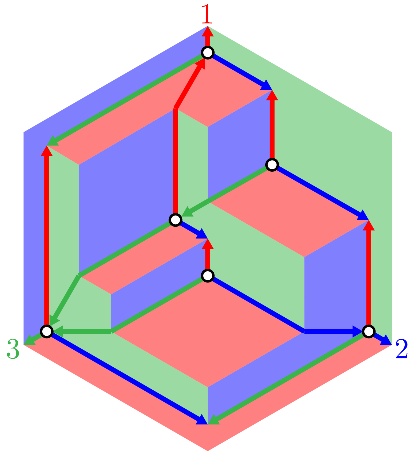
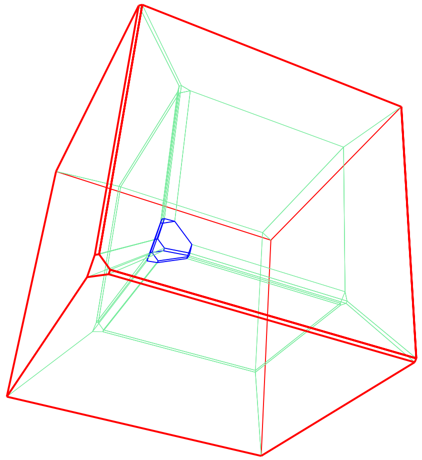
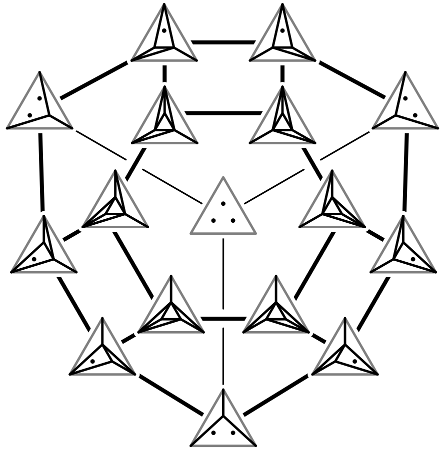
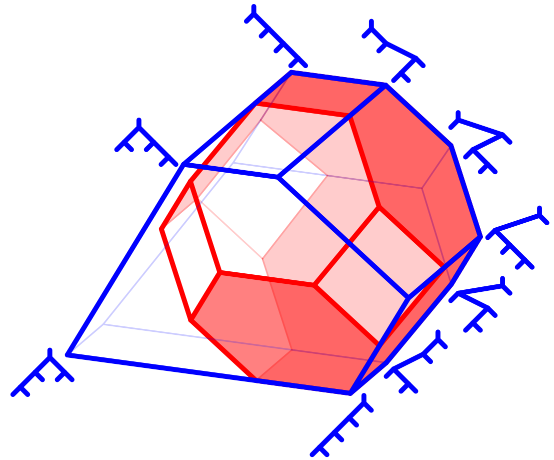
MPRI 2-38-1. Algorithms and combinatorics for geometric graphs



V. PILAUD

Thursdays 15/10, 22/10, 29/10 & 5/11, 2020

slides available at: http://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/MPRI-2-38-1-VP1.pdf

Course notes available at: https://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/notesCoursMPRI20.pdf

PLANAR & GEOMETRIC GRAPHS GRAPHS

Planar graphs are very special:

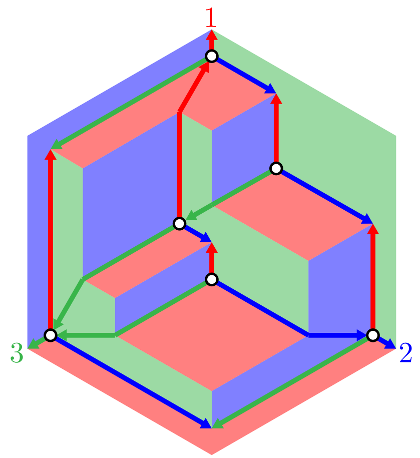
- combinatorially (few edges, Euler relation, 4-colorable, ...),
- algorithmically (use planar structure to design more efficient algorithms).

The course focusses on three aspects:

1. Graphs drawn in the plane
    - basics (combinatorial representations, topology, duality, Euler's formula),
    - embeddings (Tutte barycentric theorem, Schnyder woods),
    - algorithms (planarity testing, efficient algorithms for planar graphs)
    - crossing numbers for graphs (topological and geometrical versions)
2. Graphs on surfaces (AdM)
3. Polytopes and triangulations (VP)

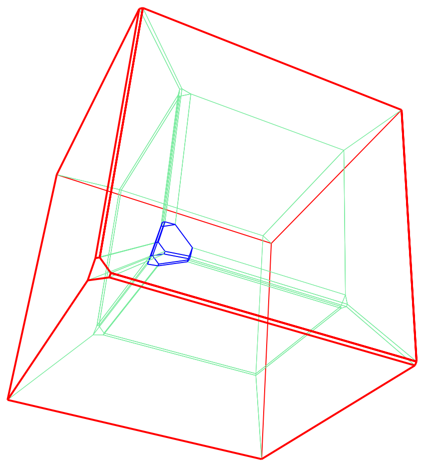
PROGRAM

Cours I
October 15



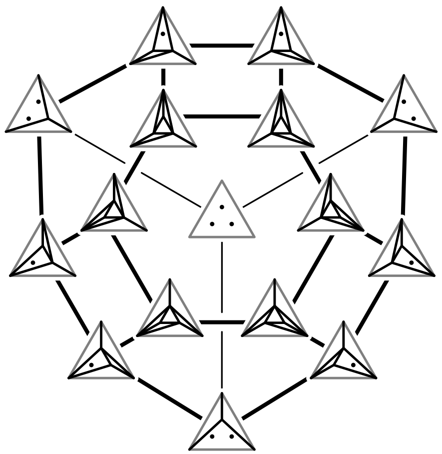
Schnyder woods
& orthogonal surfaces

Cours II
October 22



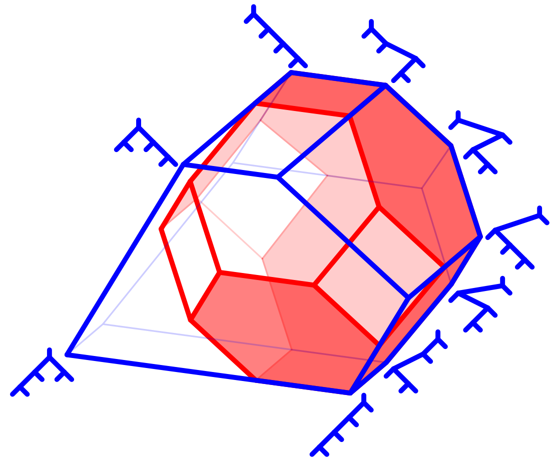
Polytopes

Cours III
October 29



Triangulations
& secondary polytope

Cours IV
November 5



Permutahedron
& associahedron

Exercice sheet on October 29th, to be sent by November 19th.
Final exam (???) on November 26th.

PROGRAM

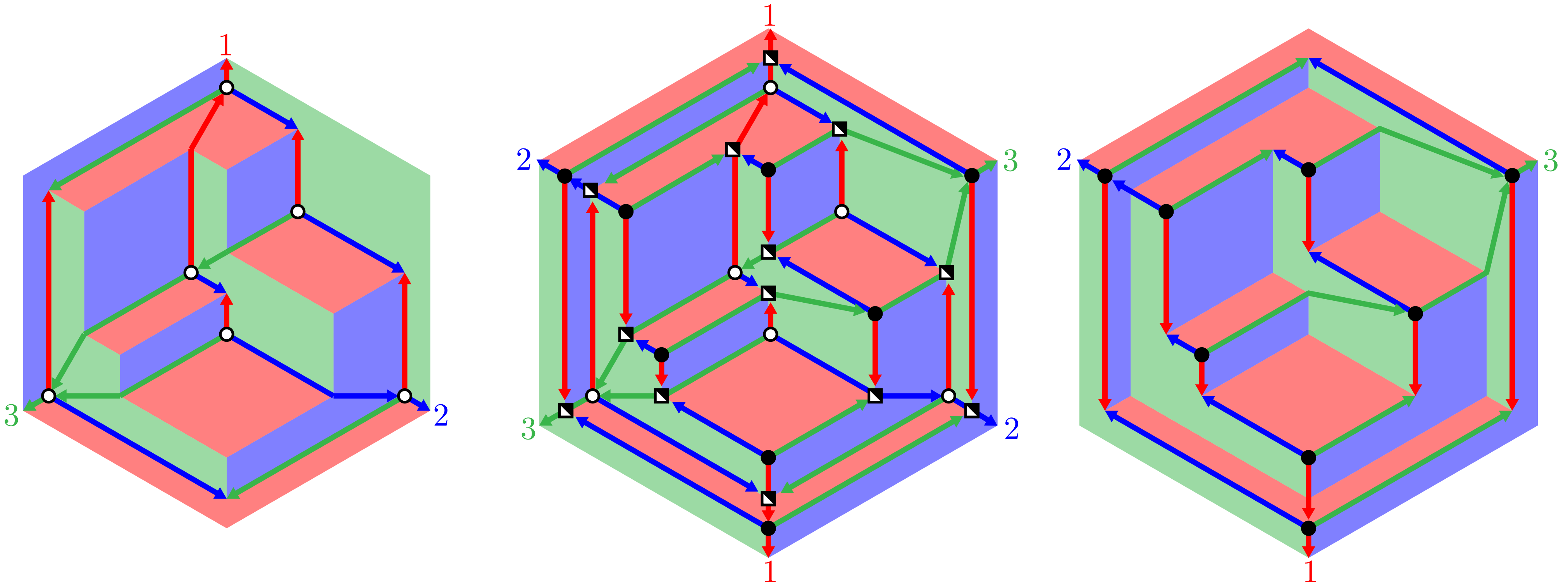
All course material is available here:

http://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/

All informations about the course are available here:

https://wikimpri.dptinfo.ens-cachan.fr/doku.php?id=cours:c-2-38-1

Schnyder woods and applications



V. PILAUD

MPRI 2-38-1. Algorithms and combinatorics for geometric graphs

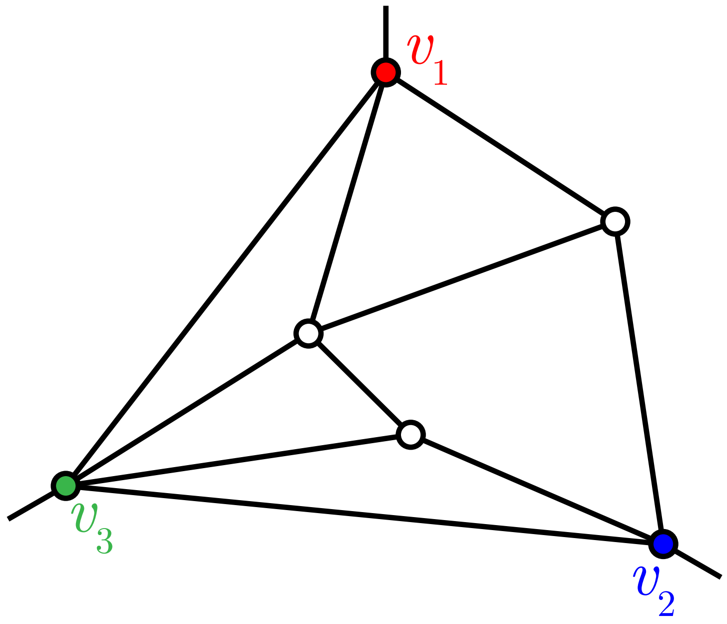
Thursday October 15th, 2020

slides available at: `http://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/MPRI-2-38-1-VP1.pdf`

Course notes available at: `https://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/notesCoursMPRI20.pdf`

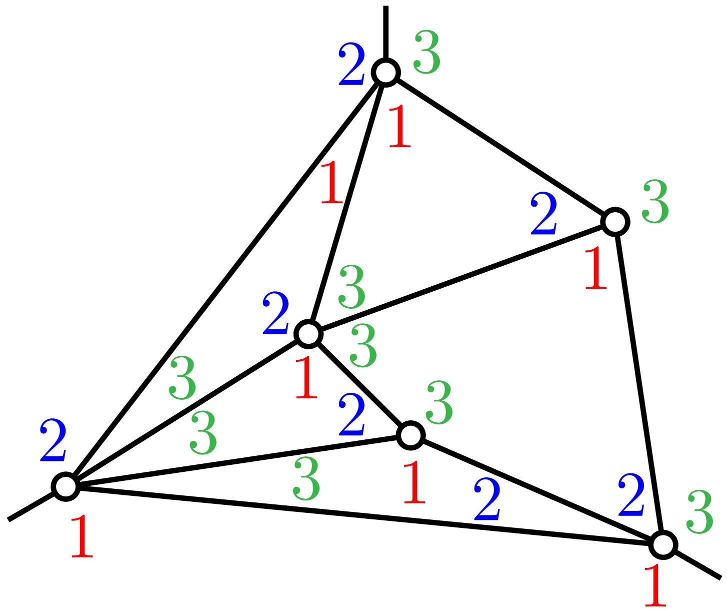
SCHNYDER LABELINGS AND WOODS

PLANAR MAP

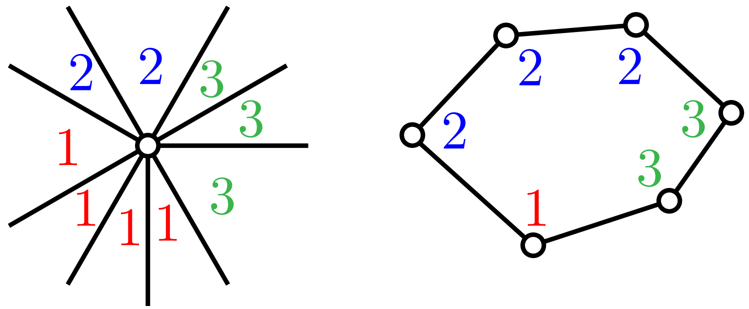


M = planar map with three distinguished vertices v_1, v_2, v_3 clockwise on the outer face where a half edge is pending in the outer face.

SCHNYDER LABELING



DEF. Schnyder labeling on M = labeling of the angles of M with labels $\{1, 2, 3\}$ st:
(L1) the angles at the half-edge of v_i are labeled $i+1$ and $i-1$ clockwise,
(L2) clockwise around each vertex, the labels form intervals of 1's, 2's and 3's,
(L3) clockwise around each face, the labels form intervals of 1's, 2's and 3's.



SCHNYDER LABELING

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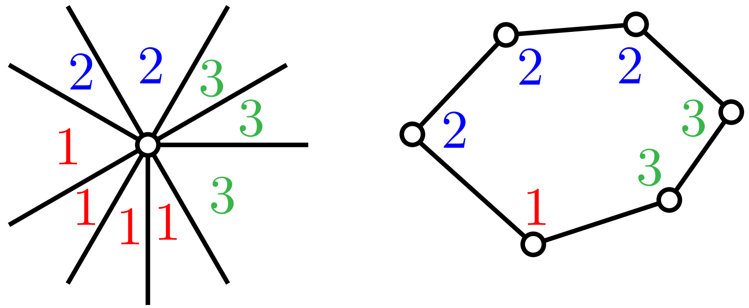


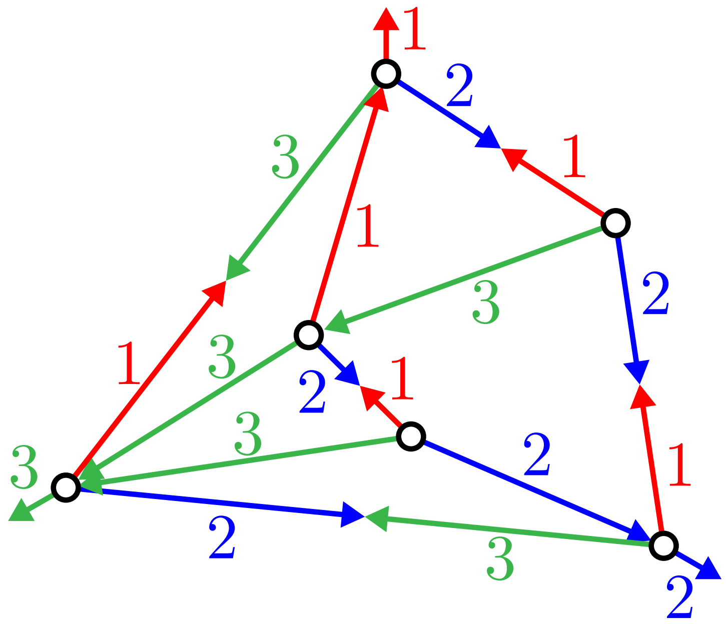
LEM. The three labels $\{1, 2, 3\}$ appear among the four angles surrounding any edge.

proof: Count the number of adjacent angles (same vertex and adjacent faces, or adjacent vertices and same face) with distinct labels. There are:

- 3 around each vertex,
- 3 around each face,
- 2 at each half-edge.

Since $3|V| + 3|F| = 3|E| + 6$ by Euler relation, there are also 3 for each edge.

SCHNYDER WOOD



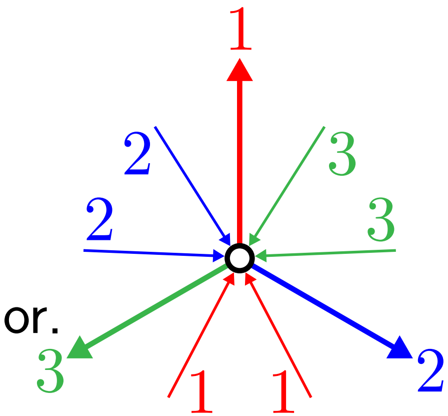
DEF. Schnyder wood on M = (bi-)orientation and (bi-)coloration of the edges of M with $\{1, 2, 3\}$ st:

(W0) bioriented edges get two distinct colors,

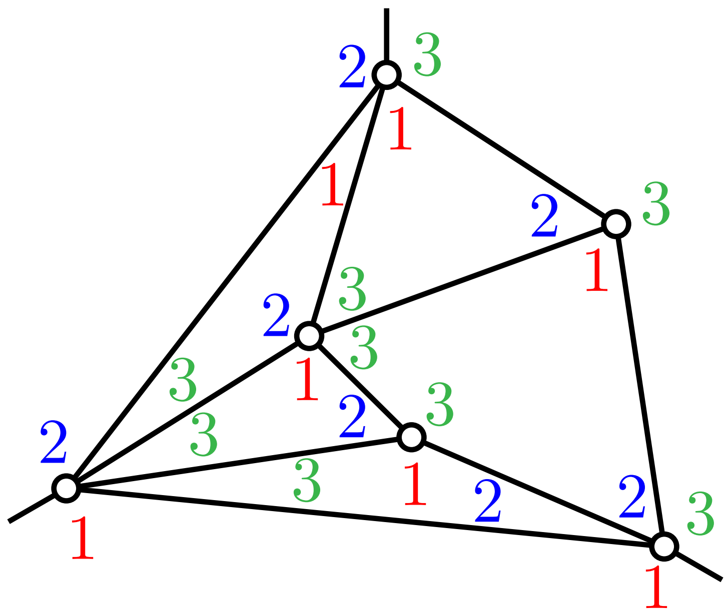
(W1) the half-edge at v_i is directed outwards and colored i,

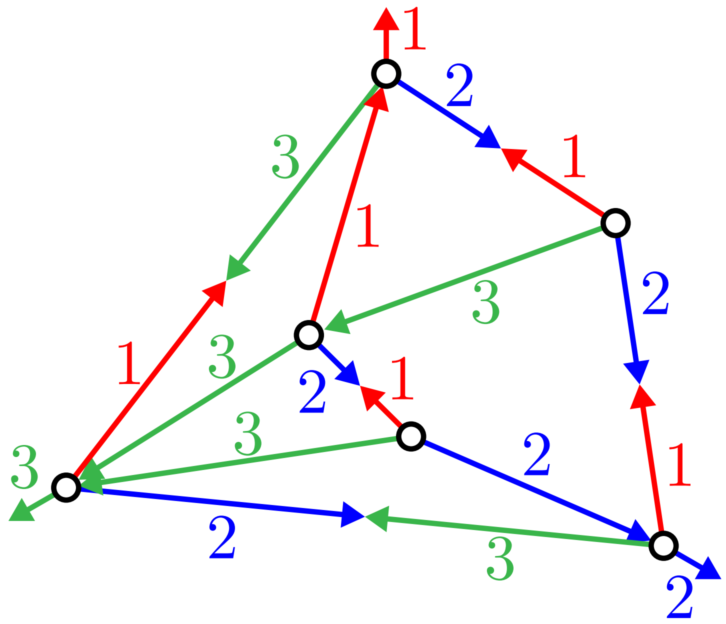
(W2) each vertex v has outdegree one in each label,

(W3) no interior face whose boundary is a directed cycle in one color.

