Introduction to the Associahedron

Thibault Manneville (LIX, Polytechnique)
supervised by **Vincent Pilaud** (LIX, CNRS)

June 3rd, 2015

What is an Associahedron?

It is altogether

- a combinatorial structure:
  - simplicial complex
  - lattice of Catalan objects (Tamari lattice)
- a geometric structure:
  - polytope
- an algebraic structure:
  - set of basis for Hopf algebras
  - index set of seeds and variables for cluster aglebras

Simplicial complexes

Definition

$\Delta \subseteq \mathcal{P}(S)$ is a **simplicial complex** if $\sigma' \subseteq \sigma \in \Delta \Longrightarrow \sigma' \in \Delta$.
$\sigma \in \Delta$: **simplex** or **face** of Δ.

Simplicial complexes

Definition

$\Delta \subseteq \mathcal{P}(S)$ is a **simplicial complex** if $\sigma' \subseteq \sigma \in \Delta \Longrightarrow \sigma' \in \Delta$.
$\sigma \in \Delta$: **simplex** or **face** of Δ.

Example $S = \{1, 2, 3, 4\}$

$$\Delta = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{3,4\},\{1,3,4\}\}$$

Simplicial complexes

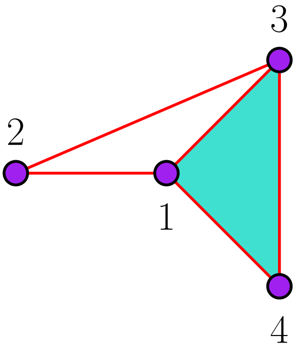
Definition

$\Delta \subseteq \mathcal{P}(S)$ is a **simplicial complex** if $\sigma' \subseteq \sigma \in \Delta \Longrightarrow \sigma' \in \Delta$.
$\sigma \in \Delta$: **simplex** or **face** of Δ.

Example $S = \{1, 2, 3, 4\}$

$\Delta = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{1, 3, 4\}\}$

geometrical representation:

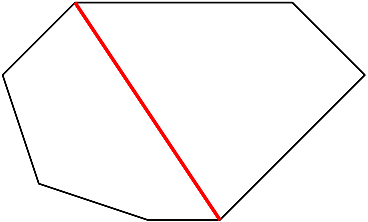


Simplicial complexes

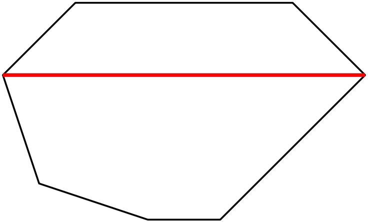
Definition

$\Delta \subseteq \mathcal{P}(S)$ is a **simplicial complex** if $\sigma' \subseteq \sigma \in \Delta \Longrightarrow \sigma' \in \Delta$.
$\sigma \in \Delta$: **simplex** or **face** of Δ.

the associahedron: $S = \{$diagonals of a convex $(n+3)$-gon$\}$



a diagonal



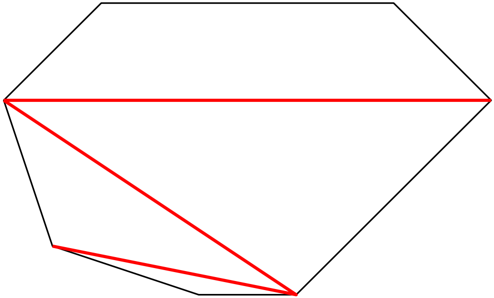
another one

Simplicial complexes

Definition

$\Delta \subseteq \mathcal{P}(S)$ is a **simplicial complex** if $\sigma' \subseteq \sigma \in \Delta \Longrightarrow \sigma' \in \Delta$.
$\sigma \in \Delta$: **simplex** or **face** of Δ.

the associahedron: $S = \{$diagonals of a convex $(n+3)$-gon$\}$
$\Delta(n) = \{$dissections of the $(n+3)$-gon$\}$



a dissection

Simplicial complexes

Definition

$$\dim(\sigma) = |\sigma| - 1 \qquad \dim(\Delta) = \max\{\dim(\sigma) | \sigma \in \Delta\}$$

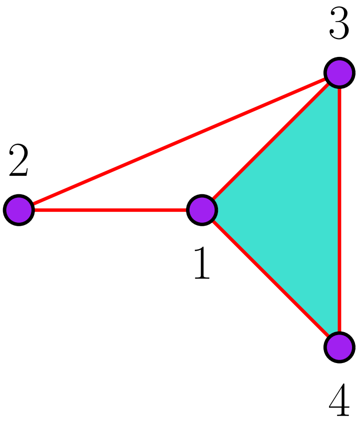
Δ is **pure**: inclusion-maximal simplexes are dimension-maximal.

