-\frac{Z_7}{p} + Z_5 - \frac{S_7}{p} + S_5\right)\log n^*,$$

$$\mathbb{V}(C_n) \sim Pd(S)n\frac{q}{p^2}$$

Note again that the dominant term of the variance corresponds to a *sum of n iid Geom pq^{l-1} RV*. Intuitively, the asymptotic distribution should be Gaussian: indeed from (4) and (7), the correction $\tilde{C}_{l,S}$ is asymptotically $\mathcal{O}(1)$, but *not independent* of the dominant term and the correction $-\Delta$ is also asymptotically $\mathcal{O}(1)$, but *independent* of the dominant term. Actually we have

Theorem 8.2

conditioned on a success,

$$\mathbb{P}\left[\frac{C_n - \mathbb{E}(C_n)}{\sqrt{\mathbb{V}(C_n)}} \leq x\right] \underset{n\to\infty}{\longrightarrow} \phi(x),$$

where $\phi(x)$ denotes the Gaussian distribution function.

Proof.

$$C_{n,S} = [[\![I=1]\!]][S_{\Gamma,1}(J,G) + JG] + [[\![I>1]\!]][S_{\Gamma,2}(J,I) + JI + \tilde{C}_{I,S}].$$

Here, $S_{\Gamma,1}, \eta_1$ are related to the case $I=1$ and $S_{\Gamma,2}, \eta_2$ are related to the case $I>1$. In the sequel, with some abuse of notation, $\mathcal{O}_V(1)$ will denote a RV, asymptotically independent of n, with finite moments.

$$C_{n,S} = \sum_j \Big(\mathbb{P}[J=j, I=1]\,[S_{\Gamma,1}(j,G) + jG] \\ + \sum_{i\geq 2} \mathbb{P}[J=j, I=i]\left[[S_{\Gamma,2}(j,i) + ji]\tilde{P}_2(i,S) + \tilde{C}_{i,S}\right]\Big).$$

We have, *conditioned on a success*, (we use the dominant success probability $Pd(S)$)

$$
\begin{aligned}
\frac{C_{n,S}}{Pd(S)} \sim \frac{\Pi_1}{Pd(S)} \sum_j \frac{\mathbb{P}[J=j, I=1]}{\Pi_1}[S_{\Gamma,1}(j,G)+jG] \\
+ \frac{S_1}{Pd(S)} \sum_j \sum_{i\geq 2} \frac{\mathbb{P}[J=j, I=i]\tilde{P}_2(i,S)}{S_1}[S_{\Gamma,2}(j,i)+ji] \\
+ \frac{S_1}{Pd(S)} \sum_j \sum_{i\geq 2} \frac{\mathbb{P}[J=j, I=i]}{S_1}\tilde{C}_{i,S} \\
= \frac{\Pi_1}{Pd(S)} \sum_j \frac{\mathbb{P}[J=j, I=1]}{\Pi_1}[S_{\Gamma,1}(j,G)+jG] \\
+ \frac{S_1}{Pd(S)} \sum_j \sum_{i\geq 2} \frac{\mathbb{P}[J=j, I=i]\tilde{P}_2(i,S)}{S_1}[S_{\Gamma,2}(j,i)+ji] + \mathcal{O}_V(1).
\end{aligned}
$$

Now, with $\Gamma_k(j)$ denoting a sequence of iid a truncated geometric RV $< j$,

$$\begin{aligned} S_{\Gamma,1}(j,G) + jG &= S_{\Gamma,1}(j,0) - \sum_{1}^{G} \Gamma_k(j) + jG \\ &= S_{\Gamma,1}(j,0) + \mathcal{O}_V(1) + (\log n^* + \eta_1)G \\ &= S_{\Gamma,1}(j,0) + \mathcal{O}_V(1) + \log n^* \mathcal{O}_V(1), \end{aligned}$$

and similarly for $S_{\Gamma,2}(j,i) + ji$. So

$$\begin{aligned} \frac{C_{n,S}}{Pd(S)} &\sim \frac{\Pi_1}{Pd(S)} \sum_j \frac{\mathbb{P}[J=j, I=1]}{\Pi_1} S_{\Gamma,1}(j,0) \\ &+ \frac{S_1}{Pd(S)} \sum_j \sum_{i\geq 2} \frac{\mathbb{P}[J=j, I=i]\tilde{P}_2(i,S)}{S_1} S_{\Gamma,2}(j,0) \\ &+ \mathcal{O}_V(1) + \log n^* \mathcal{O}_V(1). \end{aligned}$$