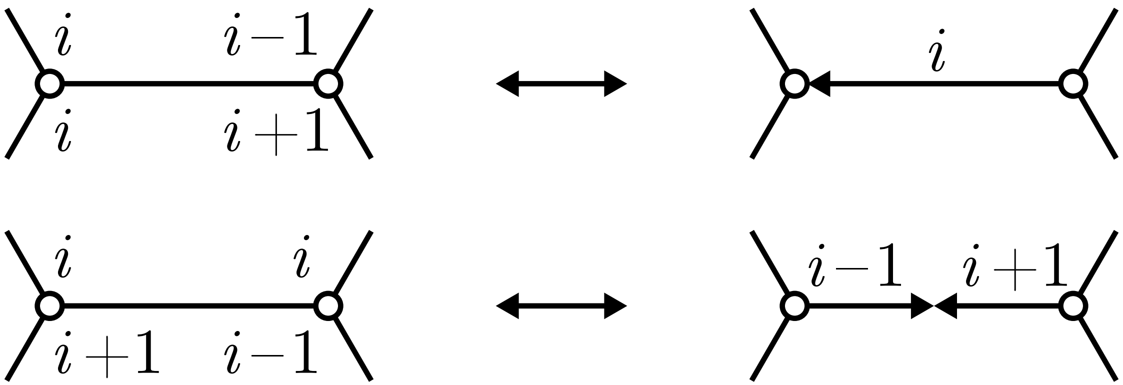


SCHNYDER LABELINGS VS SCHNYDER WOODS





THM. The transformation given by



is a bijection from Schnyder labelings to Schnyder woods.

SCHNYDER LABELINGS VS SCHNYDER WOODS

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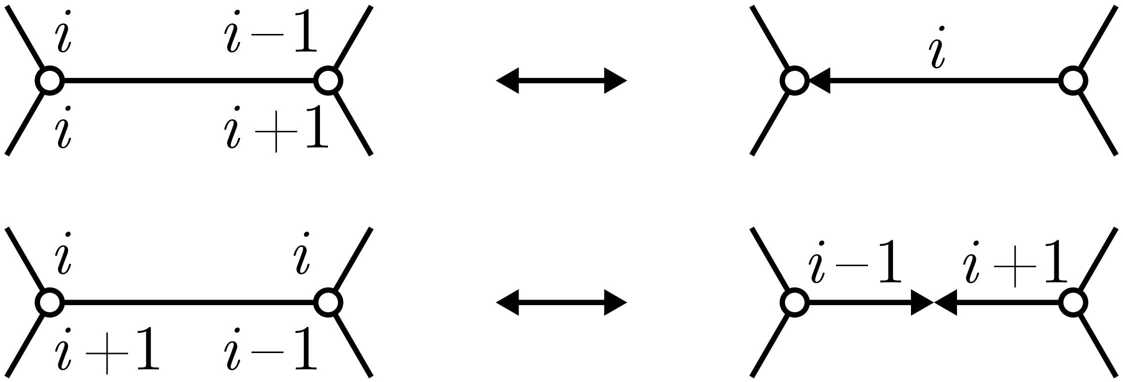


is a bijection from Schnyder labelings to Schnyder woods.

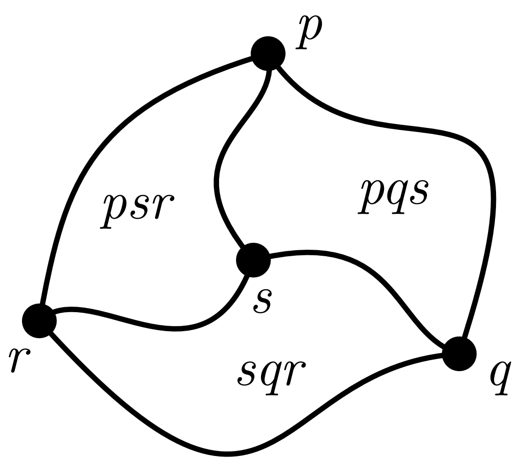
remarks:

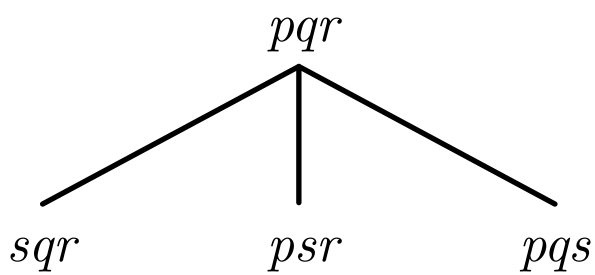
- Only two possible situations by the local rules around vertices, edges and faces.
- If M is triangulated, the second situation cannot occur except on the external face, so that there is no internal bioriented edge.

EXM: STACKED TRIANGULATIONS

DEF. stacked triangulation = triangulation obtained from an initial triangle abc by iteratively refining a triangle pqr into three triangles sqr, psr, and pqs.

construction tree = ternary tree where pqr is the parent of sqr, psr, and pqs.





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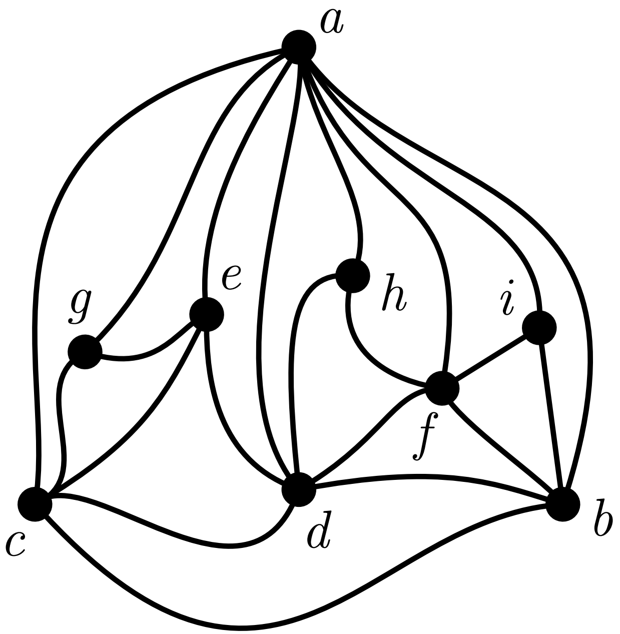

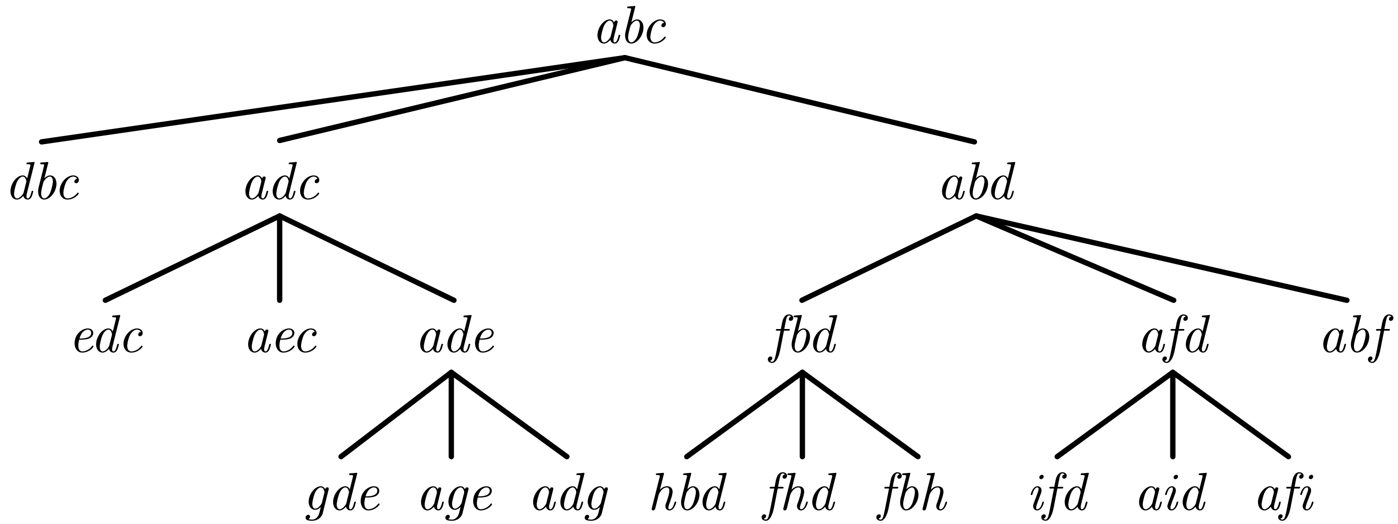

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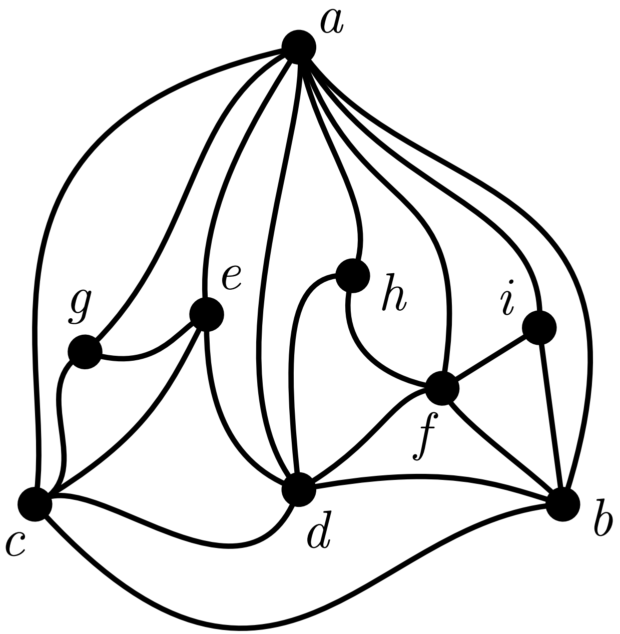

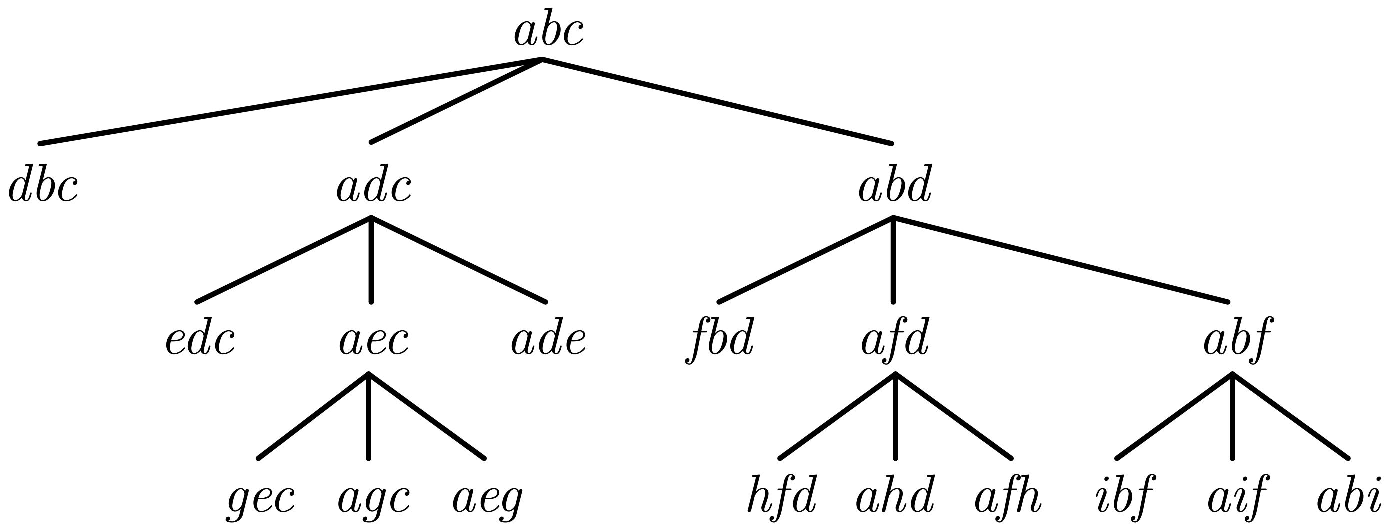

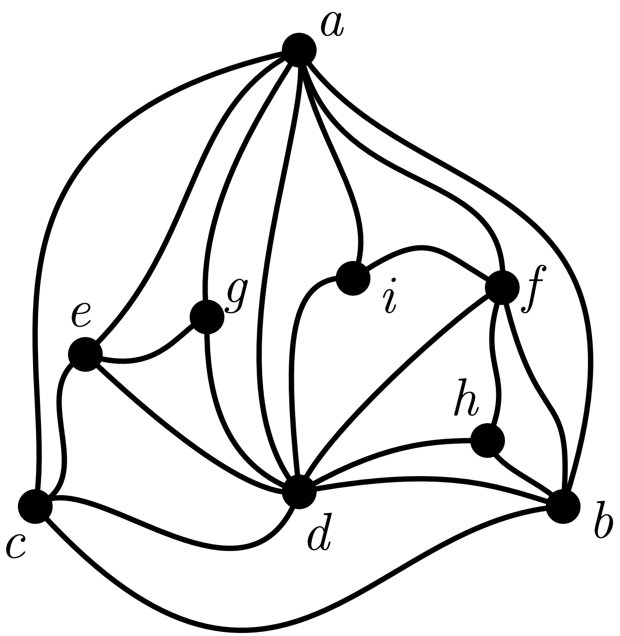

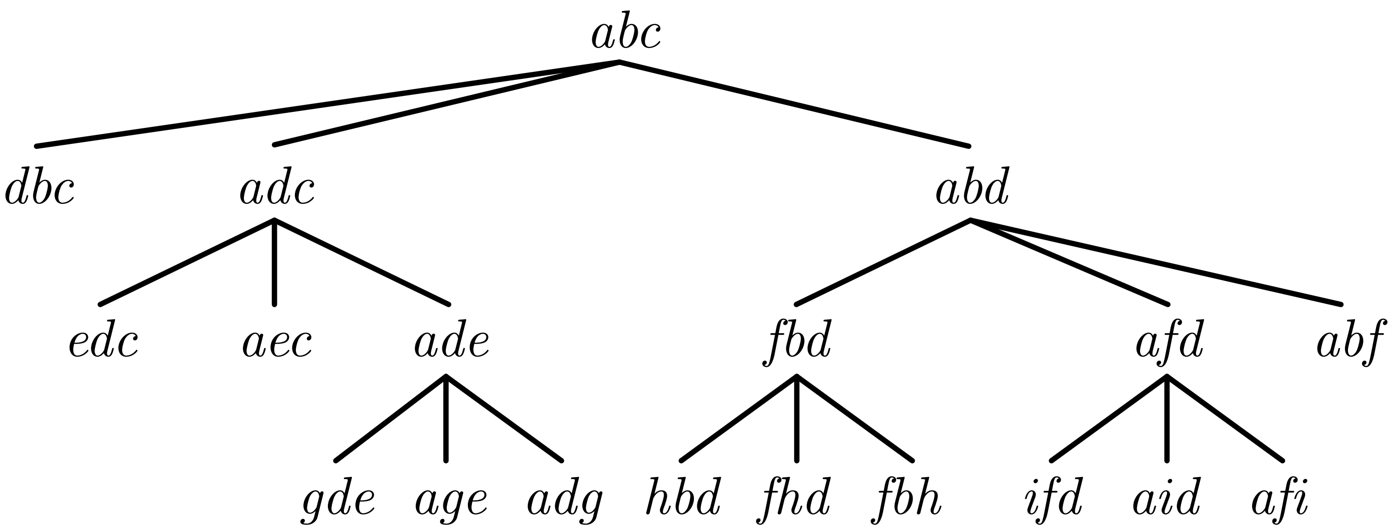

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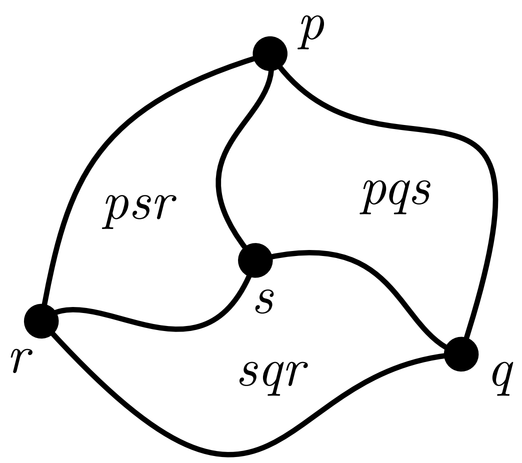

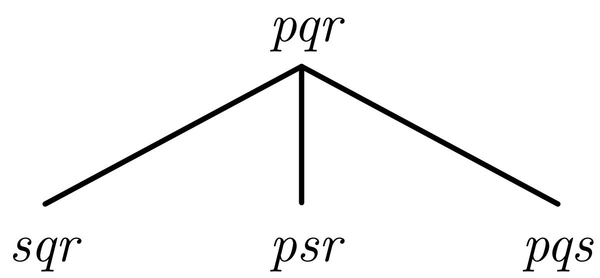

QU. Numbers of vertices, edges and faces of a stacked triangulation?

- in terms of the number of stacking operations,
- in terms of the construction tree.

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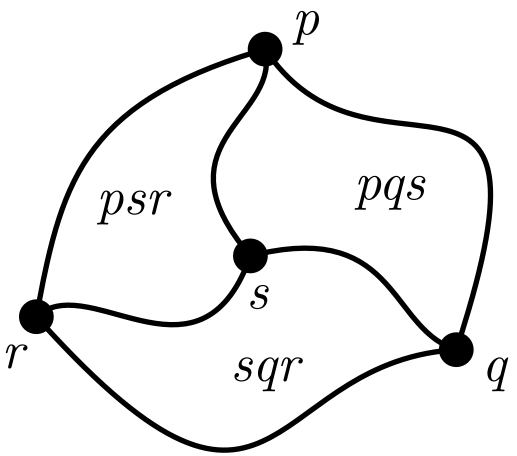

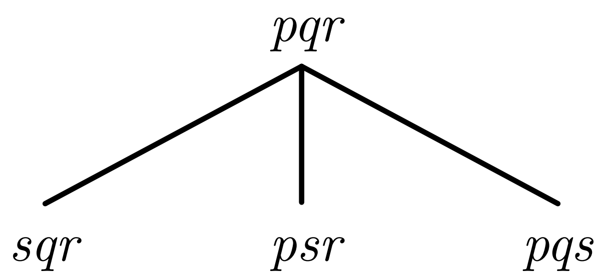

REM. In a stacked triangulation obtained after n stacking operations, and with construction tree C,

- number of vertices $= 3 + n = 3 +$ number interior nodes in C,
- number of edges $= 3(n + 1) = 3 +$ number edges in C,
- number of faces $= 2n + 1 =$ number of leaves of C.

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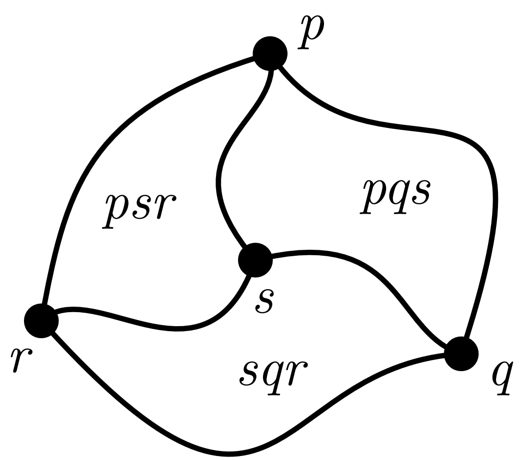

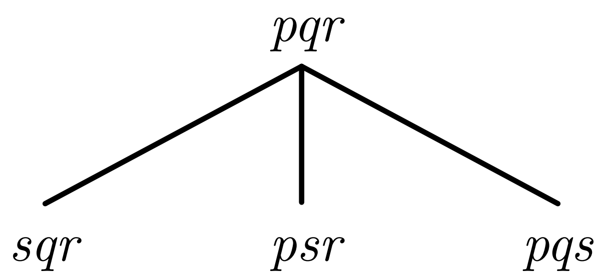

PROP. A stacked triangulation admits a unique Schnyder labeling and Schnyder wood.

