λ-terms of bounded unary height

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Journées ALÉA
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λ-terms and enriched trees

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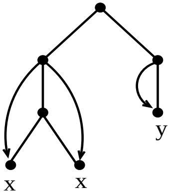

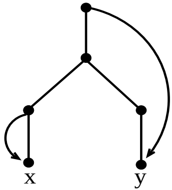

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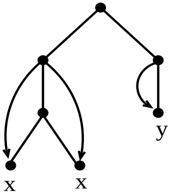
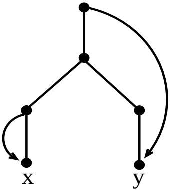
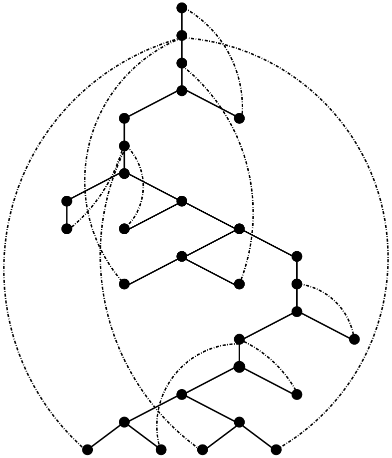




These λ-terms are closed (no free variable)

A bigger closed λ-term



λ-terms as enriched Motzkin trees

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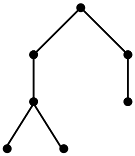

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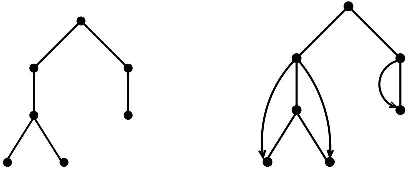

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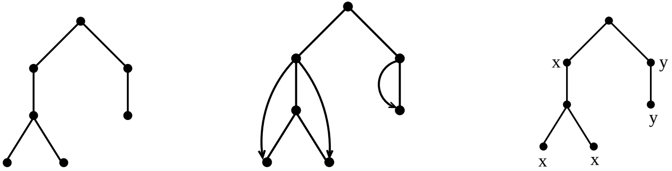

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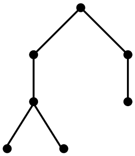
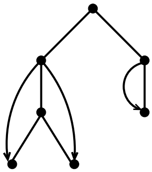
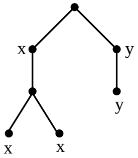


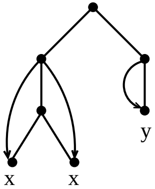




Labelling rules:

- Binary nodes are unlabelled
- Unary nodes get distinct labels (colors)
- Leaves get the label (color) of one of their unary ancestors

- Here all variables are bound



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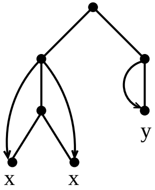
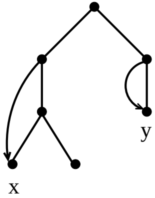


- Some variables may be free



- Recursive definition for λ-terms?
  - $\mathcal{L}$: class of λ-terms with free variables
  - $\mathcal{N}$ atomic class of binary node
  - $\mathcal{U}$ atomic class of unary node
  - $\mathcal{F}$ atomic class of free leaf
  - $\mathcal{B}$ atomic class of bound leaf

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$$\mathcal{L} = \mathcal{F} + \left(\mathcal{N} \times \mathcal{L}^2\right) + (\mathcal{U} \times \mathit{subs}(\mathcal{F} \to \mathcal{F} + \mathcal{B}, \mathcal{L}))$$

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- Generating function

$$L(z,f) = fz + zL(z,f)^2 + zL(z,f+1).$$

with $z \leftrightarrow$ size of the λ-term and $f \leftrightarrow$ free leaves
(size = total number of nodes)

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$$L(z,1) = \frac{1}{z}L(z,0) - L(z,0)^2$$

$$\begin{aligned} L(z,0) &= [f^0]L(z,f) \\ &= z^2 + 2z^3 + 4z^4 + 13z^5 + 42z^6 + 139z^7 + 506z^8 \\ &\quad +1915z^9 + 7558z^{10} + \cdots \end{aligned}$$

$$L(z,0) = \frac{1}{2z}\,\left(1 - \sqrt{\Lambda(z)}\right)$$

with $\Lambda(z)$ equal to

$$1 - 2z + 2z\sqrt{1 - 2z - 4z^2 + 2z\sqrt{....\sqrt{1 - 2z - 4nz^2 + 2z\sqrt{...}}}}$$

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- Restrict the number of pointers from an abstraction: BCI and BCK
- Restrict the number of abstractions in a path from the root towards a leaf: bounded unary height

BCI and BCK

Two classes of closed λ-terms:

- BCI: each abstraction binds *exactly* one variable
- BCK: each abstraction binds *at most* one variable

