Computational Complexity of real functions

Amaury Pouly

April 8, 2015

Outline

Examples

"computable = can be solved with a computer"

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Example (Sorting)

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ACK: given n and m, compute $A_{m,n}$ defined by

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$\Rightarrow$ "clearly computable"...but slow ? ($A_{4,2} \approx 10^{20000}$)

Examples (2)

Example (Collatz/Syracuse sequence)

COLLATZ: given n decide if this sequence converges to 1:

$$u_0 = n \qquad u_{k+1} = \begin{cases} \frac{u_k}{2} & \text{if } u_k \text{ is even} \\ 3u_k & \text{otherwise} \end{cases}$$

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$\Rightarrow$ what does a "program" mean ?

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- formalise a notion of program and “being computable”

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- Computational complexity: refine the notion of primitive recursive
- Turing degrees: refine the notion of uncomputable problems

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$\Rightarrow$ "looks quite hard"$\rightarrow$ we can check all strategies !

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$\Rightarrow$ undecidable even if we assume we can "solve" HALT

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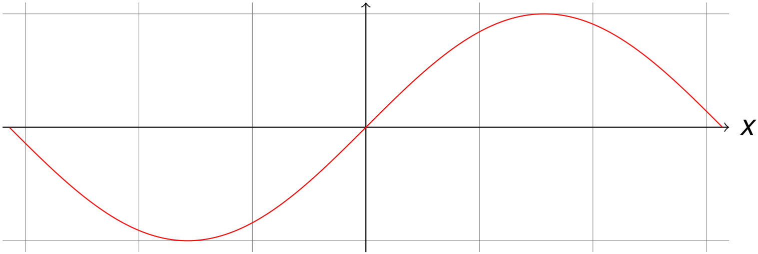

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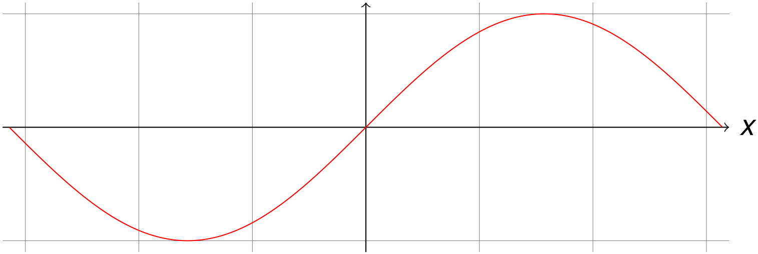

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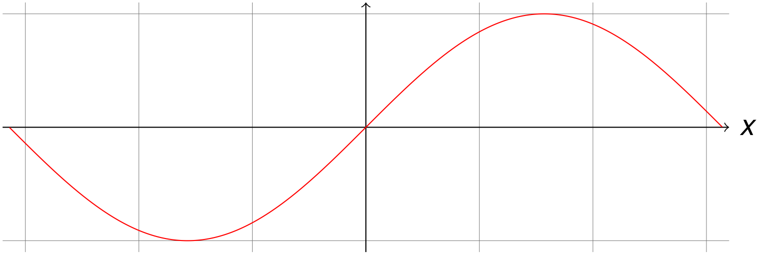

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- how do you represent a real number ? (infinite object)
- what is a program working on them ?

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$\Rightarrow$ Very analytic, approximation theory

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Example (Non-computable real)

$$r = \sum_{n=0}^{\infty} d_n 2^{-n}$$

where

$$d_n = 1 \Leftrightarrow \text{the } n^{th} \text{ Turing Machine halts on input } n$$

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$f : [a, b] \to \mathbb{R}$ is computable iff $\exists m, \psi$ computable functions s.t $\forall n \in \mathbb{N}$:

- $\forall x, y, |x - y| \leqslant 2^{-m(n)} \Rightarrow |f(x) - f(y)| \leqslant 2^{-n}$ ▶ effective continuity
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Example (Counter-Example)

$$f(x) = \lceil x \rceil \qquad \blacktriangleright \text{ not continuous}$$

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Question

Can we give a purely analog model of computation ?