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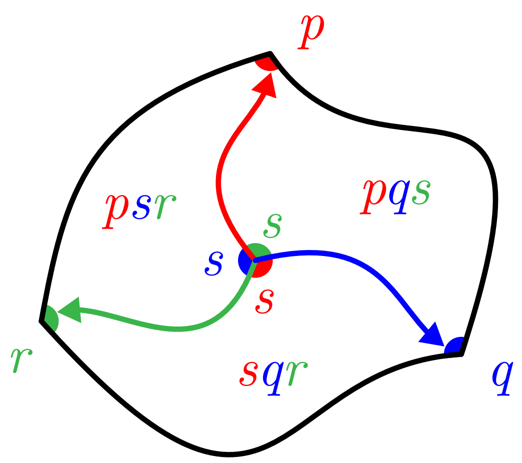
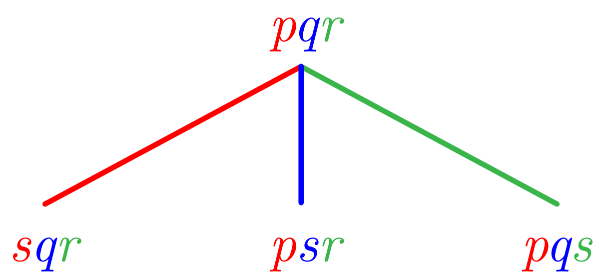
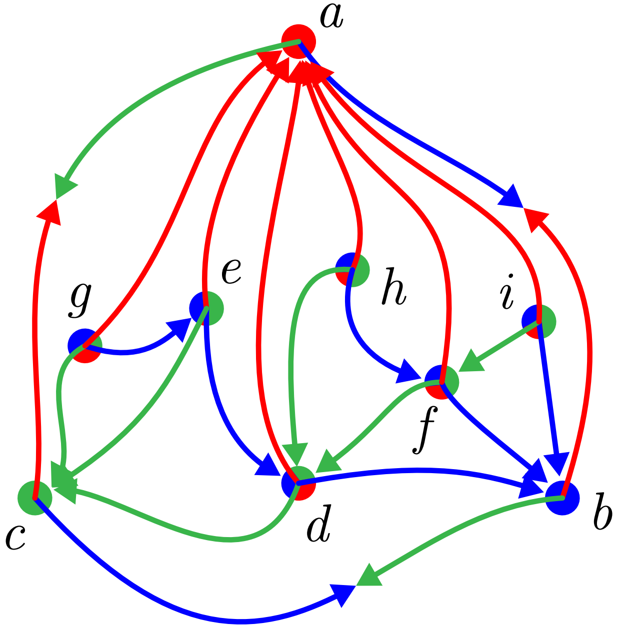


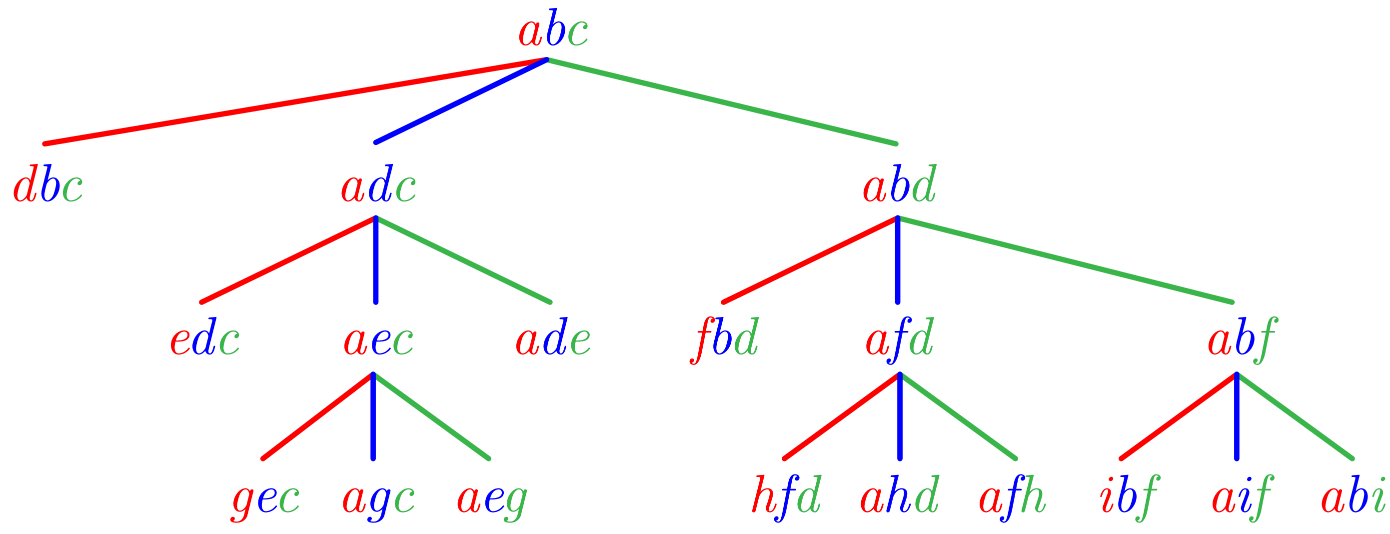


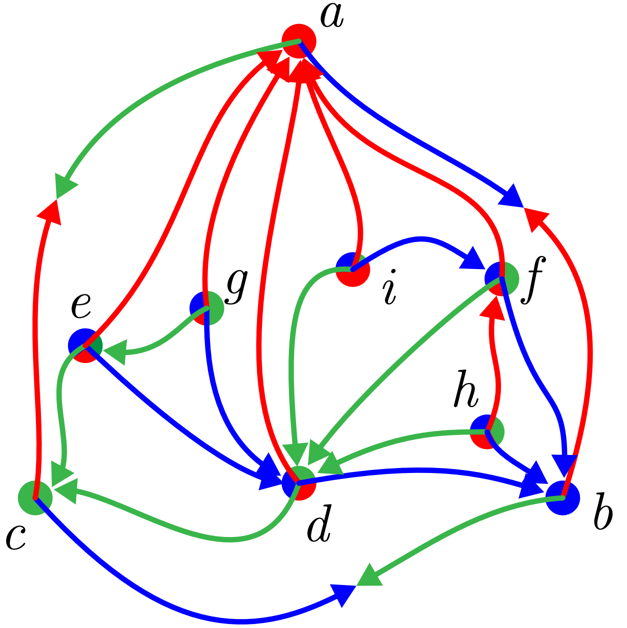
PROP. A stacked triangulation admits a unique Schnyder labeling and Schnyder wood.

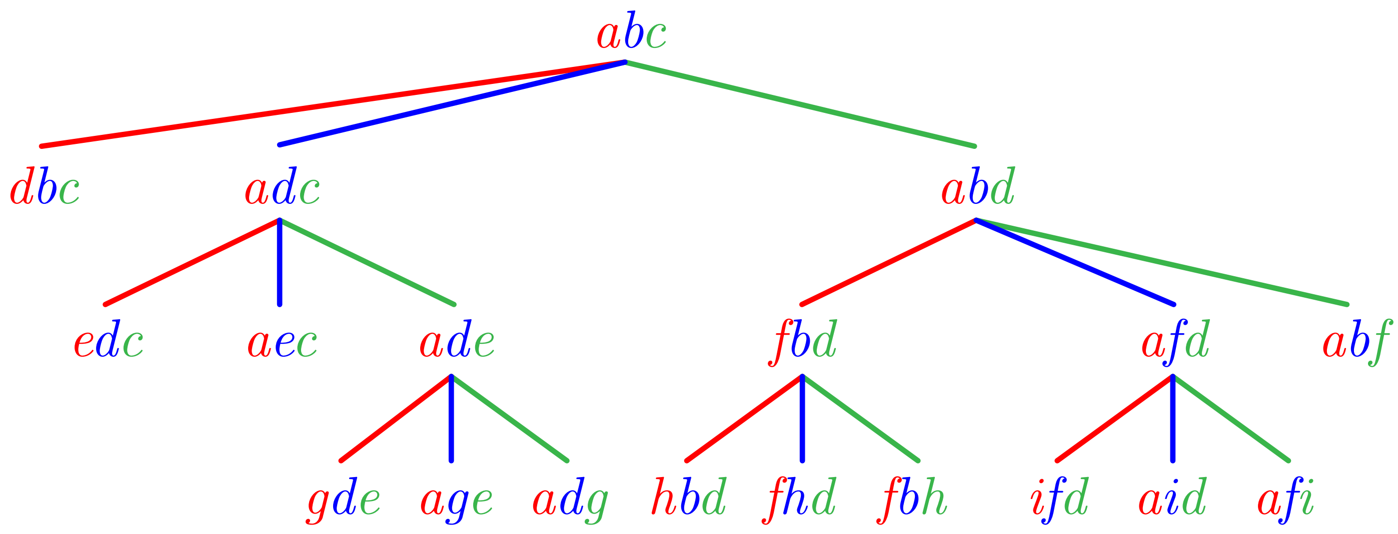
proof idea: induction.

EXM: STACKED TRIANGULATIONS



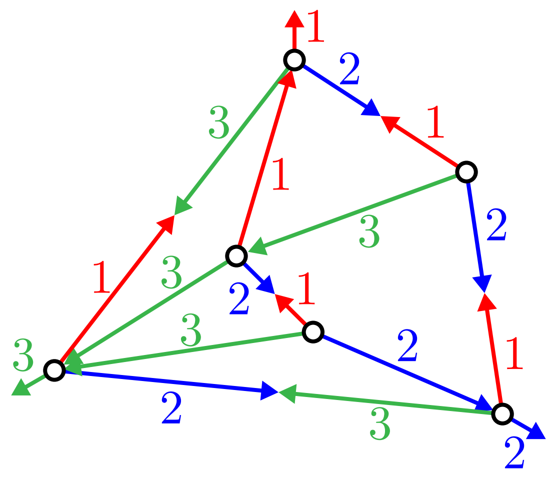






SCHNYDER EMBEDDING

DEF. M planar map with Schnyder wood.



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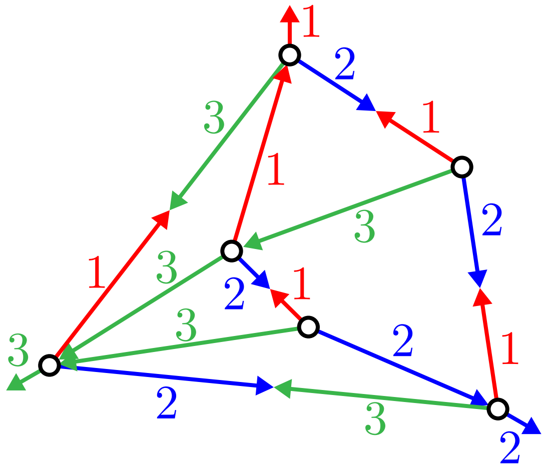
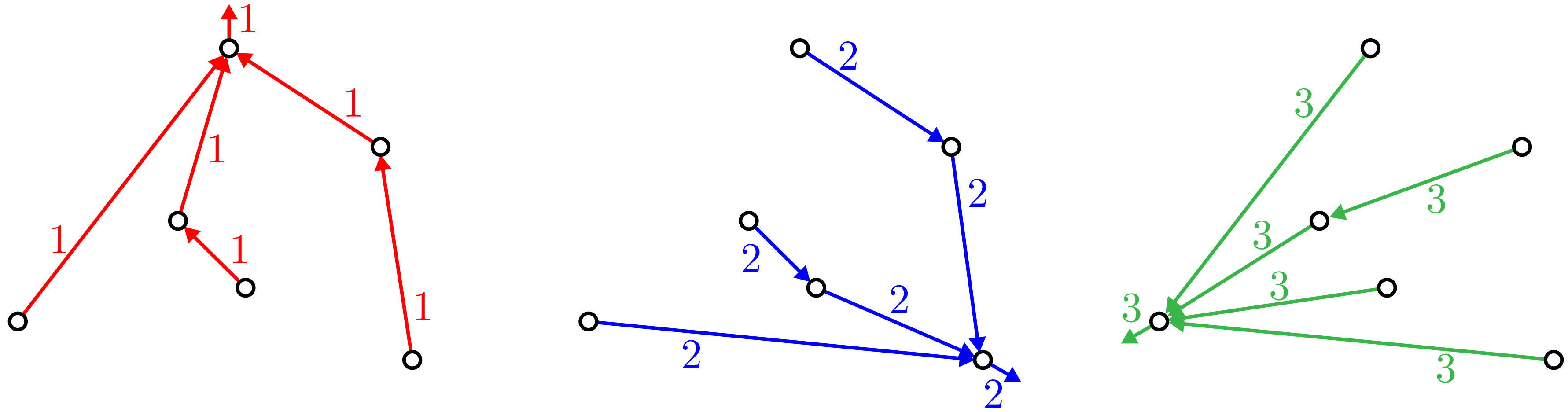


T_i = directed graph formed by edges colored i.



TREES

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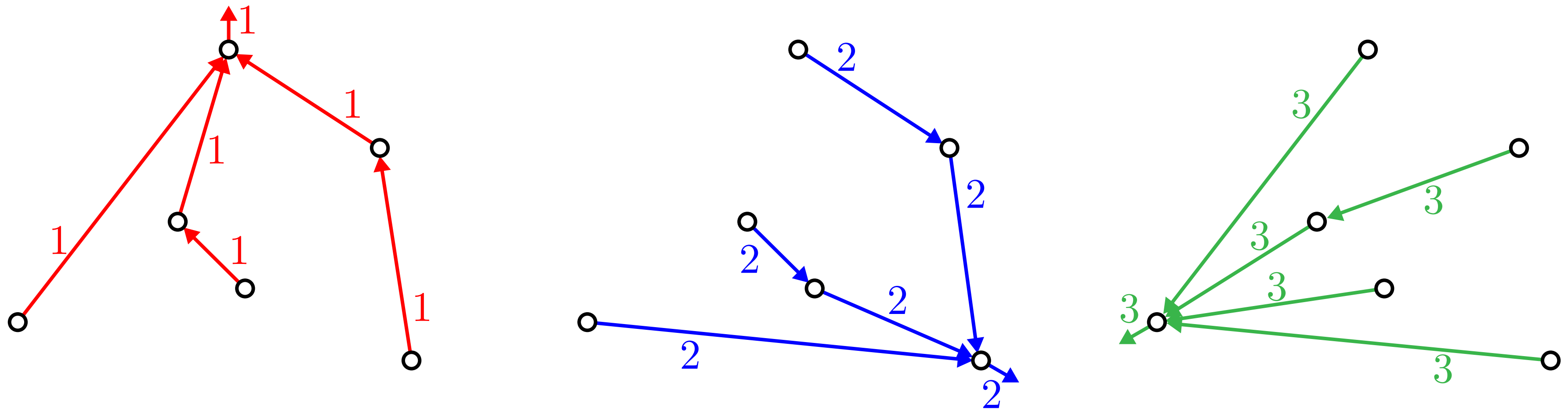


PROP. T_i is a directed tree rooted at v_i.

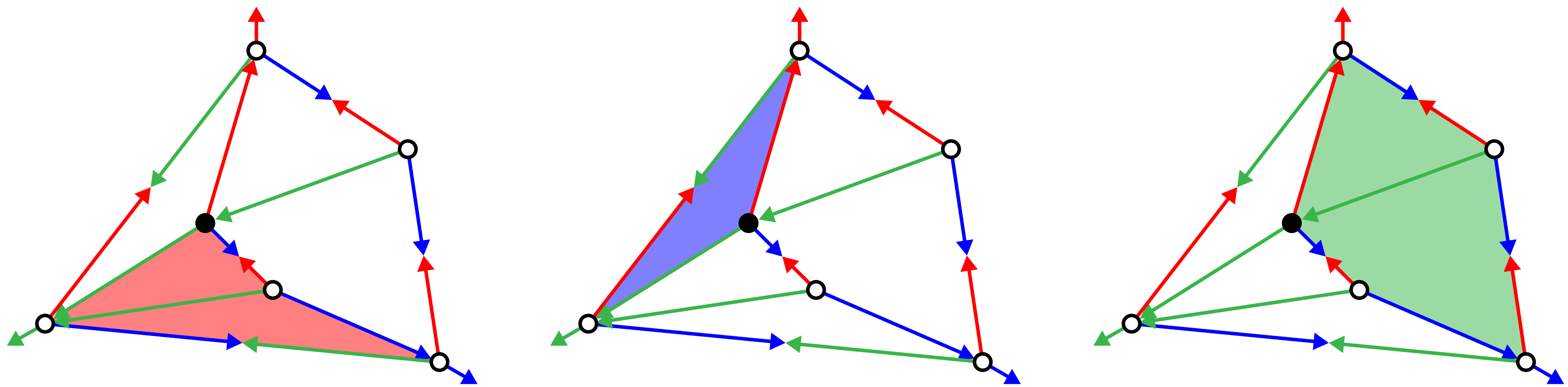
proof ideas:

- All vertices except v_i have outdegree 1, so enough to prove acyclicity.
- In fact, $D_i = T_i \cup T_{i-1}^{\text{rev}} \cup T_{i+1}^{\text{rev}}$ is already acyclic if we ignore bidirected edges or paths.
  If Z is an area minimal cycle in D_i, then:
  - Z bounds a single face F (otherwise, it has a chord or contains a vertex...),
  - if Z is clockwise, no angle of F has label $i+1$.

REGIONS

DEF. For a vertex v of M, denote:

- $P_i(v)$ = directed path in T_i to the root v_i,
- $R_i(v)$ = region bounded by the two paths $P_{i-1}(v)$ and $P_{i+1}(v)$,
- $r_i(v)$ = number of faces in region $R_i(v)$.