Simplicial complexes

Definition

$$\dim(\sigma) = |\sigma| - 1 \qquad \dim(\Delta) = \max\{\dim(\sigma) | \sigma \in \Delta\}$$

Δ is **pure**: inclusion-maximal simplexes are dimension-maximal.



is not pure.

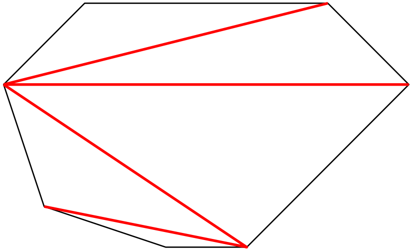
Simplicial complexes

Definition

$$\dim(\sigma) = |\sigma| - 1 \qquad \dim(\Delta) = \max\{\dim(\sigma) | \sigma \in \Delta\}$$

Δ is **pure**: inclusion-maximal simplexes are dimension-maximal.

inclusion-maximal dissections:



a triangulation

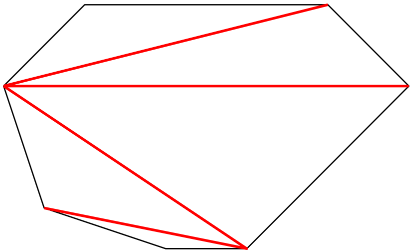
Simplicial complexes

Definition

$$\dim(\sigma) = |\sigma| - 1 \qquad \dim(\Delta) = \max\{\dim(\sigma) | \sigma \in \Delta\}$$

Δ is **pure**: inclusion-maximal simplexes are dimension-maximal.

inclusion-maximal dissections:



a triangulation

triangulations have n diagonals $\Longrightarrow$ $\Delta(n)$ is pure.

Simplicial complexes

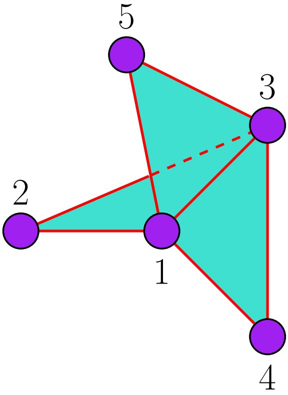
Definition

Δ is a **pseudo-manifold** if it is pure and for any σ maximal simplex of Δ and $s \in \sigma$, there is a unique $s' \in S$ such that $\sigma \setminus \{s\} \cup \{s'\}$ is a maximal simplex of Δ.

Simplicial complexes

Definition

Δ is a **pseudo-manifold** if it is pure and for any σ maximal simplex of Δ and $s \in \sigma$, there is a unique $s' \in S$ such that $\sigma \setminus \{s\} \cup \{s'\}$ is a maximal simplex of Δ.

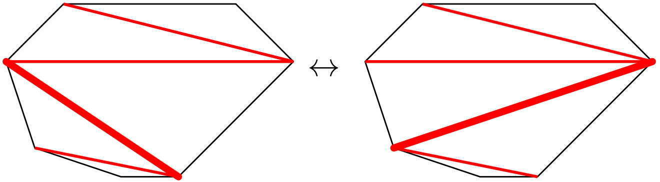


not a pseudo-manifold

Simplicial complexes

Definition

Δ is a **pseudo-manifold** if it is pure and for any σ maximal simplex of Δ and $s \in \sigma$, there is a unique $s' \in S$ such that $\sigma \smallsetminus \{s\} \cup \{s'\}$ is a maximal simplex of Δ.



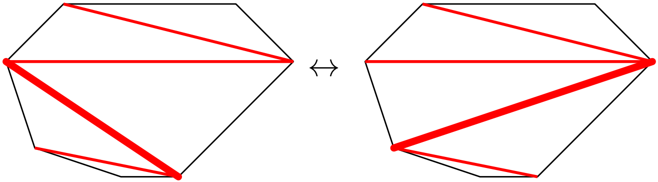
a flip

$\Longrightarrow$ $\Delta(n)$ is a pseudo-manifold.

