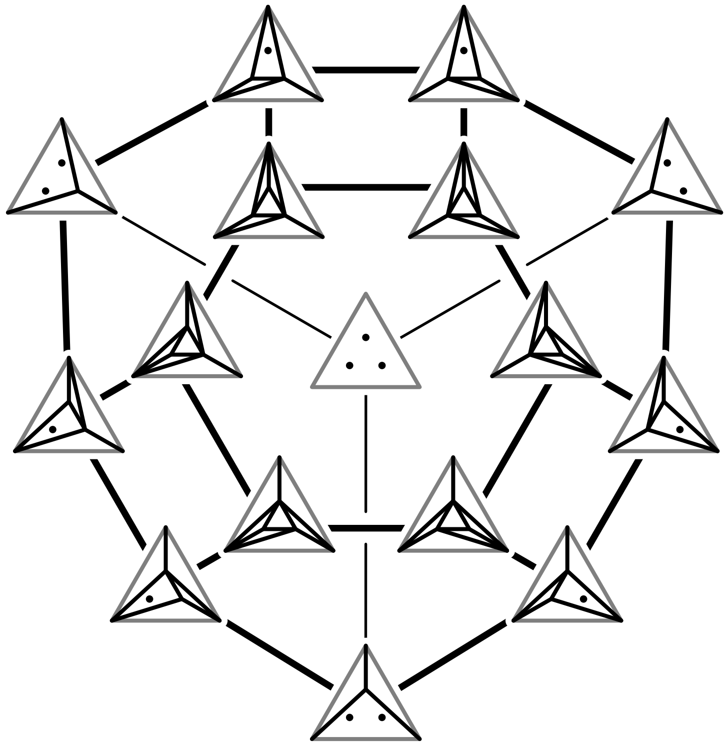
Triangulations



V. PILAUD

MPRI 2-38-1. Algorithms and combinatorics for geometric graphs

Thursday October 29th, 2020

slides available at: http://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/MPRI-2-38-1-VP3.pdf

Course notes available at: https://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/notesCoursMPRI20.pdf

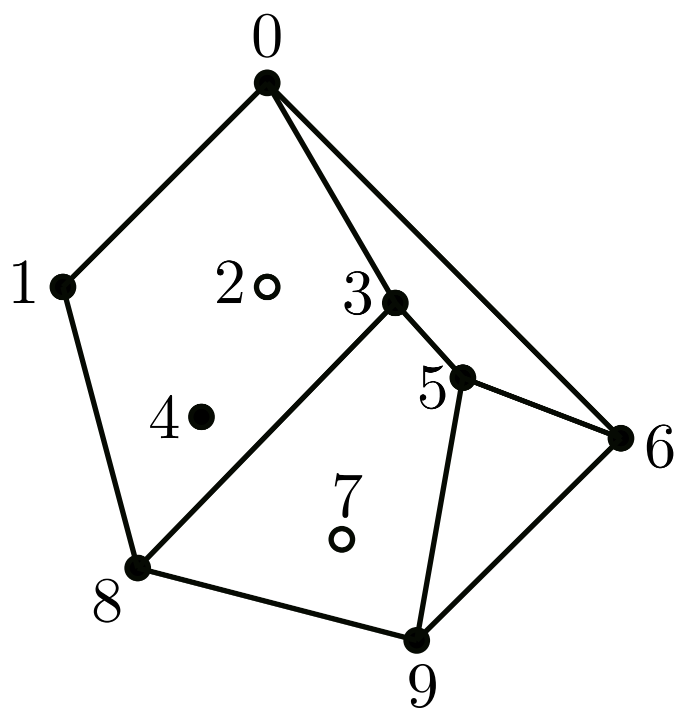
TRIANGULATIONS & SUBDIVISIONS

SUBDIVISIONS

DEF. $\boldsymbol{P}$ = point set in $\mathbb{R}^d$.

polyhedral subdivision of $\boldsymbol{P}$ = collection $\mathcal{S}$ of subsets of $\boldsymbol{P}$ st:

- closure property: if $\mathrm{conv}(\boldsymbol{X})$ is a face of $\mathrm{conv}(\boldsymbol{Y})$ and $\boldsymbol{Y} \in \mathcal{S}$, then $\boldsymbol{X} \in \mathcal{S}$,
- union property: $\mathrm{conv}(\boldsymbol{P}) = \bigcup_{\boldsymbol{X} \in \mathcal{S}} \mathrm{conv}(\boldsymbol{X})$,
- intersection property: $\mathrm{conv}(\boldsymbol{X})$ and $\mathrm{conv}(\boldsymbol{Y})$ have disjoint relative interiors and intersect along a face of both, for any $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{S}$.

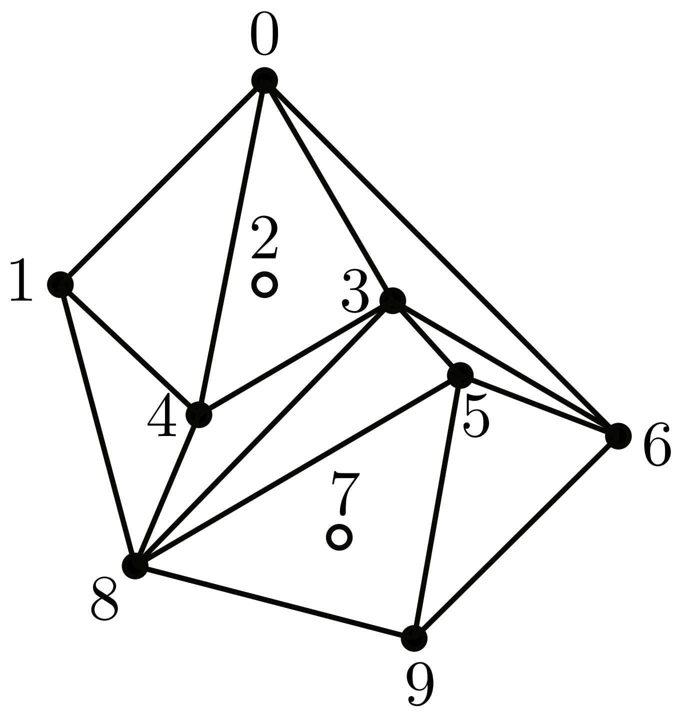


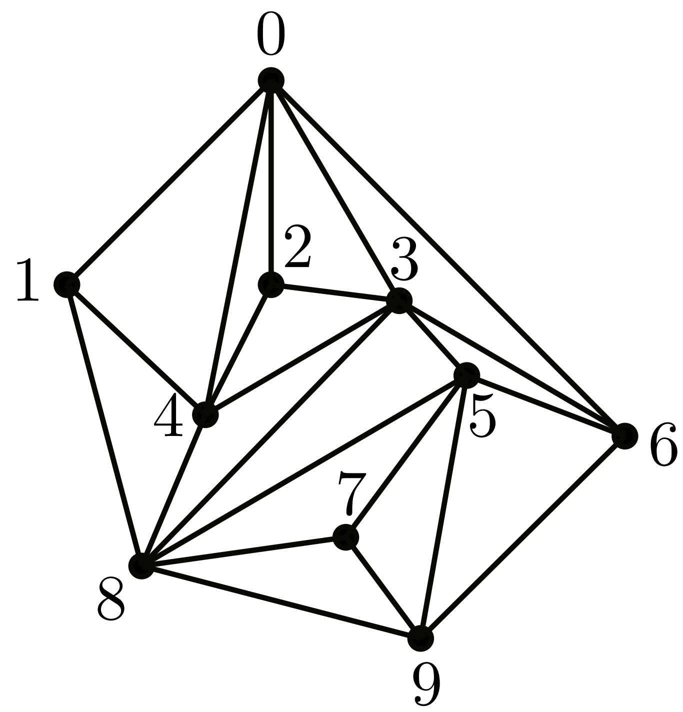
$\mathcal{S} = \{01348, 0356, 3589, 569\}$ + all faces...

TRIANGULATIONS

DEF. triangulation = subdivision $\mathcal{T}$ where all subsets are affinely independent.
(in particular, $\mathrm{conv}(\boldsymbol{X})$ is a simplex for all $\boldsymbol{X} \in \mathcal{T}$).

full triangulation = each point belongs to at least one simplex.





QU. Show that any full triangulation of a planar point set with i interior and b boundary points has $i + b$ vertices, $3i + 2b - 3$ edges, and $2i + b - 2$ triangles.

TRIANGULATIONS IN 3 DIMENSION

QU. What is the minimum / maximum number of simplices that triangulate the 3-cube?

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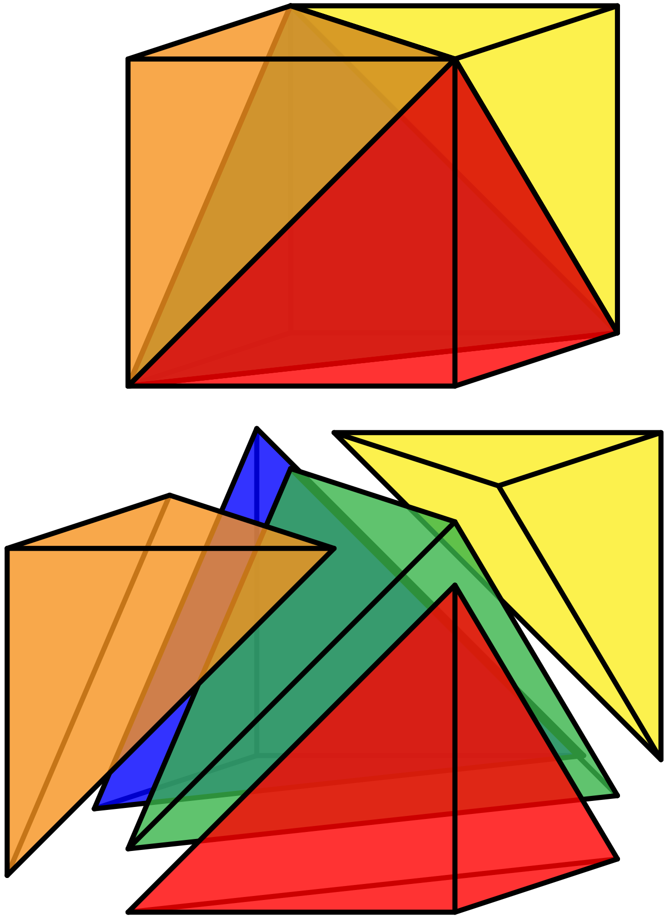
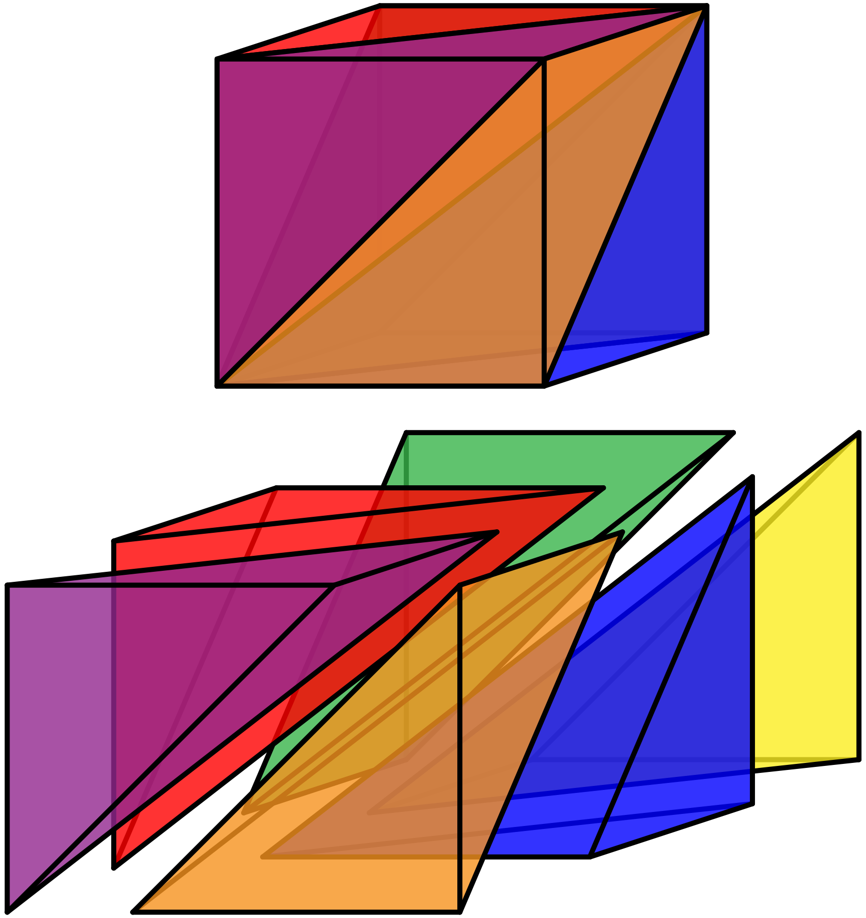


minimum $= 5$



maximum $= 6$

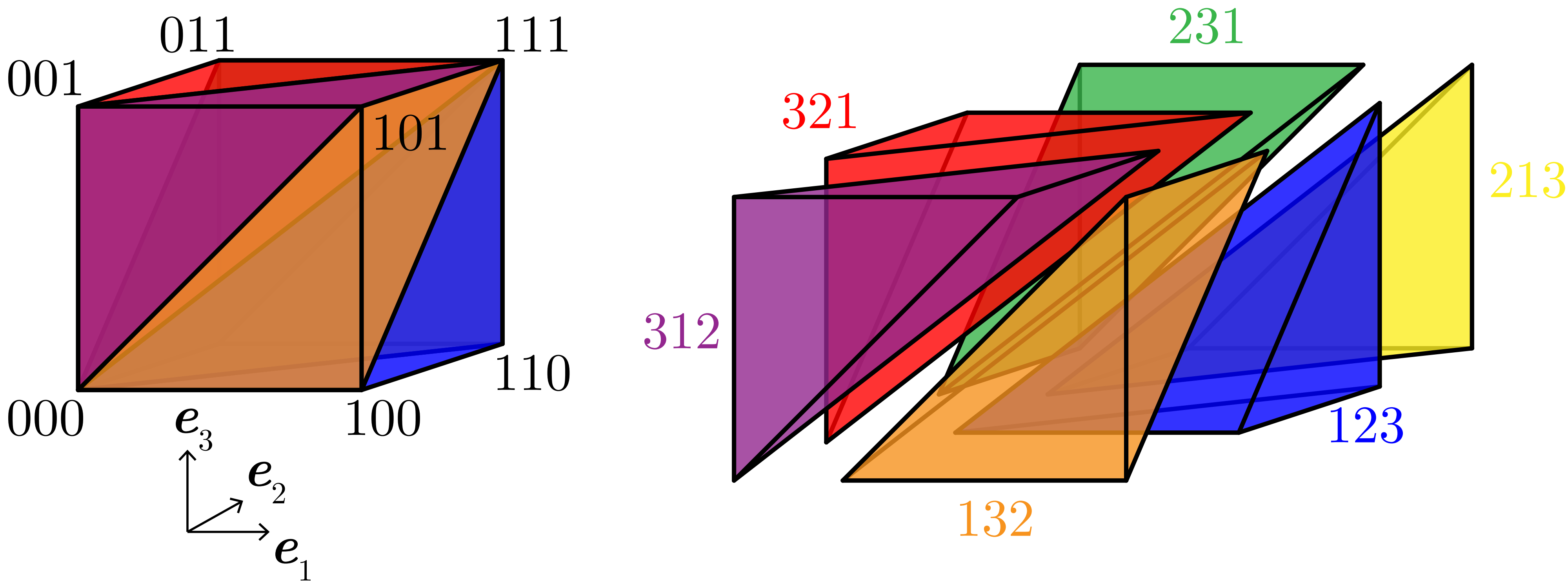
In dimension d, minimum is very difficult, maximum is $d!$

FREUDENTHAL TRIANGULATION

DEF. Freudenthal triangulation of the d-cube $\square_d$ = triangulation with a simplex

$$\triangle_\sigma = \Big\{ \sum_{i \le j} \boldsymbol{e}_{\sigma(i)} \;\Big|\; 0 \le j \le d \Big\} = \Big\{ \boldsymbol{x} \in \square_d \;\Big|\; x_{\sigma(1)} \ge x_{\sigma(2)} \ge \dots \ge x_{\sigma(d)} \Big\}$$

for each permutation $\sigma \in \mathfrak{S}_d$.

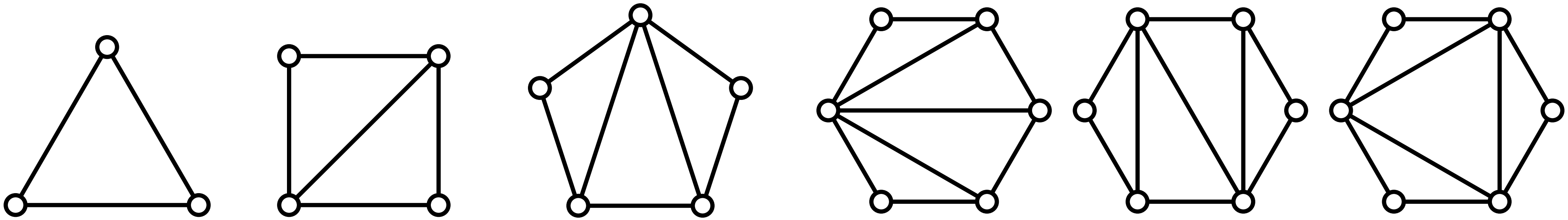


NUMBER OF TRIANGULATIONS

CONVEX POSITION & CATALAN NUMBERS

PROP. number triangulations convex n-gon = Catalan number $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
C_n	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	2674440



1 2 5 6 6 2

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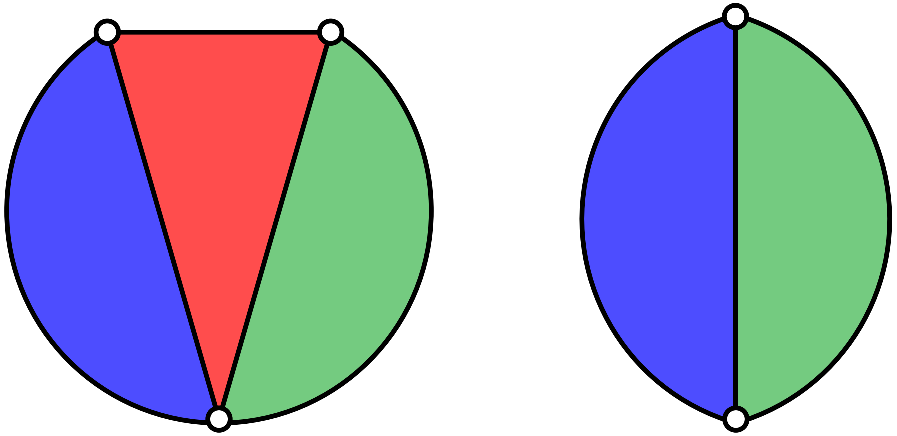
proof A: in the triangulations of the $(n-1)$-gon:

- number of edges $= 2n-5$
- average degree of a vertex $= 2(2n-5)/(n-1)$



Thus, contracting the triangle containing 1 and n, we get the induction formula

$$T_n = \frac{2(2n-5)}{n-1} T_{n-1} \qquad \text{thus} \qquad T_n = \frac{2^{n-3}(2n-5)(2n-7)\dots 3}{(n-1)(n-2)\dots 2} T_3 = \frac{1}{n-1}\binom{2n-4}{n-2}.$$

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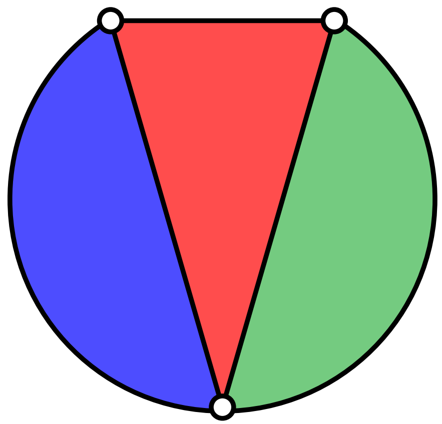


proof B: decomposing the triangulation by the triangle containing 1 and n, we have the summation formula

$$T_n = \sum_{2 \le j \le n-1} T_j \cdot T_{n-j+1}$$

For the generating function $T(x) = \sum_{j \ge 2} T_j x^{j-2}$, this gives

$$T(x) = 1 + x \cdot T(x)^2 \qquad \text{thus} \qquad T(x) = \frac{1+\sqrt{1-4x}}{2x}.$$

We then get T_j developing the series.