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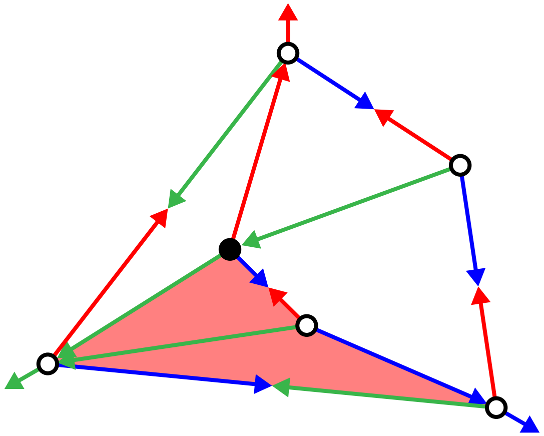
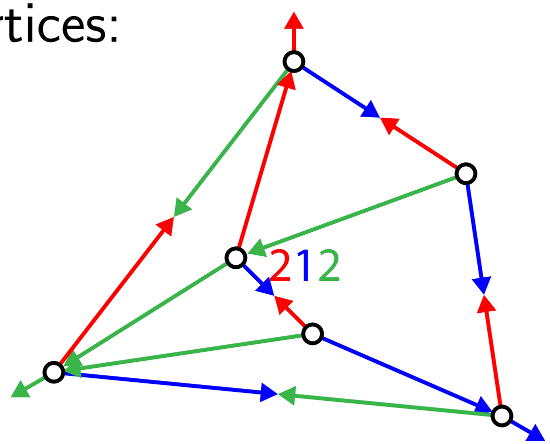


QU. Compute $r_1 r_2 r_3$ for all vertices:



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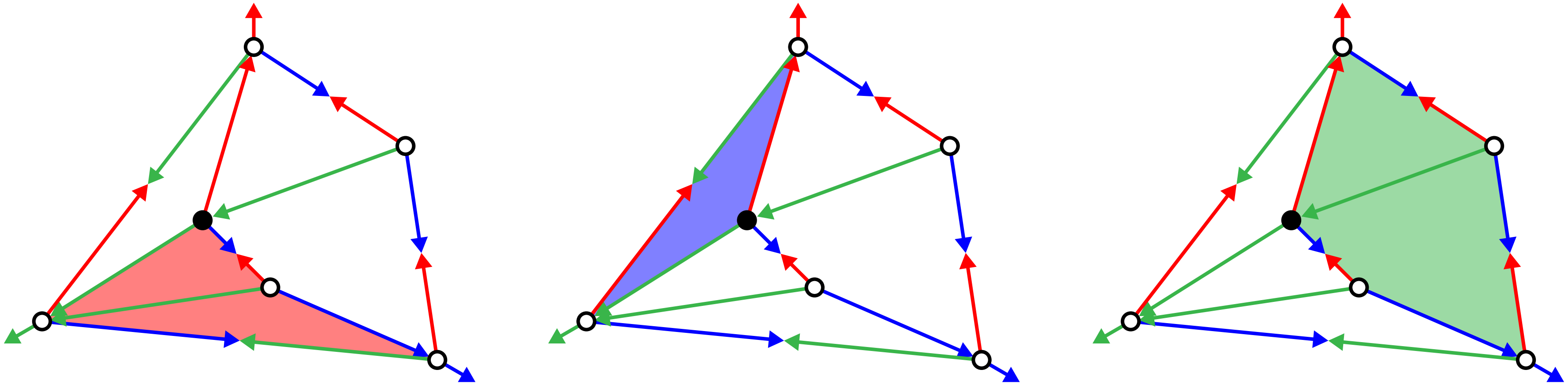
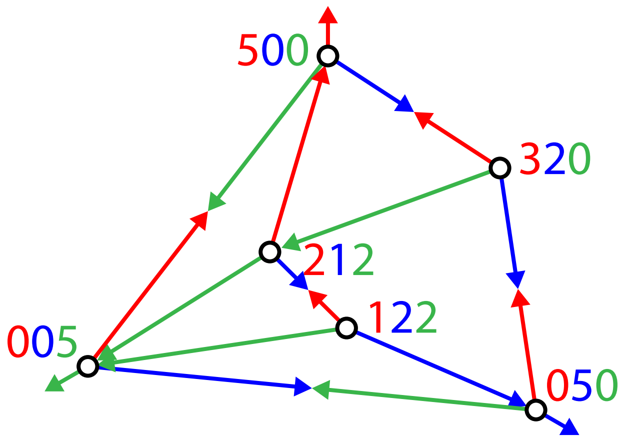


REM. $r_1 r_2 r_3$ are given by:



REGIONS

$R_i(v)$ = region bounded by the two paths $P_{i-1}(v)$ and $P_{i+1}(v)$.

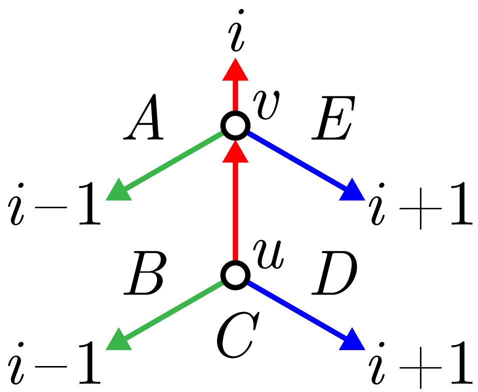
PROP. u, v = two adjacent vertices in the map M. Then:

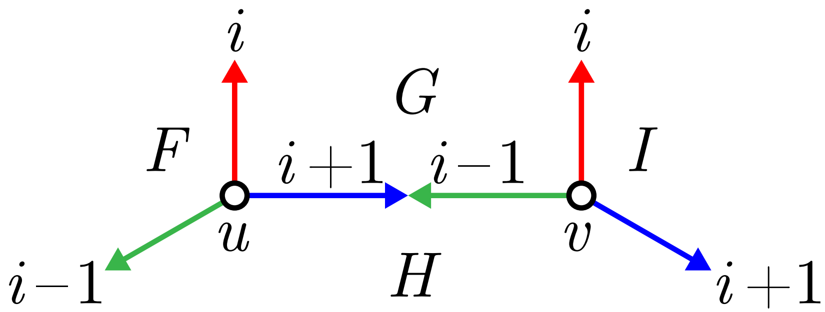
(R1) if there is a unidirected edge colored i from u to v, then

$$R_i(u) \subsetneq R_i(v), \qquad R_{i-1}(u) \supsetneq R_{i-1}(v), \qquad \text{and} \qquad R_{i+1}(u) \supsetneq R_{i+1}(v),$$

(R2) if there is a bidirected edge colored $i+1$ from u to v and $i-1$ from v to u, then

$$R_i(u) = R_i(v), \qquad R_{i-1}(u) \supsetneq R_{i-1}(v), \qquad \text{and} \qquad R_{i+1}(u) \subsetneq R_{i+1}(v).$$





$$R_i(u) = C \subsetneq B \cup C \cup D = R_i(v)$$
$$R_{i-1}(u) = D \cup E \supsetneq E = R_{i-1}(v)$$
$$R_{i+1}(u) = A \cup B \supsetneq A = R_{i+1}(v)$$

$$R_i(u) = H = R_i(v)$$
$$R_{i-1}(u) = G \cup I \supsetneq I = R_{i-1}(v)$$
$$R_{i+1}(u) = F \subsetneq F \cup G = R_{i+1}(v)$$

SCHNYDER EMBEDDING

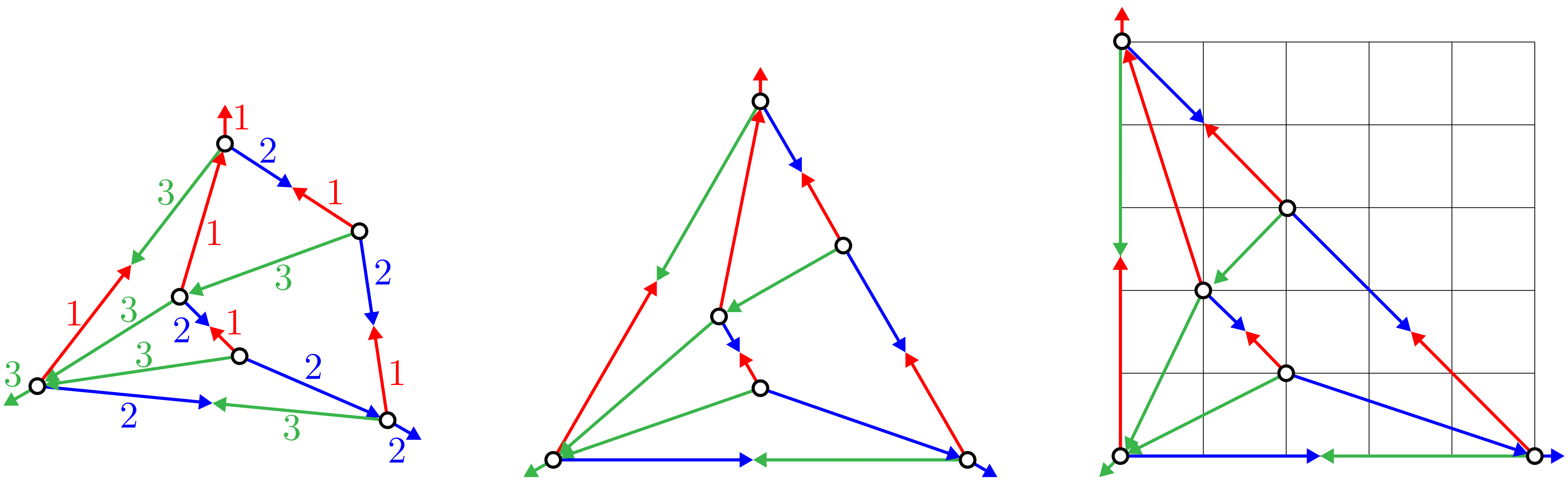
M = planar map with f faces (including the unbounded one),
endowed with a Schnyder wood.

$\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ = three arbitrary non-colinear points in the plane.

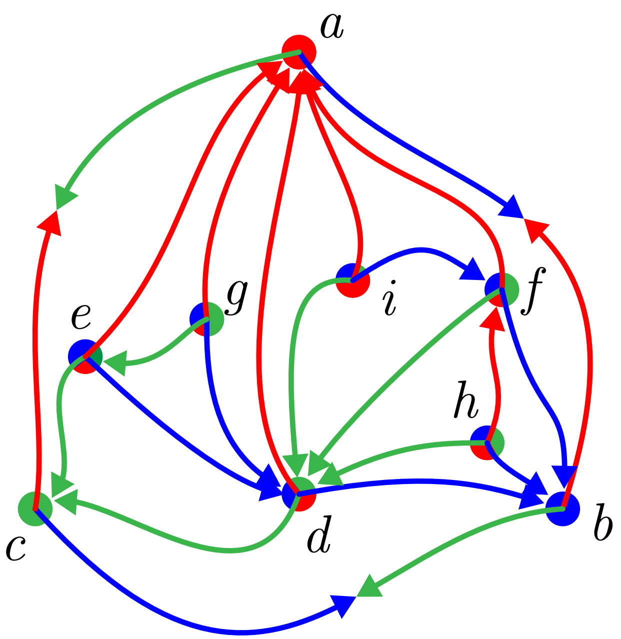
THM. The map

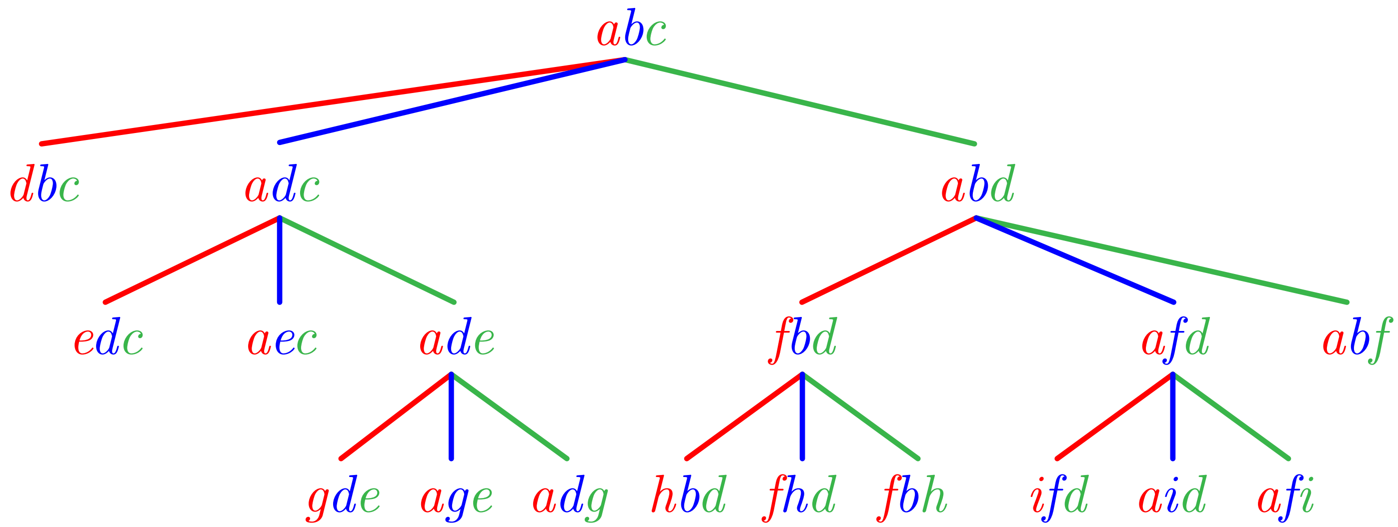
$$\mu : v \longmapsto \frac{1}{f-1}\big(r_1(v)\cdot \boldsymbol{p}_1 + r_2(v)\cdot \boldsymbol{p}_2 + r_3(v)\cdot \boldsymbol{p}_3\big)$$

defines a straightline embedding of M in the plane where all faces are convex.



EXM: STACKED TRIANGULATIONS



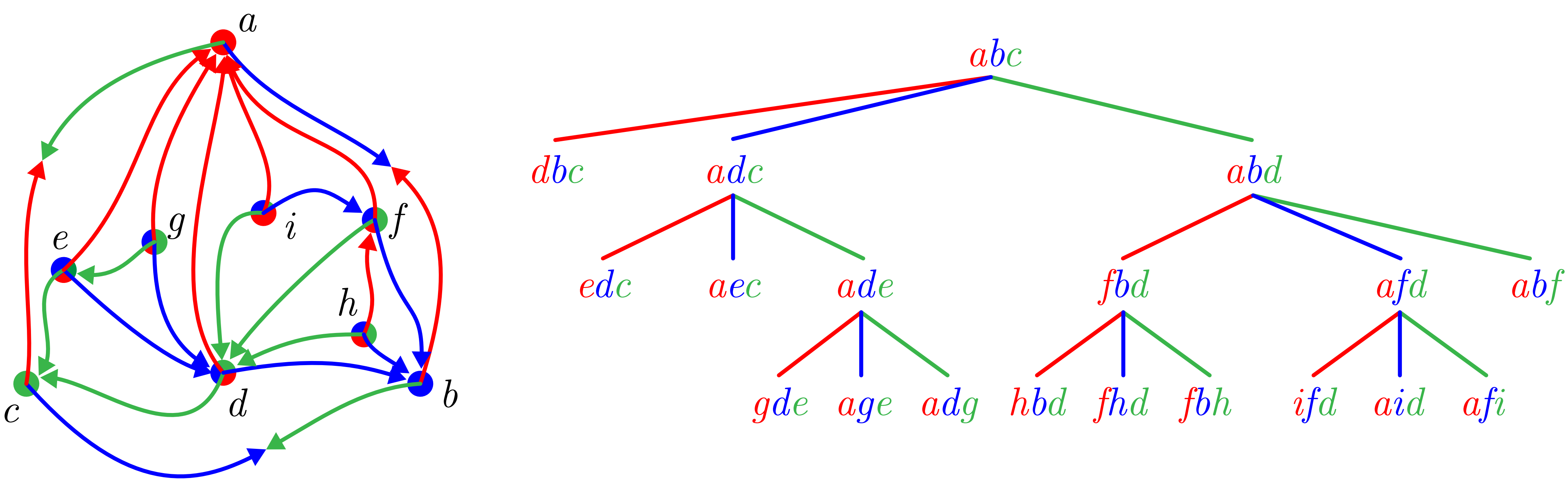


QU. Describe on the construction tree C of a stacked triangulation:

- the trees T_1, T_2 and T_3,
- the sizes $r_1(v)$, $r_2(v)$ and $r_3(v)$ of the regions of a vertex v.

Draw the Schnyder embedding for $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ being the vertices of an equilateral triangle.

EXM: STACKED TRIANGULATIONS

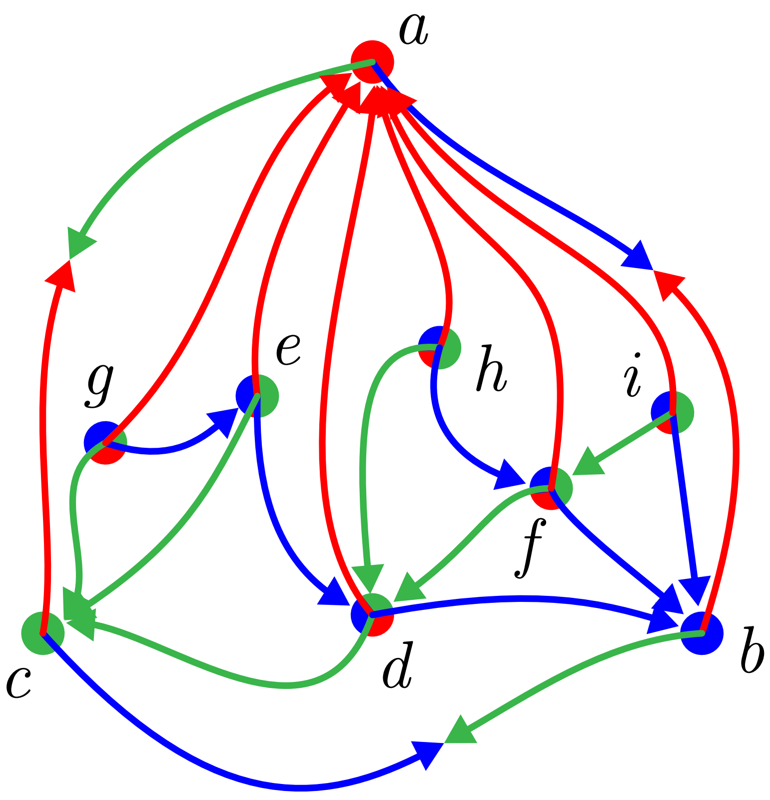


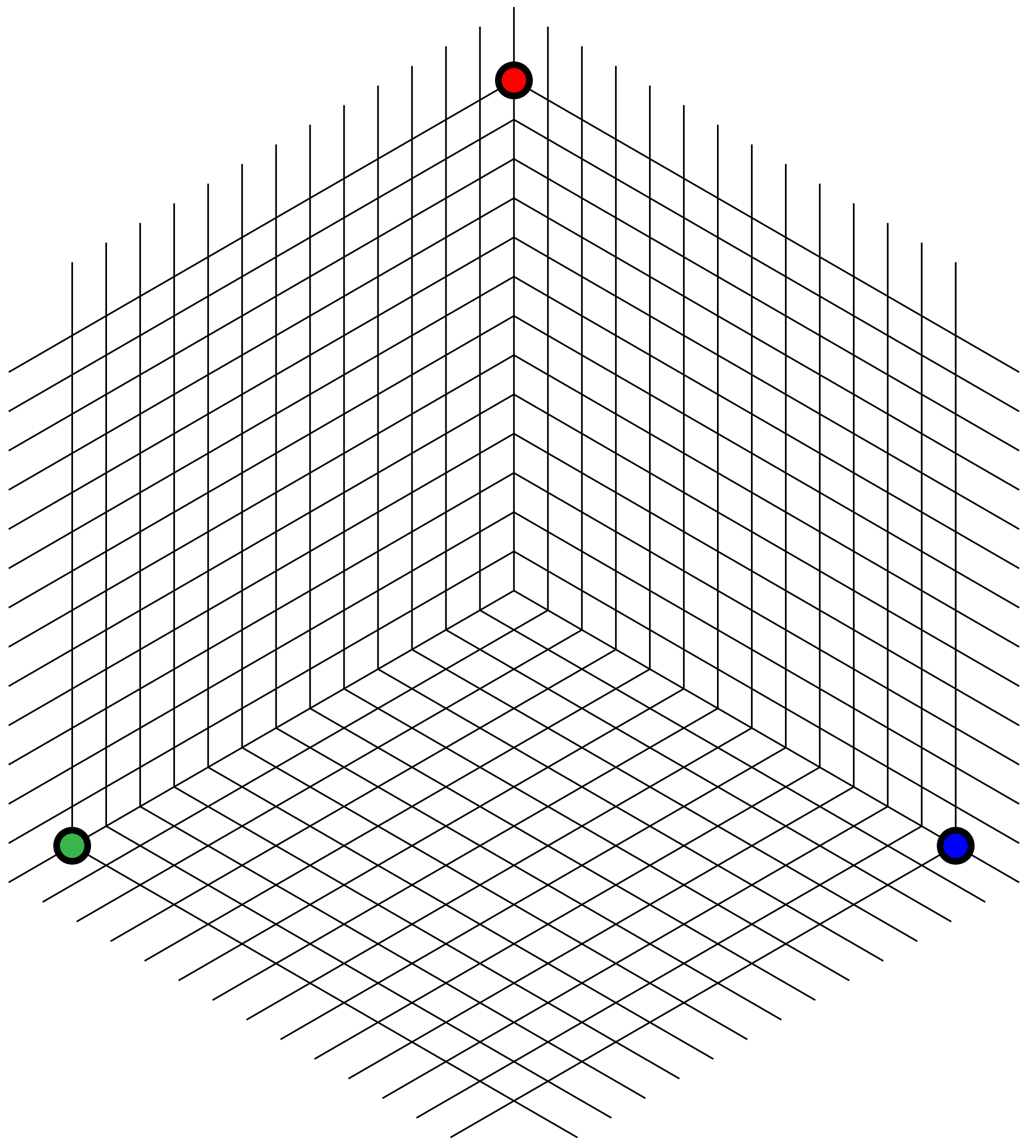
PROP. The tree T_i is obtained by contracting all edges colored $i-1$ and $i+1$ in C

PROP. Assume v is inserted in triangle t, and let γ be the path from t to the root in C. The size $r_i(v)$ is obtained by summing the number of leaves of the subtrees of the blue children of the nodes of γ that are not in γ.