O. Bodini, D. Gardy, A. Jacquot: Asymptotics and random sampling for constrained λ-terms (Gascom'10)

Class of λ-terms when each abstraction binds exactly one variable:

$$T(z,f) = zf + zT^2(z,f) + z\frac{\partial T}{\partial f}(z,f)$$

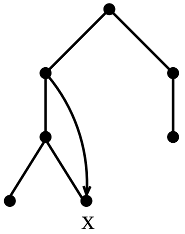
BCI: no free variables $\Rightarrow T(z,0)$

Bijection with triangular pointed diagrams, enumerated according to the number of edges (Vidal)

$$[z^{3n+2}]T(z,0) \sim \frac{6n(6n/e)^n}{\sqrt{2\pi n}}$$

λ-terms of bounded unary height

A λ-term with two free variables...



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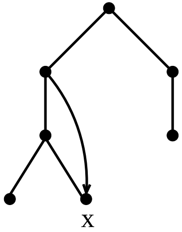
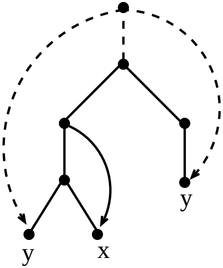
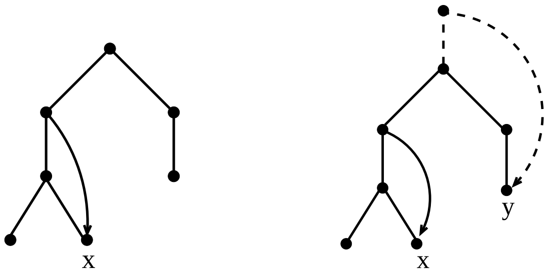




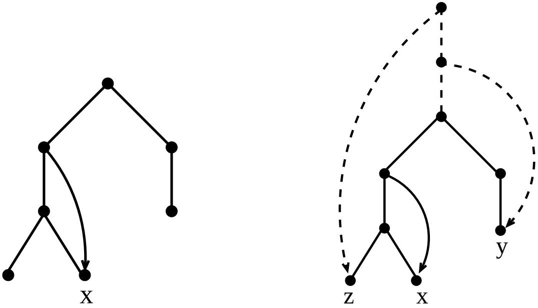
... now all its variables are bound!

A λ-term with two free variables...



... here only one variable is bound

A λ-term with two free variables...



... here again all the variables are bound

Remark: de Bruijn indices

Variables represented by numbers: backwards ranks of the unary nodes that binds them – if there are not enough unary nodes above it, a variable is free

$$(\lambda.31)(\lambda 2)$$

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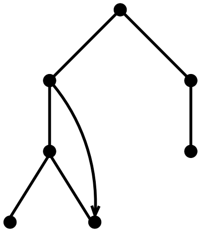

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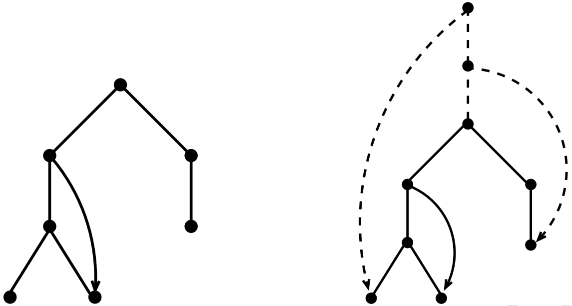

The classes $\mathcal{P}^{(i,k)}$

k: maximal number of abstractions on a path from the root to a leaf

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- ...
- $\mathcal{P}^{(i,k)}$: λ-terms with bound variables, i kinds of free variables, and unary height $\leq k - i$

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- $\mathcal{P}^{(i,k)}$: λ-terms with bound variables, i kinds of free variables, and unary height $\leq k-i$
- $\mathcal{P}^{(k,k)}$: λ-terms with bound variables, k kinds of free variables, and no unary node

The classes $\mathcal{P}^{(i,k)}$

- $i = k$

$$\mathcal{P}^{(k,k)} = k\mathcal{Z} + \mathcal{Z}{\mathcal{P}^{(k,k)}}^2$$

Generating function:

$$P^{(k,k)}(z) = kz + zP^{(k,k)}(z)^2$$

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- $i < k$

$$\mathcal{P}^{(i,k)} = i\mathcal{Z} + \mathcal{Z}{\mathcal{P}^{(i,k)}}^2 + \mathcal{Z}\mathcal{P}^{(i+1,k)}$$

Generating function:

$$P^{(i,k)}(z) = iz + zP^{(i,k)}(z)^2 + zP^{(i+1,k)}(z)$$

Solve:

$$
\begin{aligned}
P^{(k,k)}(z) &= \frac{1-\sqrt{1-4kz^2}}{2z} \\
P^{(i,k)}(z) &= \frac{1-\sqrt{1-4iz^2-4z^2P^{(i+1,k)}(z)}}{2z}
\end{aligned}
$$

We now have $S^{(k)}(z) = P^{(0,k)}(z)$: we can start the asymptotic study of its coefficients!