QU. Show that the following are Catalan families (ie. counted by Catalan numbers):

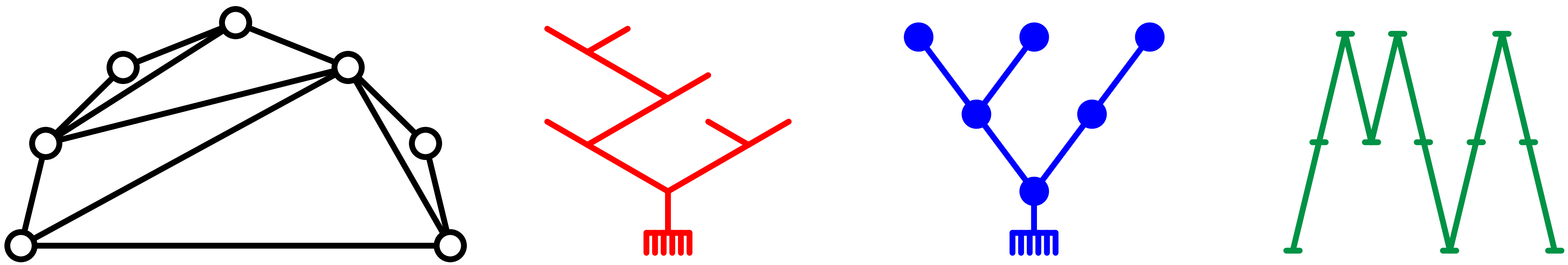
(i) triangulations of a convex n-gon,

(ii) binary trees with $n-2$ internal nodes,

(iii) rooted plane trees with $n-1$ nodes,

(iv) Dyck paths of length $2n-4$ (ie. paths with up steps $\nearrow$ and down steps $\searrow$ starting at $(0,0)$ finishing at $(2n-4,0)$ and which never go below the horizontal axis),

(v) valid bracketings of a non-associative product on $n-1$ elements.



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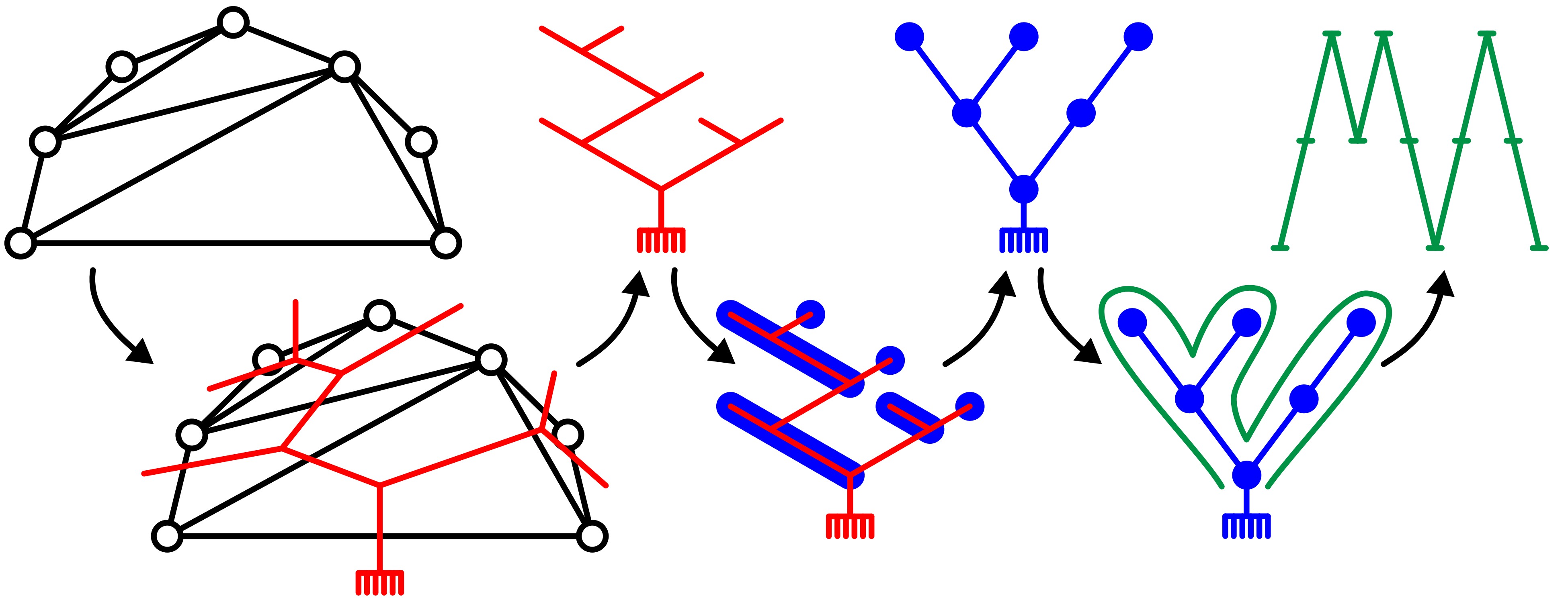
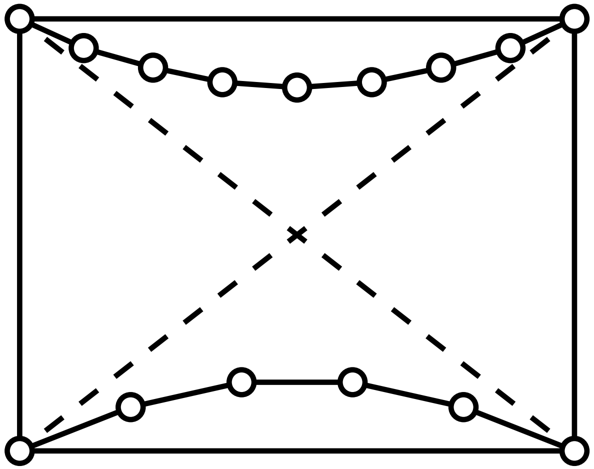
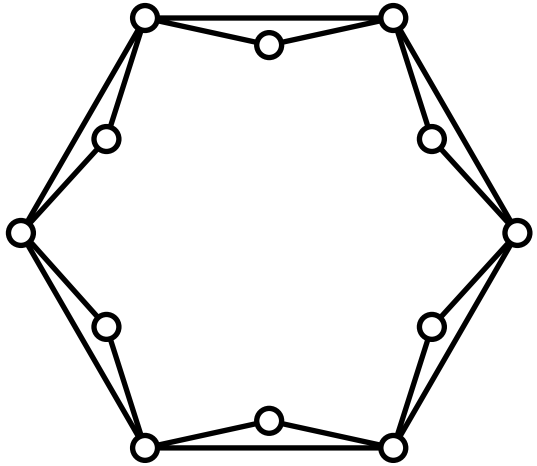


DOUBLE CHAIN AND DOUBLE CIRCLE



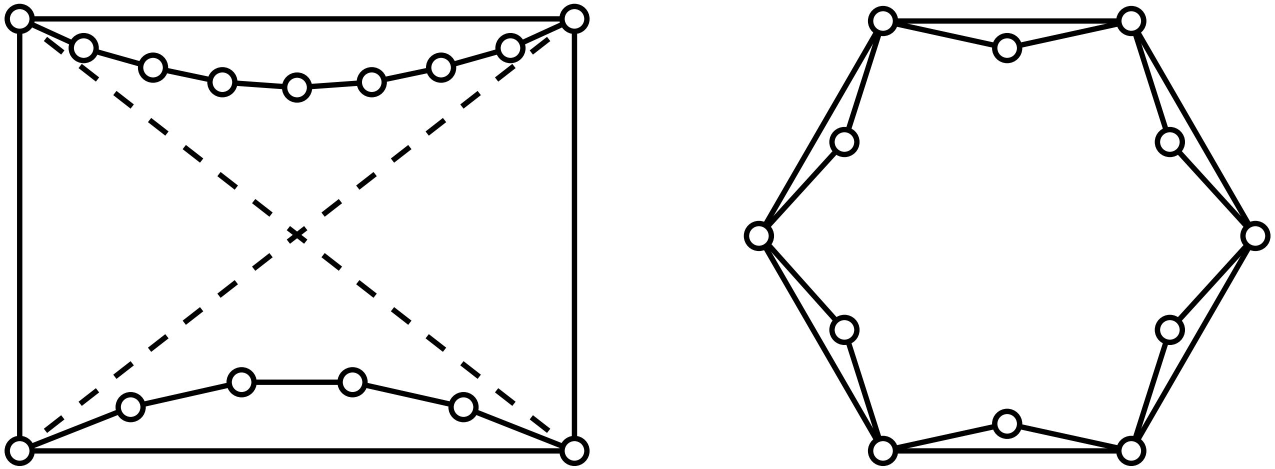
double chain



double circle

QU. Compute the numbers of full triangulations of the double chain and double circle.

DOUBLE CHAIN AND DOUBLE CIRCLE



double chain double circle

PROP. The numbers of full triangulations of the double chain and double circle are

$$C_m C_n \binom{m+n+2}{m+1} \qquad \text{and} \qquad \sum_{i \in [n]} (-1)^i \binom{n}{i} C_{n+i-2}.$$

proof:

- db chain: all edges of the chains belong to full triangulations...
- db circle: inclusion-exclusion for triangulations of convex polygon with no even ear.

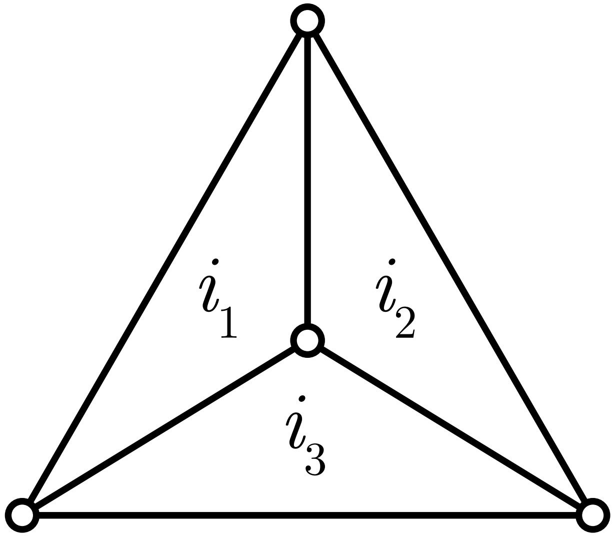
QU. What about all triangulations?

UPPER AND LOWER BOUNDS

THM. Any planar point set in general position with i interior and b boundary points has at least $C_{b-2}2^{i-b+2} = \Omega(2^n n^{-3/2})$ and at most $59^i\, 7^b / \binom{i+b+6}{6} \leq 59^n$ full triangulations.

proof: For the lower bound:

1. if $b = 3$:
   - check it for $i \leq 8$. This is a combinatorial problem!
   - use stacked triangulations:
     each point separates the triangle into three regions with $i = i_1 + i_2 + i_3 + 1$, thus defines at least $2^{i_1-1} \cdot 2^{i_2-1} \cdot 2^{i_3-1} = 2^{i-4}$ stacked triangulations
     thus in total, at least $i2^{i-4} \geq 2^{i-1}$ stacked triangulations.



2. if $b \geq 4$, choose a triangulation of the boundary, and stack in all triangles.

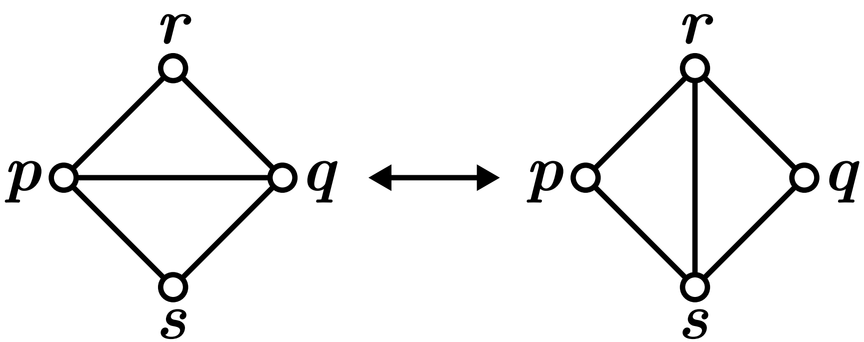
For the upper bound: see poly...

FLIPS

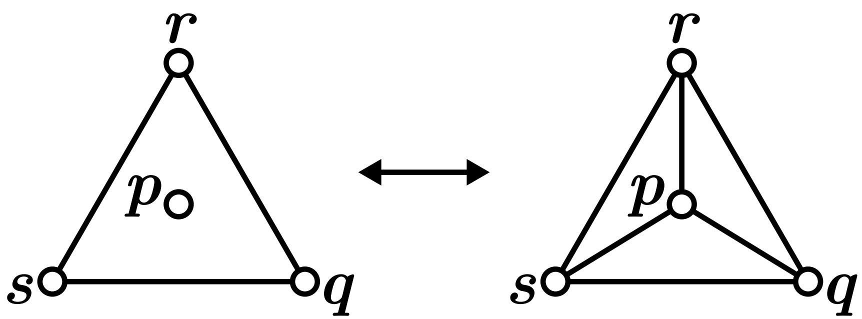
FLIPS

DEF. flip = local operation on triangulations of P defined as:

- diagonal flip = if pqr and prs form a convex quadrilateral $pqrs$, replace the diagonal pr by the other diagonal qs of $pqrs$.

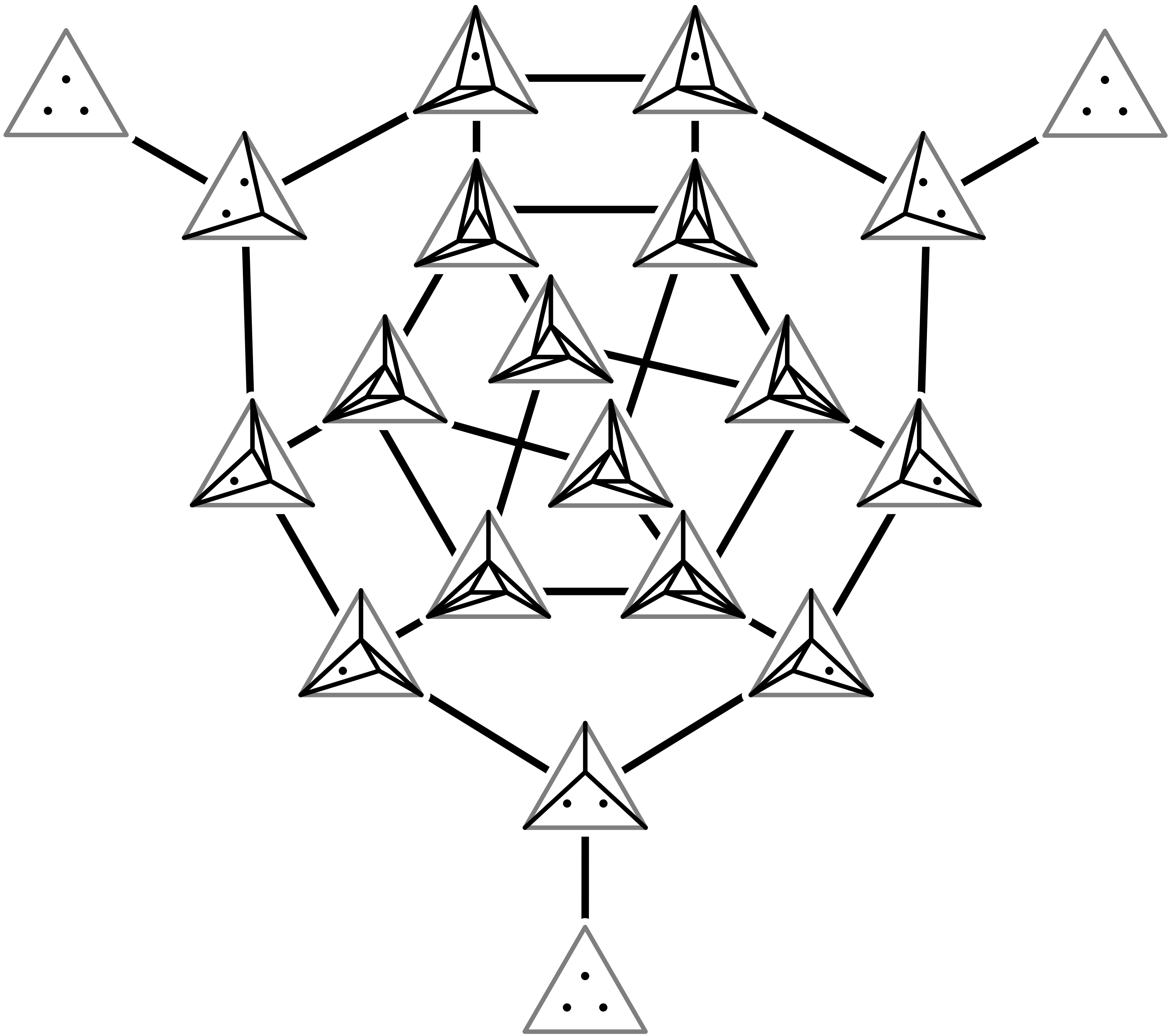


- insertion/deletion flip = if a point p is contained in the interior of a triangle uvw, then insert the edges pu, pv, and pw or vice-versa.



DEF. flip graph = graph with vertices = triangulations and edges = flips.