Now we must show that $S_\Gamma(j,0)$ is asymptotically Gaussian. We could simply use Feller [2], example $IX, 1, a$, but we want an error estimation. We will provide the first terms of our expansions, but Maple knows more. The standard deviation of $\Gamma(j)$ will be denoted by $\sigma(j)$. We have

$$\Sigma_0 = 1 - \frac{e^{-L\eta}}{np},$$

$$E(j) \sim \frac{1}{p} - \frac{e^{-L\eta}(j-1)}{np},$$

$$\sigma(j) \sim \frac{\sqrt{q}}{p} - \frac{e^{-L\eta}(j-1)^2}{2n\sqrt{q}}.$$

Now the Probability generating function (PGF) of $\Gamma(j)$ is given by

$$F(z) = \frac{\sum_1^{j-1} pq^{l-1} z^l}{\Sigma_0} = \frac{\frac{pz}{1-qz} - \frac{e^{-L\eta} z^j}{n(1-qz)}}{\Sigma_0},$$

and the PGF of $S_\Gamma(j,0)$ is given by $[F(z)]^n$. We will now use the Saddle point method (for a good introduction to this method, see Flajolet and Sedgewick [3], ch. $VIII$).

By Cauchy's theorem,

$$\mathbb{P}(S_\Gamma(j,0)=k) = \frac{1}{2\pi\mathbf{i}}\int_\Omega \frac{[F(z)]^n}{z^{k+1}}dz = \frac{1}{2\pi\mathbf{i}}\int_\Omega e^{H(z)}dz,$$

where Ω is inside the analyticity domain of the integrand and encircles the origin and

$$H(z) = n\left[\ln\left[\frac{pz}{1-qz} - \frac{e^{-L\eta}z^j}{n(1-qz)}\right] - \ln[\Sigma_0]\right] - (k+1)\ln(z).$$

Set

$$H^{(i)} := \frac{d^iH}{dz^i}.$$

First we must find the solution of

$$H^{(1)}(\tilde{z}) = 0 \tag{10}$$

with smallest module.

Set $\tilde{z} := z^* - \varepsilon$, where $z^* = \lim_{n\to\infty} \tilde{z}$. Here, it is easy to check that $z^* = 1$. Set $k = nE(j) + \sqrt{n}\sigma(j)x$, x fixed. We will soon see that $\varepsilon = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$, so we can expand z^j in $F(z)$ as

$$z^j = 1 - j\varepsilon + \frac{j(j-1)}{2}\varepsilon^2 + \dots$$

Also $j = \log n^* + \eta$. This leads, to first order (keeping only the ε term in (10)), to

$$\varepsilon := \frac{-px}{\sqrt{nq}} + \mathcal{O}\left(\frac{\log n^*}{n}\right).$$

This shows that, asymptotically, ε is given by a series of powers of $n^{-1/2}$, where each coefficient is given by a series of powers of $\log n^*$. To obtain more precision, we set again $k = nE(j) + \sqrt{n}\sigma(j)x$, expand in powers of $n^{-1/2}$, and equate each coefficient to 0.

We have, with $\tilde{z} := z^* - \varepsilon = 1 - \varepsilon$,

$$\mathbb{P}(S_\Gamma(j,0)=k) = \frac{1}{2\pi\mathbf{i}}\int_\Omega \exp\Big[H(\tilde{z}) + H^{(2)}(\tilde{z})(z-\tilde{z})^2/2! + \sum_{l=3}^{\infty} H^{(l)}(\tilde{z})(z-\tilde{z})^l/l!\Big]dz.$$

Note that the linear term vanishes. Set $z = \tilde{z} + \mathbf{i}\tau$. This gives

$$\mathbb{P}(S_\Gamma(j,0)=k) = \frac{1}{2\pi}\exp[H(\tilde{z})]\int_{-\infty}^{\infty} \exp\Big[H^{(2)}(\tilde{z})(\mathbf{i}\tau)^2/2! + \sum_{l=3}^{\infty} H^{(l)}(\tilde{z})(\mathbf{i}\tau)^l/l!\Big]d\tau. \tag{11}$$

Let us first analyze $H(\tilde{z})$. We obtain

$$H(\tilde{z}) = -x^2/2 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$$

Also,

$$H^{(2)}(\tilde{z}) = n\frac{q}{p^2} + \mathcal{O}(\sqrt{n}),$$
$$H^{(4)}(\tilde{z}) = \mathcal{O}(n)$$

We can now compute (11), for instance by using the classical trick of setting

$$S^{(2)}(\tilde{z})(i\tau)^2/2! + \sum_{l=3}^{\infty} S^{(l)}(\tilde{z})(i\tau)^l/l! = -u^2/2.$$

