AN ASYMPTOTIC THEORY FOR $K$-DOMINANT SKYLINES OF RANDOM SAMPLES

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According to Merriam-Webster

An outline (as of buildings or a mountain range) against the background of the sky
\( \mathbf{x} = (x_1, \ldots, x_d) \) dominates \( \mathbf{y} = (y_1, \ldots, y_d) \)

if \( x_i > y_i \) for \( i = 1, \ldots, d \).
Nondominated points in a point set are called *maximal points* or *maxima* of the point set.
\[ x = (x_1, \ldots, x_d) \textit{ dominates } y = (y_1, \ldots, y_d) \]

if \[ x_i < y_i \text{ for } i = 1, \ldots, d. \]

Nondominated points in a point set are called \textit{minimal points} or \textit{minima} of the point set.
SYNONYMS OF MAXIMA

- Pareto optimality
- elite
- efficiency
- skyline
- admissibility
- sink, source
Diverse terms show its popularity

- **Pareto optimality** (Pareto front, Pareto efficiency, Pareto solutions, ...) in multiobjective optimization or multicriteria decision
- **Efficiency** in econometrics
- **Sink or source** in applied probability
- **Admissibility** in decision theory
- **Elite** in evolutionary computation
- **Skyline** in databases = minima
APPLICATIONS

USEFULNESS

- Political science
- Social science
- Engineering
- Econometrics
- Operations Research
- Biology
- Education
- Computer algorithms
TRADEOFFS EVERYWHERE

Daily life is full of tradeoffs

To eat or not to eat, to buy or to sell (stock shares)
Eat this or eat that, buy this or buy that, ...
Efficiency is equivalent to Pareto optimality

Important issue: description and characterization of Pareto optimal solutions.

Vilfredo Pareto (1848-1923)
Evolutionary algorithm (from Wiki)

A subset of evolutionary computation, a generic population-based metaheuristic optimization algorithm. An EA uses some mechanisms inspired by biological evolution: reproduction, mutation, recombination, and selection.
4.1.3 The problem of the maxima of a point set

This section is devoted to the discussion of a problem that bears an intriguing resemblance to that of the convex hull and yet on a deeper level, to be elucidated later, is fundamentally different from it. This problem occurs in a large number of applications, in statistics, economics, operations research, etc. Indeed, it was originally described as the “floating-currency problem”\(^5\):

In Erehwon, every citizen has a portfolio of foreign currencies; these foreign currencies wildly fluctuate in their values, so that every evening a person whose portfolio has the largest cumulative value is declared the King of Erehwon. Which is the smallest subset of the population that is certain to contain all potential kings?
Finding the maxima

- a prototype problem with many algorithmic and practical applications;
- a component problem in several applications;
- related to many computational geometric problems.

A fundamental problem.
Examples

- the minimum independent dominating set in permutation graphs
- the largest empty rectangle
- restricted empty rectangles
- the enclosure problem for planar (rectilinear) \( d \)-gon (which in turn has several applications in CAD systems for VLSI circuits).
- Other examples include: convex hull, polygon decomposition, shortest path problem, finding empty simplices, geometric containment problem, official statistics, grid placement problem, ...
SKYLINE IN DATABASES

Introduced by Börzsönyi, Kossmann and Stocker (2001)

Travelers looking for **cheaper hotels closer** to the beach.

**Figure 1:** Skyline of Hotels

**Figure 2:** Skyline of Manhattan
The skyline operator
We propose to extend database systems by a Skyline operation. This operation filters out a set of interesting points from a potentially large set of data points. A point is interesting if it is not dominated by any other point. For example, a hotel might be interesting for somebody ...
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[PDF] The skyline operator
S Börzsönyi, D Kossmann… - Proceedings of the 17th …, 2001 - Citeseer
We propose to extend database systems by a Skyline operation. This operation filters out a set of interesting points from a potentially large set of data points. A point is interesting if it is not dominated by any other point. For example, a hotel might be interesting for somebody traveling to ...
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MININAL DIRECTED SPANNING TREES ON RANDOM POINTS


Motivation: information transmission through wireless networks, hydrology models

Figure 1: A minimal directed spanning tree. The subgraph formed by the bold edges is $L_n$. 
MULTIFACETED ASPECTS OF MAXIMA

Applications

Theory

 Algorithms
MULTIFACETED ASPECTS OF MAXIMA

Applications

Theory

Algorithms
$n$ points randomly sampled from $\mathcal{D}$

How many maxima?