FLIP GRAPH



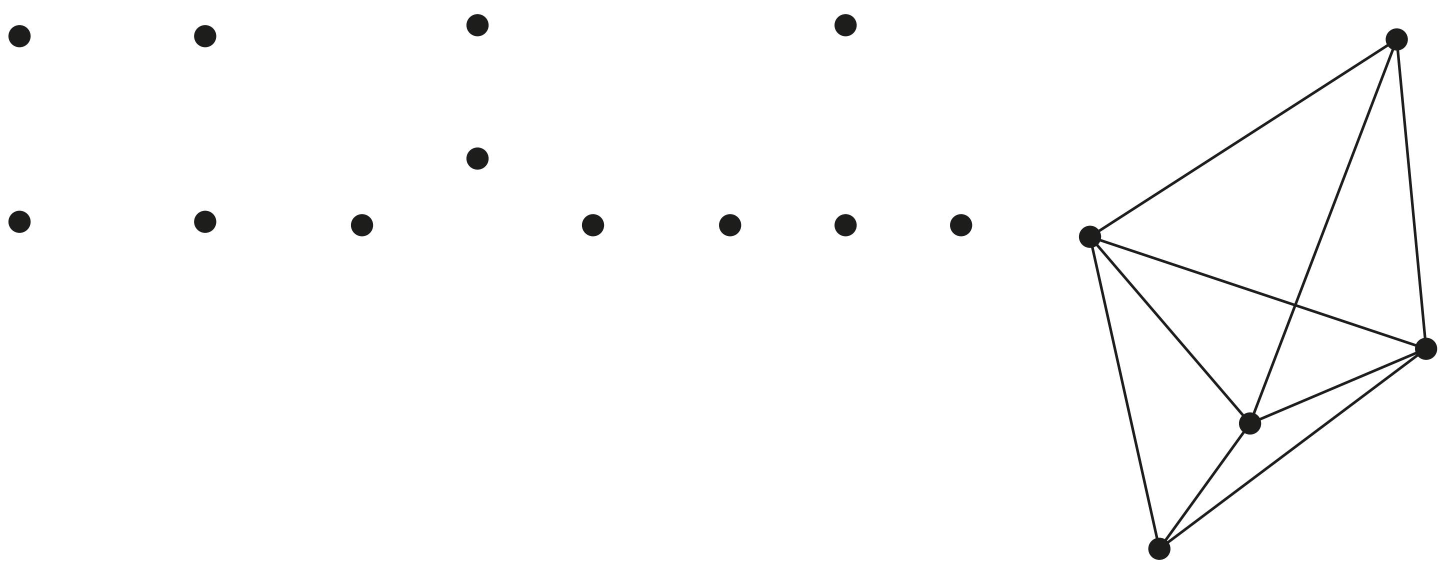
FLIPS IN HIGHER DIMENSION

THM. For any set $\boldsymbol{X}$ of $d+2$ points in $\mathbb{R}^d$, there exists a partition $\boldsymbol{X} = \boldsymbol{X}^+ \sqcup \boldsymbol{X}^- \sqcup \boldsymbol{X}^\circ$ such that $\mathrm{conv}(\boldsymbol{X}^+) \cap \mathrm{conv}(\boldsymbol{X}^-) \neq \varnothing$.

proof: There is an affine dependence $\sum_{\boldsymbol{x} \in \boldsymbol{X}} \lambda_{\boldsymbol{x}} \, \boldsymbol{x} = 0$ with $\sum_{\boldsymbol{x} \in \boldsymbol{X}} \lambda_{\boldsymbol{x}} = 0$ (to see it, linearize).

Let $\boldsymbol{X}^+ = \{\boldsymbol{x} \in \boldsymbol{X} \mid \lambda_{\boldsymbol{x}} > 0\}$ $\quad \boldsymbol{X}^- = \{\boldsymbol{x} \in \boldsymbol{X} \mid \lambda_{\boldsymbol{x}} < 0\}$ $\quad \boldsymbol{X}^\circ = \{\boldsymbol{x} \in \boldsymbol{X} \mid \lambda_{\boldsymbol{x}} = 0\}$.

Then $\Lambda = \sum_{\boldsymbol{x}^+ \in \boldsymbol{X}^+} \lambda_{\boldsymbol{x}^+} = \sum_{\boldsymbol{x}^- \in \boldsymbol{X}^-} (-\lambda_{\boldsymbol{x}^-})$ and $\dfrac{1}{\Lambda} \sum_{\boldsymbol{x}^+ \in \boldsymbol{X}^+} \lambda_{\boldsymbol{x}^+} \, \boldsymbol{x}^+ = \dfrac{1}{\Lambda} \sum_{\boldsymbol{x}^- \in \boldsymbol{X}^-} (-\lambda_{\boldsymbol{x}^-}) \, \boldsymbol{x}^-$.



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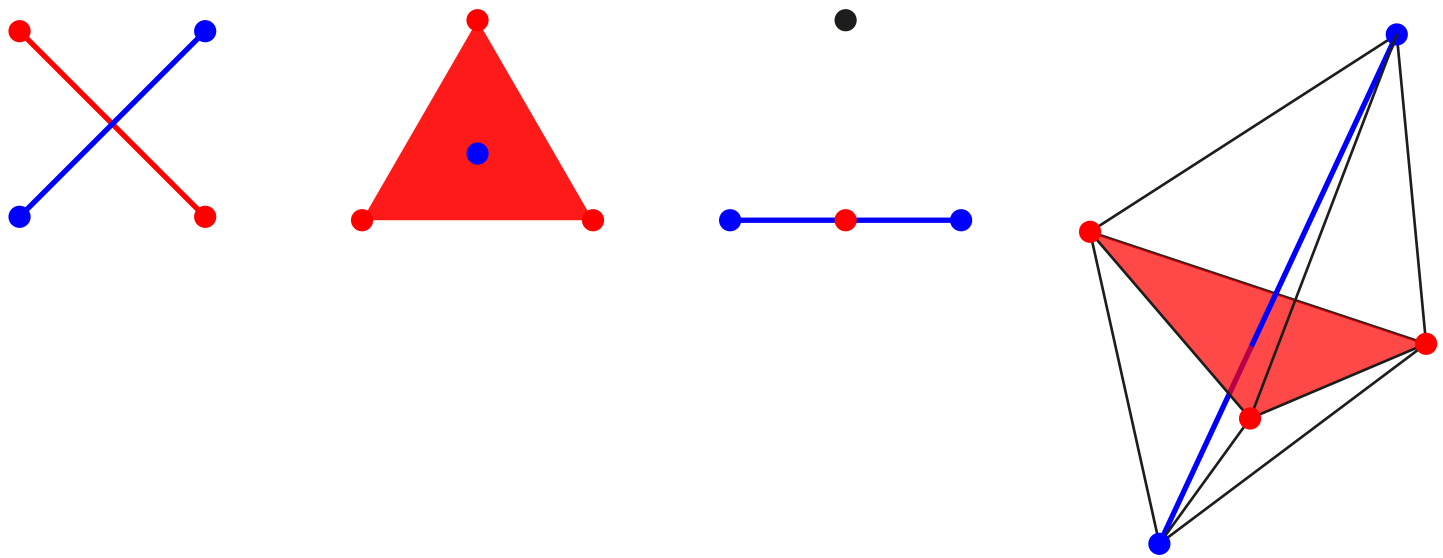

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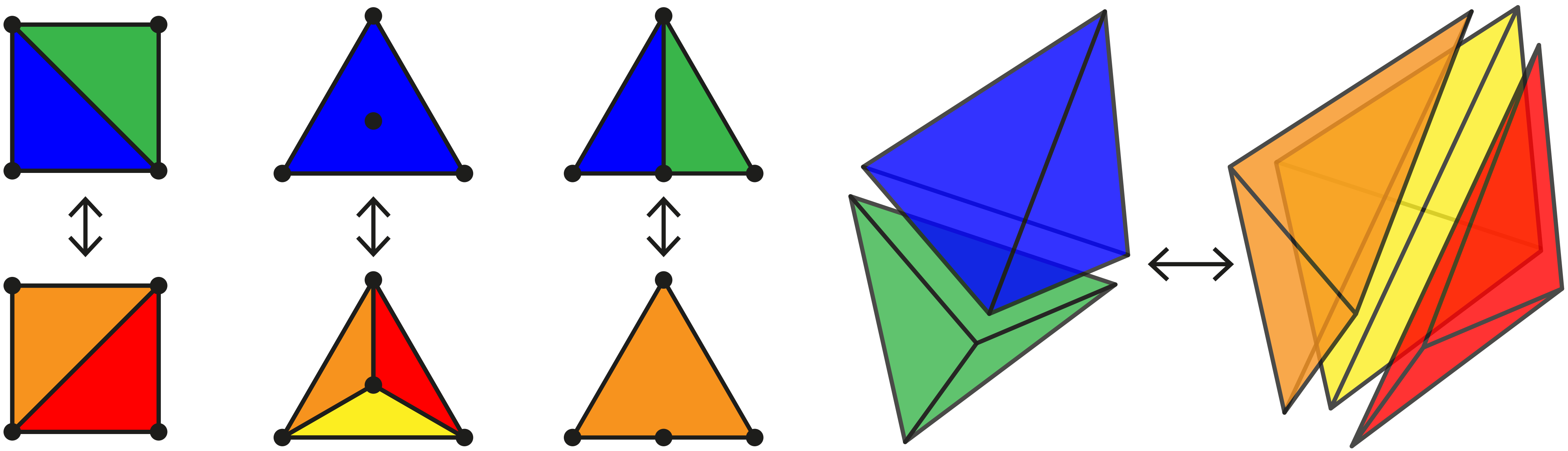
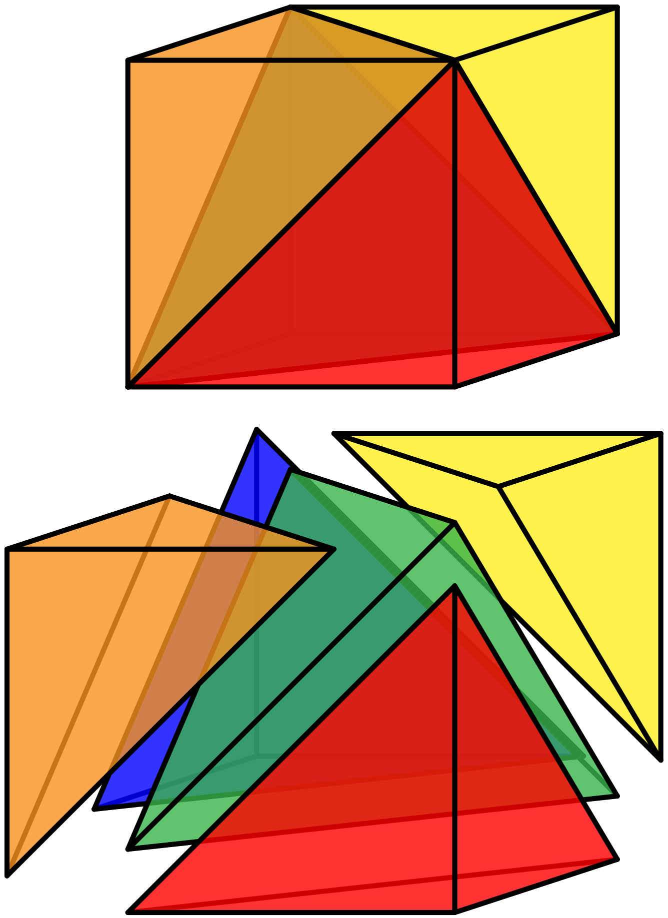
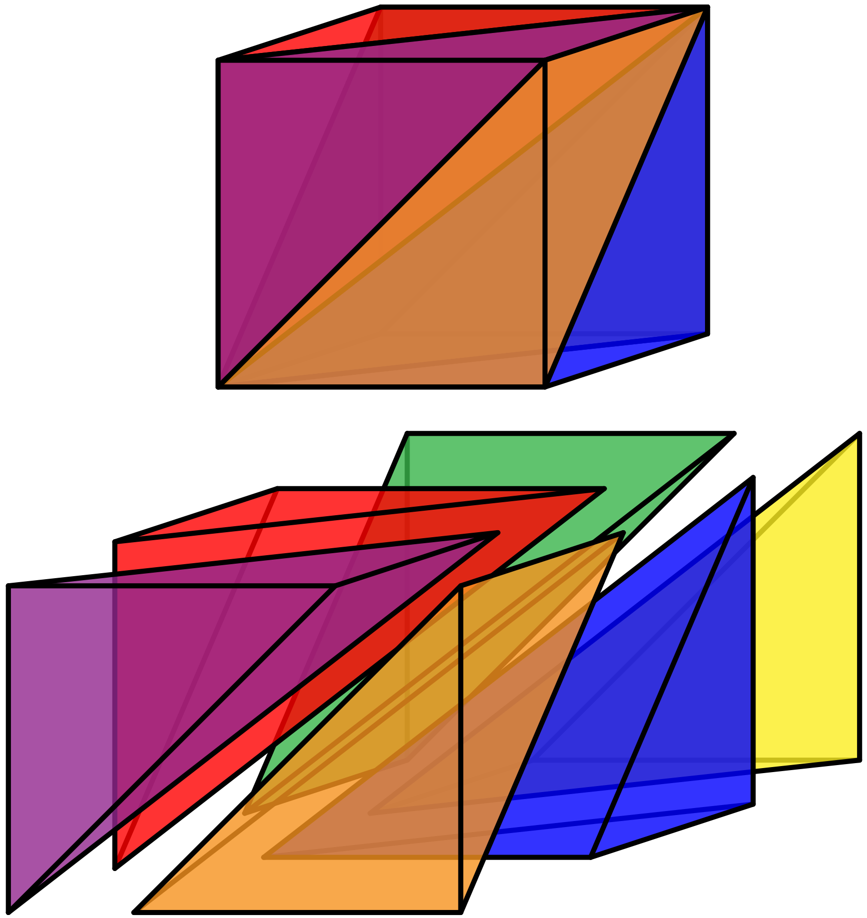
DEF. $\boldsymbol{X}$ set of $d+2$ points in $\mathbb{R}^d$.
$\boldsymbol{X} = \boldsymbol{X}^+ \sqcup \boldsymbol{X}^- \sqcup \boldsymbol{X}^\circ$ Radon partition of $\boldsymbol{X}$ with (inclusion) maximal $\boldsymbol{X}^\circ$.
Bistellar flip $= \left\{ \operatorname{conv}(\boldsymbol{X} \smallsetminus \{\boldsymbol{x}\}) \;\middle|\; \boldsymbol{x} \in \boldsymbol{X}^+ \right\} \longleftrightarrow \left\{ \operatorname{conv}(\boldsymbol{X} \smallsetminus \{\boldsymbol{x}\}) \;\middle|\; \boldsymbol{x} \in \boldsymbol{X}^+ \right\}$



FLIPS IN HIGHER DIMENSION

QU. How many flips to connect these triangulations of the 3-cube?





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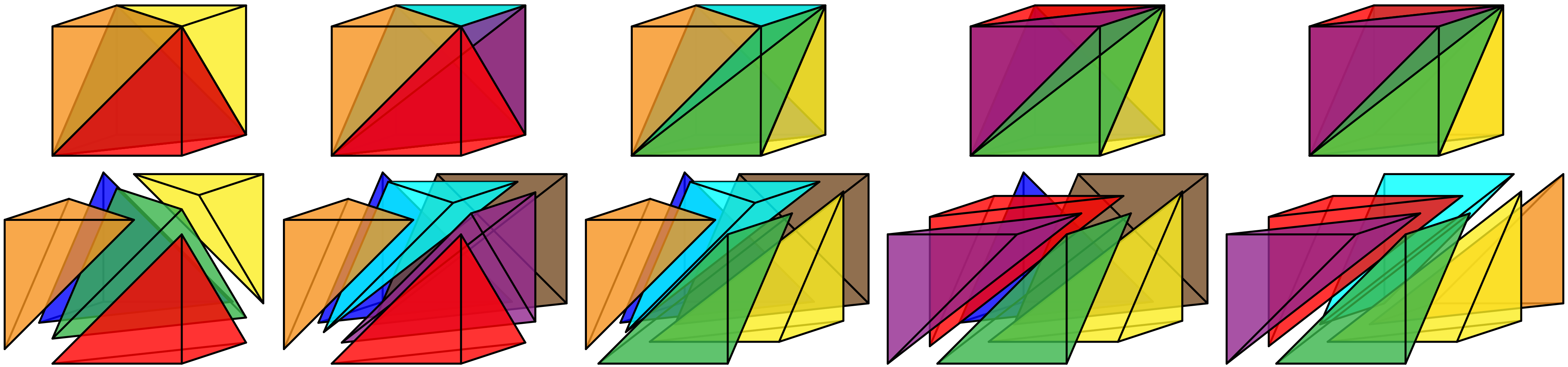


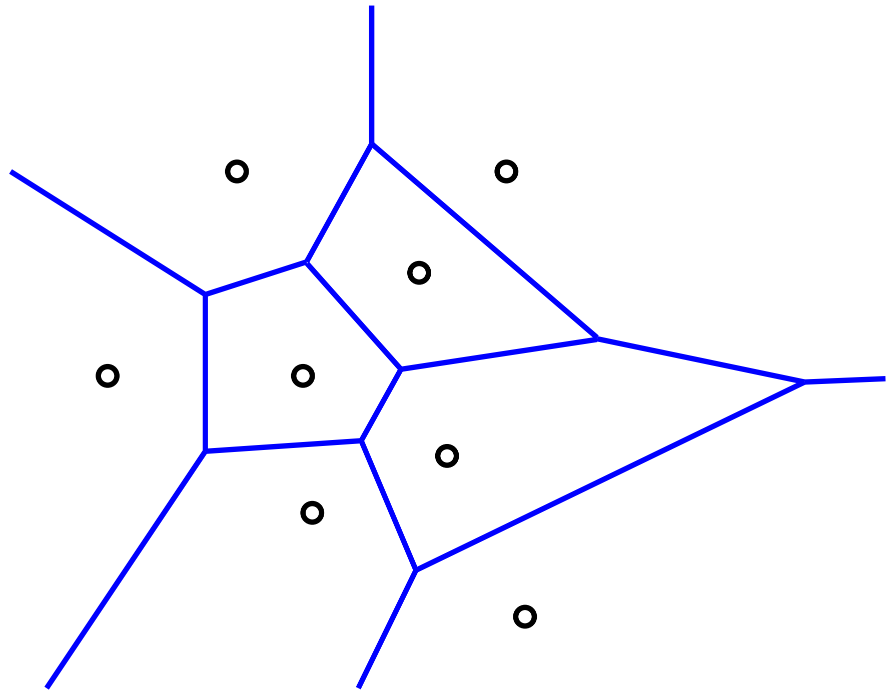
DELAUNAY TRIANGULATION (AGAIN)

VORONOI DIAGRAM

DEF. $\boldsymbol{P}$ = set of sites in $\mathbb{R}^n$.

Voronoi region $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P}) = \left\{\boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \|\boldsymbol{x} - \boldsymbol{p}\| \leq \|\boldsymbol{x} - \boldsymbol{q}\| \text{ for all } \boldsymbol{q} \in \boldsymbol{P}\right\}$.

Voronoi diagram $\mathrm{Vor}(\boldsymbol{P})$ = partition of $\mathbb{R}^n$ formed by $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P})$ for $\boldsymbol{p} \in \boldsymbol{P}$.



