EXAM COURSE 2-38-2 MPRI 2017
ALGORITHMS AND COMBINATORICS FOR GEOMETRIC GRAPHS
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The course and your personal notes are authorized. Electronic devices are forbidden.

Prepare two separated sheets for the two halves of the course.

The two problems are independent and can be treated in an arbitrary order. Clearly indicate the number of the question in front of your answer. You can skip some questions if you are stuck. However, it is recommended to treat a coherent part of the subject rather than sporadic questions.

The care in the redaction and presentation of your solution will be considered in the notation.

1. Boxicity of a graph

1.1. **Interval graphs.** We consider a finite set V and define $\binom{V}{2} := \{\{u,v\} \mid u \neq v \in V\}$. Consider a set $\mathcal{I} := \{I_v \mid v \in V\}$ where $I_v := [x_v, y_v]$ is an interval of $\mathbb{R}$. The *interval graph* of $\mathcal{I}$ is the graph $G_\mathcal{I}$ with vertex set V and edge set $\big\{\{u,v\} \in \binom{V}{2} \mid I_u \cap I_v \neq \varnothing\big\}$.

Q 1. What is the interval graph of $\{[1,4],[2,6],[3,8],[5,9],[7,10]\}$? Give a set of intervals with interval graph $G = (V,E)$ where $V = \{a,b,c,d,e\}$ and $E = \{ab, ac, bc, cd, ce, de\}$.

Q 2. Consider an interval graph $G_\mathcal{I} = (V,E)$. Show that:

- all induced cycles in $G_\mathcal{I}$ are triangles,
- there is a partial order $\prec$ on V whose comparability graph is the complement of $G_\mathcal{I}$, *i.e.* such that $\{u,v\}$ is an edge in $G_\mathcal{I}$ if and only if u and v are incomparable in $\prec$.

In fact, this is a characterization of interval graphs, but we skip the proof here.

1.2. **Boxicity.** Consider a set $\mathcal{B} := \{B_v \mid v \in V\}$ where $B_v := [x_v^1, y_v^1] \times \cdots \times [x_v^d, y_v^d]$ is a *box* in $\mathbb{R}^d$ for some $d \geq 1$. The *box graph* of $\mathcal{B}$ is the graph $G_\mathcal{B}$ with vertex set V and edge set $\big\{\{u,v\} \in \binom{V}{2} \mid B_u \cap B_v \neq \varnothing\big\}$. See Figure 1 for an example. Given a graph on V, the *boxicity* of G is the smallest possible dimension d such that there exists a set $\mathcal{B} = \{B_v \mid v \in V\}$ of boxes whose box graph $G_\mathcal{B}$ is G.

Q 3. What is the boxicity of a complete graph?

Q 4. Show that a cycle of length at least 4 has boxicity 2.

Q 5. Consider the intersection $G \cap H = (V, E \cap F)$ of two graphs $G = (V,E)$ and $H = (V,F)$. Show that the boxicity of $G \cap H$ is at most the sum of the boxicities of G and H.

Q 6. What is the boxicity of an interval graph? Show that the boxicity of $G = (V,E)$ is the minimal number d of interval graphs $G_{\mathcal{I}^1} = (V, E^1), \dots, G_{\mathcal{I}^d} = (V, E^d)$ such that $E = E^1 \cap \cdots \cap E^d$.

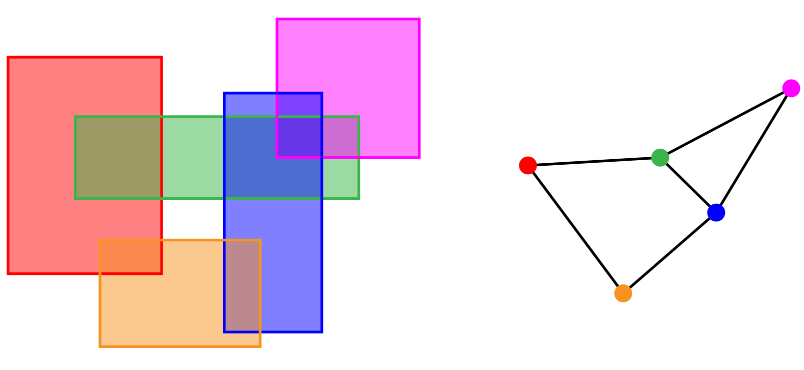

Figure 1. A set $\mathcal{B}$ of rectangles (2-dimensional boxes) and the corresponding box graph $G_\mathcal{B}$.