Commonly used models: hypercube and $d$-simplex

$n$ points uniformly and independently from $\mathcal{D} = [0, 1]^d$

or from the $d$-dimensional simplex

$$\{x_1 + \cdots + x_d \leq 1, x_1, \ldots, x_d \in [0, 1]\}$$
KNOWN RESULTS FOR MAXIMA IN HYPERCUBES

Let \( \mu_{n,d} := \mathbb{E}(M_{n,d}) \).

Barndorff-Nielsen & Sobel (1966)

For \( d \geq 2 \),

\[
\mu_{n,d} = \sum_{1 \leq k \leq n} \binom{n}{k} (-1)^{k-1} k^{1-d} \\
= \frac{(\log n)^{d-1}}{(d-1)!} (1 + O((\log n)^{-1})).
\]

Rederived or simplified in

SUMMATION FORMULAE FOR $\mu_{n,d}$

- $H_n^{(a)} := \sum_{1 \leq j \leq n} 1/j^a$

- $\mu_{n,d} = \sum_{1 \leq k \leq n} \binom{n}{k} (-1)^{k-1} k^{1-d}$

- $\mu_{n,d} = \sum_{1 \leq i_1 \leq \cdots \leq i_{d-1}} \frac{1}{i_1 \cdots i_{d-1}}$

- $\mu_{n,d} = \sum_{i_1 + 2i_2 + \cdots + (d-1)i_{d-1} = d-1, i_1, \ldots, i_{d-1} \geq 0} \frac{H_n^{i_1}(H_n^{(2)})^{i_2} \cdots (H_n^{(d-1)})^{i_{d-1}}}{i_1! \cdots i_{d-1}! 1^{i_1} \cdots (d-1)^{i_{d-1}}}$
INTEGRAL REPRESENTATIONS FOR $\mu_{n,d}$

$$\mu_{n,d} = n \int_{(0,1)^d} (1 - x_1 x_2 \cdots x_d)^{n-1} \, dx,$$

$$\mu_{n,d} = \frac{n}{(d-1)!} \int_0^1 (1 - x)^{n-1} (-\log x)^{d-1} \, dx,$$

$$\mu_{n,d} = \frac{1}{2\pi i} \oint_{|z|=r<1} z^{-d} \prod_{1 \leq j \leq n} \frac{1}{1 - z/j} \, dz,$$

$$\mu_{n,d} = \frac{(-1)^n}{2\pi i} \int_{\frac{1}{2} + i\infty}^{\frac{1}{2} - i\infty} \frac{n!}{s^d(s-1)\cdots(s-n)} \, ds.$$
For $n \geq 1$ and $d \geq 2$

\[
\mu_{n,d} = \mu_{n-1,d} + \frac{\mu_{n,d-1}}{n},
\]

\[
\mu_{n,d} = \sum_{1 \leq j \leq n} \frac{\mu_{j,d-1}}{j},
\]

\[
\mu_{n,d} = \frac{1}{d-1} \sum_{1 \leq j \leq d-1} H^{(d-j)}_{n} \mu_{n,j},
\]
\[ \mu_{n,d} = n \mathbb{E} \left[ (1 - U_1 U_2 \cdots U_d)^{n-1} \right], \]
\[ \mu_{n,d} = n \mathbb{P}(G_2 + \cdots + G_n < d), \]

where \( U_1, U_2, \ldots, U_d \) are uniform \([0, 1]\) RVs and

\[ \mathbb{E}[z^{G_j}] = \frac{1 - 1/j}{1 - z/j} \quad (2 \leq j \leq n); \]

Flajolet et al. (1995): \( \frac{\mu_{n,d}}{n} \) is the probability that the first subtree in a random quadtree of \( n \) nodes is empty.
VARIANCE OF $M_{n,d}$