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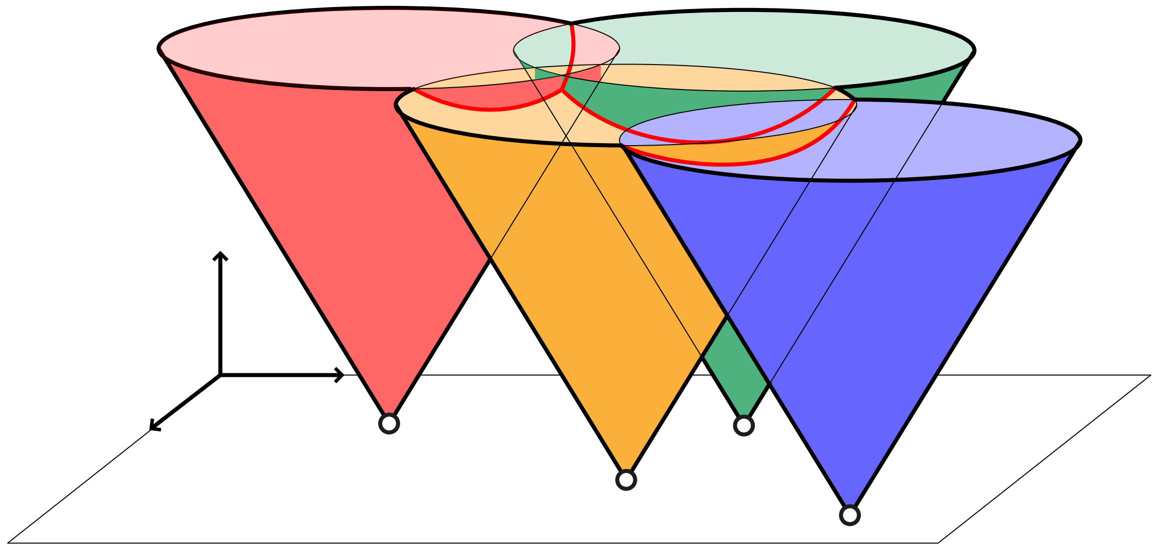
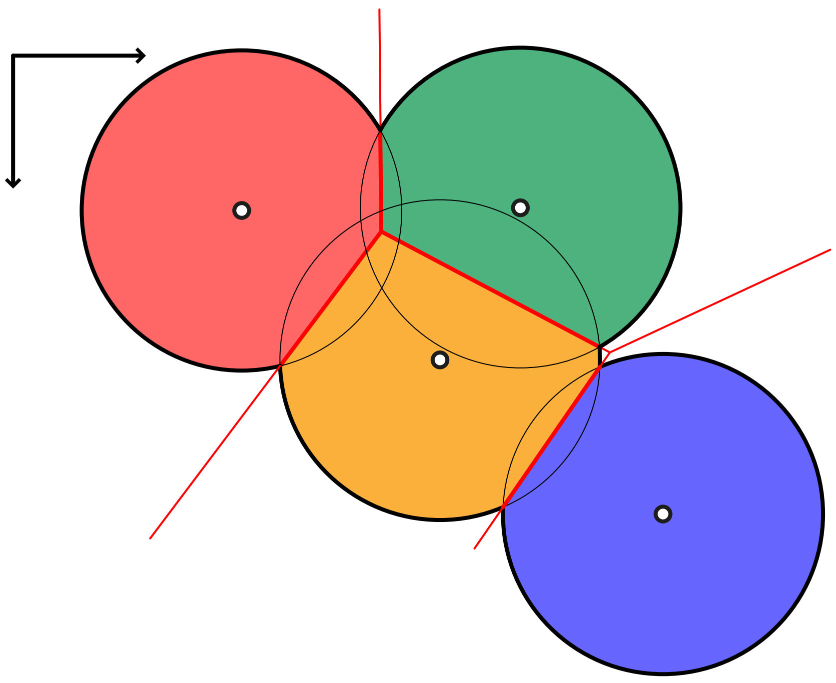
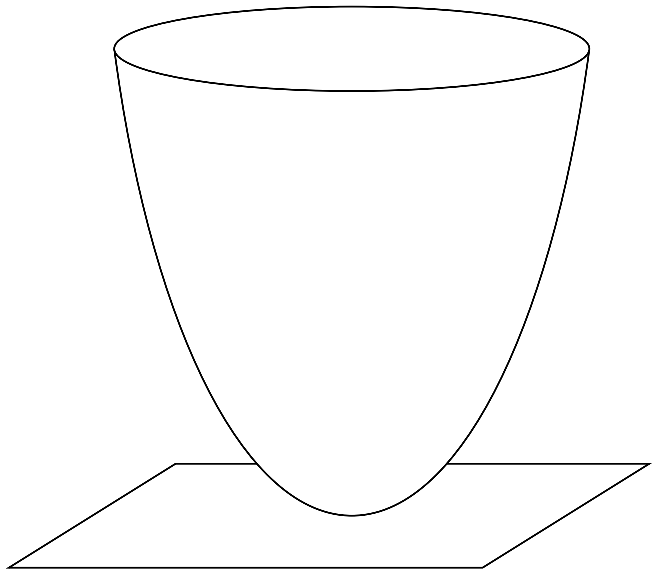




LIFTING POINTS ON THE PARABOLOID

paraboloïd $\mathcal{P}$ with equation $x_{d+1} = \sum_{i \in [d]} x_i^2$.



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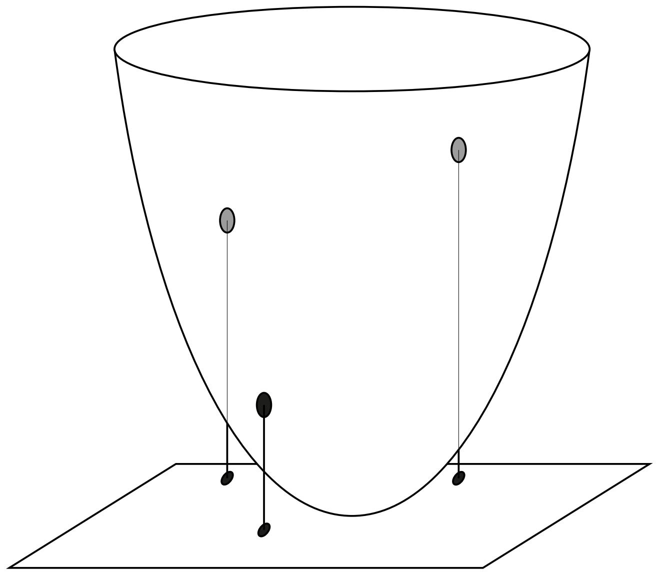

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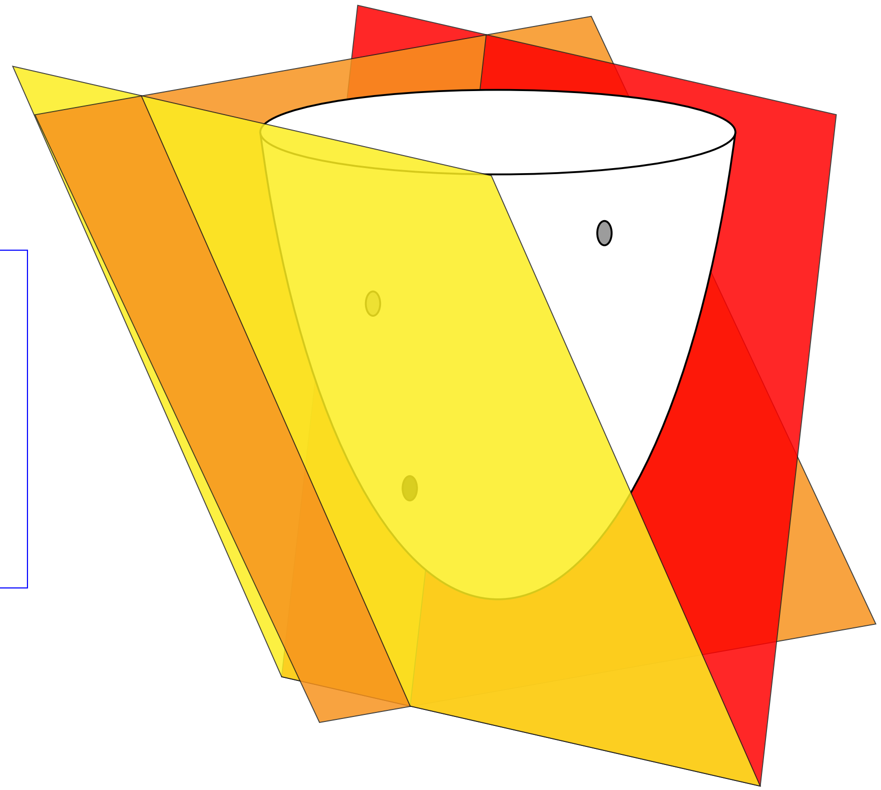

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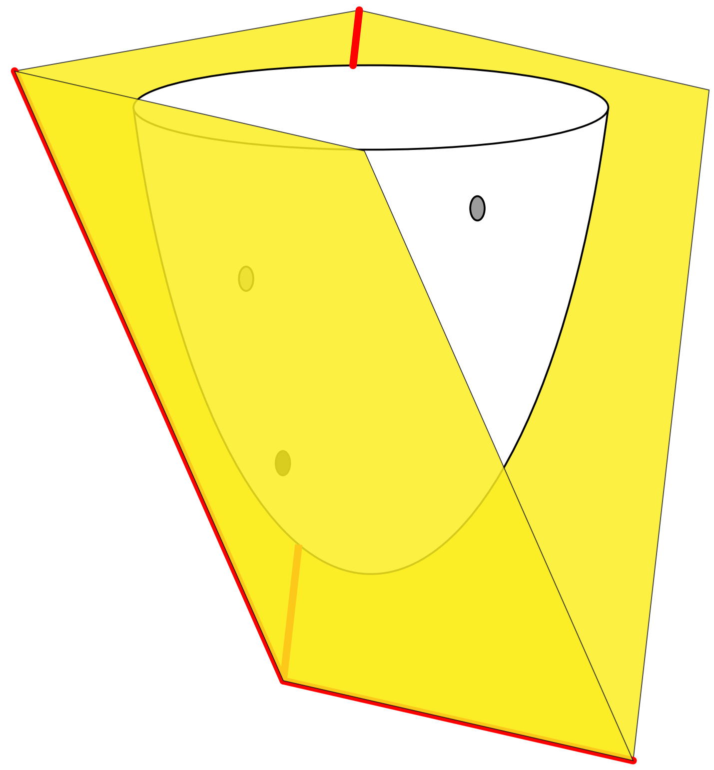

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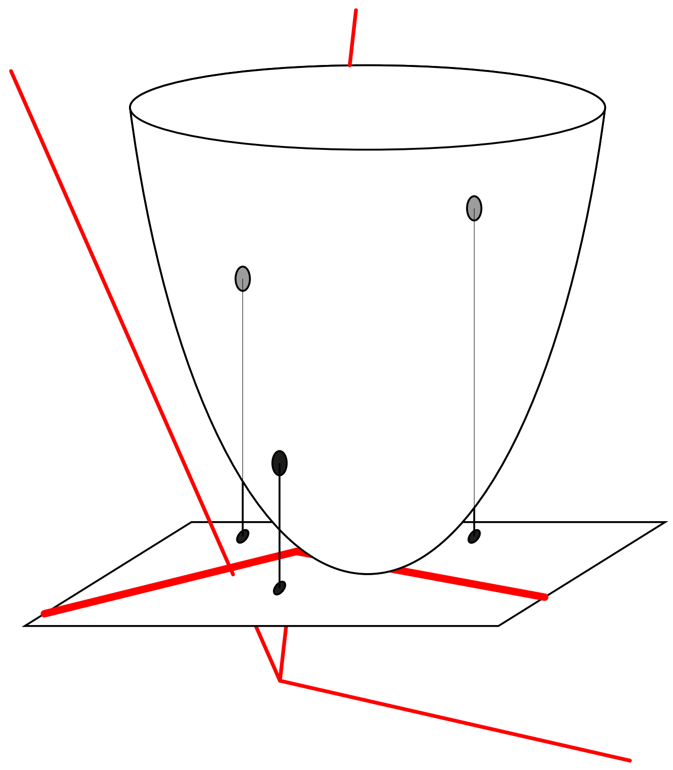

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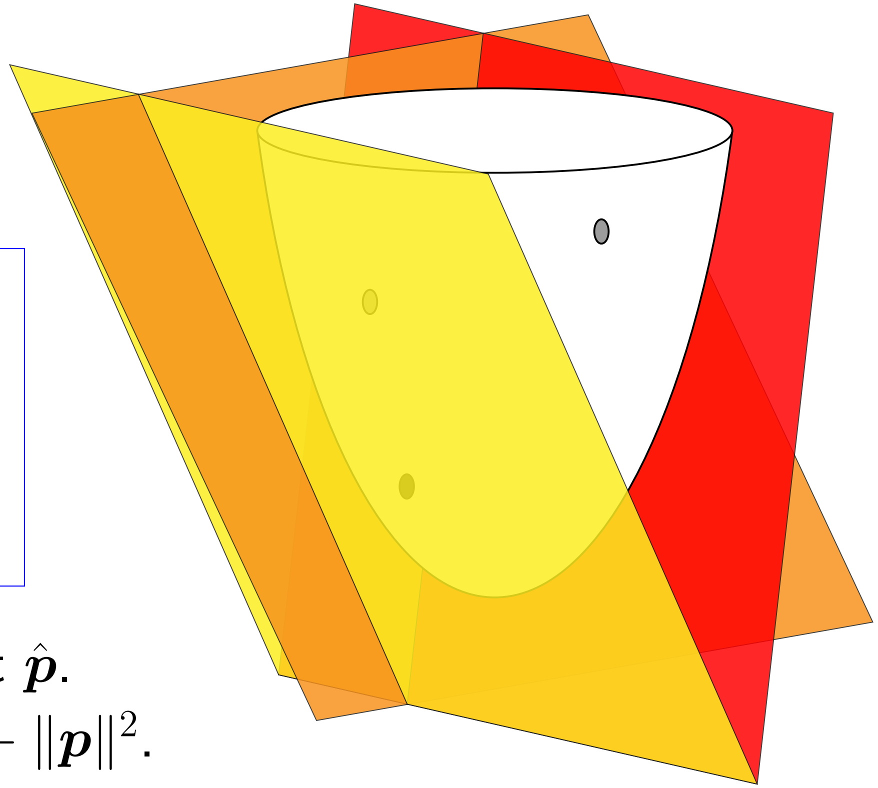

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PROP. The Voronoi diagram $\mathrm{Vor}(\boldsymbol{P})$ is the vertical projection of the upper enveloppe of the planes tangent to the paraboloïd $\mathcal{P}$ at the lifted points $\hat{\boldsymbol{p}}$ for $\boldsymbol{p} \in \boldsymbol{P}$.

proof: $H(\boldsymbol{p})$ = tangent plane to the paraboloïd $\mathcal{P}$ at $\hat{\boldsymbol{p}}$.
= plane of equation $x_{d+1} = 2\langle\, \boldsymbol{p} \mid \boldsymbol{x} \,\rangle - \|\boldsymbol{p}\|^2$.

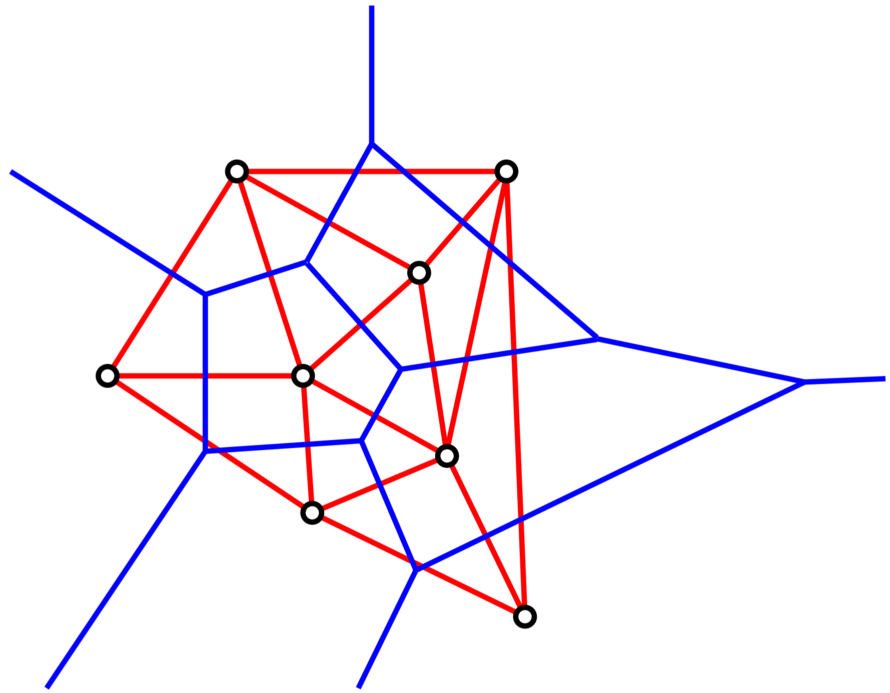
Therefore, $H(\boldsymbol{p})$ above $H(\boldsymbol{q})$ at point $\boldsymbol{x} \iff \|\boldsymbol{x} - \boldsymbol{p}\| \le \|\boldsymbol{x} - \boldsymbol{q}\|$.

DELAUNAY COMPLEX

DEF. $\boldsymbol{P}$ = set of sites in $\mathbb{R}^n$.

Voronoi region $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P}) = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \|\boldsymbol{x} - \boldsymbol{p}\| \leq \|\boldsymbol{x} - \boldsymbol{q}\| \text{ for all } \boldsymbol{q} \in \boldsymbol{P} \right\}$.

Voronoi diagram $\mathrm{Vor}(\boldsymbol{P})$ = partition of $\mathbb{R}^n$ formed by $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P})$ for $\boldsymbol{p} \in \boldsymbol{P}$.



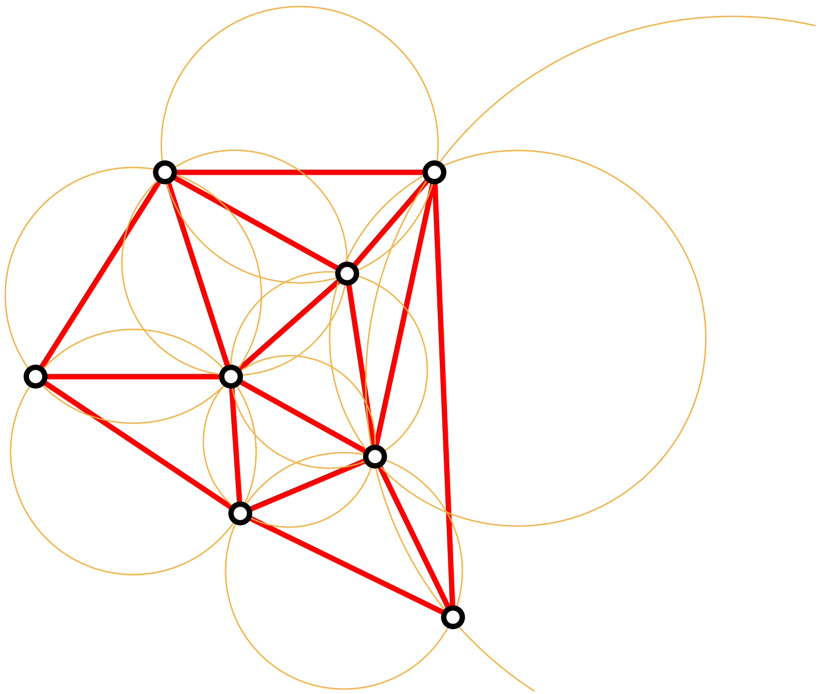
DEF. Delaunay complex $\mathrm{Del}(\boldsymbol{P})$ = intersection complex of $\mathrm{Vor}(\boldsymbol{P})$

$$\mathrm{Del}(\boldsymbol{P}) = \left\{ \mathrm{conv}(\boldsymbol{X}) \;\middle|\; \boldsymbol{X} \subseteq \boldsymbol{P} \text{ and } \bigcap_{\boldsymbol{p} \in \boldsymbol{X}} \mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P}) \neq \varnothing \right\}.$$

EMPTY CIRCLES

PROP. For any three points $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ of $\boldsymbol{P}$,

- $\boldsymbol{pq}$ is an edge of $\mathrm{Del}(\boldsymbol{P}) \iff$ there is an empty circle passing through $\boldsymbol{p}$ and $\boldsymbol{q}$,
- $\boldsymbol{pqr}$ is a triangle of $\mathrm{Del}(\boldsymbol{P}) \iff$ the circumcircle of $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ is empty.