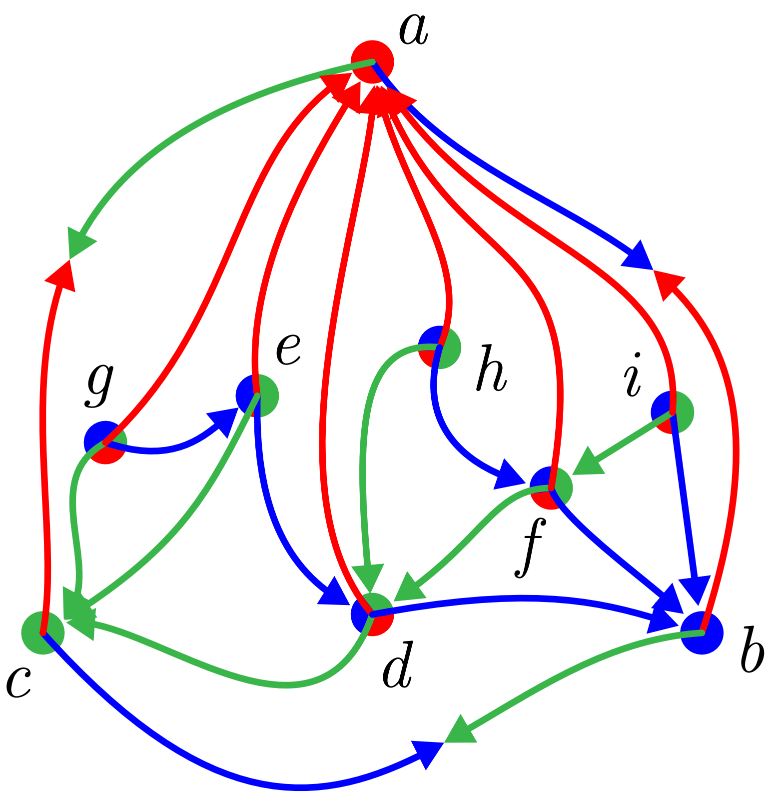
<u>proof idea:</u> induction.

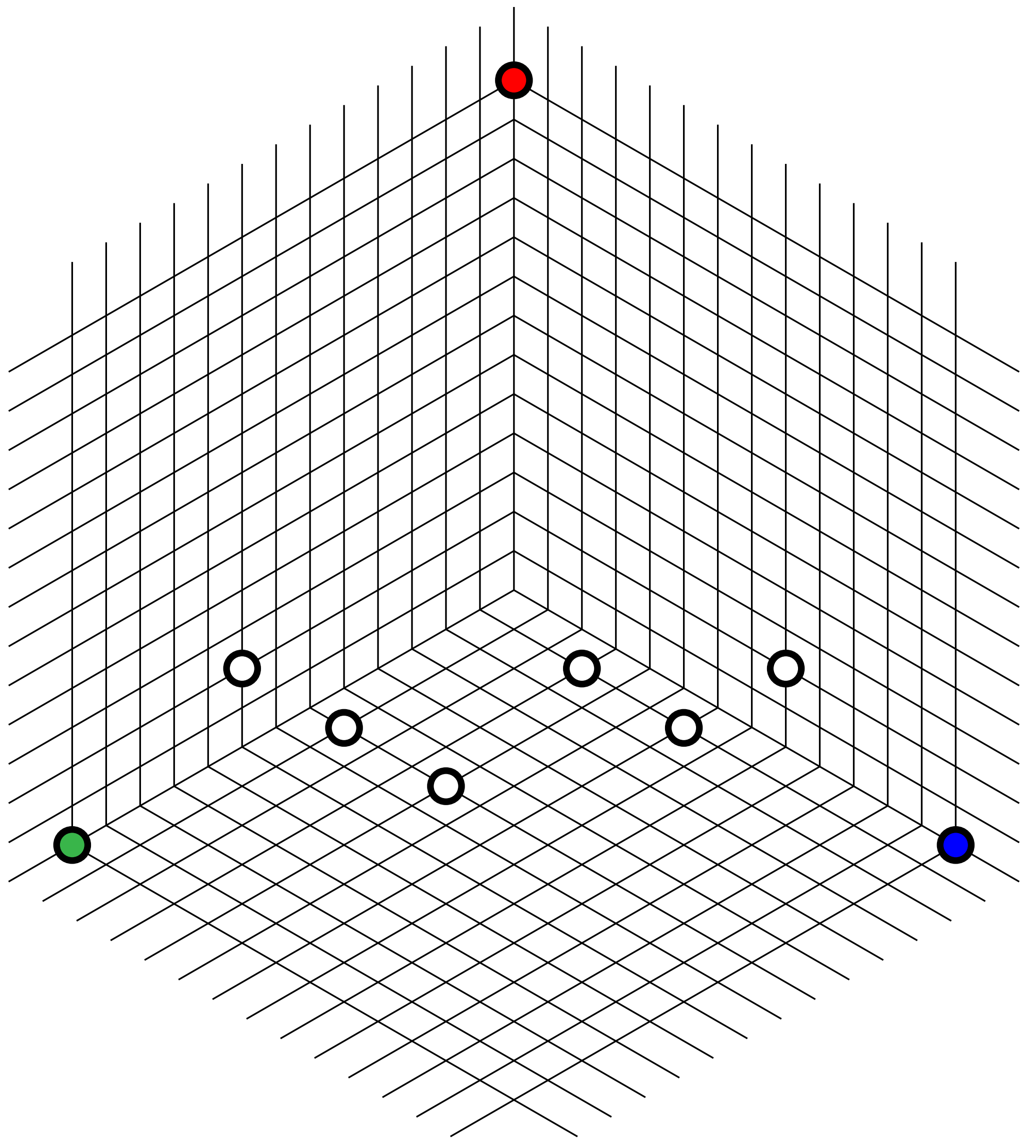
EXM: STACKED TRIANGULATIONS



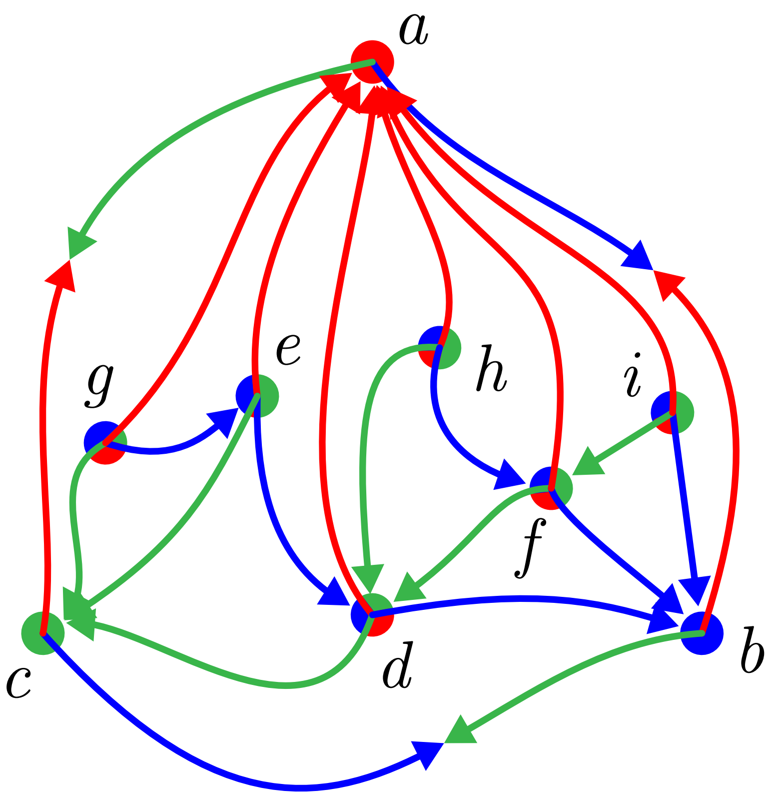


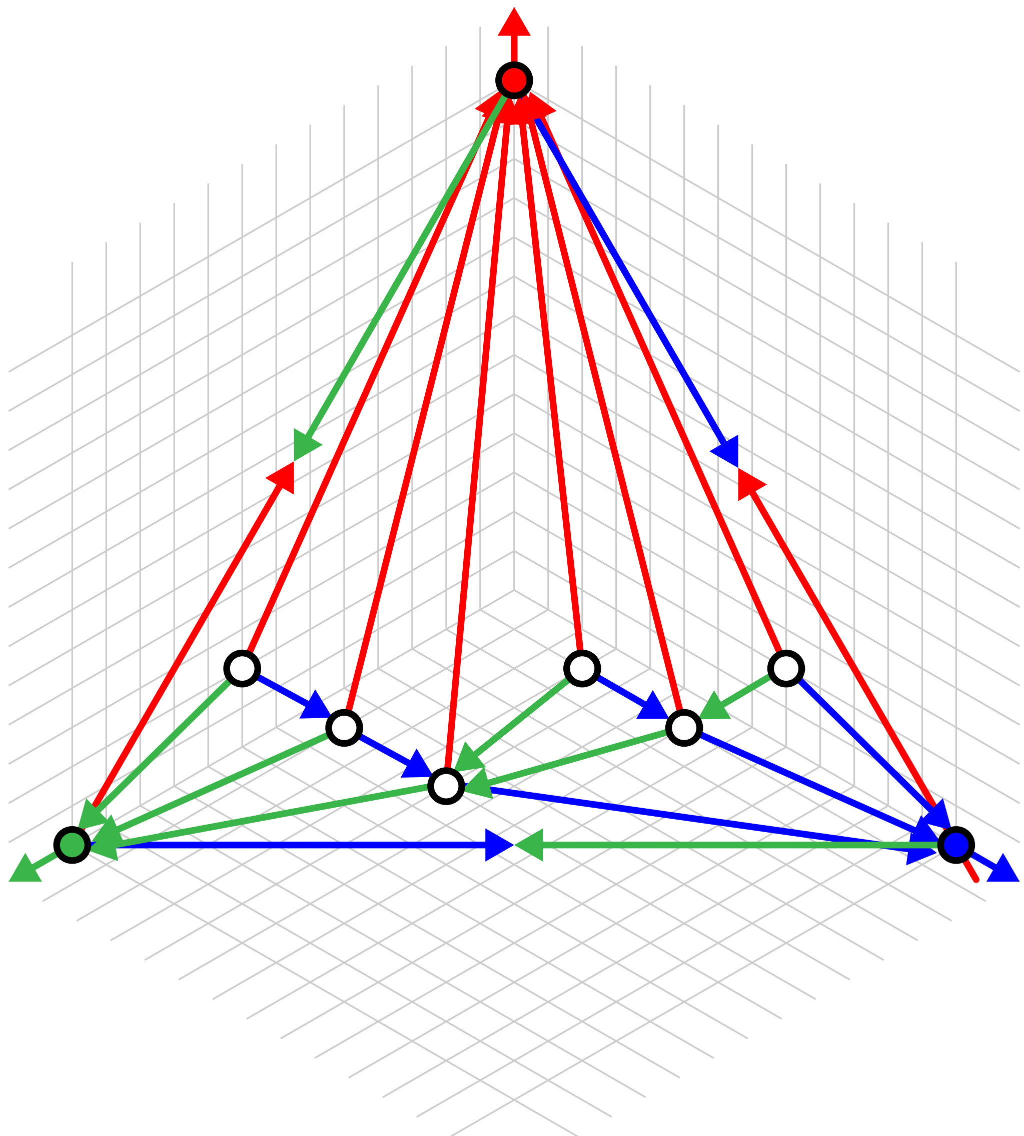
EXM: STACKED TRIANGULATIONS





EXM: STACKED TRIANGULATIONS





GEODESIC MAPS ON ORTHOGONAL SURFACES

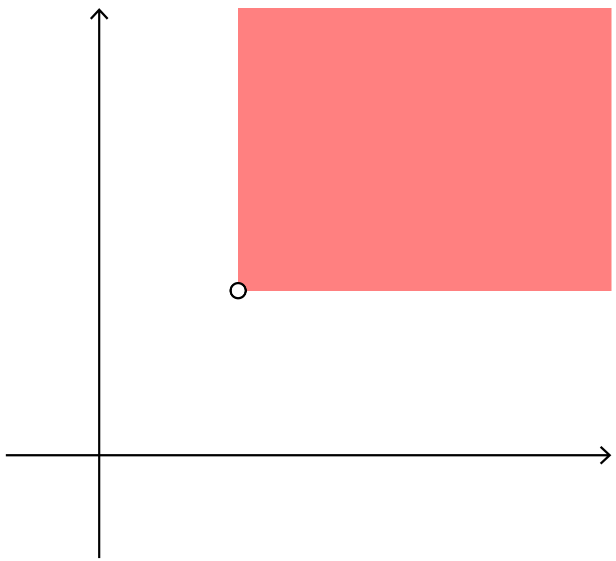
DOMINANCE ORDER

DEF. dominance order in $\mathbb{R}^3$ = $\boldsymbol{u} \le \boldsymbol{v} \iff u_i \le v_i$ for all $i \in [3]$ (componentwise).

DEF.

cone dominating $\boldsymbol{y} \in \mathbb{R}^3$

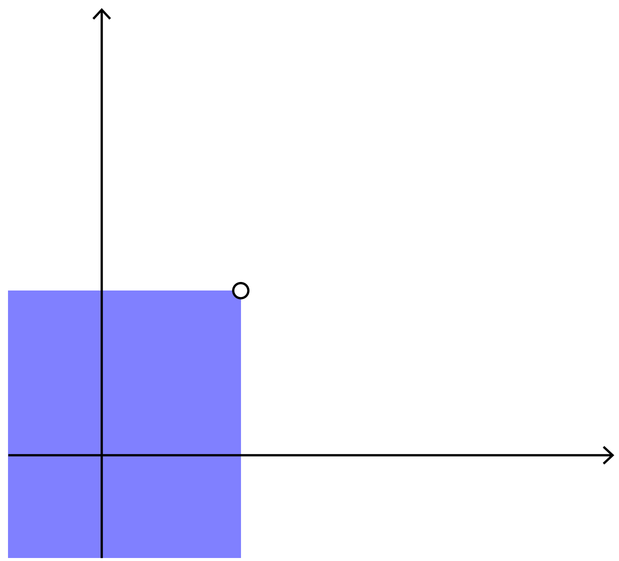
$$\Delta_{\boldsymbol{y}} = \left\{ \boldsymbol{z} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{y} \le \boldsymbol{z} \right\}$$



(= upper ideal of $\boldsymbol{y}$)

cone dominated by $\boldsymbol{y} \in \mathbb{R}^3$

$$\nabla_{\boldsymbol{y}} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{x} \le \boldsymbol{y} \right\}$$



(= lower ideal of $\boldsymbol{y}$)

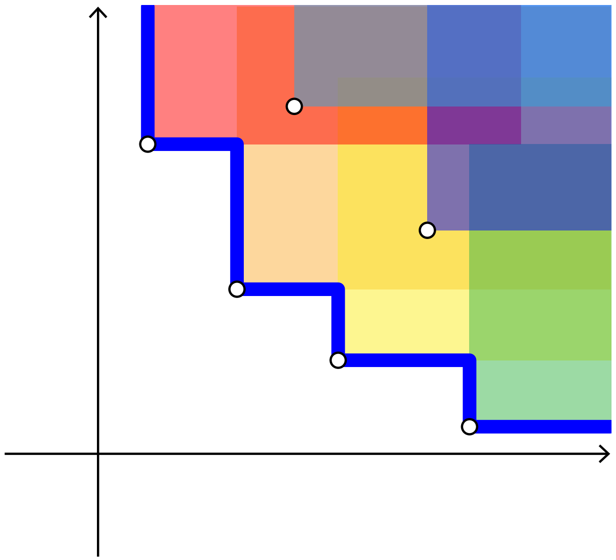
ORTHOGONAL SURFACE

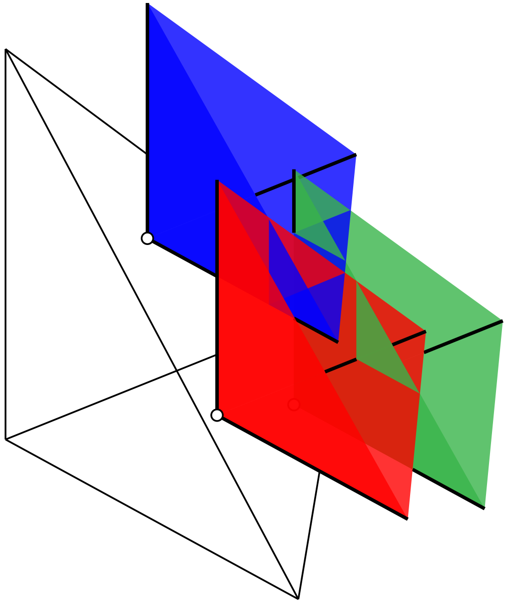
DEF. dominance order in $\mathbb{R}^3$ = $\boldsymbol{u} \leq \boldsymbol{v} \iff u_i \leq v_i$ for all $i \in [3]$ (componentwise).

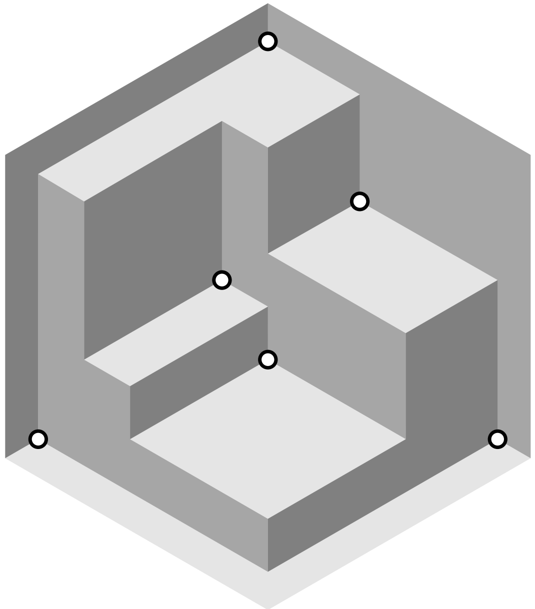
DEF. cone dominating $\boldsymbol{y} \in \mathbb{R}^3$: $\Delta_{\boldsymbol{y}} = \left\{ \boldsymbol{z} \in \mathbb{R}^3 \;\middle|\; \boldsymbol{y} \leq \boldsymbol{z} \right\}$

cone dominated by $\boldsymbol{y} \in \mathbb{R}^3$: $\nabla_{\boldsymbol{y}} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \boldsymbol{x} \leq \boldsymbol{y} \right\}$

DEF. $\langle \boldsymbol{V} \rangle = \left\{ \boldsymbol{z} \in \mathbb{R}^3 \;\middle|\; \boldsymbol{v} \leq \boldsymbol{z} \text{ for some } \boldsymbol{v} \in \boldsymbol{V} \right\} = \bigcup_{\boldsymbol{v} \in \boldsymbol{V}} \Delta_{\boldsymbol{v}}$.