1.3. **General upper bound.** We now show an upper bound on the boxicity of any graph G.

Q 7. According to Q 3, we can consider a graph $G = (V, E)$ that is not complete. Let u, v be two non-adjacent vertices of G and let $H = G \smallsetminus \{u, v\}$ be the graph G where u and v where deleted. Assume that H is the intersection of d interval graphs $G_{\mathcal{J}^1}, \dots, G_{\mathcal{J}^d}$. Define $d+1$ sets of intervals $\mathcal{I}^1, \dots, \mathcal{I}^{d+1}$ as follows:

- For all $i \in [d]$, let $\mathcal{I}^i := \mathcal{J}^i \cup \{J_u, J_v\}$ where J_u and J_v are intervals that are large enough to intersect all intervals in $\mathcal{J}^1 \cup \dots \cup \mathcal{J}^d$.
- Let $\mathcal{I}^{d+1} := \{I_w \mid w \in V\}$ where

$$I_w := \begin{cases} \{-1\} & \text{if } w = u \\ \{1\} & \text{if } w = v \\ \{0\} & \text{if } (u, w) \notin E \text{ and } (v, w) \notin E \\ [-1, 0] & \text{if } (u, w) \in E \text{ but } (v, w) \notin E \\ [0, 1] & \text{if } (u, w) \notin E \text{ but } (v, w) \in E \\ [-1, 1] & \text{if } (u, w) \in E \text{ and } (v, w) \in E. \end{cases}$$

Show that G is the intersection of the interval graphs $G_{\mathcal{I}^1}, \dots, G_{\mathcal{I}^{d+1}}$.

Q 8. Deduce from the previous question that a graph on n vertices has boxicity at most $n/2$.

Q 9. Consider the graph U_p on $p = 2q$ vertices obtained by deleting a perfect matching M from the complete graph K_{2q}. Show that the boxicity of U_p is at least $q = p/2$. (Hint: Assume that U_p is the intersection of d interval graphs $G_{\mathcal{I}^1}, \dots, G_{\mathcal{I}^d}$. Show that each edge of the matching M is missing in at least one of the interval graphs $G_{\mathcal{I}^k}$ and that two edges of the matching M cannot be missing in the same interval graph $G_{\mathcal{I}^k}$.)

1.4. **Schnyder woods and boxicity.** Thomassen proved that planar graphs have boxicity at most 3. The goal of this section is to prove this result for triangulations using Schnyder woods.

Consider a triangulation $T = (V, E)$ endowed with a Schnyder wood (T^1, T^2, T^3). In other words, T^1, T^2, T^3 are three spanning trees of T, which partition the edges of T (except the edges of the outer face which are all contained in two of these trees), and which fulfill Schnyder's local conditions around each vertex. Note that in contrast to the general case seen in the course, only the edges of the external face are bioriented since T is a triangulation. Consider a vertex $v \in V$. We denote by $R^i(v)$ the region of T bounded by the paths from v to the root of the trees T^{i-1} and T^{i+1}, and we let $r^i(v) = |R^i(v)|$. We define $x^i_v := r^i(v)$ and $y^i_v := r^i(v^i)$, where v^i is the parent of v in the tree T^i. Note that when v is the root of T^i, the vertex v^i is not defined, but we let $y^i_v := r^i(v)+1$. Consider the box $B_v := [x^1_v, y^1_v] \times [x^2_v, y^2_v] \times [x^3_v, y^3_v]$. Finally, let $\mathcal{B} := \{B_v \mid v \in V\}$. We have represented an example in Figure 2.