Simplicial complexes

Definition

Δ is a **pseudo-manifold** if it is pure and for any σ maximal simplex of Δ and $s \in \sigma$, there is a unique $s' \in S$ such that $\sigma \smallsetminus \{s\} \cup \{s'\}$ is a maximal simplex of Δ.



a flip

$\Longrightarrow$ $\Delta(n)$ is a pseudo-manifold.

flip graph:
- vertices: triangulations (**dual graph** of $\Delta(n)$)
- edges: flips

Simplicial complexes

Much more other properties:

some combinatorial: manifolds, spheres, homoligical spheres...

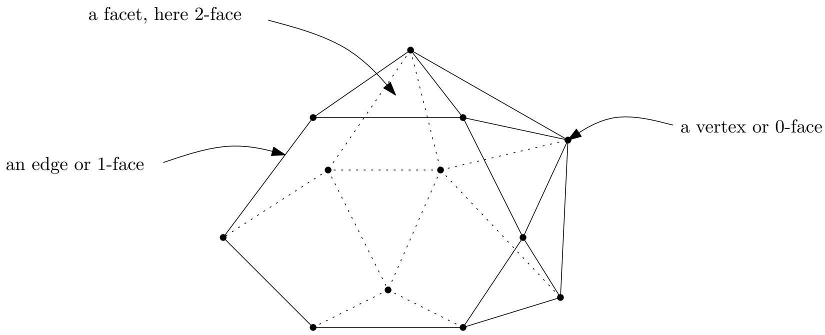
some geometrical: realizable by fans, polytopes...

Polytopes

$P \in \mathbb{R}^d$ is a **polytope** if $P = \text{conv}(S)$ with $|S| < \infty$.

Polytopes

$P \in \mathbb{R}^d$ is a **polytope** if $P = \text{conv}(S)$ with $|S| < \infty$.



Polytopes

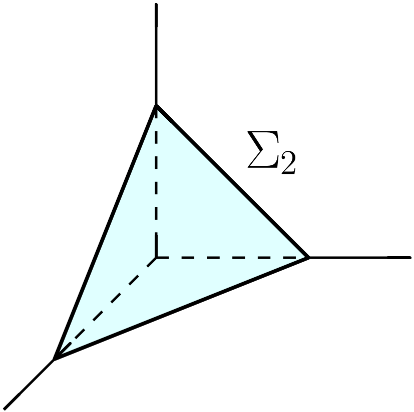
P is **simplicial** if its faces are simplexes.

Geometrical **simplex** of dimension n: $\Sigma_n = \text{conv}\{e_i\}_{i \in [n+1]}$

Polytopes

P is **simplicial** if its faces are simplexes.

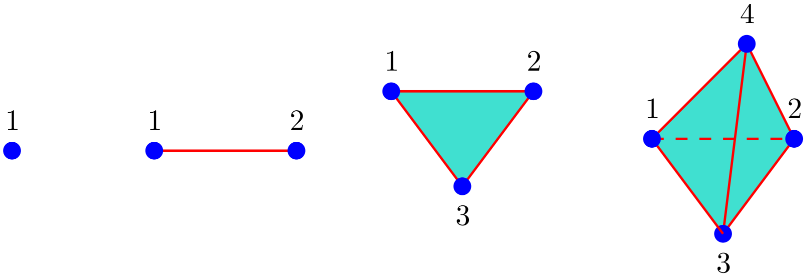
Geometrical **simplex** of dimension n: $\Sigma_n = \mathrm{conv}\{e_i\}_{i \in [n+1]}$



Polytopes

P is **simplicial** if its faces are simplexes.