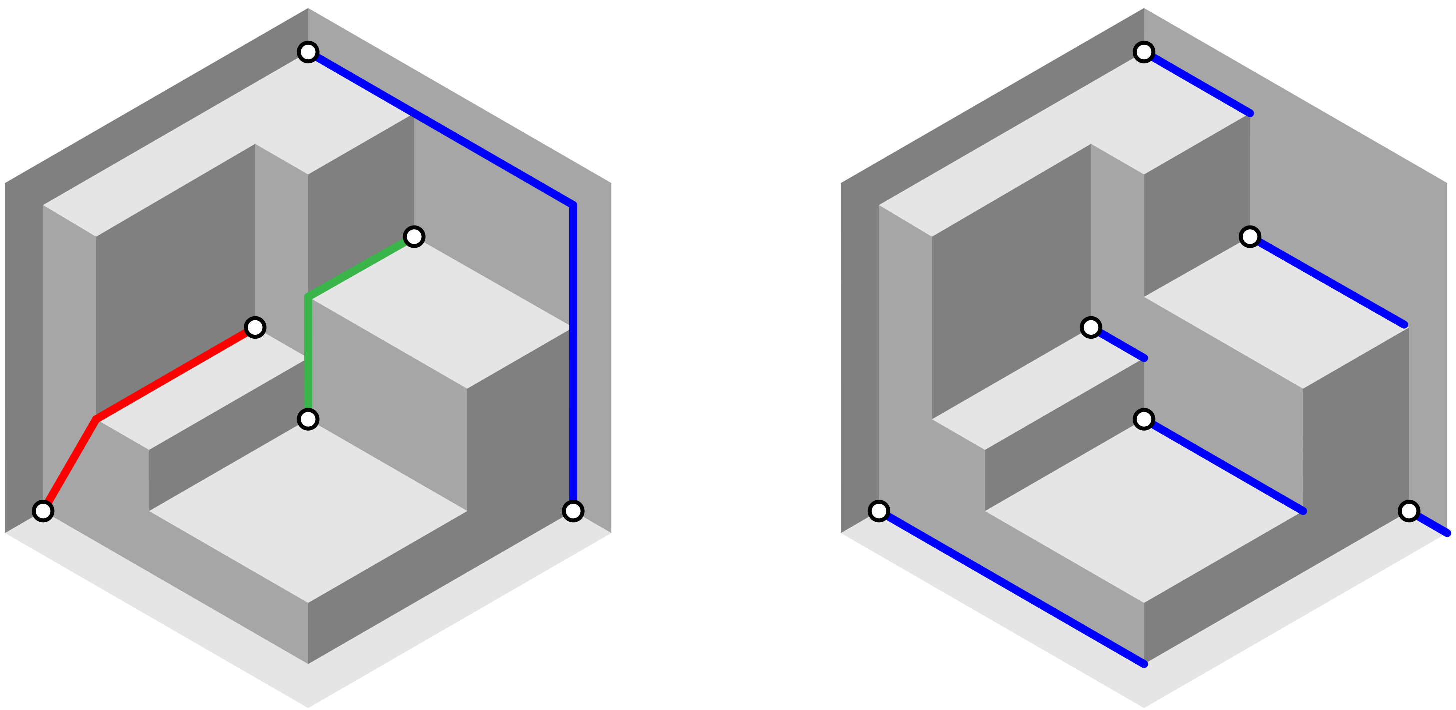
orthogonal surface $\mathcal{S}_{\boldsymbol{V}}$ = boundary of $\langle \boldsymbol{V} \rangle$

(assume now that $\boldsymbol{V}$ = antichain)

ELBOW GEODESICS AND COORDINATE ARCS

DEF. On an orthogonal surface $\mathcal{S}_{\boldsymbol{V}}$, define

- elbow geodesic $=$ union of the segments from $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$ to $\boldsymbol{u} \vee \boldsymbol{v} = [\max(u_i, v_i)]_{i \in [n]}$,
- coordinate arcs $=$ (not always bounded) segments from $\boldsymbol{v} \in \boldsymbol{V}$ in an axis direction.



elbow geodesics　　　　coordinate arcs

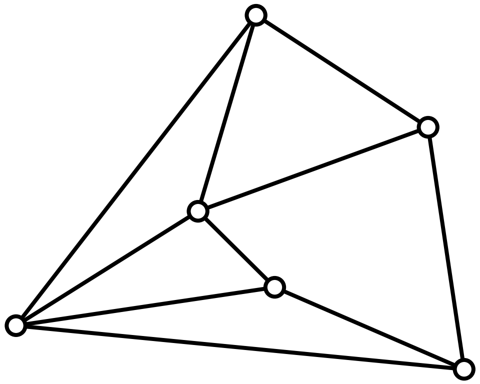
GEODESIC EMBEDDING

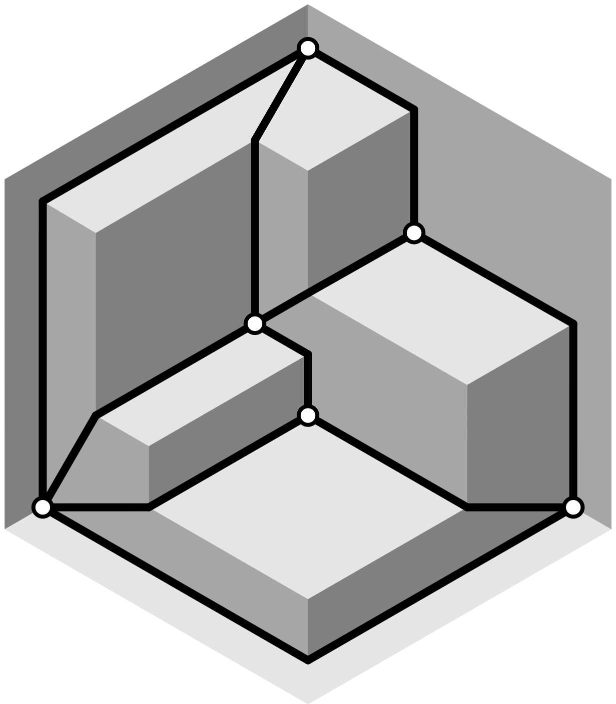
DEF. <u>geodesic embedding</u> of a map M on a surface $\mathcal{S}_{\boldsymbol{V}}$ = drawing of M on $\mathcal{S}_{\boldsymbol{V}}$ st:

(G1) there is a bijection between the points of $\boldsymbol{V}$ and the vertices of M,

(G2) every edge of M is an elbow geodesic in $\mathcal{S}_{\boldsymbol{V}}$ and every bounded coordinate arc is part of an edge of M,

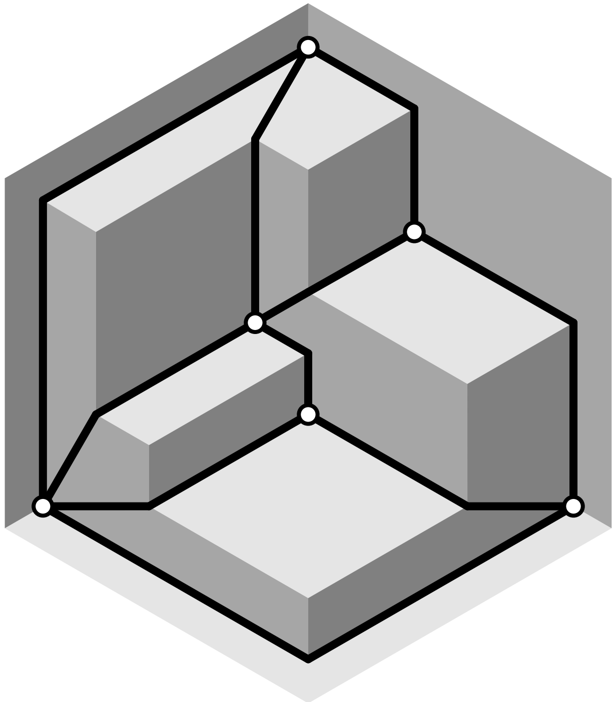
(G3) the drawing is crossing-free.





GEODESIC EMBEDDINGS VS SCHNYDER WOODS

THM. If $\boldsymbol{V}$ is an axial antichain, then a geodesic embedding of a map M on $\mathcal{S}_{\boldsymbol{V}}$ induces a Schnyder wood on M.



proof idea:

- label the angles according to the color of the flat region containing it,
- orient and color the edges according to the three axis. An elbow geodesic can get one or two colors depending on whether it contains one or two bounded coordinate arcs.

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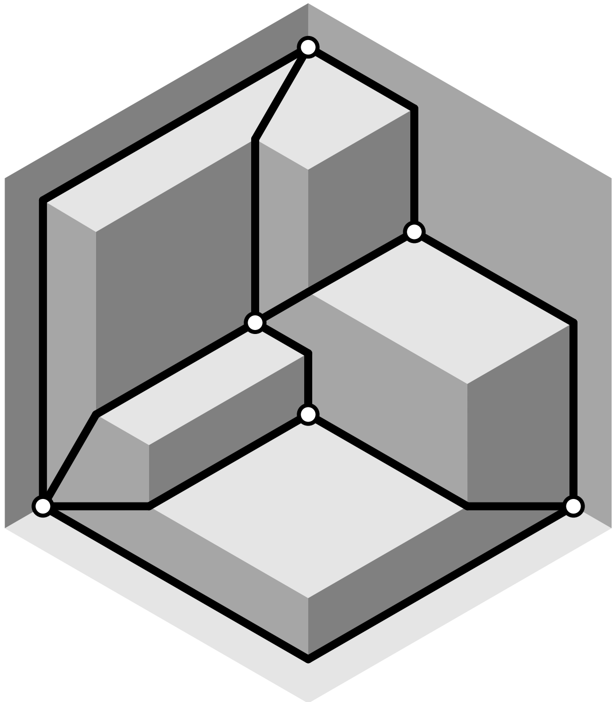

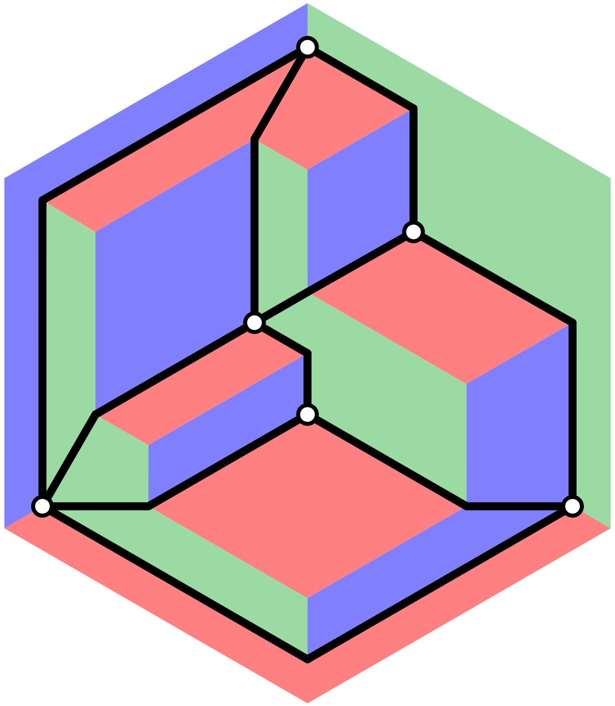

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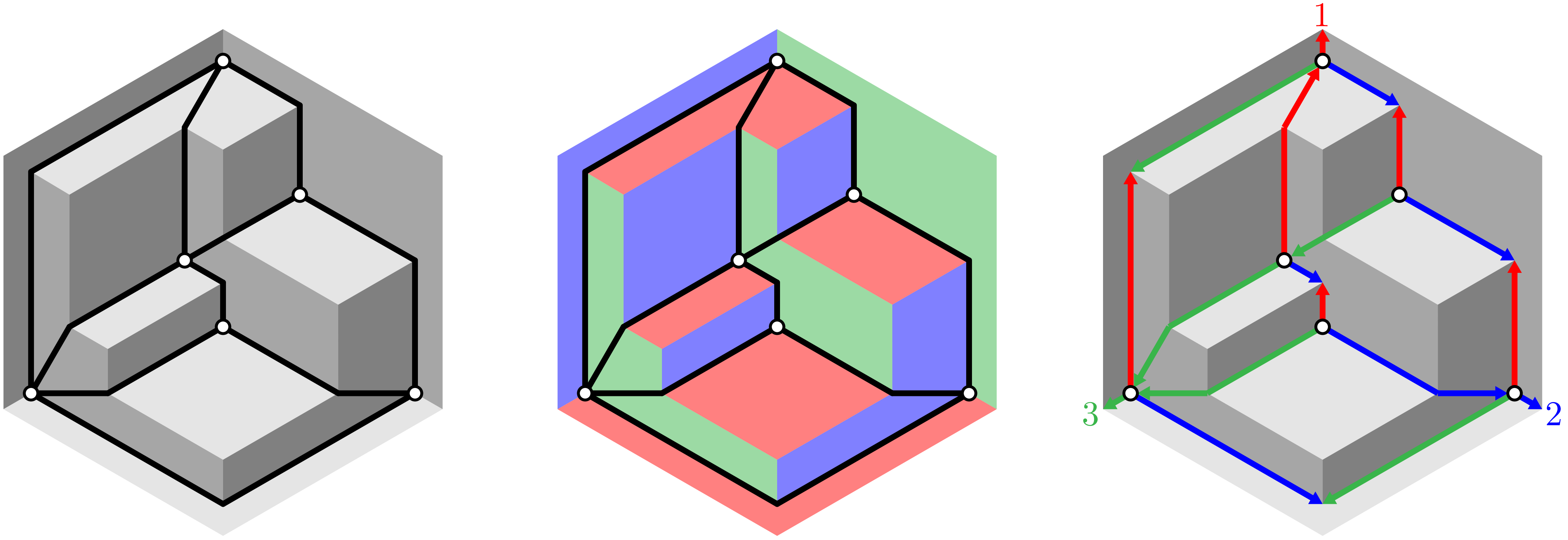

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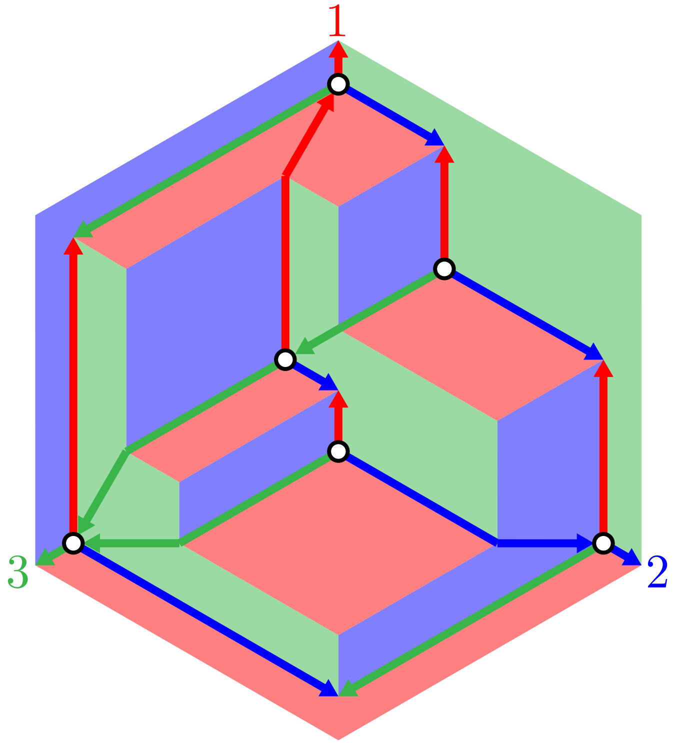

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GEODESIC EMBEDDINGS VS SCHNYDER WOODS

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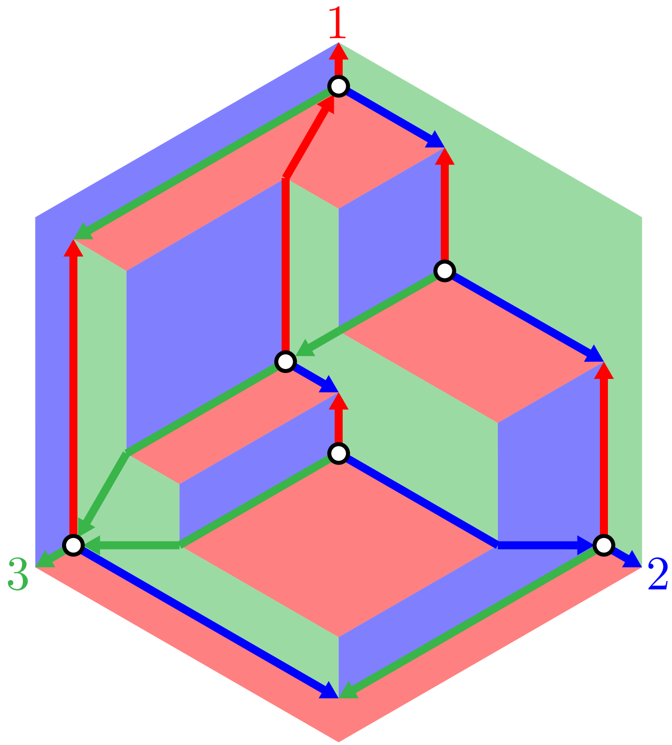
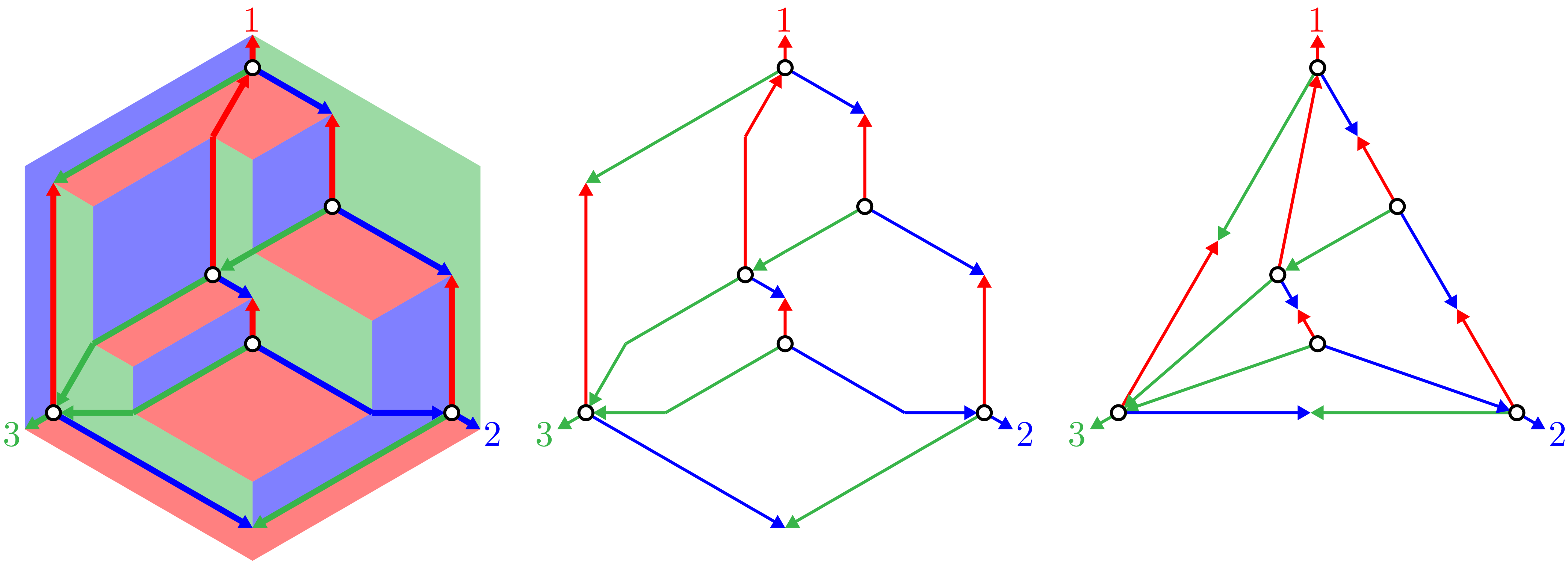


THM. Given a Schnyder wood W on a planar map M, the region vectors of the vertices of M with respect to W form an axial antichain $\boldsymbol{V}$ inducing a geodesic embedding of M on $\mathcal{S}_{\boldsymbol{V}}$.

FROM GEODESIC EMBEDDINGS TO SCHNYDER EMBEDDINGS

THM. The projection of the geodesic embedding onto the plane $v_1 + v_2 + v_3 = f - 1$ gives a planar drawing of M whose edges are bended segments. Replacing them by straight segments preserves the non-crossing-freeness.



proof idea: when straightening the geodesic embedding, the elbow geodesic joining $\boldsymbol{u}$ and $\boldsymbol{v}$ is controled by $\nabla_{\boldsymbol{u} \vee \boldsymbol{v}}$.