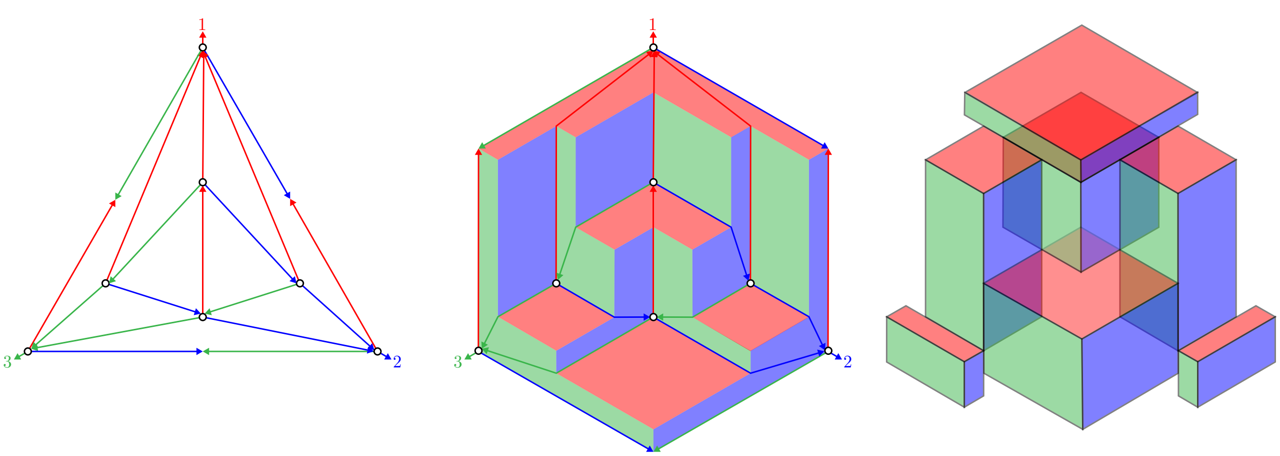

FIGURE 2. A triangulation with a Schnyder wood (left), the corresponding orthogonal surface (middle), and the corresponding box representation (right). The three external boxes have been reduced to let the other ones apparent.

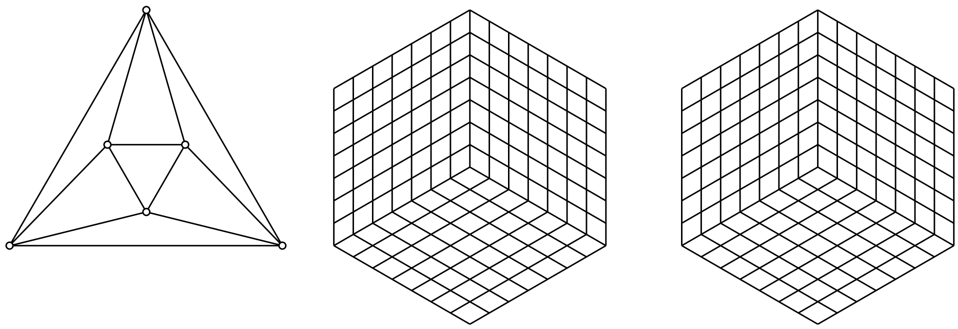

FIGURE 3. The octahedral triangulation (left), and some framework to draw box representations corresponding to two Schnyder woods on O (middle and right).

Q 10. Consider the triangulation O of Figure 3 (left) (note that it is the graph of an octahedron). Compute all possible Schnyder woods on O (Hint: starting from one Schnyder wood, all the other are obtained by returning oriented cycles surrounding a face of O.) For each Schnyder wood of O, compute the boxes B_v for all vertices v of O. Finally, draw these boxes as in Figure 2. You can use the framework of Figure 3 (middle and right) to help your drawing (don't forget to insert this in your exam).

Q 11. Consider two adjacent vertices $u, v \in V$, and assume that u is a child of v in T^i. Show that

$$x_v^{i-1} \le x_u^{i-1} \le y_v^{i-1} \qquad y_u^i = x_v^i \qquad x_v^{i+1} \le x_u^{i+1} \le y_v^{i+1}$$

and conclude that $\{u, v\} \in G_{\mathcal{B}}$.

Q 12. Consider now two non-adjacent vertices $u, v \in V$, let i be such that $u \in R_i(v)$ and let u^i be the parent of u in T^i. Show that $y_u^i < x_v^i$ when

- v lies on the path from u to the root of T^i, or
- u does not lie on the paths from v to the root of the trees T^{i-1} and T^{i+1}.

Deduce that if u and v are non-adjacent vertices in T, then $B_u \cap B_v = \varnothing$ so that $\{u, v\} \notin G_{\mathcal{B}}$.

Q 13. Conclude from the two previous questions that $T = (V, E)$ is the box graph $G_{\mathcal{B}}$ for the set of boxes $\mathcal{B} := \{B_v \mid v \in V\}$.

Q 14. We now consider the planar graph of Figure 4 (left) with two marked vertices u, v. Computing $x^3(u)$ and $y^3(v)$, show that the recipe given for triangulations does not directly work for arbitrary 3-connected planar graphs. This issue is illustrated in Figure 4 where you can see that the boxes corresponding to u and v are disjoint, while u and v should be adjacent.