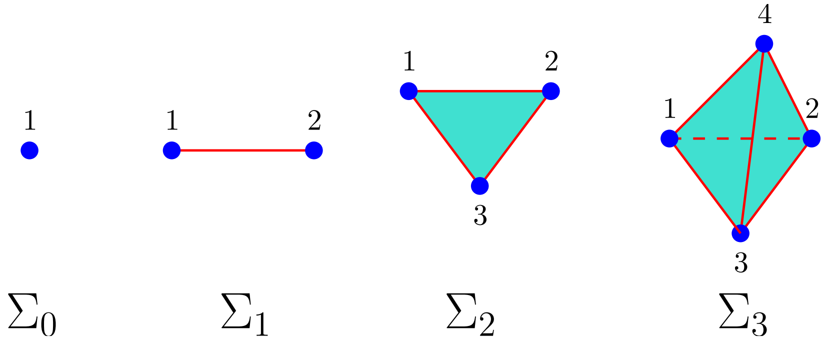
Geometrical **simplex** of dimension n: $\Sigma_n = \text{conv}\{e_i\}_{i \in [n+1]}$



Σ_0 Σ_1 Σ_2 Σ_3

Polytopes

P is **simplicial** if its faces are simplexes.
Geometrical **simplex** of dimension n: $\Sigma_n = \text{conv}\{e_i\}_{i \in [n+1]}$



$\forall I \subseteq [n+1], \text{conv}\{e_i\}_{i \in I}$ is a face of Σ_n.
Simplexes: only polytopes with this property.

Polytopes

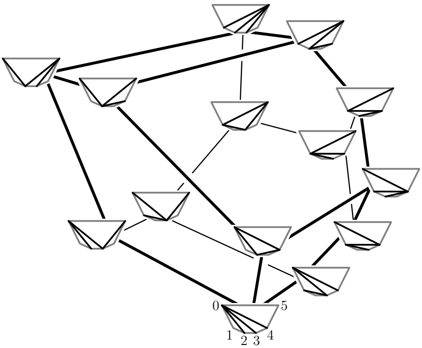
Theorem (Lee,Loday,Hohlweg-Lange,Ceballos-Santos-Ziegler...)

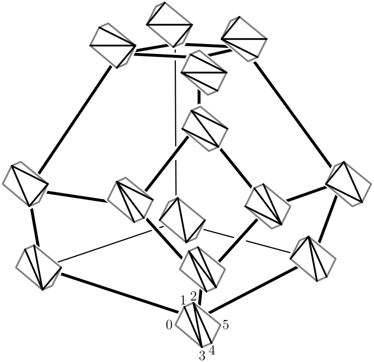
The associahedron $\Delta(n)$ is realizable as a convex polytope.

Polytopes

Theorem (Lee,Loday,Hohlweg-Lange,Ceballos-Santos-Ziegler...)

The associahedron $\Delta(n)$ is realizable as a convex polytope.



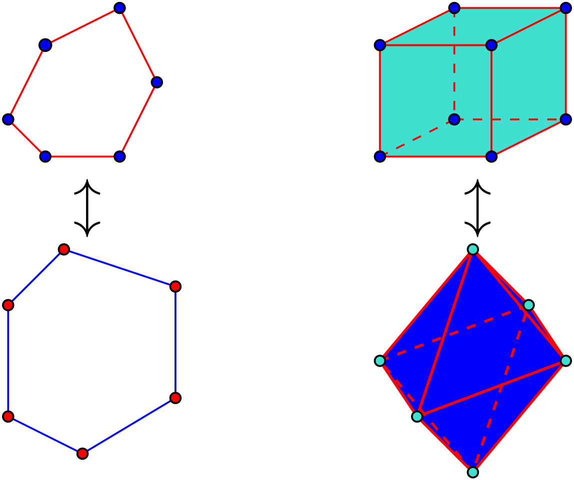


Polytopes

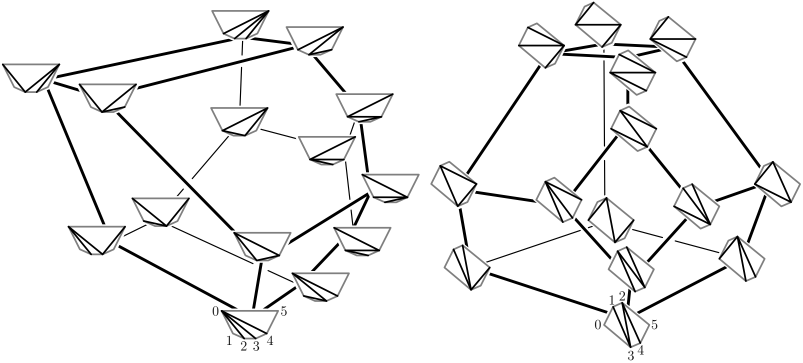
duality on polytopes: reversing the inclusion order on faces.

