EXAM COURSE 2-38-2 MPRI 2018
ALGORITHMS AND COMBINATORICS FOR GEOMETRIC GRAPHS
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The course and your personal notes are authorized. Electronic devices are forbidden.

Prepare two separated sheets for the two halves of the course.

The three problems in this part are independent and can be treated in an arbitrary order. Clearly indicate the number of the question in front of your answer. You can skip some questions if you are stuck. However, it is recommended to treat a coherent part of the subject rather than sporadic questions.

The care in the redaction and presentation of your solution will be considered in the notation.

1. Crossing-free spanning trees of a geometric graph

In this problem, we want to estimate the expectation of the number of crossings in a spanning tree on a given point set of $\mathbb{R}^2$.

1.1. Spanning trees containing a spanning forest.

Recall that:

- A *spanning forest* of a graph (V, E) is a graph (V, F) with $F \subseteq E$ and no cycle. A *spanning tree* is a connected spanning forest, or equivalently a spanning forest with $|V| - 1$ edges.
- A *rooted forest* $F^\bullet$ is a forest F with one distinguished root vertex per connected component. We say that a rooted forest $F^\bullet$ contains a rooted forest $G^\bullet$ if
  - all edges of G are edges of F, and
  - the root of each connected component T of $G^\bullet$ lies on the path between any other vertex of T and the root of the connected component of $F^\bullet$ containing T.

In this problem, we only consider spanning forests and spanning trees of the complete graph K_n on $\{1, \dots, n\}$. We fix a spanning forest F of K_n with k connected components of size $n_1, \dots, n_k$.

Q1. In how many ways can we root a tree with n vertices? In how many ways can we root the forest F?

We now consider a spanning forest $F^\bullet$ obtained by rooting the forest F. We denote by $T(F^\bullet)$ the number of rooted spanning trees of K_n containing the rooted spanning forest $F^\bullet$. To compute $T(F^\bullet)$, we will make a double counting of the set $X(F^\bullet)$ of pairs $(T^\bullet, \pi)$, where $T^\bullet$ is a rooted spanning tree of K_n containing the rooted spanning forest $F^\bullet$, and π is a permutation of the edges of $T^\bullet$ which are not in $F^\bullet$. The first counting is easy:

Q2. Express $|X(F^\bullet)|$ in terms of $T(F^\bullet)$ and k.

For the second counting, we construct an element of $X(F^\bullet)$ using the following algorithm. We construct a sequence $F_0^\bullet, \dots, F_{k-1}^\bullet$ of rooted forests such that $F_0^\bullet = F^\bullet$, and for each $i \in [k-1]$,

- $F_i^\bullet$ has $k - i$ connected components, and
- the rooted forest $F_i^\bullet$ contains the rooted forest $F_{i-1}^\bullet$ (with the definition given above).

By definition, $F_{k-1}^\bullet = T^\bullet$ is a spanning tree containing the spanning forest $F^\bullet$. We obtain $F_i^\bullet$ from $F_{i-1}^\bullet$ by adding an edge e_i connecting two connected components of $F_{i-1}^\bullet$. To choose this connecting edge e_i, we pick any vertex $v \in [n]$ and connect it to the root of one of the $k - i$ connected component of $F_{i-1}^\bullet$ not containing v. Note that this choice is imposed by the fact that the rooted forest $F_i^\bullet$ contains the rooted forest $F_{i-1}^\bullet$. Finally, the order in which we have added $e_1, \dots, e_{k-1}$ is the permutation π of $T^\bullet \smallsetminus F^\bullet$.

Q3. Using this algorithm, show that $|X(F^\bullet)| = n^{k-1}(k-1)!$.

Q4. Deduce from the last three questions that the number $T(F)$ of spanning trees of K_n containing the spanning forest F (nothing is rooted here) is given by

$$T(F) = n^{k-2} \prod_{i \in [k]} n_i.$$

Q5. Consider a spanning forest F_m with m connected components of size 2 and $n - 2m$ isolated vertices. What is the number of spanning trees of K_n containing F_m and the number of spanning trees of K_n not containing F_m?