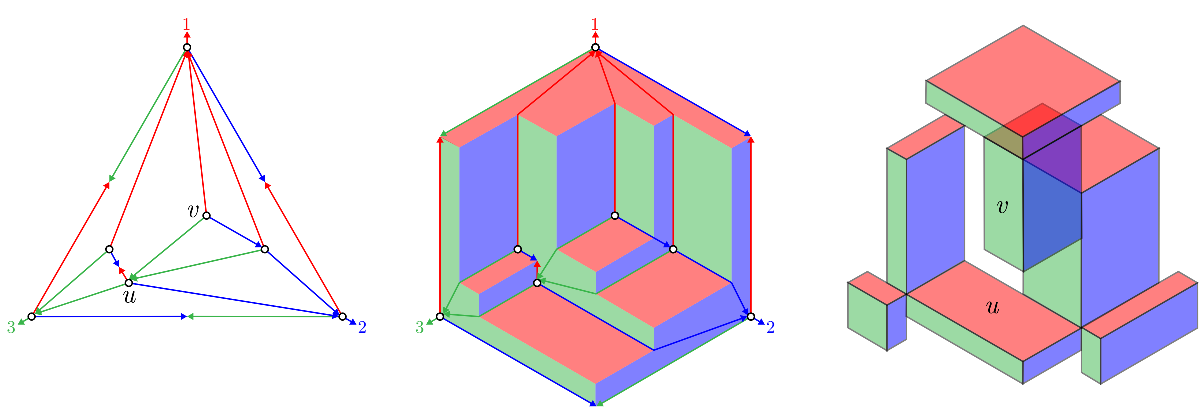

FIGURE 4. A planar graph with a Schnyder wood (left), the corresponding orthogonal surface (middle), and the corresponding incorrect box representation (right). The three external boxes have been reduced to let the other ones apparent.

2. Hamiltonicity of the permutahedron, the associahedron and the cube

A graph $G = (V, E)$ is *Hamiltonian* if there exists a cycle in E that passes exactly once through each vertex of V. The objective of this problem is to find Hamiltonian cycles in the graphs of the permutahedron, the associahedron and the cube.

2.1. **The cube.** We start with the graph of the cube.

Q 15. Draw an Hamiltonian cycle in the d-dimensional cube for $d \in \{2, 3, 4\}$. Show that the graph of the d-dimensional cube is Hamiltonian for $d \geq 2$. (Hint: By induction, construct a Hamiltonian cycle from Hamiltonian cycles of the $(d-1)$-dimensional cubes given by two opposite facets.)

Q 16. You are in front of d interrupters and try to switch on a light. You know that there is a single position of all interrupters that will switch the light on. You pay each time you switch a single interrupter. Design a strategy to switch as few interrupters as possible and still be sure that you found the position of all interrupters that switches the light on. How much will you pay?

2.2. **The permutahedron.** Recall that the graph of d-dimensional permutahedron is the graph whose vertices are the permutations of $[d+1]$ and whose edges correspond to switching two consecutive positions in a permutation.

Q 17. Using the strategy of Figure 5, describe a Hamiltonian cycle in the d-dimensional permutahedron.

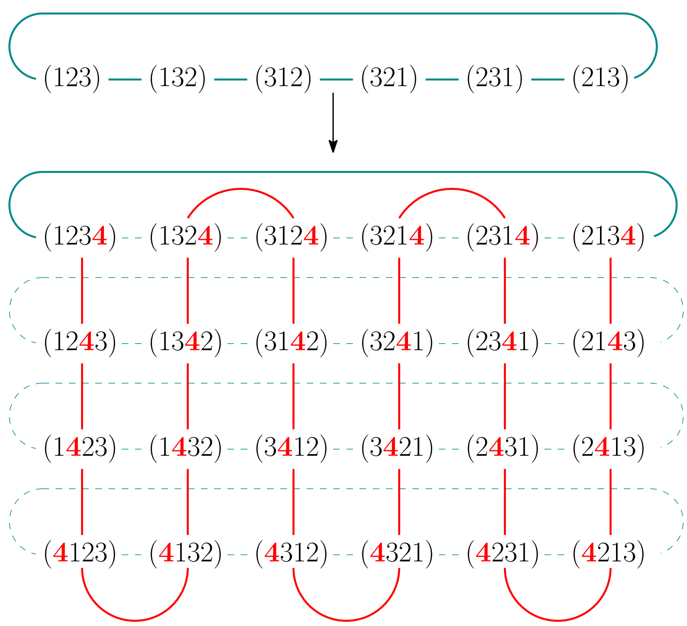

Figure 5. A Hamiltonian cycle in the graph of the permutahedron.