GPAC

General Purpose Analog Computer

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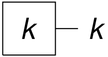
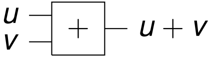
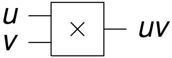


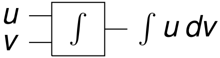
A constant unit



An adder unit



An multiplier unit



An integrator unit

GPAC: beyond the circuit approach

Theorem

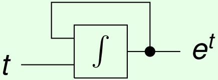
y is generated by a GPAC iff it is a component of the solution $y = (y_1, \ldots, y_d)$ of the ordinary differential equation (ODE):

$$\begin{cases} y' = p(y) \\ y(t_0) = y_0 \end{cases}$$

where p is a vector of polynomials.

GPAC: examples

Example (One variable, linear system)



$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$

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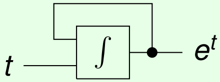
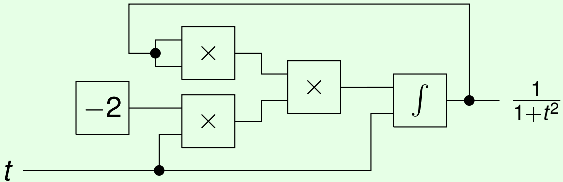


$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$

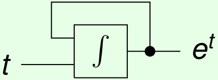
Example (One variable, nonlinear system)



$$\begin{cases} y' = -2ty^2 \\ y(0) = 1 \end{cases}$$

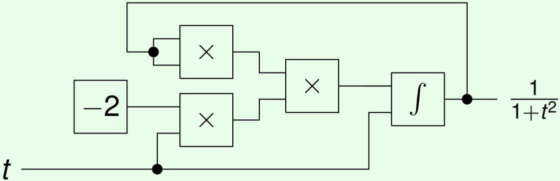
GPAC: examples

Example (One variable, linear system)



$$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$

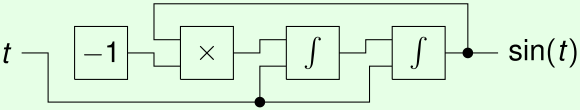
Example (Two variable, nonlinear system)



$$\begin{cases} y' = -2ty^2 \\ y(0) = 1 \\ t' = 1 \\ t(0) = 0 \end{cases}$$

GPAC: examples

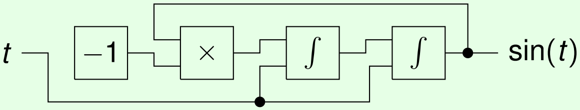
Example (Two variables, linear system)



$$\begin{cases} y' &= z \\ z' &= -y \\ y(0) &= 0 \\ z(0) &= 1 \end{cases}$$

GPAC: examples

Example (Two variables, linear system)



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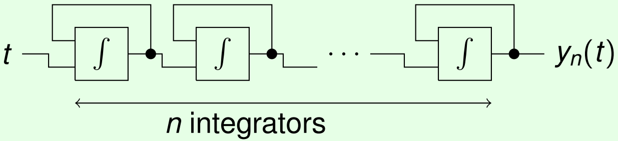
Exercice (Tear your mind apart)



$$\begin{cases} y_1' = y_1 \\ y_2' = y_2 y_1' \\ \vdots \\ y_n' = y_n y_{n-1}' \end{cases}$$

GPAC: examples

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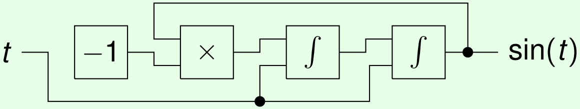