For $d \geq 2$,

$$\frac{\nabla(M_{n,d})}{(\log n)^{d-1}} = \frac{1}{(d-1)!} + \kappa_d + O\left((\log n)^{-1}\right).$$

where

$$\kappa_d = \frac{1}{(d-1)!} \sum_{\ell \geq 1} \frac{1}{\ell^2} \sum_{1 \leq p, q \leq \ell} \binom{\ell}{p} \binom{\ell}{q} (-1)^{p+q} pq$$

$$\times \left( (p^{-1} + q^{-1})^{d-1} - p^{1-d} - q^{1-d} \right).$$

Bai, Devroye, H., Tsai (2005): an asymptotic expansion.
\[ \kappa_d \text{ FOR } 2 \leq d \leq 9 \]

\[ \zeta(s) \text{: Riemann's zeta function} \]

\[
\begin{align*}
\kappa_2 &= 0, \\
\kappa_3 &= \zeta(2), \\
\kappa_4 &= 2\zeta(3), \\
\kappa_5 &= \frac{33}{16}\zeta(4), \\
\kappa_6 &= \frac{5}{4}\zeta(5) + \frac{1}{6}\zeta(2)\zeta(3), \\
\kappa_7 &= \frac{1451}{1728}\zeta(6) + \frac{7}{72}\zeta(3)^2, \\
\kappa_8 &= \frac{1729}{5760}\zeta(7) + \frac{13}{360}\zeta(2)\zeta(5) + \frac{181}{1440}\zeta(3)\zeta(4), \\
\kappa_9 &= \frac{1891}{89600}\zeta(2)^4 + \frac{\zeta(2)\zeta(3)^2}{320} + \frac{11}{160}\zeta(3)\zeta(5) \\
&\quad - \frac{17}{1920} \sum_{k \geq 1} k^{-2} \sum_{j > k} j^{-6}.
\end{align*}
\]

Bai et al. (1998, 2005): \( \{\kappa_d\}_{2 \leq d \leq 8} \)

Costermans’s PhD Thesis (2008): \( \{\kappa_d\}_{9 \leq d \leq 13} \)
For others, see Costermans (2008), Costermans and Minh (2008).
CENTRAL LIMIT THEOREM FOR $M_{n,d}$


A CLT for $M_{n,d}$.

**Bai, Devroye, H., Tsai (2005)**

Prove a CLT for $M_{n,d}$ by Poisson processes and Stein’s method with a rate

$$O \left( \left( \log n \right)^{-\left( d-1 \right)/4} \left( \log \log n \right)^d \right).$$

**Open**

Prove the Berry-Esseen bound $O(\left( \log n \right)^{-\left( d-1 \right)/2})$.

Same theory can be developed for multivariate records.
THE CURSE OF DIMENSIONALITY

Phase transition at \( d = \log n \)

\[
\mu_{n,d} \sim \begin{cases} 
\Gamma \left( 1 - \frac{d - 1}{\log n} \right) \frac{(\log n)^{d-1}}{(d-1)!}, & \text{if } \frac{d - \log n}{\sqrt{\log n}} \to -\infty, \\
\Phi \left( \frac{d - \log n}{\sqrt{\log n}} \right) n, & \text{if } \frac{d - \log n}{\sqrt{\log n}} = O(1), \\
n, & \text{if } \frac{d - \log n}{\sqrt{\log n}} \to \infty,
\end{cases}
\]

where \( \Gamma = \) the Gamma function and \( \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-t^2/2} \, dt \) the standard normal distribution function.
TOO MANY MAXIMA TO USE

Approximate values of $\mu_{n,d}$

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<th>$d$</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>164.7</td>
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<td>$10^5$</td>
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<td>304.9</td>
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<td>5239.4</td>
</tr>
</tbody>
</table>

Other alternatives are needed
Skylines = Minima

\[ x = (x_1, \ldots, x_d) \text{ dominates } y = (y_1, \ldots, y_d) \]

if \( x_i < y_i \) for \( i = 1, \ldots, d \).

Idea: change full dominance to partial

The point \( x \) \textbf{\( k \)-dominates} \( y \) if there are \( k \) \((1 \leq k \leq d)\) dimensions in which \( x_j \) is not greater than \( y_j \) and is less than in at least one of these \( k \) dimensions.