Dominant singularity of $S^{(k)}$

The function $S^{(k)}$

$$S^{(k)} = \frac{1 - \sqrt{1 - 2z + 2z\sqrt{1 - 2z - 4z^2 + 2z\sqrt{\ldots\sqrt{1 - 4z^2}}}}}{2z}$$

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- $S^{(k)}$ is algebraic and written with $k + 1$ iterated radicands
- Its singularities are the values that cancel its radicands
- Which radicant has smallest root?

Determination of the dominant radicand for $k = 1$

$$S^{(1)}(z) = \frac{1 - \sqrt{1 - 2z + 2z\sqrt{1 - 4z^2}}}{2z}$$

Algebraic singularities:

- $\pm 1/2$: cancels the innermost radicand
- $1/2$: cancels the external radicand

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Dominant singularity: $1/2$ (type $1/4$)
Other singularity: $-1/2$ (type $1/2$)

Asymptotics:

$$[z^n]S^{(1)}(z) \sim \frac{1}{4} \frac{2^{\frac{1}{4}} 2^n n^{-\frac{5}{4}}}{\Gamma(\frac{3}{4})}$$

Determination of the dominant radicand for $k = 2$

$$S^{(2)}(z) = \frac{1 - \sqrt{1 - 2z + 2z\sqrt{1 - 2z - 4z^2 + 2z\sqrt{1 - 8z^2}}}}{2z}$$

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Asymptotics:

$$[z^n]S^{(2)}(z) \sim \frac{1}{\Gamma(\frac{1}{2})} h_2 n^{-\frac{3}{2}} \rho^{-n}$$

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- $k = 1$: second innermost radicand
- $k = 2$: second innermost radicand
- $k = 3, 4, \ldots$: second innermost radicand
- $k = 9$: third innermost radicand!

Let $R_{i,k}(z)$ be the i^{th} radicand of $S^{(k)}(z)$ $(1 \leq i \leq k+1)$

$$P^{(i,k)}(z) = \frac{1 - \sqrt{R_{k-i,k}(z)}}{2z}.$$

- For $k = 1$:

$$R_{1,k}(z) = 1 - 4kz^2$$

- For $i \geq 2$:

$$R_{i,k}(z) = 1 - 4\,(k - i + 1)z^2 - 2z + 2z\sqrt{R_{i-1,k}(z)}$$

$$R_{i,k}(z) = 1 - 4\,(k - i + 1)z^2 - 2z + 2z\sqrt{R_{i-1,k}(z)}$$

- On the real positive axis, $R_{i,k}(z)$ is decreasing and has a single positive root $\rho_{i,k}$

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- The dominant singularity of $R_{i,k}$ is the smallest of the $\rho_{j,k}$
- The sequence of the roots $\rho_{j,k}$ is decreasing...
- ... but sometimes two consecutive values are equal!

We then have two dominant radicands

Function	Radicand	Singularity
$S^{(1)}$	{1,2}	0.5
$S^{(2)}$	2	0.3438
$S^{(3)}$	2	0.2760
...	...	...
$S^{(8)}$	{2,3}	0.1667
$S^{(9)}$	3	0.1571
...	...	...
$S^{(134)}$	3	0.0418
$S^{(135)}$	{3,4}	0.0417
$S^{(136)}$	4	0.0415
...	...	...

Values of k which give two dominant radicands?

- Define $(u_k)_{k\geq 0}$ with $u_0 = 0$ and

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- The sequence $(u_k)_{k\geq 0}$ is doubly exponential
- $\lim_{k\to\infty} u_k^{1/2^k} \simeq \chi = 1.36660956...$

Define $N_k = u_k^2 - u_k + k$.
First values: $N_1 = 1$, $N_2 = 8$, $N_3 = 135$, $N_4 = 21760$, $N_5 = 479982377$, ...

Theorem

Let i such that $k \in [N_i, N_{i+1}[$.

- *If $k \neq N_i$, the dominant radicand of $S^{(k)}(z)$ is the i-th radicand; the dominant singularity is algebraic of type $1/2$.*
- *If $k = N_i$, the radicands with ranks i and $(i+1)$ both cancel for the same value; both are dominant; the dominant singularity is algebraic of type $1/4$.*

(Radicands are ranked from the innermost to the outermost.)