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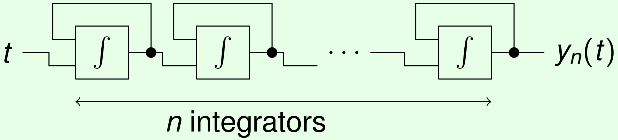
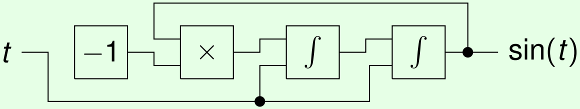


$$\begin{cases} y_1' = y_1 \\ y_2' = y_2 y_1 \\ \vdots \\ y_n' = y_n y_{n-1} \cdots y_2 y_1 \end{cases}$$

GPAC: examples

Example (Two variables, linear system)



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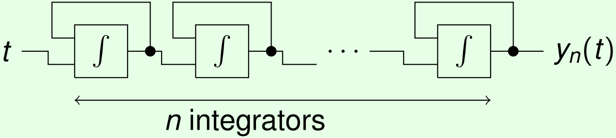

$$\begin{cases} y_1(t) = e^t \\ y_2(t) = e^{e^t} \\ \quad \cdots \\ y_n(t) = e^{e^{\cdot^{\cdot^{\cdot^{t}}}}} \end{cases}$$

Slight issue is...

- the GPAC generated functions are analytical

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- the computable functions from Computable Analysis are continuous

Question

Can we bridge the gap ? Why should we ?

The case of discrete computations

Many models:

- Recursive functions
- Turing machines
- λ-calculus
- circuits
- ...

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And

Church Thesis

All reasonable discrete models of computation are equivalent.

GPAC: back to the basics

Definition

f is **generated** by a GPAC iff it is a component of the solution y of:

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satisfies $f(x) = \lim_{t\to\infty} y_1(t)$.

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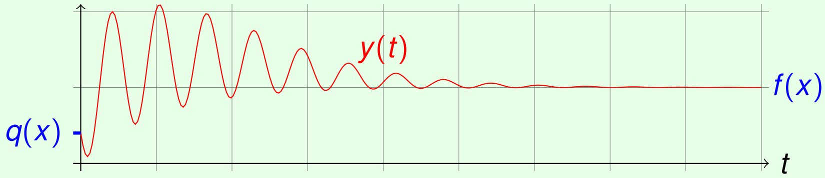

Computable Analysis = GPAC ? (again)

Theorem (Bournez, Campagnolo, Graça, Hainry)

f is GPAC-computable functions iff it is computable (in the sense of Computable Analysis).

Time Scaling

System	#1	#2
PIVP	$\begin{cases} y'(t) = p(y(t)) \\ y(1) = q(x) \end{cases}$	$\begin{cases} z'(t) = u(t)p(z(t)) \\ u'(t) = u(t) \\ z(t_0) = q(x) \\ u(1) = 1 \end{cases}$

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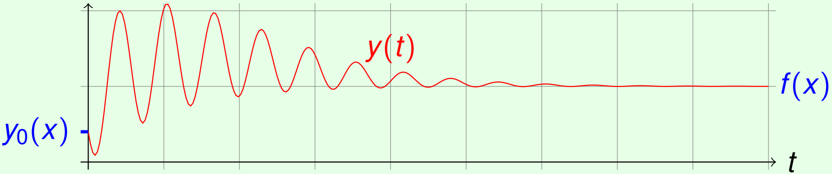
Remark

Same curve, different speed: $u(t) = e^t$ and $z(t) = y(e^t)$

Example



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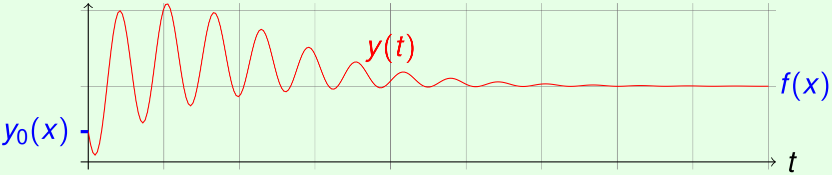
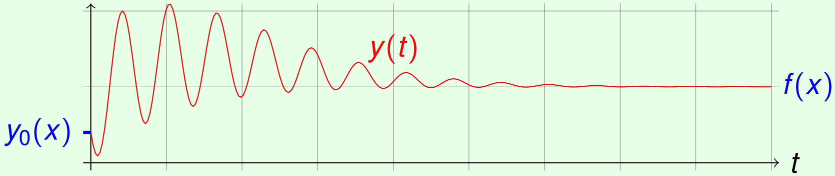