[*Hint: $T(F_0)$ gives the total number of spanning trees of K_n.*]

1.2. **Crossing free spanning trees.** We now consider a set $P = \{p_1, \dots, p_n\}$ of n points in $\mathbb{R}^2$. We denote by $\mathrm{cr}(P)$ the crossing number of P, *i.e.* the number of pairs of crossing edges with endpoints in P. We pick randomly uniformly a spanning tree T on K_n and consider the tree T_P embedded by P (meaning that we consider the geometric graph formed by the segments $p_i p_j$ for all edges ij in T).

Q6. Consider for instance the following two configurations of 4 points in $\mathbb{R}^2$: the configuration A where the 4 points are in convex position, and the configuration B where one point lies in the triangle formed by the other 3 points. Describe all spanning trees of these two configurations, and deduce the expected number of crossings in a random spanning tree of these configurations.

Q7. Consider a given crossing ξ formed by two edges with endpoints in P. What is the probability that ξ belongs to the embedded spanning tree T_P?

Q8. Express the expectation of the number of crossings of the embedded spanning tree T_P in terms of $\mathrm{cr}(P)$ and n.

[*Hint: No problem of dependencies among random variables when computing expectations.*]

2. Almost simplicial polytopes

Recall that a polytope is *simplicial* when all its facets are simplices. In this problem, we are interested in polytopes that are not simplicial, but almost. A d-dimensional polytope P is called

- *k-simplicial* if all its faces of dimension k are simplices,
- *s-almost simplicial* if all its facets are simplices, except one which has $d + s$ vertices.

Q9. What is a d-simplicial polytope? Explain the equivalences:

P is simplicial $\iff$ P is $(d-1)$-simplicial $\iff$ P is 0-almost simplicial.

The goal of the problem is to construct k-simplicial and s-almost simplicial polytopes with many faces, using constructions similar to that of the cyclic polytope seen in the course.

2.1. **$(d-k)$-simplicial polytope.** In this section, we construct a $(d-k)$-simplicial polytope with many faces (generalizing the cyclic polytope seen in the course).

Let $\mathbf{p} = (p_1, \dots, p_k)$ be a k-tuple of continuous functions $p_i : \mathbb{R} \to \mathbb{R}$. Define a curve $\chi_{\underline{p}} : \mathbb{R} \to \mathbb{R}^d$ by $\chi_{\underline{p}}(t) := \big(t, t^2, t^3, \dots, t^{d-k}, p_1(t), \dots, p_k(t)\big)$. We fix some numbers $t_1 < \dots < t_n$ and consider the polytope $Q := \mathrm{conv}(\{\chi_{\underline{p}}(t_1), \dots, \chi_{\underline{p}}(t_n)\})$.

Q10. Show that any $d - k + 1$ points on the curve $\chi_{\underline{p}}$ are affinely independent, and deduce that Q is $(d-k-1)$-simplicial.

[*Hint: compute the rank of the $(d+1) \times (d-k+1)$-matrix* $\begin{bmatrix} 1 & \dots & 1 \\ \chi_{\underline{p}}(t_1) & \dots & \chi_{\underline{p}}(t_{d-k+1}) \end{bmatrix}$ *and conclude.*]

Q11. Show that any subset of at most $\lfloor (d-k)/2 \rfloor$ vertices of Q form a face of Q.

[*Hint: use a well choosen polynomial to define a supporting hyperplane of this face.*]

2.2. **Almost simplicial polytope.** In this section, we construct an s-almost simplicial polytope with many faces, using some results of the previous questions (which can now be admitted if needed).

We consider the real function $p(t) := (n-1)^{(t-1)(d-1)} t(t+1) \dots (t+d+s-1)$, we define the curve $\xi(t) := (t, t^2, ..., t^{d-1}, p(t))$, and we consider the polytope $Q := \operatorname{conv}(\{\xi(t_1), \dots, \xi(t_n)\})$, where we have chosen this time $t_i := -s - d + i$ for all $i \in [n]$.