Computing τ as a truncated series in u, this gives, by inversion,

$$\tau = \frac{u}{\sqrt{nq/p^2}} + u^2\mathcal{O}\left(\frac{1}{n}\right)$$

Setting $d\tau = \frac{d\tau}{du}du$, and integrating on $[u = -\infty..\infty]$, this gives

$$\frac{1}{\sqrt{2\pi nq/p^2}}\left[1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right].$$

Finally (11) leads to

$$\mathbb{P}(S_\Gamma(j,0) = k) \sim \frac{1}{\sqrt{2\pi nq/p^2}} e^{-x^2/2}\left[1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right],$$

Now we consider

$$\mathbb{P}\left(\frac{\frac{C_{n,S}}{Pd(S)}-\frac{\mathbb{E}(C_{nS})}{Pd(S)}}{\sqrt{nq/p^2}}\leq x\right)$$

$$\sim\mathbb{P}\left(\left[\frac{\Pi_1}{Pd(S)}\sum_j\frac{\mathbb{P}[J=j,I=1]}{\Pi_1}[S_{\Gamma,1}(j,0)-nE_1(j)]\right.\right.$$

$$\left.+\frac{S_1}{Pd(S)}\sum_{j,i\geq 2}\frac{\mathbb{P}[J=j,I=i]\tilde{P}_2(i,S)}{S_1}[S_{\Gamma,2}(j,0)-nE_2(j)]\right]\Big/\sqrt{nq/p^2}$$

$$\left.+\frac{\mathcal{O}_V(1)+\log n^*\mathcal{O}_V(1)}{\sqrt{nq/p^2}}\leq x\right)$$

$$\sim \mathbb{P}\left(\frac{\frac{\Pi_1}{Pd(S)}\sum_j \frac{\mathbb{P}[J=j,I=1]}{\Pi_1}[S_{\Gamma,1}(j,0)-nE_1(j)]}{\sqrt{n}\sigma_1(j)}\right.$$

$$\left.+\frac{\frac{S_1}{Pd(S)}\sum_j\sum_{i\geq 2}\frac{\mathbb{P}[J=j,I=i]\tilde{P}_2(i,S)}{S_1}[S_{\Gamma,2}(j,0)-nE_2(j)]}{\sqrt{n}\sigma_2(j)}\leq x\right)$$

as

$$\frac{\sigma(j)}{\sqrt{q/p^2}} \underset{n\to\infty}{\longrightarrow} 1.$$

Now

$$\frac{\mathbb{V}(C_n)}{Pd(S)nq/p^2} \underset{n\to\infty}{\longrightarrow} 1,$$

which concludes the proof ■

See also Kalpathy et al. [5], for an election leader which stops if $I > 1$. C_n is shown to be asymptotically Gaussian.

Other cases.

The following other cases are asymptotically analyzed in our full report:

- Distribution of C_n (number coins flipped), with failure, M1.
- Moments of $K_n - \log n^*$, failure case, M1
- Model 2

L. Bondesson, T. Nilsson, and G. Wikstrand.
Probability calculus for silent elimination : A method for medium access control.
Technical report, Technical Report 3-2007, Umea University, Mathematics and Mathematical Statistics. Available at http://www8.cs.umu.se/ nilsson/, 2007.

W. Feller.
Introduction to Probability Theory and its Applications. Vol II.
Wiley, 1971.

P. Flajolet and R. Sedgewick.
Analytic combinatorics.
Cambridge University press, 2009.

P. Hitczenko and G. Louchard.
Distinctness of compositions of an integer: a probabilistic analysis.
Random Structures and Algorithms, 19(3,4):407–437, 2001.

R. Kalpathy, H.M. Mahmoud, and M.D. Ward.
Asymptotic properties of a leader election algorithm.
Technical report, 2010.

M. Loève.
Probability Theory.
D. Van Nostrand, 1963.

G. Louchard and H. Prodinger.
Asymptotics of the moments of extreme-value related distribution functions.
Algorithmica, 46:431–467, 2006.
Long version:
http://www.ulb.ac.be/di/mcs/louchard/moml.ps.

G. Louchard and H. Prodinger.
On gaps and unoccupied urns in sequences of geometrically distributed random variables.
Discrete Mathematics, 308,9:1538–1562, 2008.

Long version:
http://www.ulb.ac.be/di/mcs/louchard/gaps18.ps.

G. Louchard and H. Prodinger.
The asymmetric leader election algorithm.
Annals of Combinatorics, 12:449–478, 2009.

G. Louchard, H. Prodinger, and M.D. Ward.
The number of distinct values of some multiplicity in sequences of geometrically distributed random variables.
Discrete Mathematics and Theoretical Computer Science, AD:231–256, 2005.
2005 International Conference on Analysis of Algorithms.