2.3. **The associahedron.** Recall that the graph of the d-dimensional associahedron is the graph whose vertices are the triangulations of a labeled $(d+3)$-gon and whose edges are flips between them. To obtain a Hamiltonian cycle in the associahedron, we first organize all triangulations in a tree as illustrated in Figure 6:

- a triangulation T of the $(d+3)$-gon will appear at level d in this tree,
- the parent of T in the tree is the triangulation $p(T)$ obtained by contracting the triangle containing the edge $[d+2, d+3]$ of the polygon,
- conversely the children of T are the triangulations $p_i(T)$ for all edges $(i, d+3)$ in T, where $p_i(T)$ is obtained by inflating the edge $(i, d+3)$ of T into a triangle.

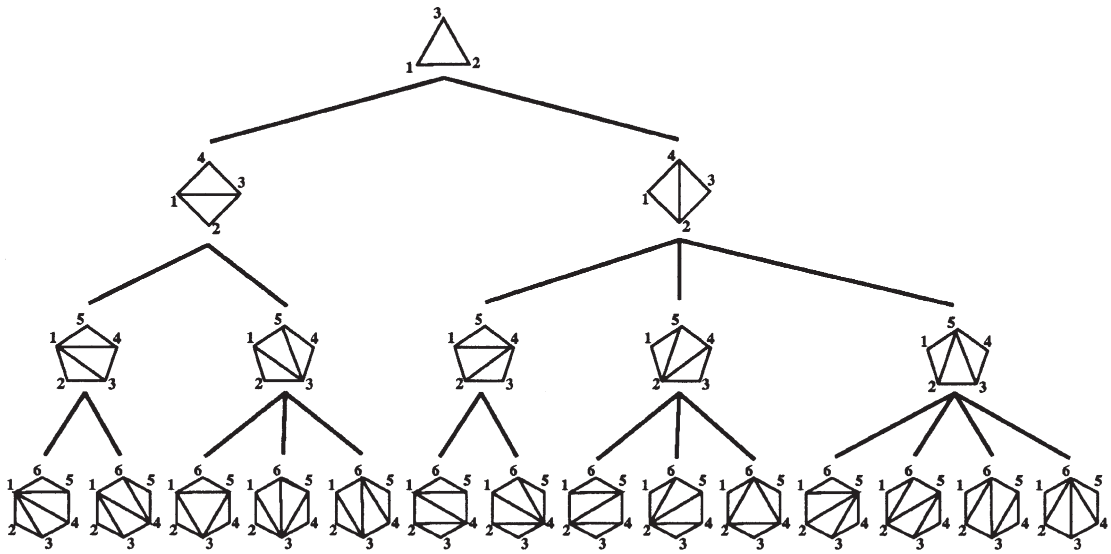

FIGURE 6. The tree of triangulations.

Q 18. Given a triangulation T, show that there is a path starting at $p_1(T)$, passing through all children $p_i(T)$ for $(i, d+3)$ in T, and finishing at $p_{d+2}(T)$.

Q 19. Show that if T and T' are related by a flip, then their children $p_1(T)$ and $p_1(T')$ are also related by a flip (and similarly $p_{d+2}(T)$ and $p_{d+2}(T)$ are also related by a flip).

Q 20. Consider the triangulation X_d of the $(d+3)$-gon whose diagonals are all incident to the vertex $d+3$, and the triangulation Y_d obtained from X_d by flipping the diagonal $[2, d+3]$. Check that $p_1(X_d) = X_{d+1}$ and $p_2(X_d) = Y_{d+1}$.

Q 21. Recall the recurrence relation on the number t_d of triangulations of the $(d+3)$-gon. Deduce that if t_d is odd, then d is even.

Q 22. Deduce by induction from the previous questions that there is a Hamiltonian cycle in the d-dimensional associahedron where X_d and Y_d are neighbors. (Hint: No difficulty when t_d is even. When t_d is odd, d is even so that X_d has an even number of children while Y_d has an odd number of children.)