To analyse this polytope, for any d-tuple of indices $\underline{i} = (i_1, \dots, i_d) \in [n]$ and for any d-tuple of variables $\underline{z} = (z_1, \dots, z_d)$, we define the determinant

$$D(\underline{i}, \underline{z}) := \det \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ \xi(t_{i_1}) & \xi(t_{i_2}) & \dots & \xi(t_{i_d}) & \underline{z} \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ t_{i_1} & t_{i_2} & \dots & t_{i_d} & z_1 \\ t_{i_1}^2 & t_{i_2}^2 & \dots & t_{i_d}^2 & z_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{i_1}^{d-1} & t_{i_2}^{d-1} & \dots & t_{i_d}^{d-1} & z_{d-1} \\ p(t_{i_1}) & p(t_{i_2}) & \dots & p(t_{i_d}) & z_d \end{bmatrix}.$$

and the half-space

$$H_{\underline{i}} := \left\{ \underline{z} \in \mathbb{R}^d \;\middle|\; D(\underline{i}, \underline{z}) \geq 0 \right\}.$$

We denote by $V(\underline{i})$ the Vandermonde determinant

$$V(\underline{i}) := \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ t_{i_1} & t_{i_2} & \dots & t_{i_d} \\ t_{i_1}^2 & t_{i_2}^2 & \dots & t_{i_d}^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_{i_1}^{d-1} & t_{i_2}^{d-1} & \dots & t_{i_d}^{d-1} \end{bmatrix} = \prod_{k<\ell} (t_{i_\ell} - t_{i_k}),$$

Q12. Observe that $p(t_1) = p(t_2) = \dots = p(t_{d+s}) = 0$ and $p(t_i) > 0$ for $d + s + 1 \leq i \leq n$. Deduce that the hyperplane $H_{(1,\dots,d)}$ defines a facet of the polytope Q containing precisely the vertices $\xi(t_1), \dots, \xi(t_{d+s})$.

Q13. Consider now $i_1 < i_2 < \dots < i_d < i_{d+1}$ with $i_{d+1} > d + s$. For any $j \in [d+1]$, we consider the Vandermonde determinant $W_j := V(i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_{d+1})$. Show that

$$D(\underline{i}, \xi(t_{i_{d+1}})) = \sum_{j=1}^{d+1} (-1)^{d+1-j} p(t_{i_j}) W_j.$$

To evaluate this sum, we group terms two by two (leaving the first alone when $d+1$ is odd) and thus consider the term $p(t_{i_{d+1-2k}})W_{d+1-2k} - p(t_{i_{d-2k}})W_{d-2k}$ for any $0 \leq k \leq \lfloor (d+1)/2 \rfloor$. Observe that the definition of $t_i := -s - d + i$ implies that $1 \leq t_{i_q} - t_{i_p} \leq n-1$ for any $1 \leq p < q \leq d+1$. Use these inequalities to show that for any $1 < j \leq d+1$, we have

- $p(t_{i_j})/p(t_{i_{j-1}}) \geq (n-1)^{d-1}$ with a strict inequality when $j = d+1$,
- $W_{j-1}/W_j \leq (n-1)^{d-1}$,

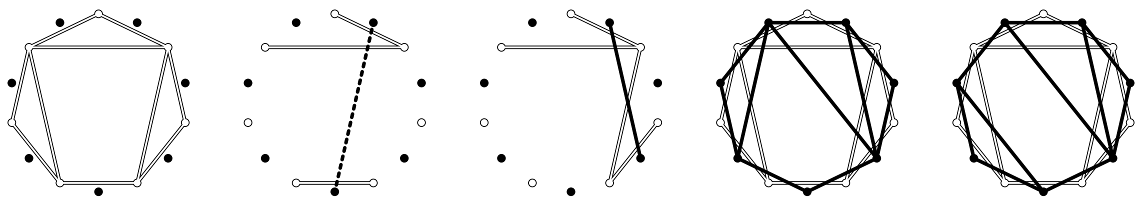
and conclude that $D(\underline{i}, \xi(t_{i_{d+1}})) > 0$ for any choice of $i_1 < i_2 < \dots < i_d < i_{d+1}$ with $i_{d+1} > d + s$.

Q14. Deduce from Question Q13 that except the facet of Question Q12, all other facets of the polytope Q are simplices, and conclude that the polytope Q is a s-almost simplicial polytope.