<u>proof idea:</u> consider the circle centered at the intersection of the Voronoi regions and passing through the Voronoi sites.

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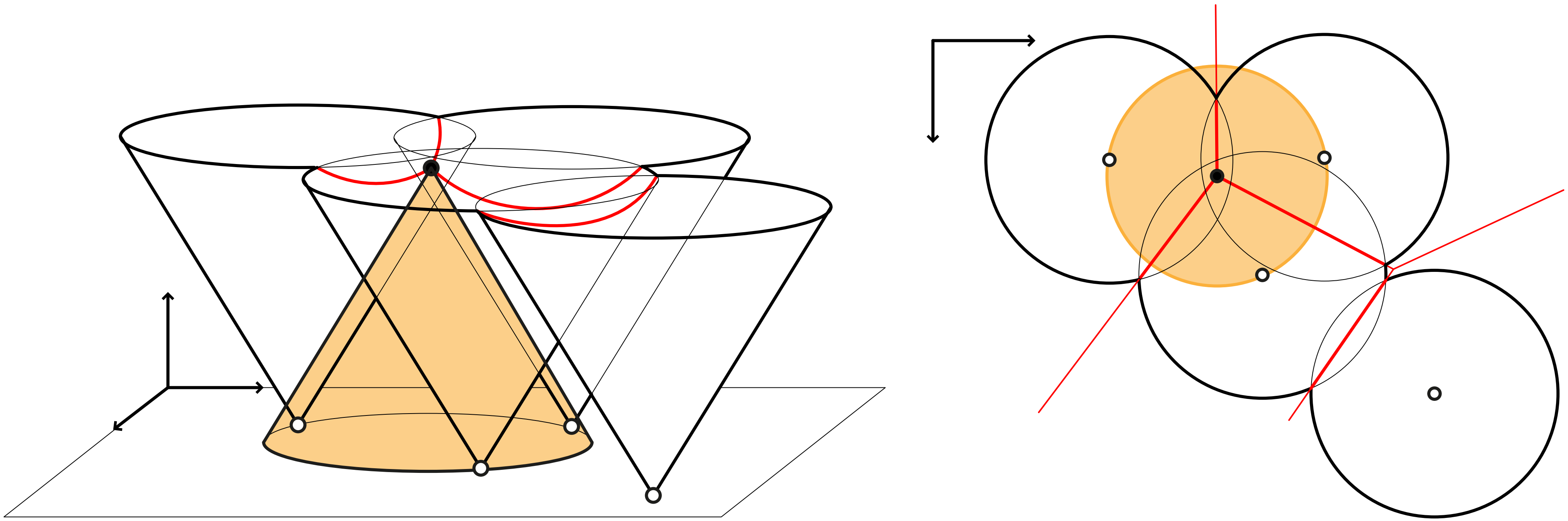
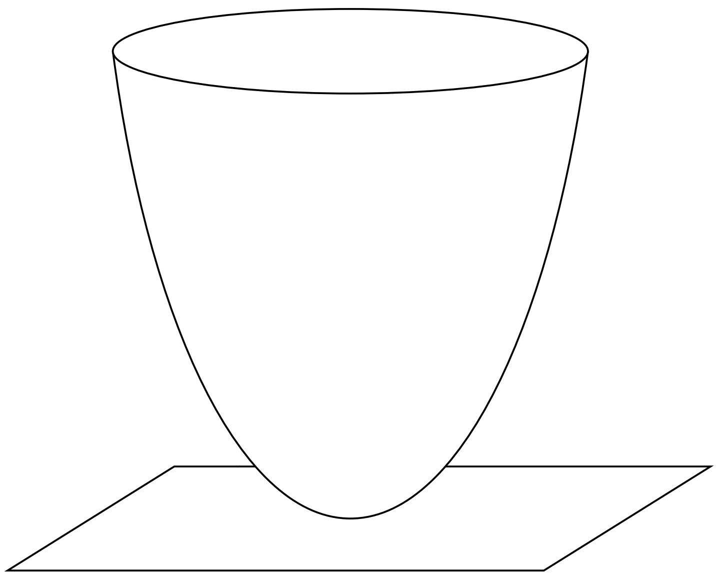


LIFTING POINTS ON THE PARABOLOID

paraboloïd $\mathcal{P}$ with equation $x_{d+1} = \sum_{i \in [d]} x_i^2$.



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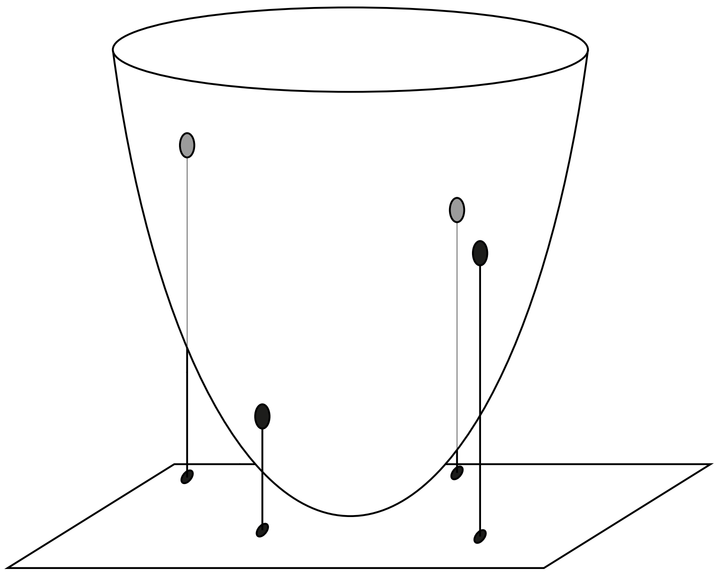

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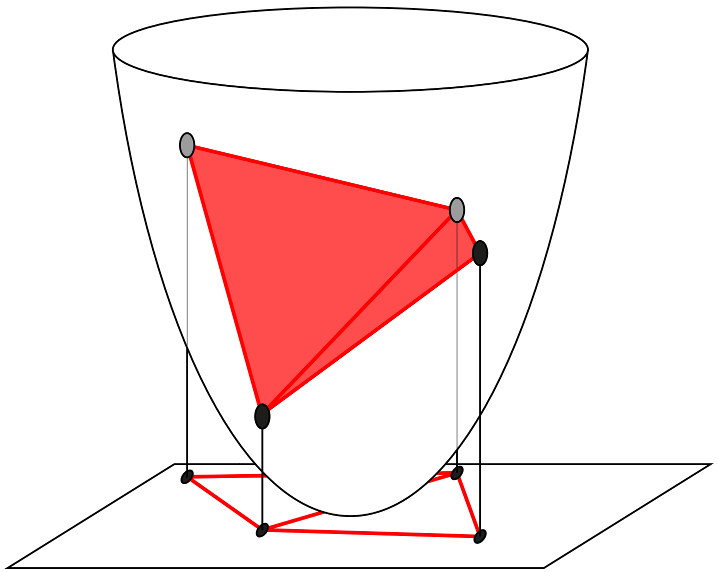

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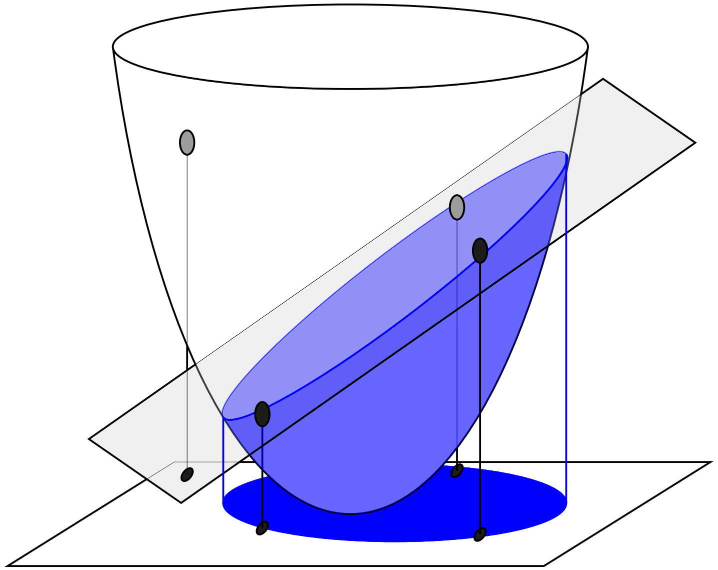

proof: Paraboloïd cap below a hyperplane:

$$x_{d+1} = \sum_{i \in [d]} x_i^2 \quad \text{and} \quad x_{d+1} \le \sum_{i \in [d]} \lambda_i \, x_i.$$

Projection of this cap:

$$\sum_{i \in [d]} (x_i - \lambda_i/2)^2 \le \sum_{i \in [d]} \lambda_i^2/4.$$

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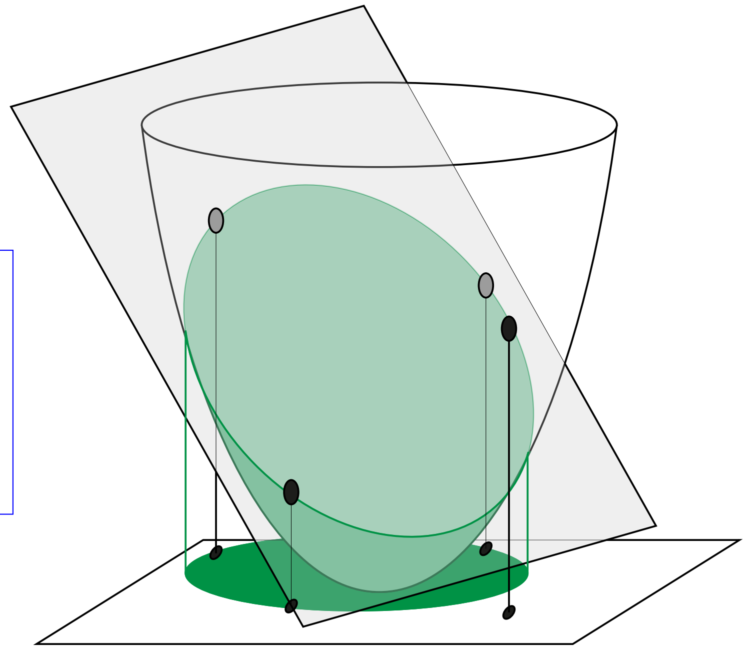

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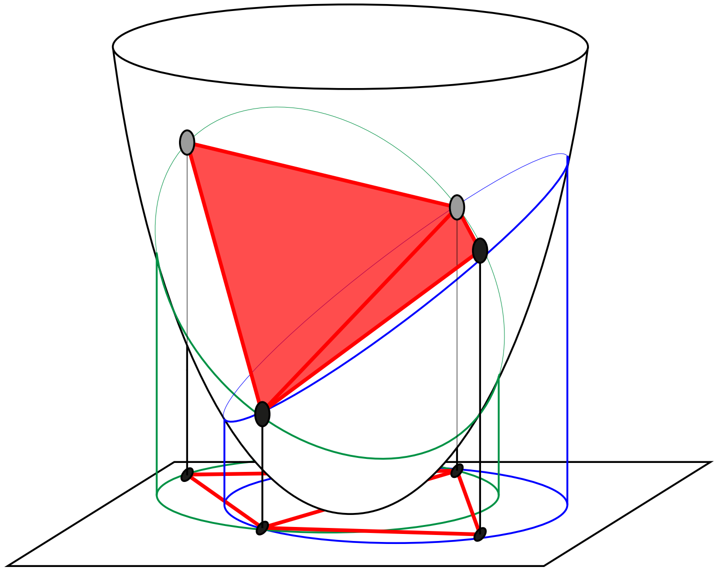

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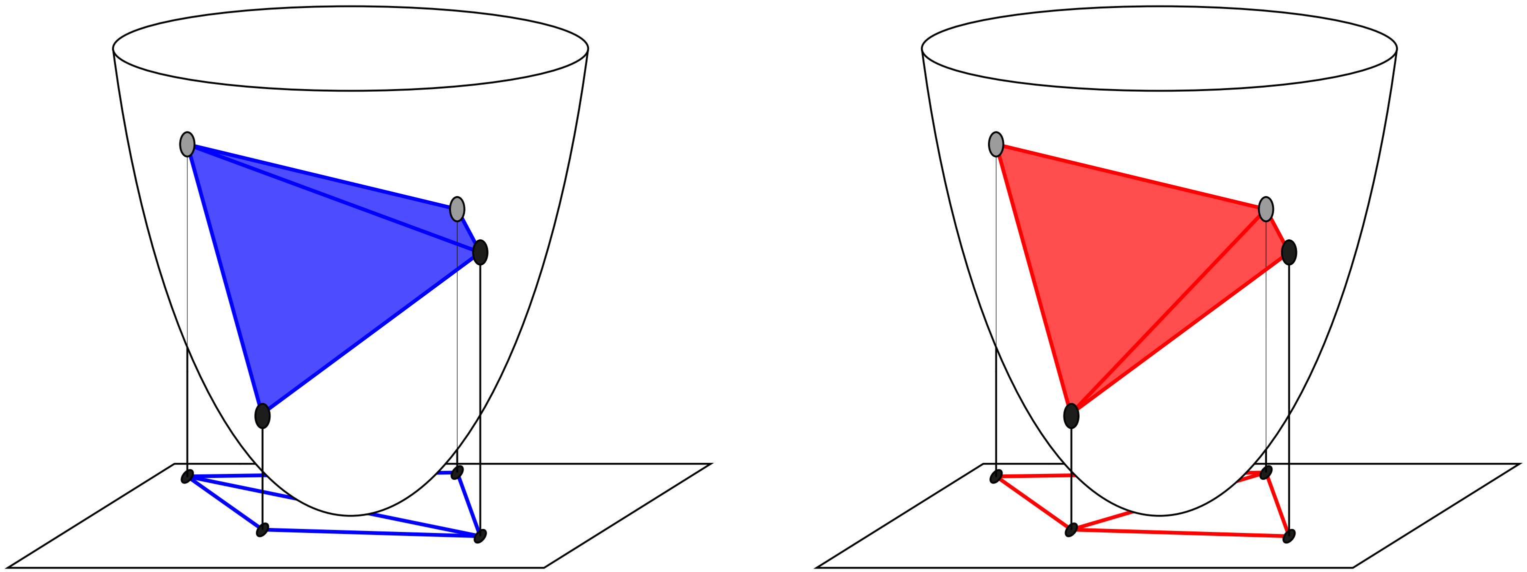
LAWSON FLIPS IN DIMENSION 2

DEF. Lawson flip = flip of an edge pq contained in two triangles pqr and pqs such that s is inside the circumcircle of pqr and r is inside the circumcircle of pqs.

PROP. Lawson flips are always possible, and lead to the Delaunay triangulation.



CORO. For any 2-dimensional point configuration, the flip graph is connected.

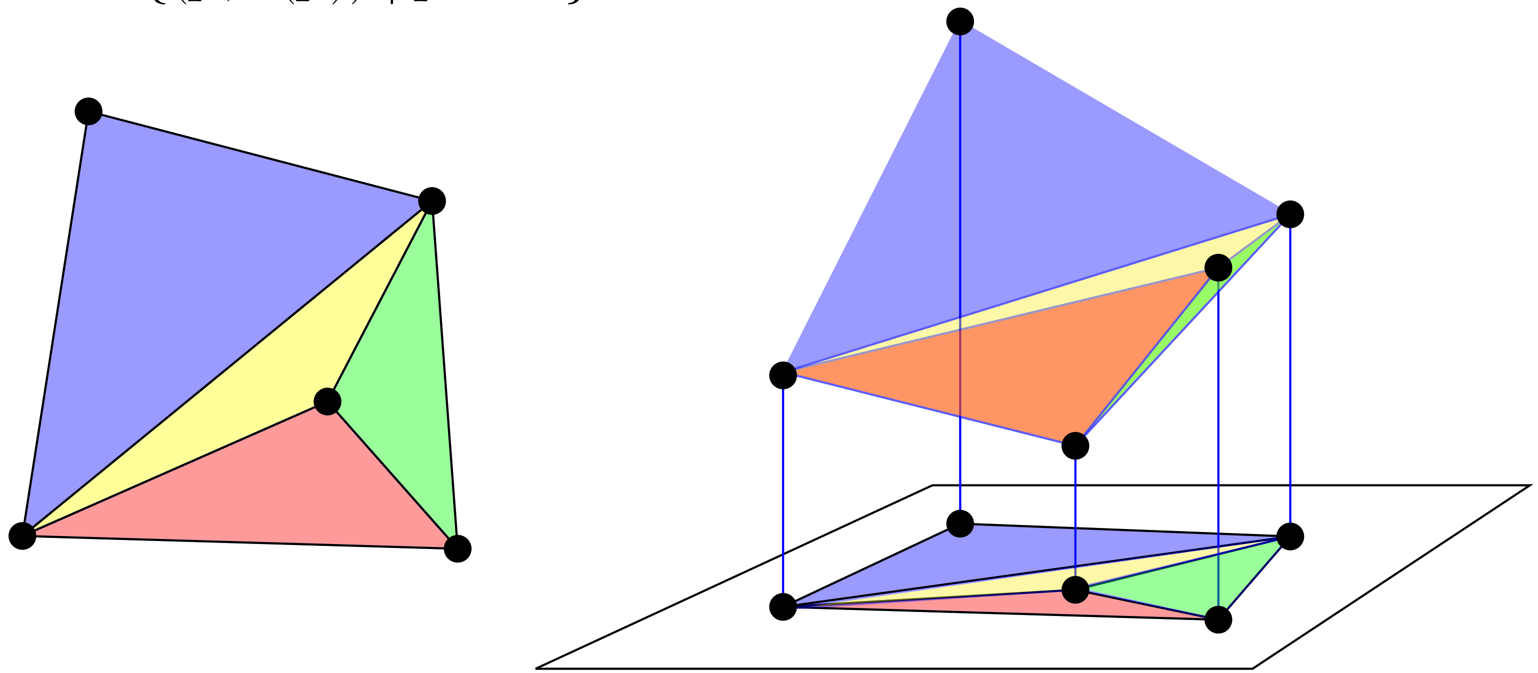
THM. (Santos) In dimension ≥ 5, some point sets have disconnected flip graphs.