Time Scaling

PIVP	$y' = p(y)$	$z(t) = y(e^t) \to \begin{cases} z' = up(z) \\ u' = u \end{cases}$
Computed Function	Same	
Convergence		Exponentially faster

Example



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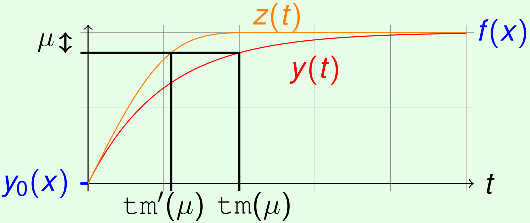
PIVP	$y' = p(y)$	$z(t) = y(e^t) \rightarrow \begin{cases} z' = up(z) \\ u' = u \end{cases}$
Computed Function	Same	
Time for precision μ	$\mathtt{tm}(\mu)$	$\mathtt{tm}'(\mu) = \log(\mathtt{tm}(\mu))$

Example



$$\|y_1(\mathtt{tm}(\mu)) - f(x)\| \leqslant \mu$$

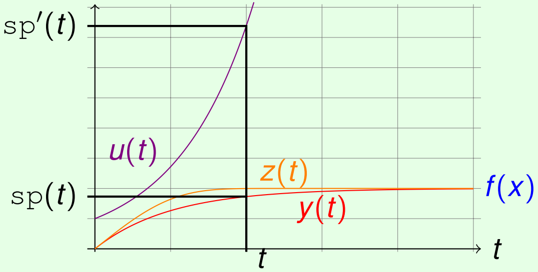
Remark

$\mathtt{tm}$ is not a good measure of complexity.

Time Scaling

PIVP	$y' = p(y)$	$z(t) = y(e^t) \rightarrow \begin{cases} z' = up(z) \\ u' = u \end{cases}$
Computed Function	Same	
Time for precision μ	$\mathtt{tm}(\mu)$	$\mathtt{tm}'(\mu) = \log(\mathtt{tm}(\mu))$
Bounding box for PIVP at time t	$\mathtt{sp}(t)$	$\mathtt{sp}'(t) = \max(\mathtt{sp}(e^t), e^t)$

Example



$$\mathtt{sp}(t) = \sup_{\xi\in[1,t]} \|y(\xi)\|$$

$$\mathtt{sp}'(t) = \sup_{\xi\in[1,t]} \|z(\xi), u(\xi)\|$$

Time Scaling

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Bounding box for PIVP at time t	$\mathtt{sp}(t)$	$\mathtt{sp}'(t) = \max(\mathtt{sp}(e^t), e^t)$
Bounding box for PIVP at precision μ	$\mathtt{sp}(\mathtt{tm}(\mu))$	$\max(\mathtt{sp}(\mathtt{tm}(\mu)), \mathtt{tm}(\mu))$

Remark

- $\mathtt{tm}(\mu)$ and $\mathtt{sp}(t)$ depend on the convergence rate
- $\mathtt{sp}(\mathtt{tm}(\mu))$ seems not

Proper Measures

Proper measures of “complexity”:

- time scaling invariant
- property of the curve

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- time scaling invariant
- property of the curve

Possible choices:

- Bounding Box at precision $\mu \Rightarrow$ Ok but geometric interpretation ?
- Length of the curve until precision $\mu \Rightarrow$ Much more intuitive

Questions ?

- Do you have any questions ?