SCHNYDER EMBEDDING

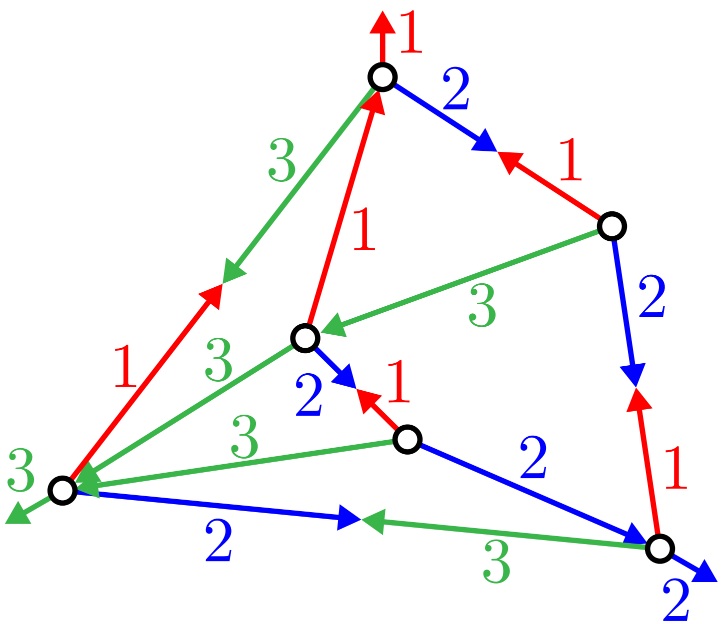
M = planar map with f faces (including the unbounded one), endowed with a Schnyder wood.

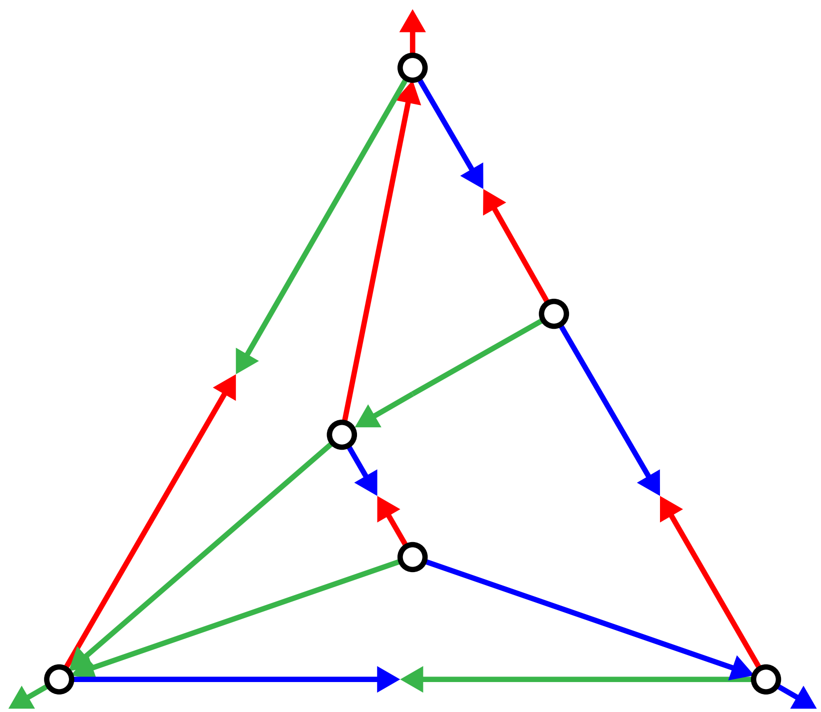
$\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ = three arbitrary non-colinear points in the plane.

THM. The map

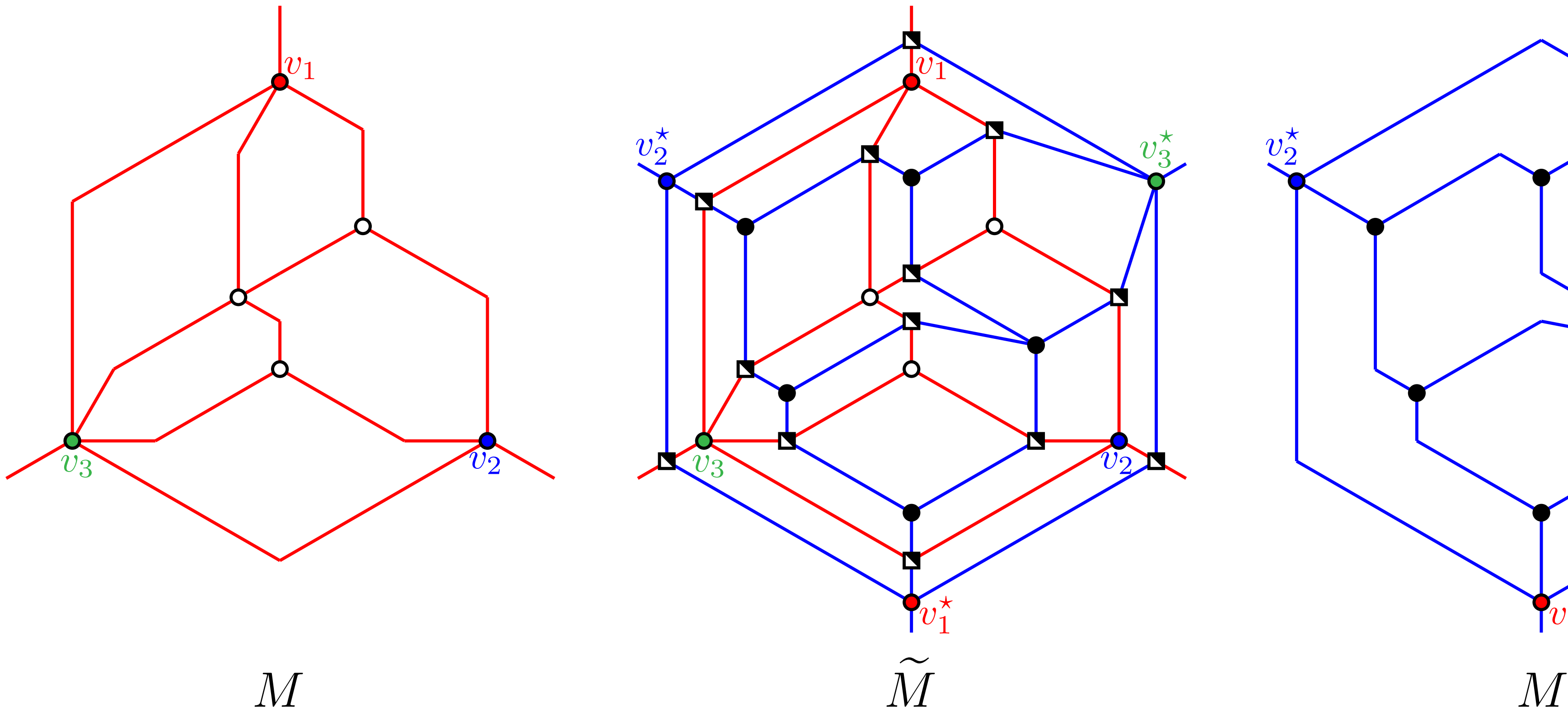
$$\mu : v \longmapsto \frac{1}{f-1}\left(r_1(v)\cdot \boldsymbol{p}_1 + r_2(v)\cdot \boldsymbol{p}_2 + r_3(v)\cdot \boldsymbol{p}_3\right)$$

defines a straightline embedding of M in the plane where all faces are convex.





PRIMAL-DUAL MAP

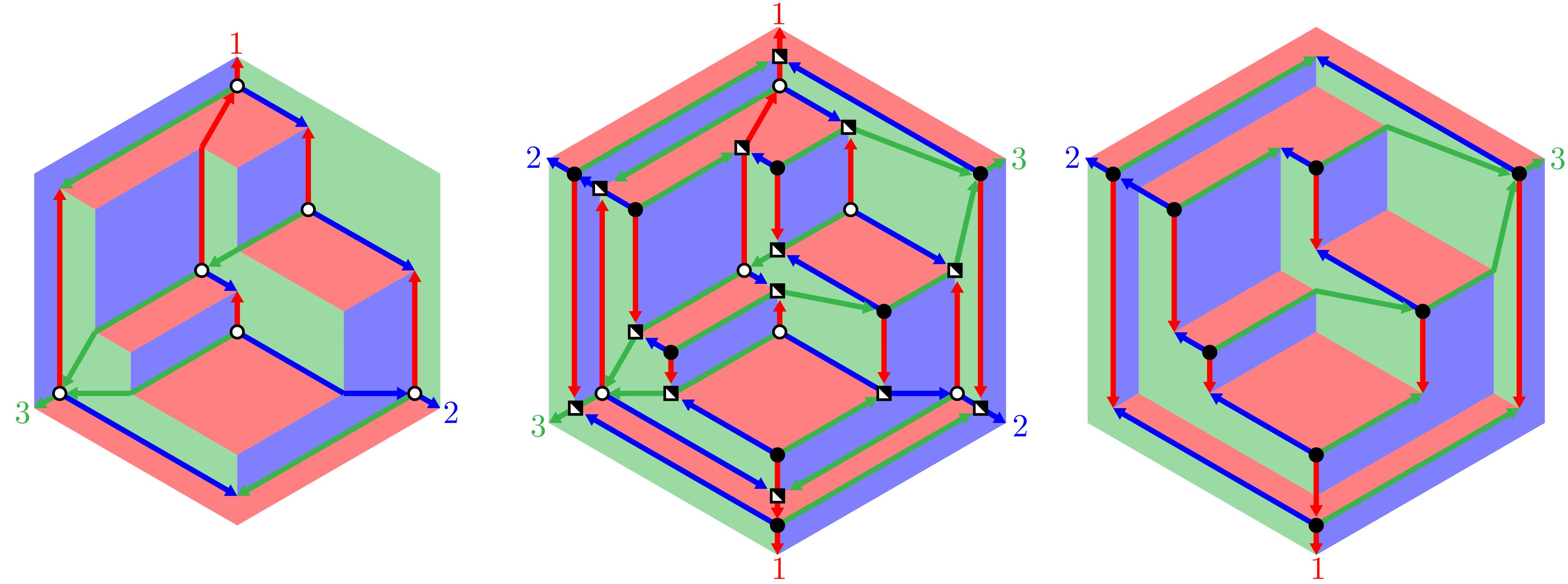


DEF. dual map of M = exchange vertices $\longleftrightarrow$ faces.

suspended dual map $M^\star$ = dual map of M where the vertex corresponding to the external face is split into three vertices.

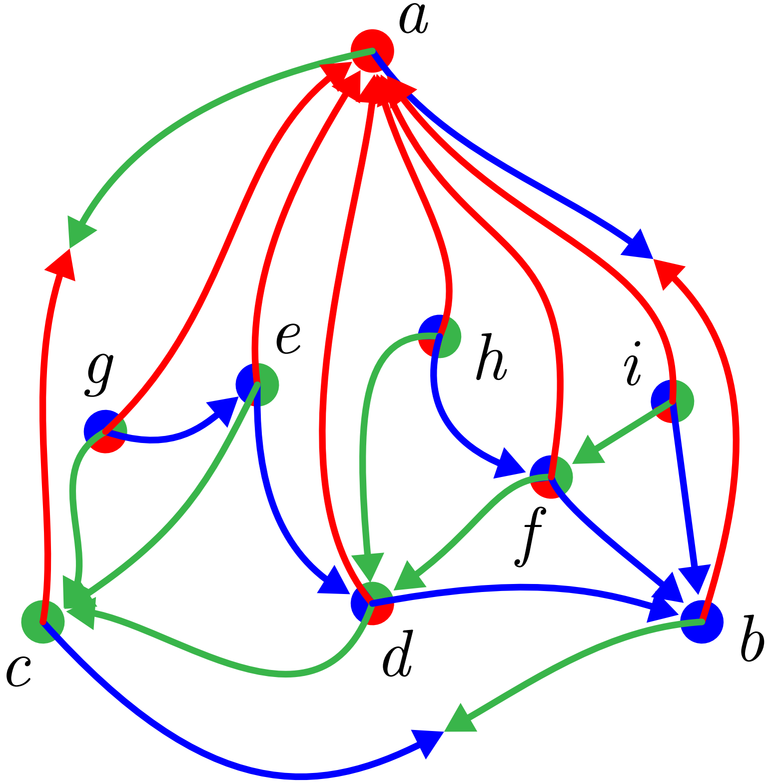
primal-dual map $\widetilde{M}$ = superimposition of the map M and its suspended dual map $M^\star$ with additional vertices at the edge intersections.

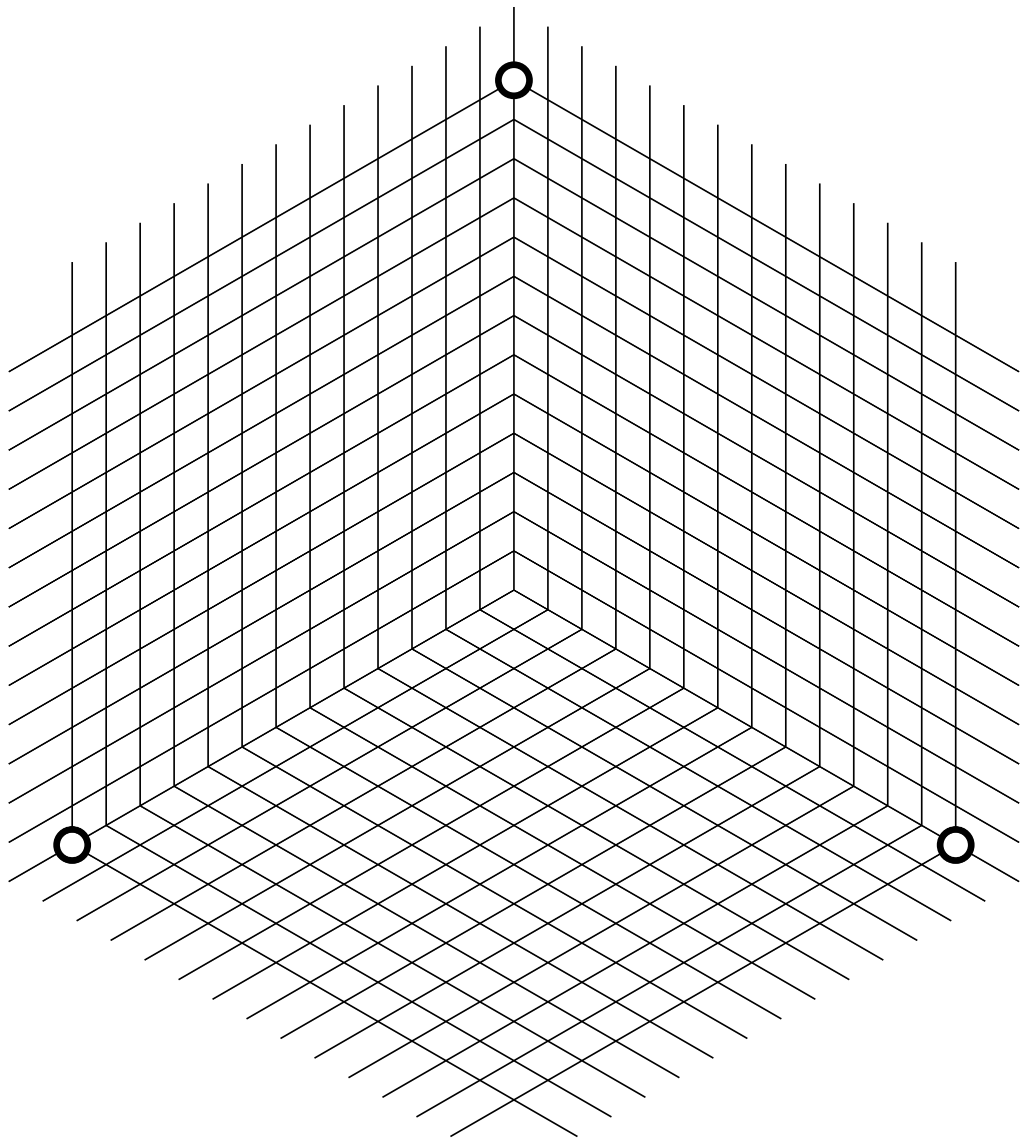
PRIMAL-DUAL GEODESIC EMBEDDING



THM. Reversing the orientation, the same orthogonal surface admits a geodesic embedding of the map M, of its suspended dual map $M^\star$, and of its primal-dual map $\widetilde{M}$.

EXM: STACKED TRIANGULATIONS





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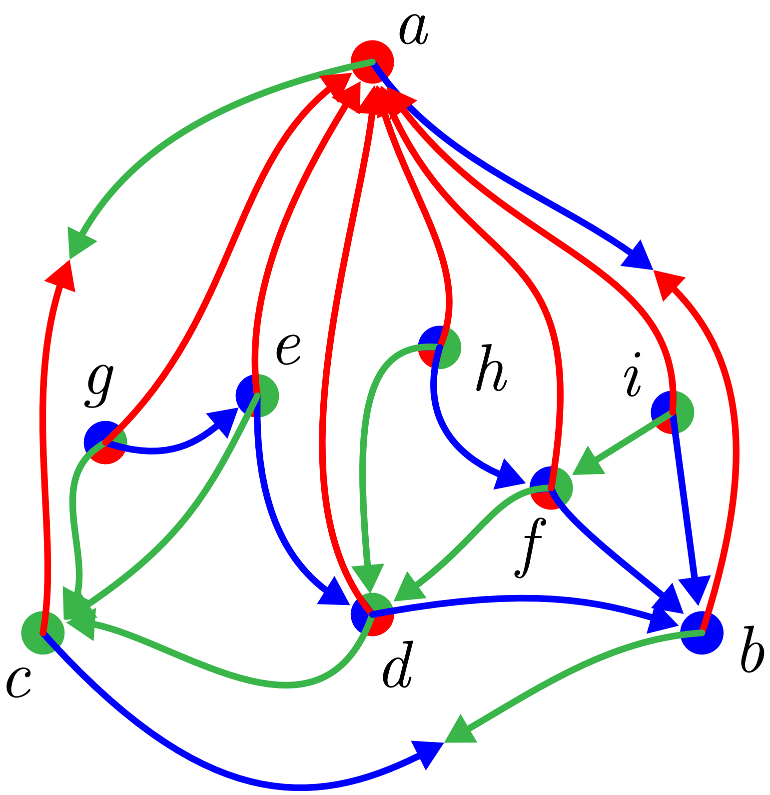