REGULAR TRIANGULATIONS & SUBDIVISIONS

LIFTINGS AND REGULAR SUBDIVISIONS

DEF. $\boldsymbol{P}$ = point configuration. $h : \boldsymbol{P} \to \mathbb{R}$ height function.

$\mathcal{S}(\boldsymbol{P}, h)$ = subdivision of $\boldsymbol{P}$ obtained as the projection of the lower convex hull of the lifted point set $\{(\boldsymbol{p}, h(\boldsymbol{p})) \mid \boldsymbol{p} \in \boldsymbol{P}\}$.



A subdivision $\mathcal{S}$ is regular if there is a height function $h : \boldsymbol{P} \to \mathbb{R}$ st $\mathcal{S} = \mathcal{S}(\boldsymbol{P}, h)$.

PROP. If $g : \mathbb{R}^n \to \mathbb{R}$ is affine, then $\mathcal{S}(\boldsymbol{P}, g + h) = \mathcal{S}(\boldsymbol{P}, h)$ for any $h : \boldsymbol{P} \to \mathbb{R}$.

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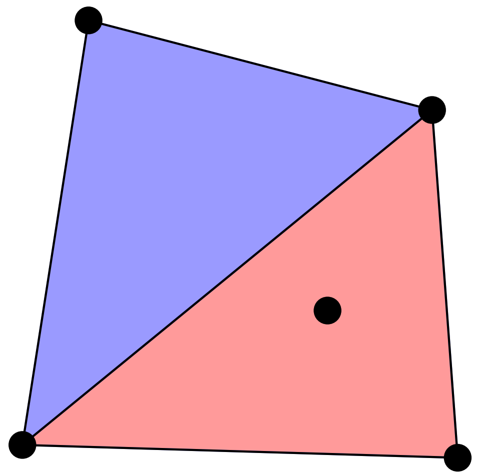

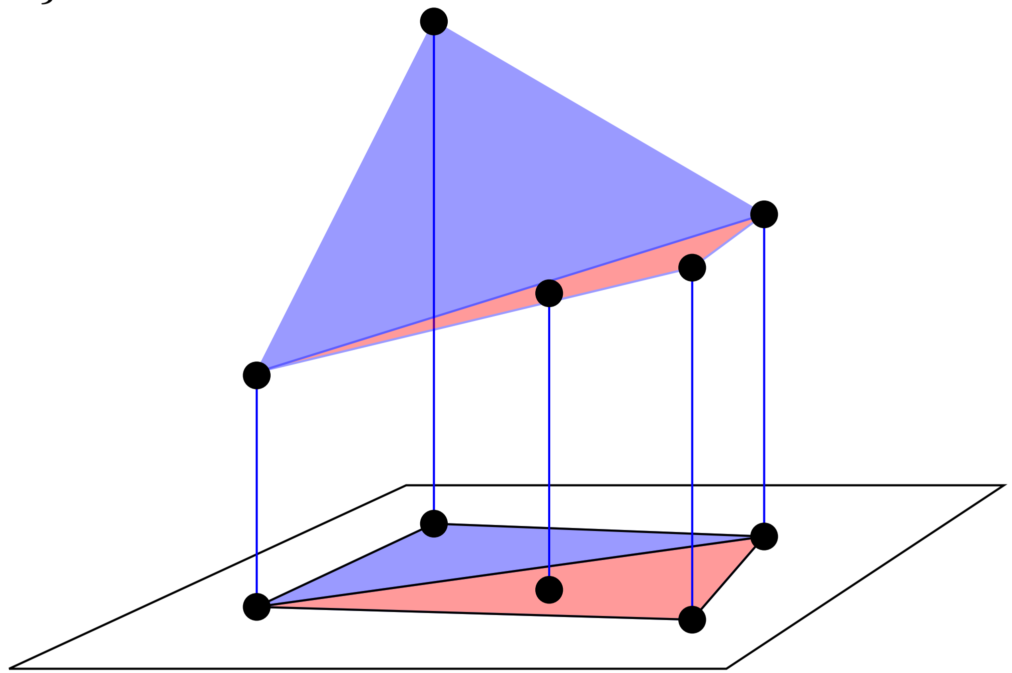

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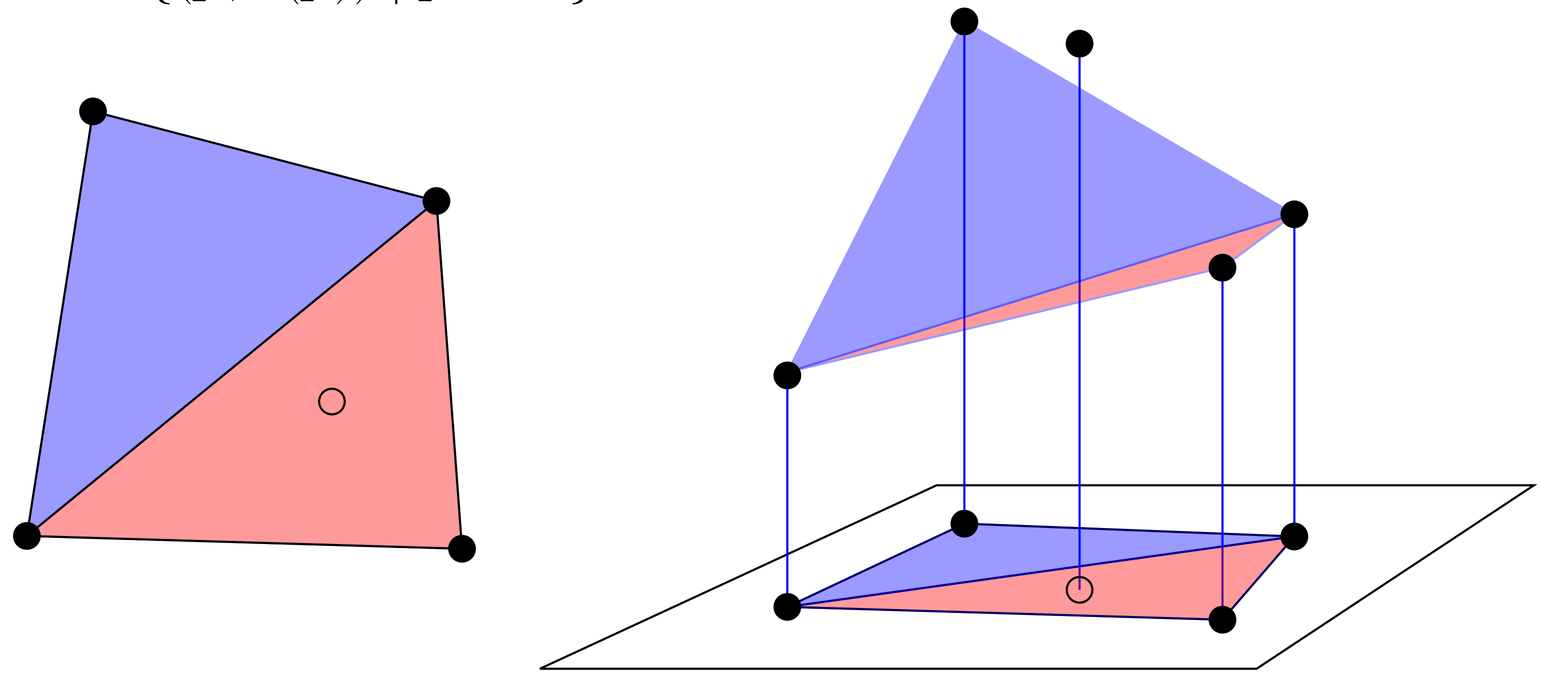

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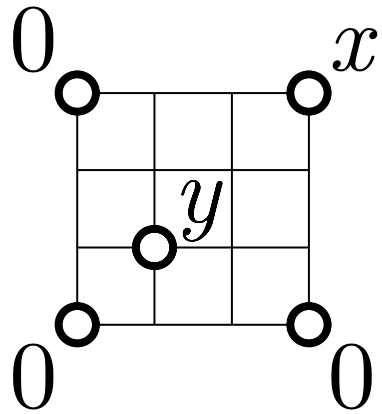
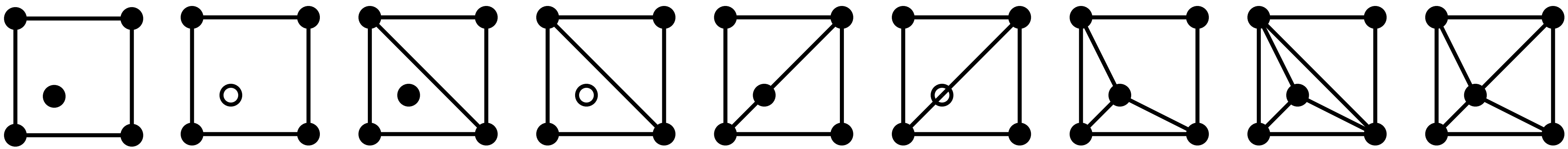
EXM OF REGULAR SUBDIVISIONS

Point configuration $\boldsymbol{P} = \{(0,0),(3,0),(0,3),(3,3),(1,1)\}$.
Restrict to height functions h with $h((0,0)) = h((3,0)) = h((0,3)) = 0$.
Let $x = h((3,3))$ and $y = h((1,1))$.



QU. Give conditions on x and y to obtain the following regular subdivisions:



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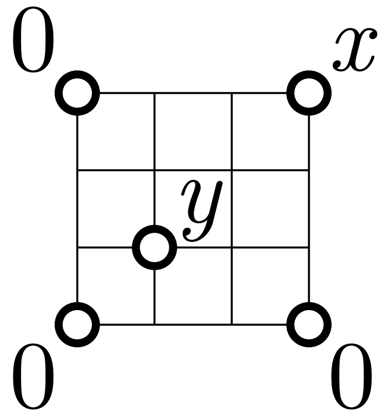

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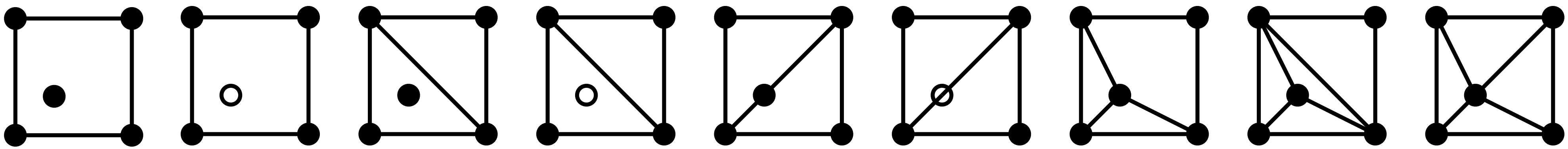

$x = 0$	$x = 0$
$y = 0$	$y > 0$

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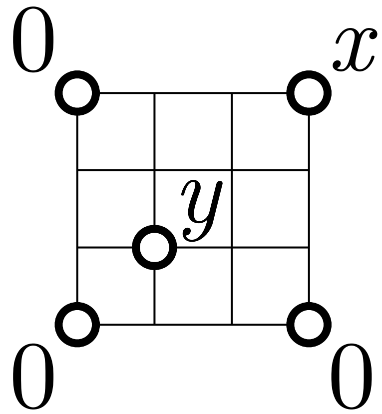

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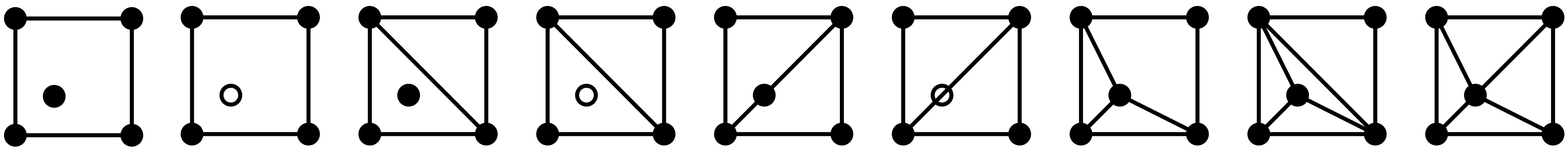

$x = 0$	$x = 0$	$x > 0$	$x > 0$					
$y = 0$	$y > 0$	$y = 0$	$y > 0$					

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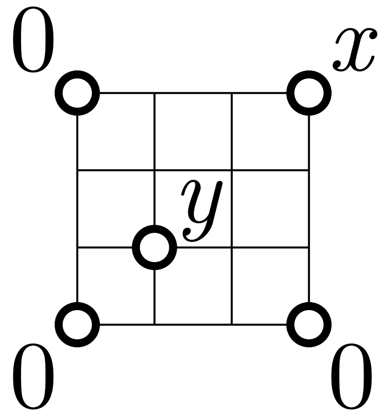

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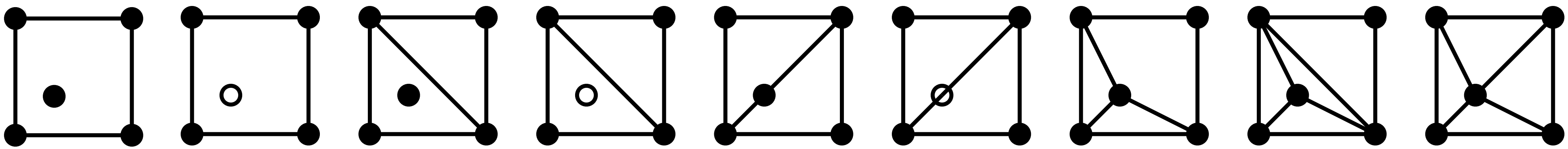

$x=0$	$x=0$	$x>0$	$x>0$	$x<0$	$x<0$			
$y=0$	$y>0$	$y=0$	$y>0$	$x-3y=0$	$x-3y<0$			

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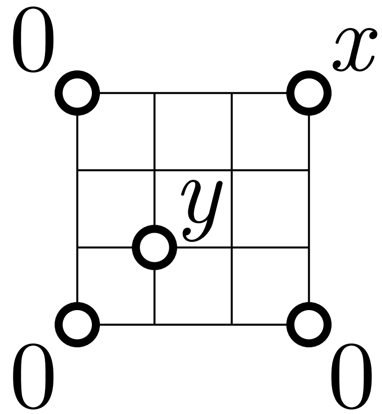

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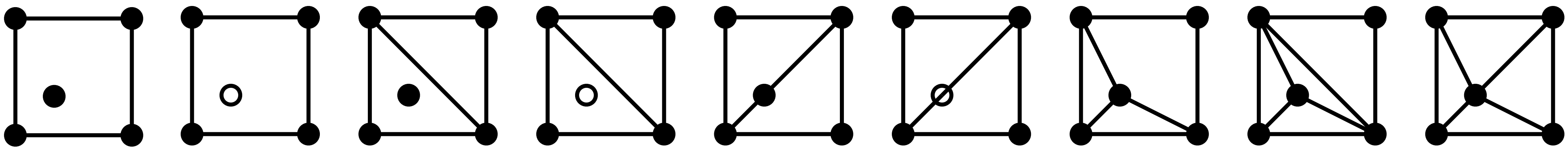
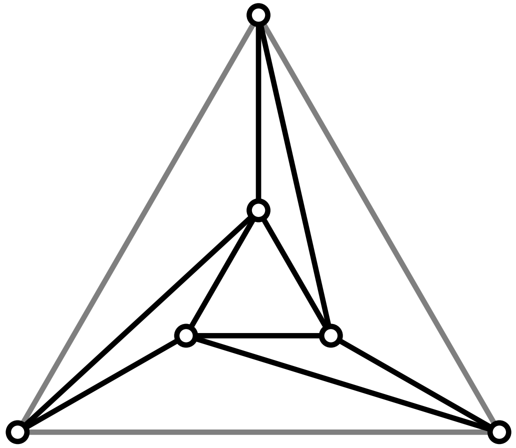
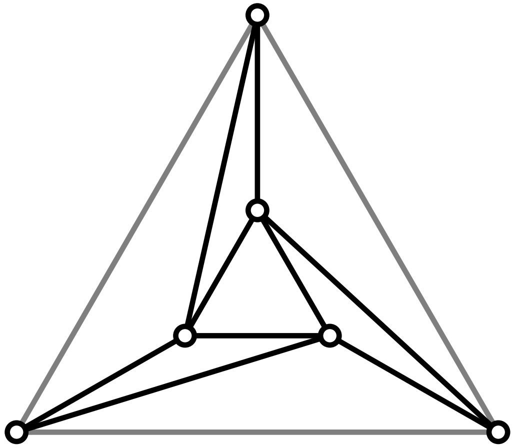


$x=0$	$x=0$	$x>0$	$x>0$	$x<0$	$x<0$	$x+3y=0$	$x+3y>0$	$x+3y<0$
$y=0$	$y>0$	$y=0$	$y>0$	$x-3y=0$	$x-3y<0$	$y<0$	$y<0$	$x-3y>0$

NON REGULAR TRIANGULATIONS

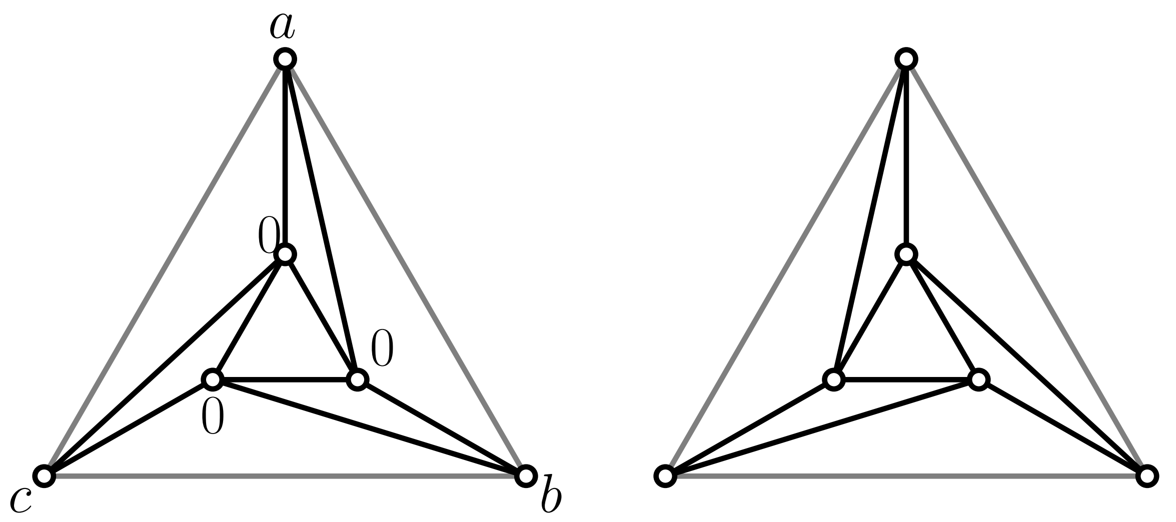
QU. Show that the following two triangulations are not regular:





NON REGULAR TRIANGULATIONS

PROP. The following two triangulations are not regular:



proof: assume the left one regular,
and pick a height function.
Up to an affine function, height 3
for the 3 internal vertices.
The heights of the 3 external
vertices satisfy: $a < b < c < a$.
Contradiction.