Polytopes

duality on polytopes: reversing the inclusion order on faces.

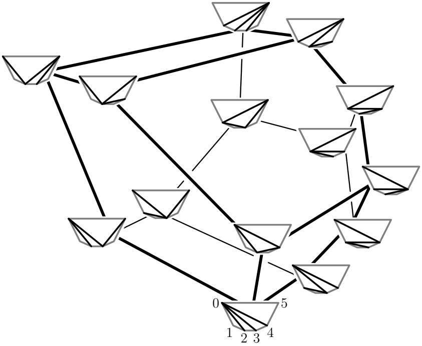


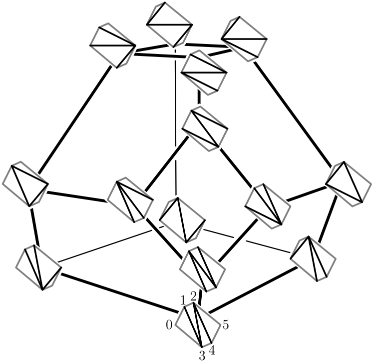
Polytopes



both the dual of an associahedron.

Polytopes





both the dual of an associahedron.

interest $\Rightarrow$ the graph is the flip graph.

Associahedra

interest for the flip graph:

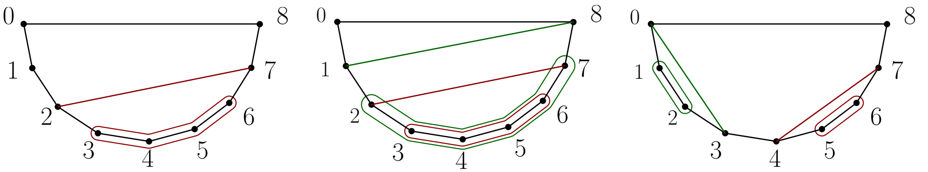
Theorem (Sleator-Tarjan-Thruston,Pournin)

The diameter of the flip graph of the n-dimensional associahedron is $2n - 10$ for $n > 9$.

Theorem (Lucas,Hurtado-Noy)

The flip graph of any associahedron is Hamiltonian.

Generalization: graph associahedra



allows to define the associahedron from a path.

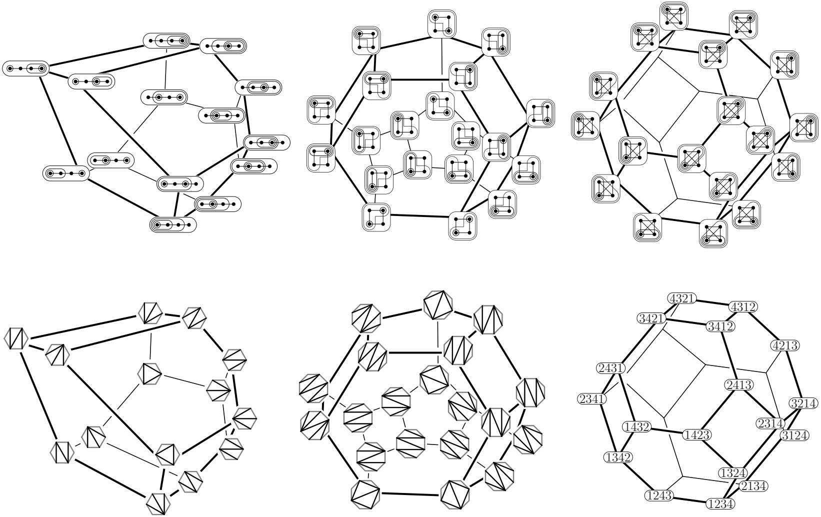
Generalization: graph associahedra

Theorem (Carr-Devadoss)

For a graph G, there is a polytope, the **graph associahedron** of G, encoding a certain simplicial complex associated to G.

$G =$ path on $n+1$ vertices $\longrightarrow$ usual associahedron.

Generalization: graph associahedra



Generalization: graph associahedra

Theorem (M-Pilaud)

Any graph associahedron is Hamiltonian.

Theorem (M-Pilaud)

The diameter of any graph associahedron satisfies:

$$\max(|E|, 2|V| - 18) \leq \delta(Asso_G) \leq \binom{|V|+1}{2}$$

THANK YOU FOR

YOUR WONDERED

ATTENTION!