The points that are not \( k \)-dominated by any other points define the \textbf{\( k \)-dominant skyline}.
$k$-DOMINANT SKYLINES

proposed by Chan-Jagadish-Tan-Tung-Zhang, 2006

Finding $k$-dominant skylines in high dimensional space

CY Chan, HV Jagadish, KL Tan… - Proceedings of the …, 2006 - portal.acm

ABSTRACT Given a d-dimensional data set, a point $p$ dominates another point $q$ than or equal to $q$ in all dimensions and better than $q$ in at least one dimension. Skyline queries, v

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Probabilistic skylines on uncertain data
J Pei, B Jiang, X Lin... - ... conference on Very large data bases, 2007 - portal.acm.org
ABSTRACT Uncertain data are inherent in some important applications. Although a considerable amount of research has been dedicated to modeling uncertain data and answering some types of queries on uncertain data, how to conduct advanced analysis on uncertain data ...
Cited by 150 - Related articles - All 14 versions

Selecting stars: The k most representative skyline operator
X Lin, Y Yuan, Q Zhang... - Data Engineering, 2007. ..., 2007 - ieeexplore.ieee.org
Page 1. Selecting Stars: The k Most Representative Skyline Operator Xuemin Lin 1 Yidong Yuan 1 Qing Zhang 2 Ying Zhang 1 1 School of Computer Science and Engineering 2 E-Health Research Center The University of New South Wales & NICTA, Australia ...
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Efficient skyline computation over low-cardinality domains
M Morse, JM Patel... - Proceedings of the 33rd ..., 2007 - portal.acm.org
ABSTRACT Current skyline evaluation techniques follow a common paradigm that eliminates data elements from skyline consideration by finding other elements in the dataset that dominate them. The performance of such techniques is heavily influenced by the underlying data ...
Cited by 82 - Related articles - All 9 versions

Approaching the skyline in Z order
KCK Lee, B Zheng, H Li... - Proceedings of the 33rd ..., 2007 - portal.acm.org
ABSTRACT Given a set of multidimensional data points, skyline query retrieves a set of data points that are not dominated by any other points. This query is useful for multi-preference analysis and decision making. By analyzing the skyline query, we observe a close connection be ...
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Monotonicity of $M_{d,k}(n)$ in $k$

$M_{d,1}(n) \leq M_{d,2}(n) \leq \cdots \leq M_{d,d-1}(n) \leq M_{d,d}(n)$

d = 2: $k = 2$ (left) and $k = 1$ (right)
AN ASYMPTOTIC VANISHING PROPERTY

Theorem. ([0, 1]^d, uniform, independent)

Let \( M_{d,k}(n) := \# \text{ points in the } k\text{-dominant skyline of a random sample.} \) Then

\[
\mathbb{E}[M_{d,k}(n)] \to 0 \quad (1 \leq k < d; d \geq 2),
\]

as \( n \to \infty \) and \( d = O(1) \).
Idea of proof

\[
\mathbb{E}[M_{d,d-1}(n)] \\
= n \mathbb{P}(p_1 \text{ is a } (d - 1)\text{-dominant skyline point}) \\
= n \int_{[0,1]^d} (1 - |B_{d-1}(x)|)^{n-1} \, dx,
\]

where \( B_k(x) \) denotes the region of the points in \([0,1]^d\) that \( k \)-dominate \( x \).
$$\mathbb{E}[M_{d,k}(n)] \rightarrow \begin{cases} 0, & \text{if } 1 \leq k < d; \\ \infty, & \text{if } k = d. \end{cases}$$
\[
\mathbb{E}[M_{d,k}(n)] = \begin{cases} 
O\left(n^{1-d/k}\right), & \text{if } 1 \leq k < d; \\
O\left(\frac{\left(\log n\right)^{d-1}}{(d-1)!}\right), & \text{if } k = d.
\end{cases}
\]

Lose too many from full to partial dominance
Theorem. ([0, 1]^d, uniform, independent)

Let \( L_{d,k}(n, j) := \# \) points in the random sample that are \( k \)-dominated by exactly \( j \) points \( (L_{d,k}(n, 0) = M_{d,k}(n)) \).

\[ \mathbb{E}[L_{d,k}(n, j)] \rightarrow 0 \quad (1 \leq k < d; d = O(1)), \]

uniformly for \( j = O(n^{(1-\varepsilon)/d}) \), as \( n \rightarrow \infty \).