Asymptotics and random generation

Theorem

Two asymptotic behaviours according to the value of k

- *For unary height* N_k, $[z^n]S^{(N_k)} \sim C_k n^{-5/4} \rho_k^n$ *with* $\rho_k = 1/2u_k$
- *If* k *does not belong to the set* $\{N_i\}$, *then* $[z^n]S^{(k)} \sim C_k n^{-3/2} \rho_k^n$

Observations

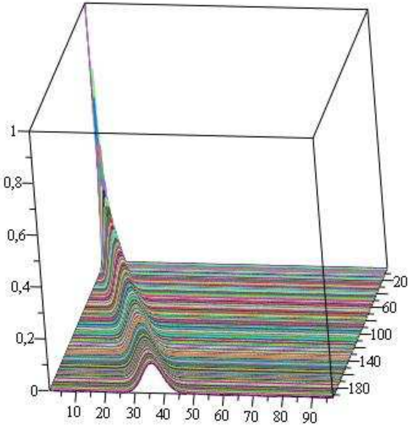
- The constants C_k become small very quickly. Variation of C_k as a function of k?
  - $[z^n]S^{(1)}(z) \sim 0.2426128012 \cdot \left(\frac{1}{n}\right)^{5/4} \cdot 2^n$
  - $[z^n]S^{(8)}(z) \sim 9.318885373 \cdot 10^{-5} \left(\frac{1}{n}\right)^{5/4} 6^n$
  - $[z^n]S^{(135)}(z) \sim 7.116999389 \cdot 10^{-158} \left(\frac{1}{n}\right)^{5/4} 24^n$
- We cannot observe the asymptotic behaviour for "reasonably-sized" terms

Random generation of λ-terms with bounded unary height k?

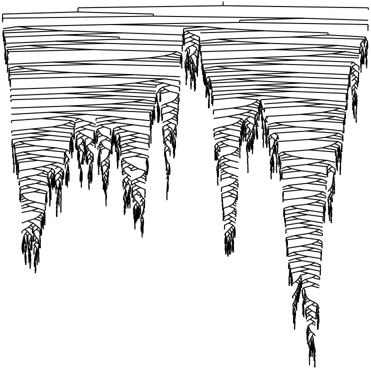
Random generation of λ-terms with bounded unary height k?

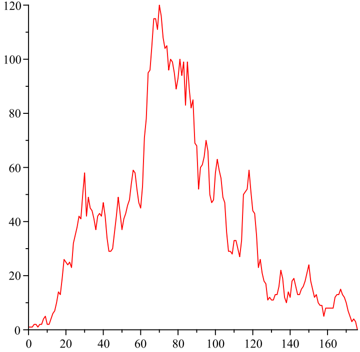
- Boltzmann generation: difficulties due to the (very small) value of the constant C_k
- Recursive method: easier!

Number of λ-terms: $n \in [1, ..., 198]$; unary height $k \in [1, ..., 98]$

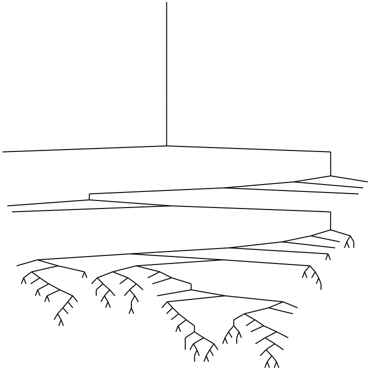


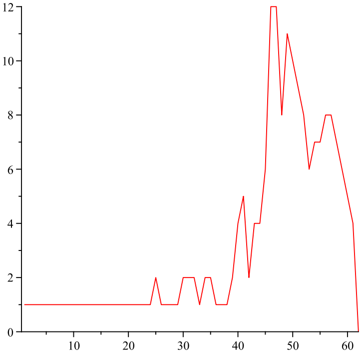
A random λ-term of unary height ≤ 8 and its profile



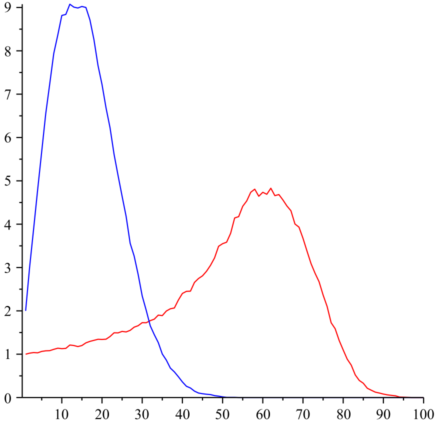


A random λ-term of size 200 and its profile





Comparing the average profiles of a random λ-term (red) and a random binary tree (blue)



What next?

How does a random λ-term of bounded unary height look like?

- Random unary height?
- Total height? (seems linear)
- Width? (seems of order $\log n$)
- Profile?

What if the unary height is not bounded?

- Consider other classes of λ-terms: BCI(p), ...
- Enumerate (unrestricted) λ-terms according to their size
- Caracterize the parameters of a random λ-term
- Logical properties (strong normalizing, ...)

A suivre...