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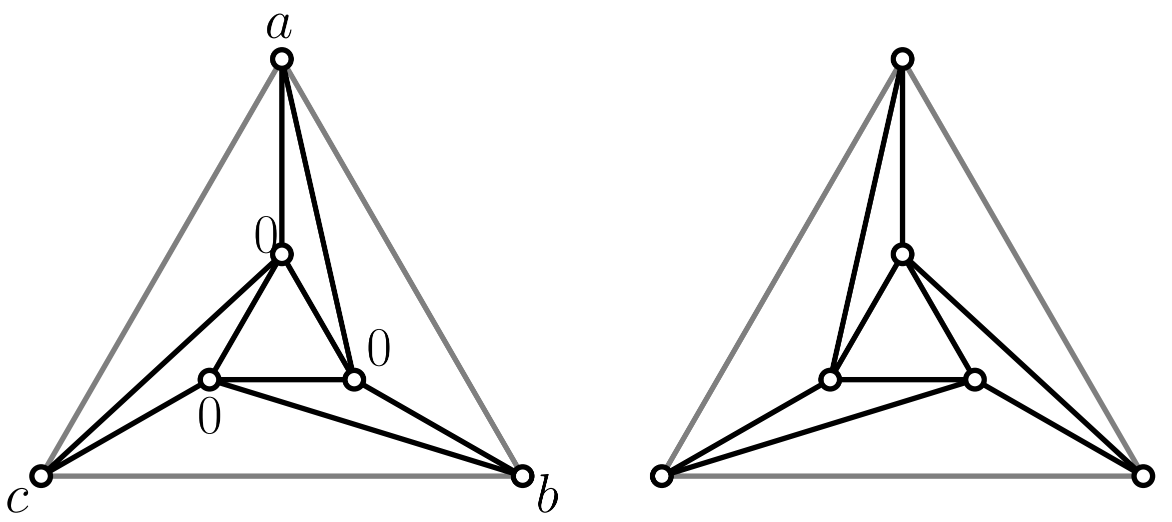

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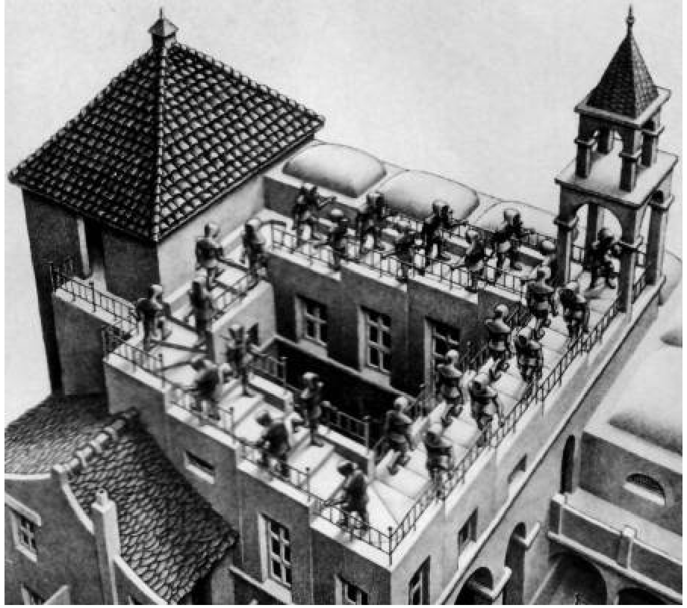
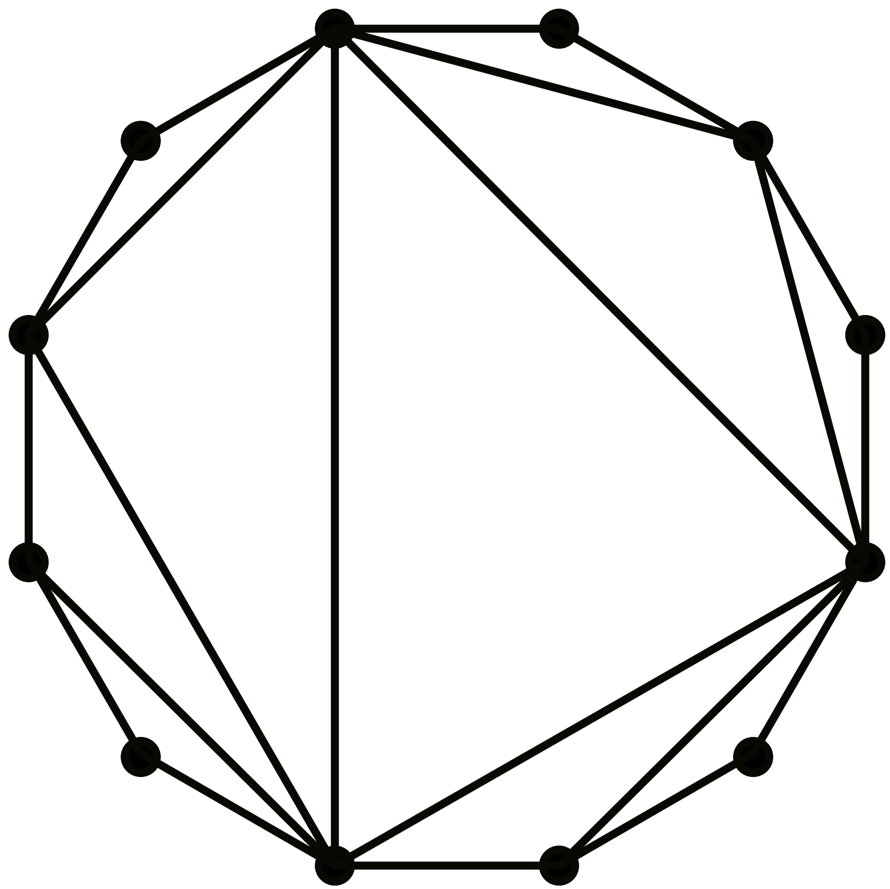


CONVEX POSITION

QU. Show that all subdivisions of a planar point set in convex position are regular.



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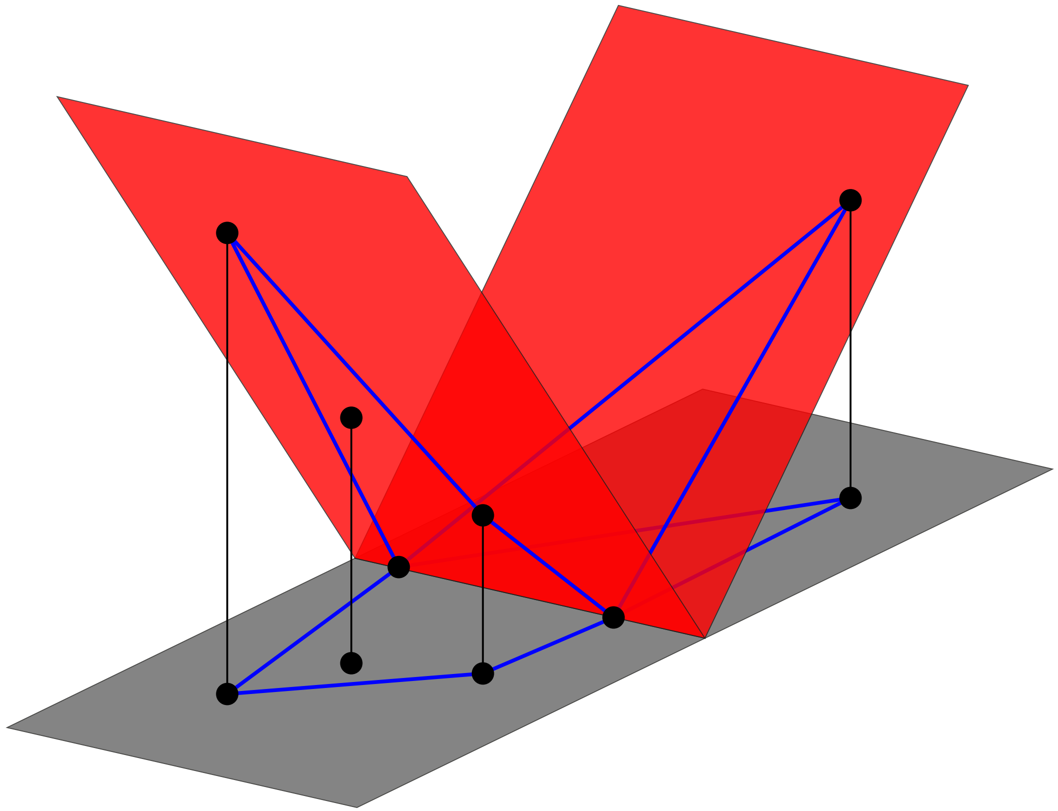
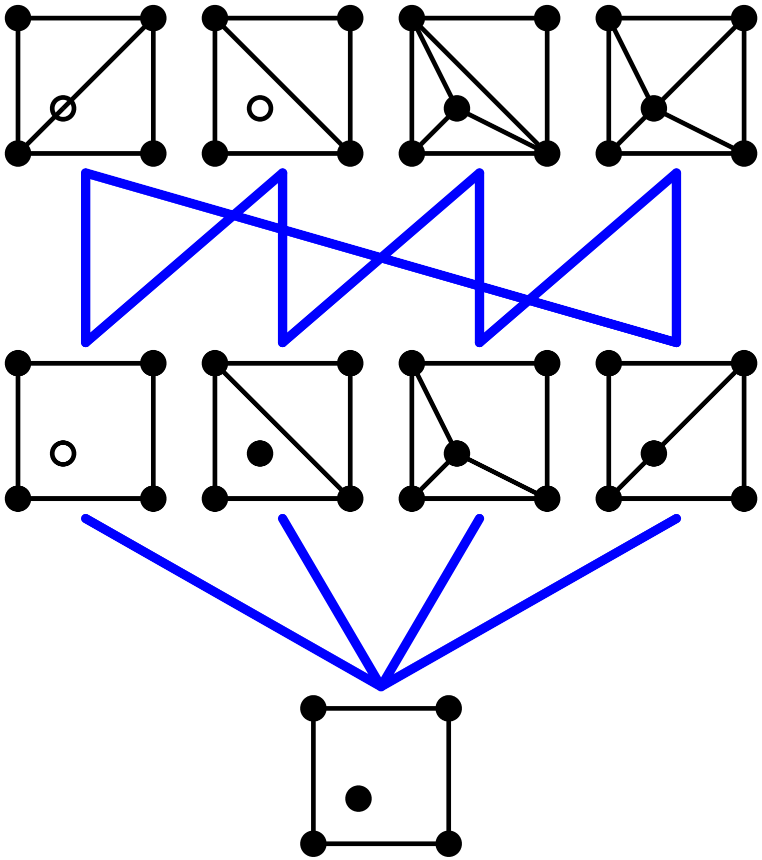


Use $h(\boldsymbol{p}) = \sum_{\delta \in \mathcal{S}} d(\delta, \boldsymbol{p})$ where $d(\delta, \boldsymbol{p})$ is the distance of $\boldsymbol{p}$ to the line spanned by δ.

REGULAR SUBDIVISION LATTICE

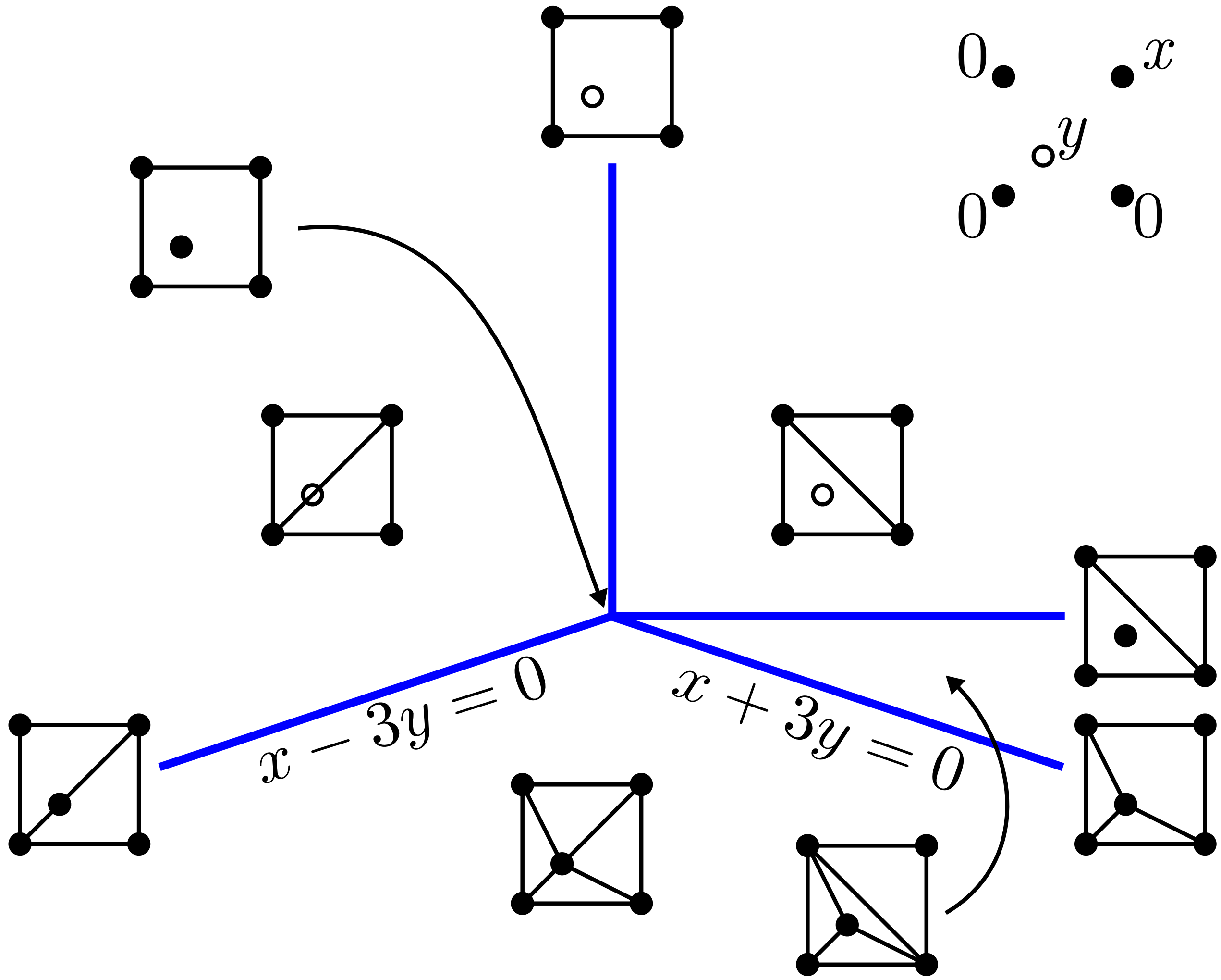
DEF. $\mathcal{S}$ refines $\mathcal{S}'$ when for any $\boldsymbol{X} \in \mathcal{S}$, there is $\boldsymbol{X}' \in \mathcal{S}'$ st $\boldsymbol{X} \subseteq \boldsymbol{X}'$.
regular subdivision lattice $=$ regular subdivisions of $\boldsymbol{P}$ ordered by refinement.



SECONDARY FAN AND POLYTOPE

SECONDARY FAN

DEF. <u>secondary cone</u> of subdivision $\mathcal{S}$ of $\boldsymbol{P} = \Sigma\mathbb{C}(\mathcal{S}) = \overline{\{\boldsymbol{h} \in \mathbb{R}^{\boldsymbol{P}} \mid \mathcal{S}(\boldsymbol{P}, \boldsymbol{h}) = \mathcal{S}\}}$.
<u>secondary fan</u> of $\boldsymbol{P}$ = fan formed by the secondary cones of all (regular) subdivisions.



SECONDARY POLYTOPE

DEF. $\mathcal{T}$ triangulation of a point set $\boldsymbol{P} \subseteq \mathbb{R}^d$.

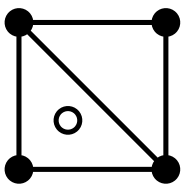
volume vector of $\mathcal{T}$:

$$\Phi(\mathcal{T}) = \left(\sum_{\boldsymbol{p} \in \triangle \in \mathcal{T}} \mathrm{vol}(\triangle) \right)_{\boldsymbol{p} \in \boldsymbol{P}}$$

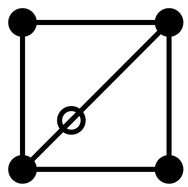
secondary polytope of $\boldsymbol{P}$:

$$\Sigma\mathbb{P}(\boldsymbol{P}) := \mathrm{conv}\,\{\Phi(\mathcal{T}) \mid \mathcal{T} \text{ triangulation of } \boldsymbol{P}\}\,.$$

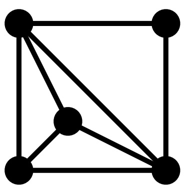
exm:



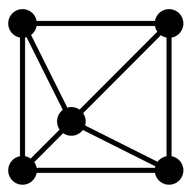
(9, 18, 18, 9, 0)



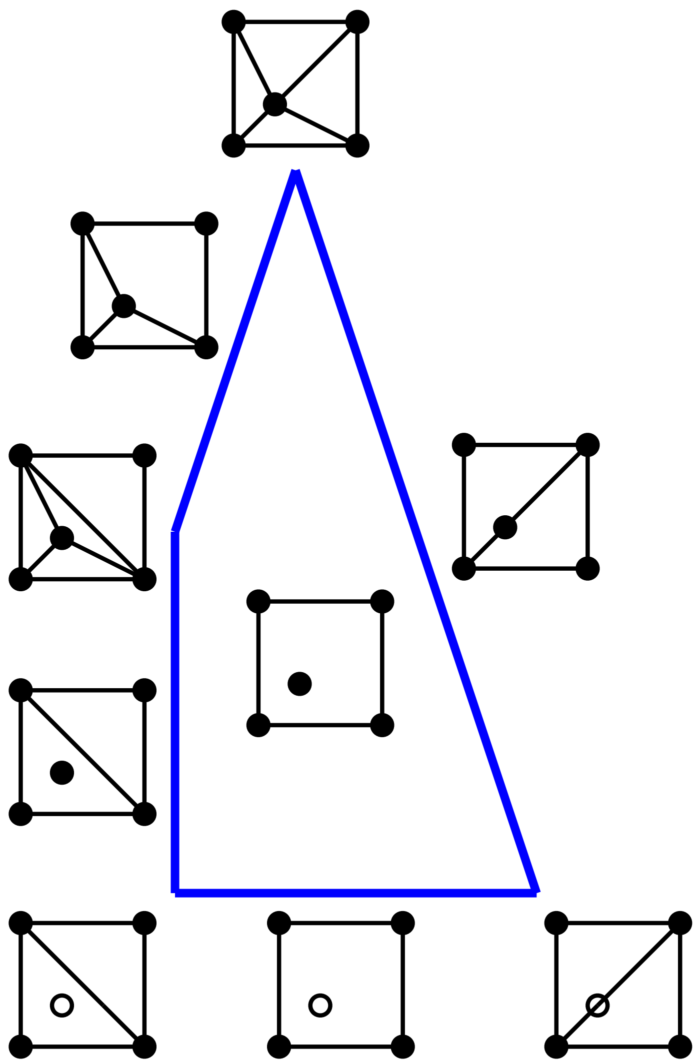
(18, 9, 9, 18, 0)