Useless even allowing more points

same zero-infinity property holds
Theorem. \((d\text{-simplex, uniform, independent})\)

Let \(M_{d,k}^{[s]}(n) := \#\) points in the \(k\)-dominant skyline of a random sample. Then

\[ \mathbb{E}[M_{d,k}(n)] \to 0 \quad (1 \leq k < d; d \geq 2), \]

as \(n \to \infty\) and \(d = O(1)\).

Again either zero or infinity

The phase change is more abrupt here:

\[ \mathbb{E}[M_{d,k}^{[s]}(n)] = \begin{cases} 
O\left(n \left(1 - d^{1-d}\right)^n\right), & \text{if } 1 \leq k < d; \\
O\left(n^{1-1/d}\right), & k = d.
\end{cases} \]
\[ \mathcal{D} := \{(-t, -2t, 3t, 4t) : 1 \leq t \leq 2\} \quad (d = 4, k = 3) \]

Any two points in \( A \) are incomparable (none \( k \)-dominating the other).

Sample \( n \) points uniformly and independently in \( \mathcal{D} \). Then

\[ M_{d,k}(n) \sim n \quad \text{(almost surely)}. \]
A categorical model

Take \( n \) points uniformly and independently from the product space

\[
\mathcal{P} := \prod_{1 \leq j \leq d} \{1, 2, \ldots, C_j\}, \quad (C_j \geq 2, 1 \leq j \leq d).
\]

Theorem. (Categorical model)

The expected number of \( k \)-dominant skylines satisfies

\[
\frac{\mathbb{E}[M_{d,k}^{[c]}(n)]}{n} \rightarrow \frac{1}{C}, \quad (1 \leq k \leq d; d \geq 2),
\]

as \( n \rightarrow \infty \), where \( C := \prod_{1 \leq j \leq d} C_j \).
Theorem. (Categorical model)

Since the possible number of points is finite,

\[ M_{d,k}^{[c]}(n) \sim \text{Binomial} \left( n; \frac{1}{C} \right). \]

*Now too many candidate points!!*
Why still so popular?

Gap between theory and practice
$n$ never $\infty$ in practice

If $n = 10^5$, $d = 10$, then $d = O(1)$ or $O(\log n)$ or $O(n^{1/4})$?

Consider $n \to \infty$ and $d$ varies with $n$

Lemma (Hypercube model, $k = d - 1$)

If $d = o\left(\frac{\log n}{\log \log n}\right)$, then

$$\mathbb{E}[M_{d,d-1}(n)] \sim \Gamma\left(\frac{1}{d-1}\right)^d \frac{n^{-1/(d-1)}}{d-1},$$

uniformly in $d$ for large $n$.

$\Gamma(x) \sim x^{-1}$ as $x \to 0$
n never \( \infty \) in practice

If \( n = 10^5 \), \( d = 10 \), then \( d = O(1) \) or \( O(\log n) \) or \( O(n^{1/4}) \)?

Consider \( n \to \infty \) and \( d \) varies with \( n \)

Lemma (Hypercube model, \( k = d - 1 \))

If \( d = o \left( \frac{\log n}{\log \log n} \right) \), then

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\mathbb{E}[M_{d,d-1}(n)] \sim \Gamma \left( \frac{1}{d-1} \right)^d \frac{n^{-1/(d-1)}}{d-1},
\]

uniformly in \( d \) for large \( n \).

\( \Gamma(x) \sim x^{-1} \) as \( x \to 0 \)
Theorem. (Hypercube model, $k = d - 1$)

Let

$$d_0 = \left\lfloor \sqrt{\frac{2 \log n}{W(2 \log n)}} \right\rfloor + 1,$$

where $W$ denotes the Lambert-$W$ function ($W(x)e^{W(x)} = x$). Then the expected number of $(d - 1)$-dominant skyline points satisfies

$$E[M_{d,d-1}(n)] \rightarrow \begin{cases} 
0, & \text{if } d \leq d_0 - 1; \\
0, & \text{if } d = d_0 \text{ and } n \neq i^2; \\
1, & \text{if } d = d_0 \text{ and } n = i^2 (i \geq 2); \\
\infty, & \text{if } d \geq d_0.
\end{cases}$$

$W(\log x) \sim \log x - \log \log x$
$d_0$ GROWS VERY SLOWLY

Let $a_i := i^2$ with $a_1 := 2$.