Q15. Using the computation of determinant of Question Q13, show that a subset $I := \{i_1 < \dots < i_d\}$ with $i_d > d + s$ defines a facet of Q if and only if the number of elements of I between any two elements of $[n] \smallsetminus I$ is even.

3. Accordion complex

Let $n \geq 3$ and consider a $2n$-gon with vertices alternatingly colored black and white. Let $P_\bullet$ and $P_\circ$ denote the n-gons whose vertices are the black and the white vertices respectively. Fix a reference dissection $D_\circ$ of $P_\circ$. A diagonal $\delta_\bullet$ of $P_\bullet$ is a *$D_\circ$-accordion* if the edges of $D_\circ$ crossed by $\delta_\bullet$ form a connected graph. A *$D_\circ$-accordion dissection* is a set of pairwise non-crossing $D_\circ$-accordions.



Reference dissection $D_\circ$ | Not a $D_\circ$-accordion | A $D_\circ$-accordion | A $D_\circ$-accordion dissection $D_\bullet$ | A $D_\circ$-accordion dissection $D'_\bullet$

Figure 1. Examples of $D_\circ$-accordions.

Q16. Assume in this question that $D_\circ$ is a triangulation of $P_\circ$. What are the $D_\circ$-accordions, the $D_\circ$-accordion dissections, and the maximal $D_\circ$-accordion dissections?

We admit here that

- all inclusion-maximal $D_\circ$-accordion dissections contain as many diagonals as $D_\circ$,
- there exists a flip operation illustrated in Figure 1 (right): for any internal diagonal $\delta_\bullet$ in a maximal $D_\circ$-accordion dissection $D_\bullet$, there exists a unique other $D_\circ$-accordion $\delta'_\bullet$ in another maximal $D_\circ$-accordion dissection $D'_\bullet$ such that $D_\bullet \smallsetminus \{\delta_\bullet\} = D'_\bullet \smallsetminus \{\delta'_\bullet\}$.

We consider the flip graph $F(D_\circ)$ on maximal $D_\circ$-accordion dissections. We admit that this flip graph is connected

Q17. Draw the flip graph $F(D_\circ)$ for the reference dissection of Figure 1 (left). You can use two colors instead of black and white, but say explicitly which color you use for the reference dissection $D_\circ$ and which color you use for the $D_\circ$-accordion dissections.

[*Hint: This flip graph has 12 vertices.*]

Q18. Assume that $D_\circ$ contains a cell $C_\circ$ with p edges on the boundary of $P_\circ$, and let $C_\circ^1, \dots, C_\circ^p$ denote the p (possibly empty) connected components of $P_\circ \smallsetminus C_\circ$. For $i \in [p]$, let $D_\circ^i$ denote the dissection formed by the cell $C_\circ$ together with the cells of $D_\circ$ contained in $C_\circ^i$. An example is shown in Figure 2. Prove that the flip graph $F(D_\circ)$ is isomorphic to the Cartesian product $F(D_\circ^1) \times \cdots \times F(D_\circ^p)$.

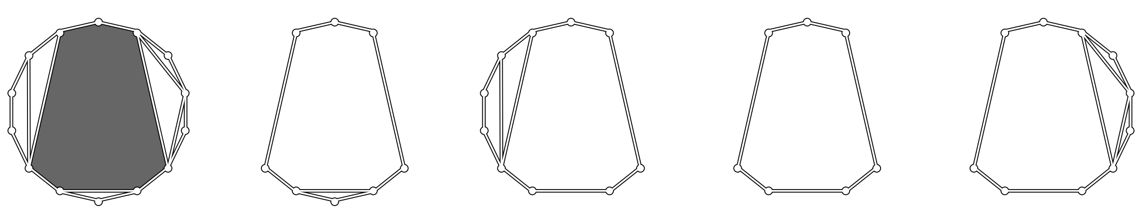


Figure 2. A reference dissection $D_\circ$ with a shaded cell $C_\circ$ (left) and the corresponding four reference dissections $D_\circ^1, D_\circ^2, D_\circ^3, D_\circ^4$ (right).