(6, 15, 15, 9, 9)



(6, 9, 9, 12, 18)



SECONDARY FAN AND POLYTOPE

THM. (Gelfand, Kapranov, and Zelevinsky) For $\boldsymbol{P}$ in general position in $\mathbb{R}^d$,

- $\Sigma\mathbb{P}(\boldsymbol{P})$ has dimension $|\boldsymbol{P}| - d - 1$,
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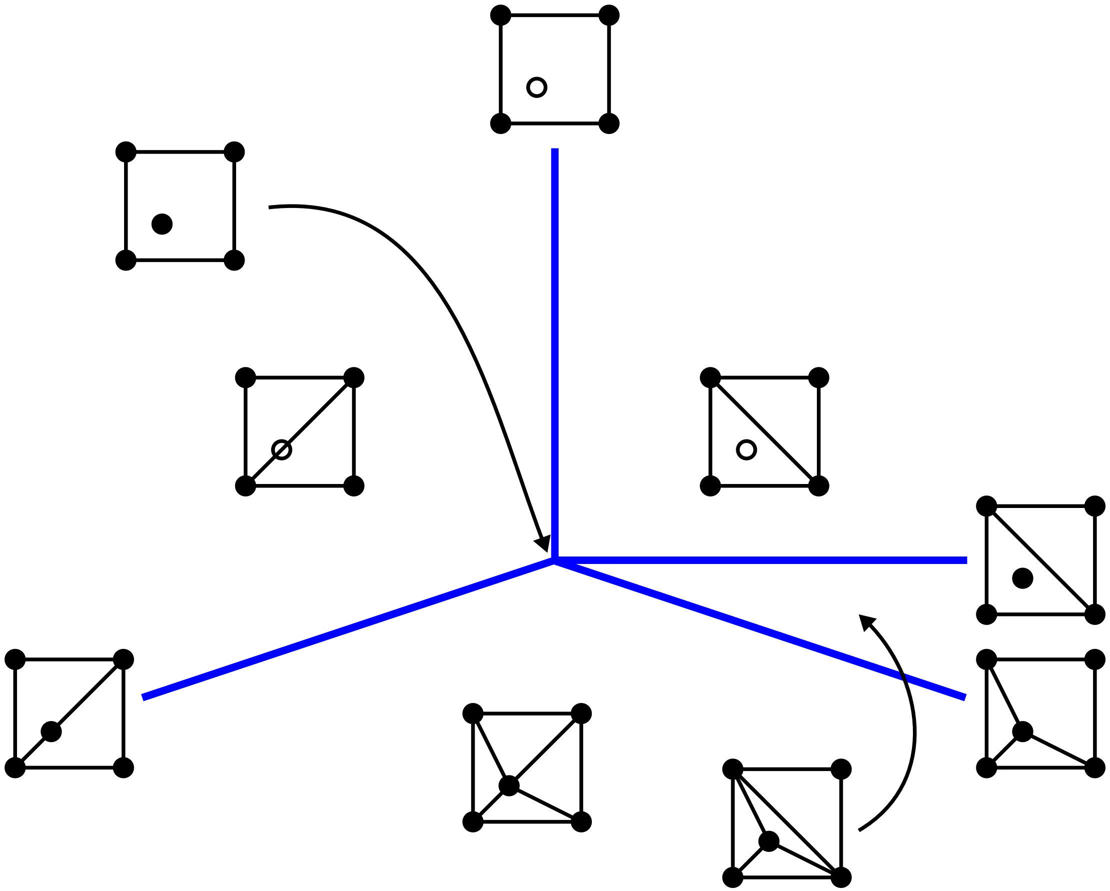

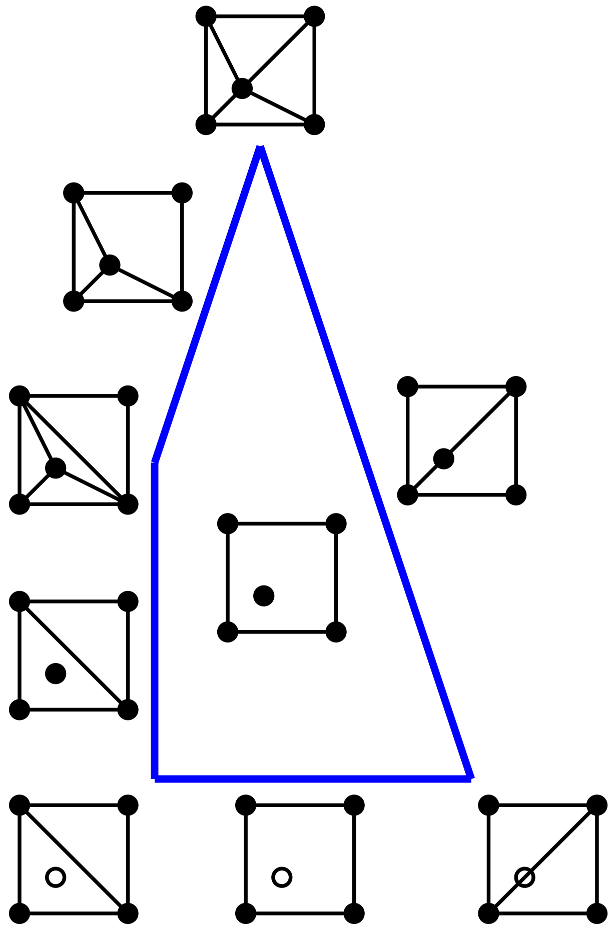

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proof: lower bound on $\dim(\Sigma\mathbb{P}(\boldsymbol{P}))$ by induction on $|\boldsymbol{P}|$:

- when $|\boldsymbol{P}| = d$, $\Sigma\mathbb{P}(\boldsymbol{P})$ is a single point,
- for $|\boldsymbol{P}| \ge d+1$ and any $\boldsymbol{p} \in \boldsymbol{P}$, $\Sigma\mathbb{P}(\boldsymbol{P} \smallsetminus \boldsymbol{p}) = \Sigma\mathbb{P}(\boldsymbol{P}) \cap \left\{ \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{P}} \;\middle|\; x_{\boldsymbol{p}} = \alpha \right\}$ where

$$\alpha = \begin{cases} 0 & \text{if } \boldsymbol{p} \text{ inside } \mathrm{conv}(\boldsymbol{P}), \\ \mathrm{vol}(\mathrm{conv}(\boldsymbol{P})) - \mathrm{vol}(\mathrm{conv}(\boldsymbol{P} \smallsetminus \boldsymbol{p})) & \text{if } \boldsymbol{p} \text{ on the boundary of } \mathrm{conv}(\boldsymbol{P}). \end{cases}$$

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upper bound on $\dim(\Sigma\mathbb{P}(\boldsymbol{P}))$ from the volume and center of mass of $\mathrm{conv}(\boldsymbol{P})$:

$$\mathrm{vol}(\boldsymbol{P}) = \sum_{\triangle \in \mathcal{T}} \mathrm{vol}(\triangle) = \sum_{\triangle \in \mathcal{T}} \sum_{\boldsymbol{p} \in \triangle} \frac{\mathrm{vol}(\triangle)}{d+1} = \frac{1}{d+1} \sum_{\boldsymbol{p} \in \boldsymbol{P}} \sum_{\boldsymbol{p} \in \triangle \in \mathcal{T}} \mathrm{vol}(\triangle) = \frac{1}{d+1} \sum_{\boldsymbol{p} \in \boldsymbol{P}} \Phi(\mathcal{T})_{\boldsymbol{p}}.$$

$$\mathrm{vol}(\boldsymbol{P}) \cdot \mathrm{cm}(\boldsymbol{P}) = \sum_{\triangle \in \mathcal{T}} \mathrm{vol}(\triangle) \cdot \mathrm{cm}(\triangle) = \sum_{\triangle \in \mathcal{T}} \mathrm{vol}(\triangle) \cdot \left(\frac{1}{d+1} \sum_{\boldsymbol{p} \in \triangle} \boldsymbol{p} \right) = \frac{1}{d+1} \sum_{\boldsymbol{p} \in \boldsymbol{P}} \Phi(\mathcal{T})_{\boldsymbol{p}} \cdot \boldsymbol{p}.$$

SECONDARY FAN AND POLYTOPE

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proof: $\mathcal{T}$ triangulation of $\boldsymbol{P}$ and a height vector $\boldsymbol{h} \in \mathbb{R}^{\boldsymbol{P}}$.

$f_{\mathcal{T},\boldsymbol{h}} : \mathrm{conv}(\boldsymbol{P}) \to \mathbb{R}$ = piecewise linear map on the simplices of $\mathcal{T}$ such that $f_{\mathcal{T},\boldsymbol{h}}(\boldsymbol{p}) = \boldsymbol{h}_{\boldsymbol{p}}$.

Then the volume below the hypersurface defined by $f_{\mathcal{T},\boldsymbol{h}}$ is

$$\int_{\mathrm{conv}(\boldsymbol{P})} f_{\mathcal{T},\omega}(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{\triangle\in\mathcal{T}} \int_{\triangle} f_{\mathcal{T},\omega}(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{\triangle\in\mathcal{T}} \frac{\mathrm{vol}(\triangle)}{d+1} \sum_{\boldsymbol{p}\in\triangle} \boldsymbol{h}_{\boldsymbol{p}}$$

$$= \frac{1}{d+1} \sum_{\boldsymbol{p}\in\boldsymbol{P}} \boldsymbol{h}_{\boldsymbol{p}} \cdot \sum_{\boldsymbol{p}\in\triangle\in\mathcal{T}} \mathrm{vol}(\triangle) = \frac{\langle\, \Phi(\mathcal{T}) \mid \boldsymbol{h} \,\rangle}{d+1}.$$

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$$= \frac{1}{d+1} \sum_{\boldsymbol{p}\in\boldsymbol{P}} \boldsymbol{h}_{\boldsymbol{p}} \cdot \sum_{\boldsymbol{p}\in\triangle\in\mathcal{T}} \mathrm{vol}(\triangle) = \frac{\langle\, \Phi(\mathcal{T}) \mid \boldsymbol{h} \,\rangle}{d+1}.$$

Therefore, if $\mathcal{T} = \mathcal{S}(\boldsymbol{P}, \boldsymbol{h}) \neq \mathcal{T}'$ then

$$\langle\, \Phi(\mathcal{T}) \mid \boldsymbol{h} \,\rangle < \langle\, \Phi(\mathcal{T}') \mid \boldsymbol{h} \,\rangle\,.$$

In other words, the normal cone of $\Phi(\mathcal{T})$ in $\Sigma\mathbb{P}(\boldsymbol{P})$ is the secondary cone of $\mathcal{T}$.

SECONDARY FAN AND POLYTOPE

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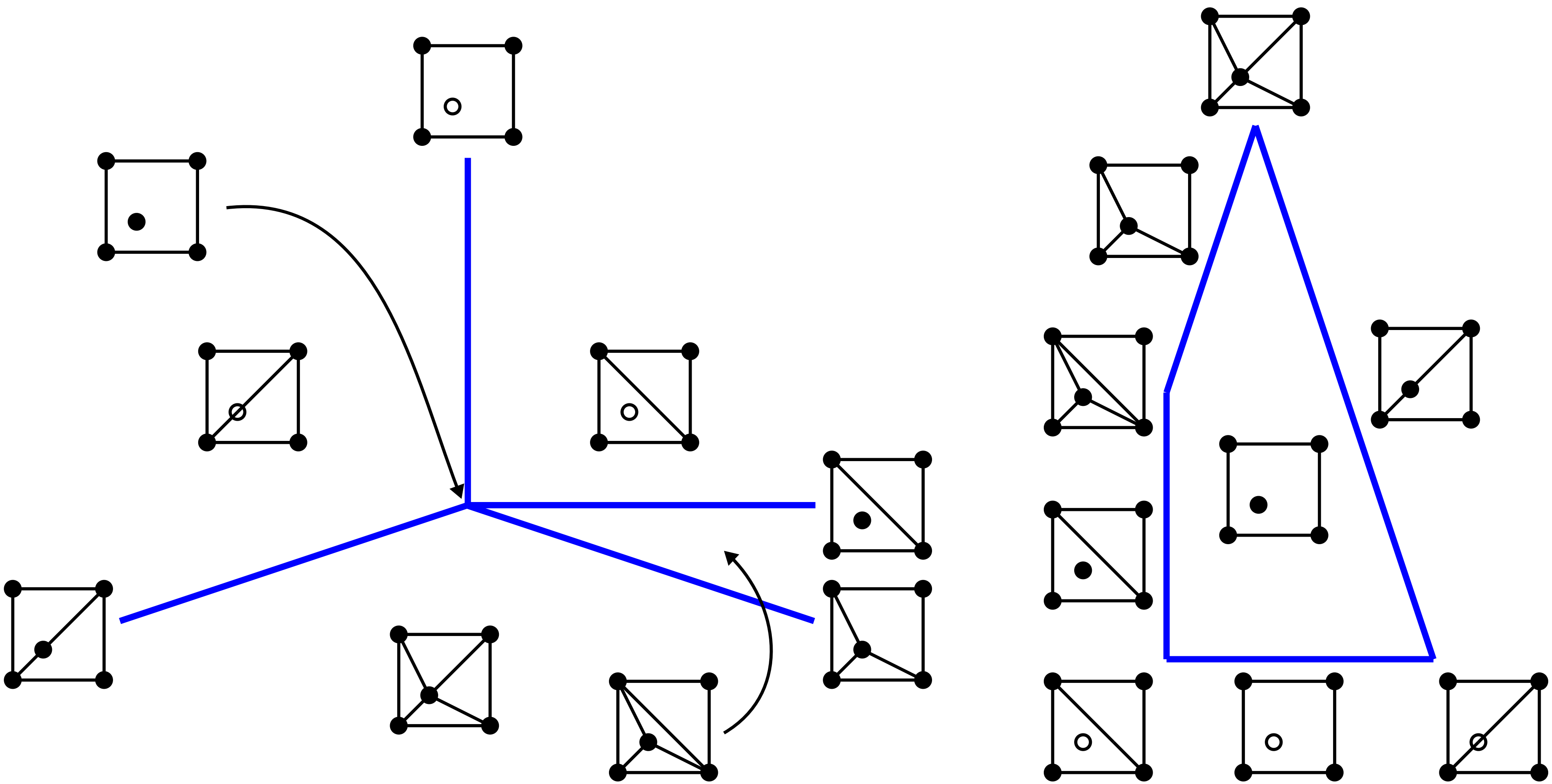

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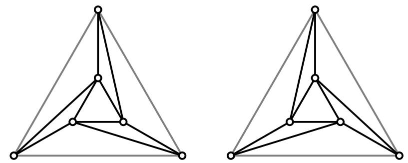
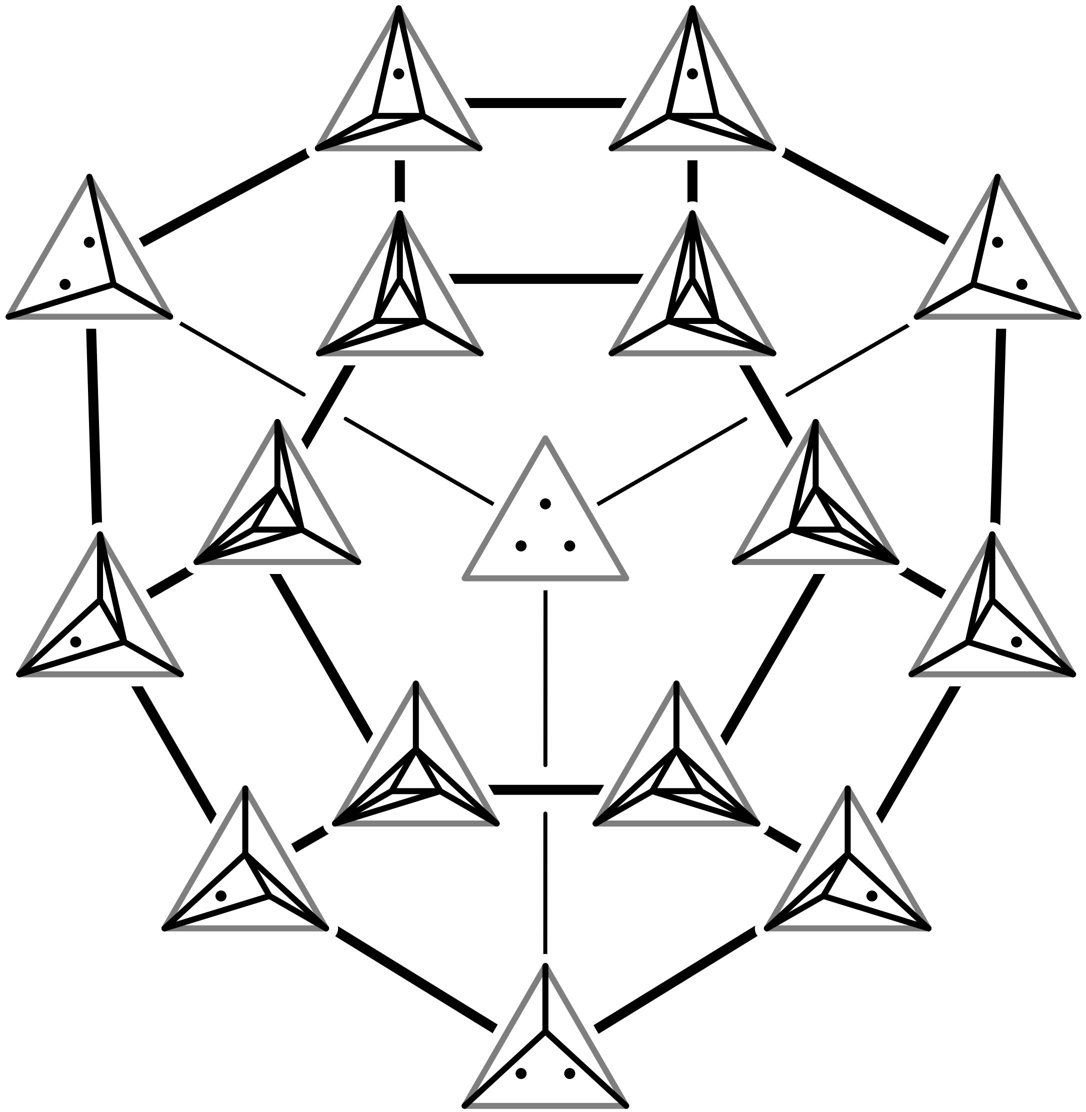
CORO. For any point set $\boldsymbol{P}$ in $\mathbb{R}^d$ (arbitrary dimension), the flip graph on regular triangulations is connected.

SECONDARY FAN AND POLYTOPE

QU. Locate the volume vectors of the non-regular triangulations  in



SOME REFERENCES

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