Then

$$d_0 = \left\lceil \sqrt{\frac{2 \log n}{W(2 \log n)}} \right\rceil + 1 = i + 1 \text{ for } a_i \leq n < a_{i+1}.$$

$$d_0 = \begin{cases} 
2, & \text{if } 2 \leq n \leq 15; \\
3, & \text{if } 16 \leq n \leq 19682; \\
4, & \text{if } 19683 \leq n \leq 42949 67295; \\
5, & \text{if } 42949 67296 \leq n \leq 2.98 \cdots \times 10^{17}; \\
6, & \text{if } 2.98 \cdots \times 10^{17} \leq n \leq 1.03 \cdots \times 10^{28}; \\
7, & \text{if } 1.03 \cdots \times 10^{28} \leq n \leq 2.56 \cdots \times 10^{41}. 
\end{cases}$$
THE_THRESHOLD PHENOMENON

Theorem. \((d\text{-simplex, } k = d - 1)\)

Let

\[d_1 := \left\lfloor \frac{\log n}{W(\log n)} \right\rfloor.\]

Then

\[E[M_{d,d-1}^{[s]}(n)] \to \begin{cases} 0, & \text{if } d \leq d_1; \\ \infty, & \text{if } d \geq d_1 + 1. \end{cases}\]

Lemma. Uniformly for \(2 \leq d \leq d_1 + 3\)

\[E[M_{d,d-1}^{[s]}(n)] \sim \alpha_d n^{-(d-1)/2} (1 - d^{1-d})^n,\]

where

\[\alpha_d := \frac{(d - 2)!}{\sqrt{d}} \left(\frac{2\pi(d^{d-1} - 1)}{d - 1}\right)^{(d-1)/2}.\]
A SKETCH OF PROOF

\[ M_{d,k}(n) \leq M_{d,k+1}(n) \]

\[ \mathbb{E}[M_{d,d-1}(n)] = n \int_{[0,1]^d} (1 - |B_{d-1}(x)|)^{n-1} \, dx, \]

where \( B_{d-1}(x) \) denotes the region of the points in \([0, 1]^d\) that \( k \)-dominate \( x \)

\[ B_{d-1}(x) = \sum_{1 \leq \ell \leq d} \prod_{j \neq \ell} x_j - (d - 1) \prod_{1 \leq j \leq d} x_j. \]
A NUMERICAL EXAMPLE

Hypercube model, \( k = d - 1 \)

The expected number of \((d - 1)\)-skyline points are given approximately by

<table>
<thead>
<tr>
<th>( n )</th>
<th>( d )</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^4 )</td>
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<td>0.8</td>
<td>15.7</td>
<td>296.8</td>
<td>5945.2</td>
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<tr>
<td>( 10^5 )</td>
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<td>0.4</td>
<td>8.8</td>
<td>187.3</td>
<td>4050.4</td>
</tr>
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</table>

Choose \( d = 5 \)
OPEN QUESTIONS

Asymptotic estimates for $\mathbb{E}[M_{d,k}(n)]$?

$$\mathbb{E}[M_{d,k}(n)] \asymp n^{1-d/k}$$

Uniform asymptotic approximations in $d, k$?

Threshold phenomenon for other $k$?

difficult in general, $k > d/2$?

Variance?
also hard
A LOWER BOUND FOR $\mathbb{E}[M_{d,k}(n)]$

$n = 1000$, $d = 100$ and $k$ from 50 to 100

$J_n(x) := x \int_x^1 \frac{(1 - t)^{n-1}}{t^2} \, dt$

$E[M_k(n)]$

$nJ_n \left( \sum_{1 \leq j \leq d-k} \binom{d}{j} 2^{-d} \right)$