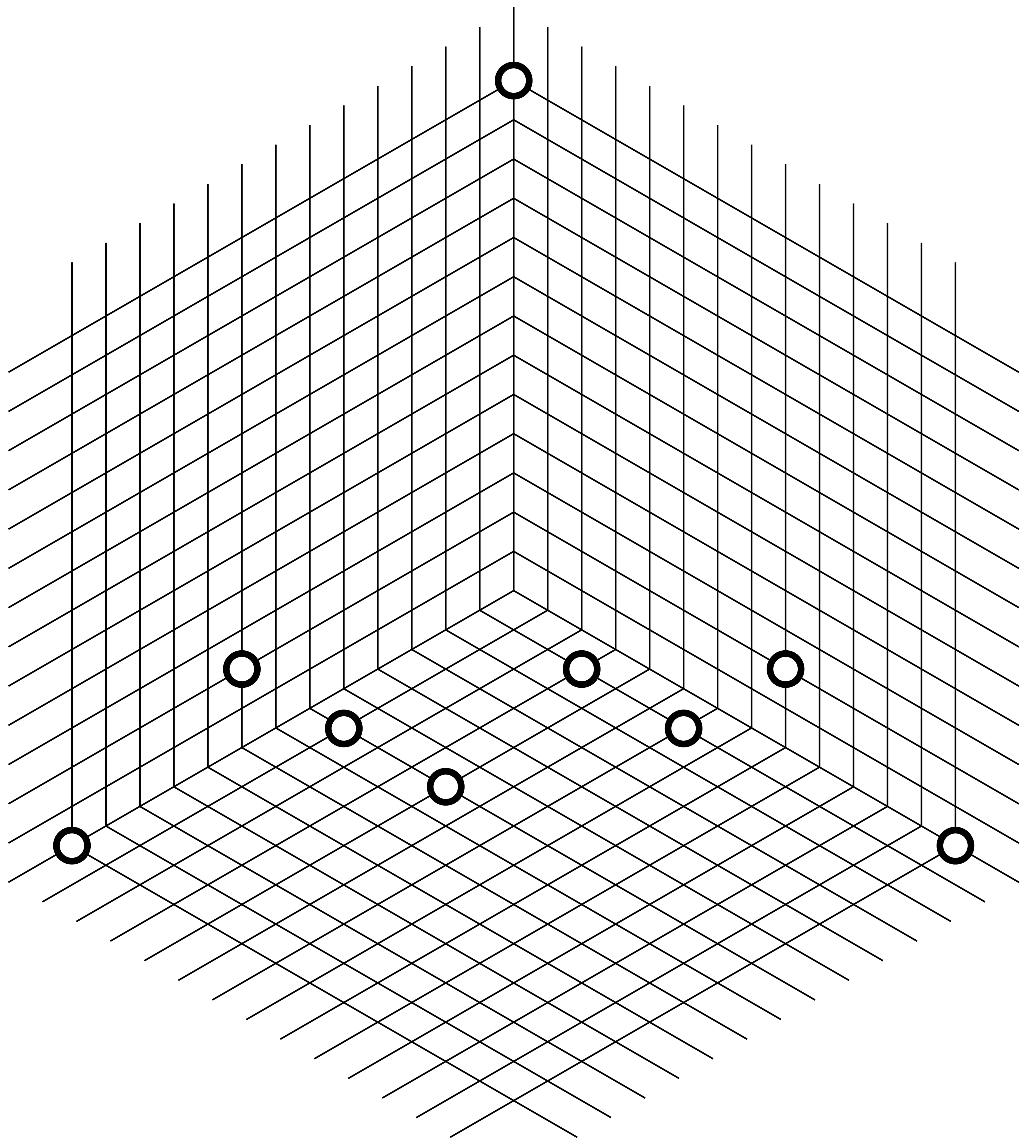

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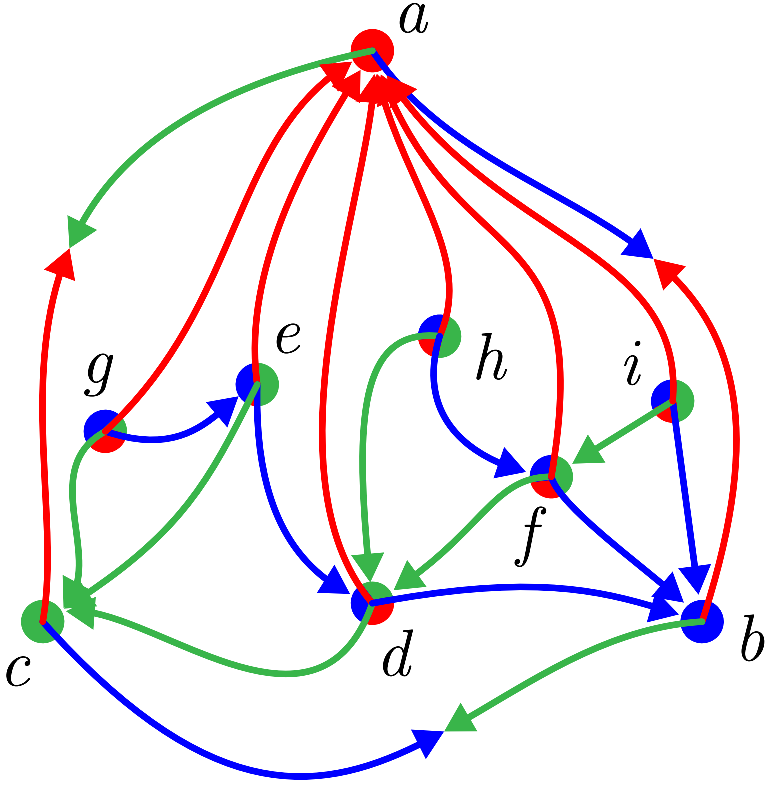

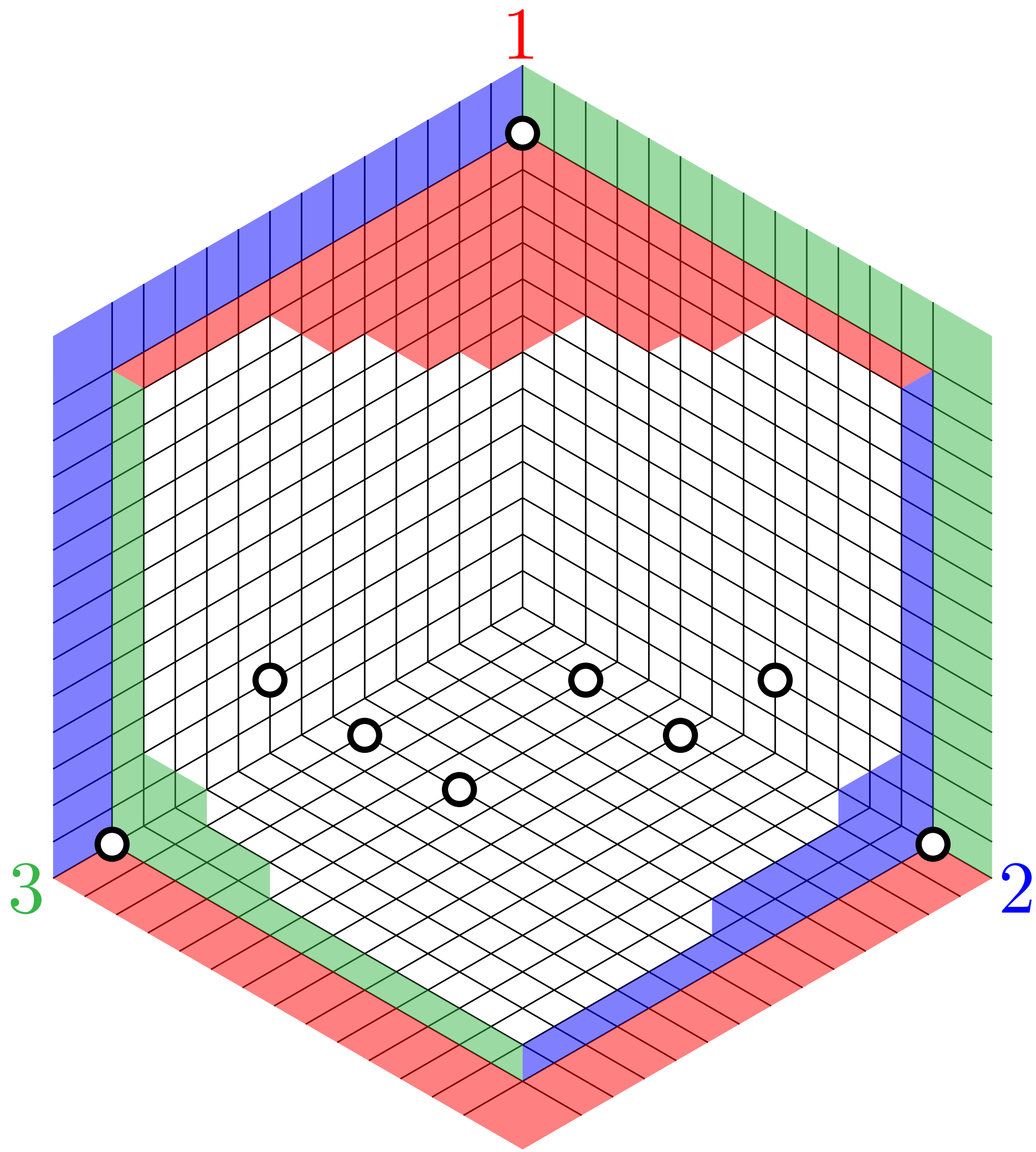

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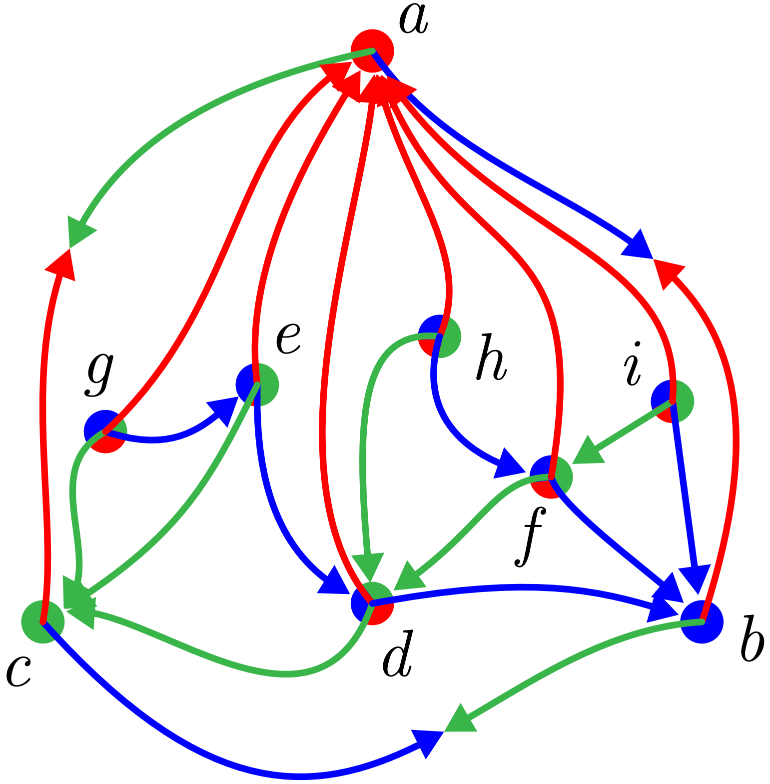

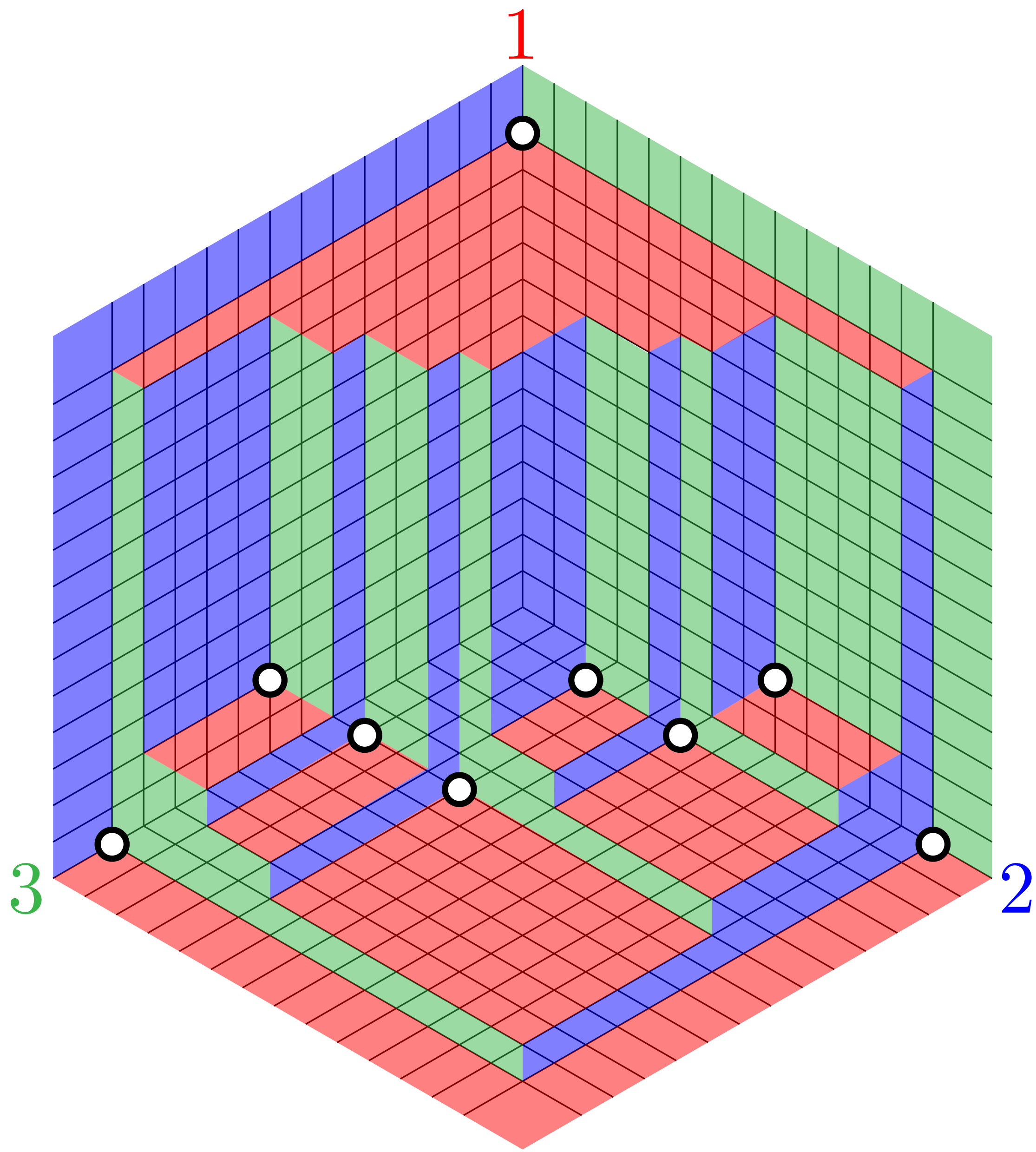

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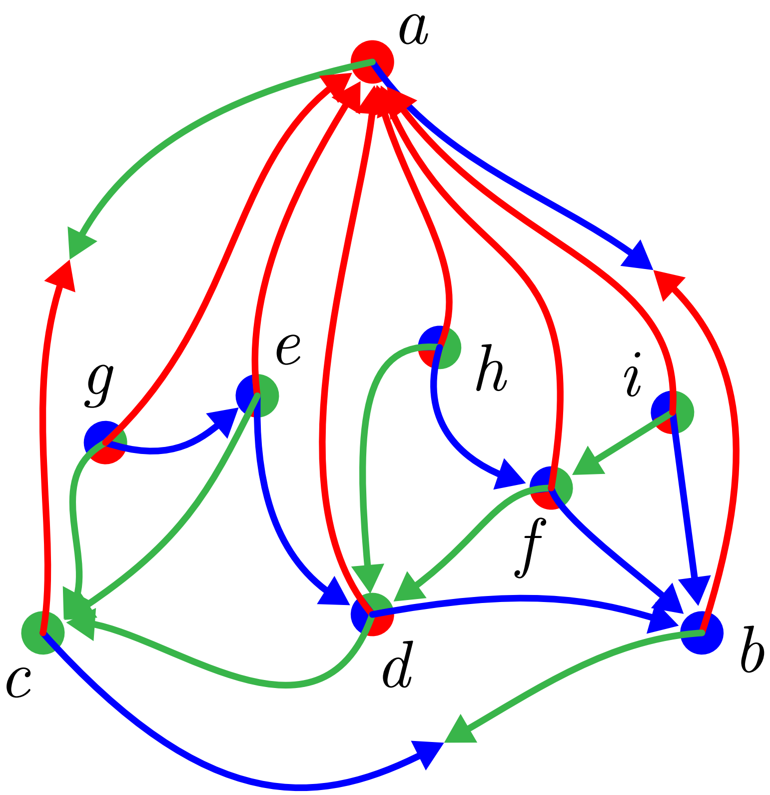

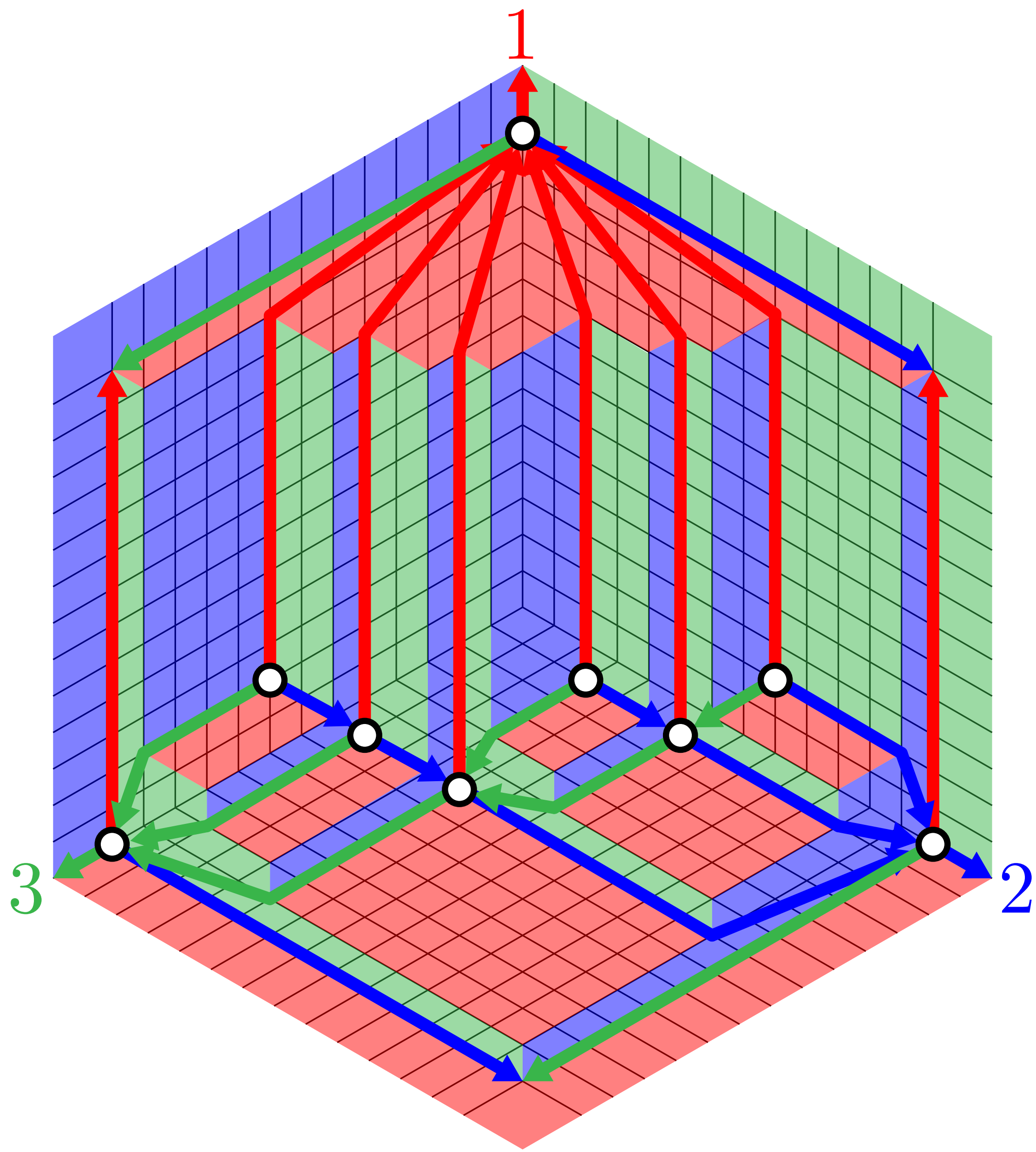

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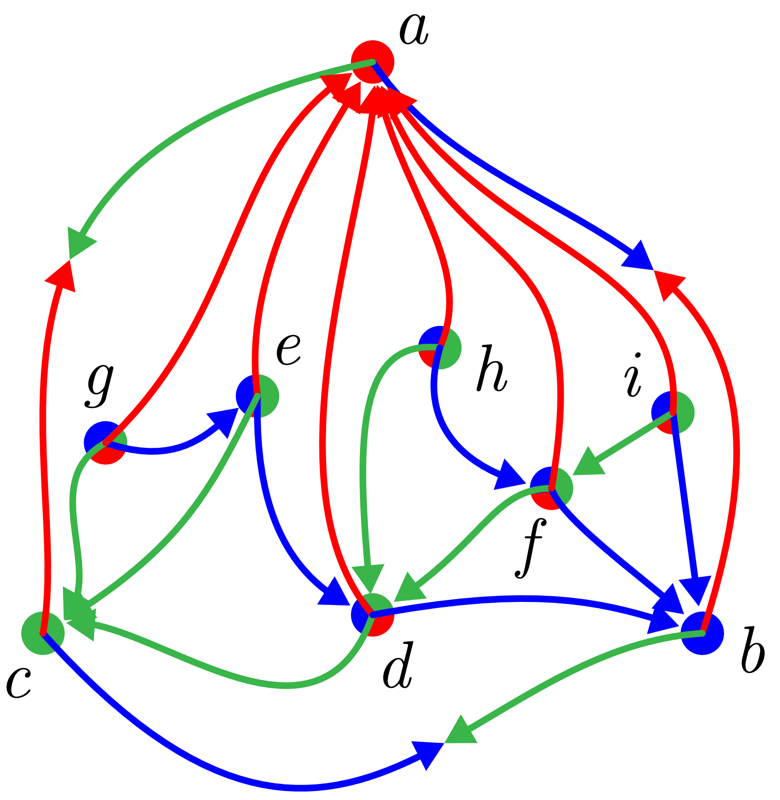
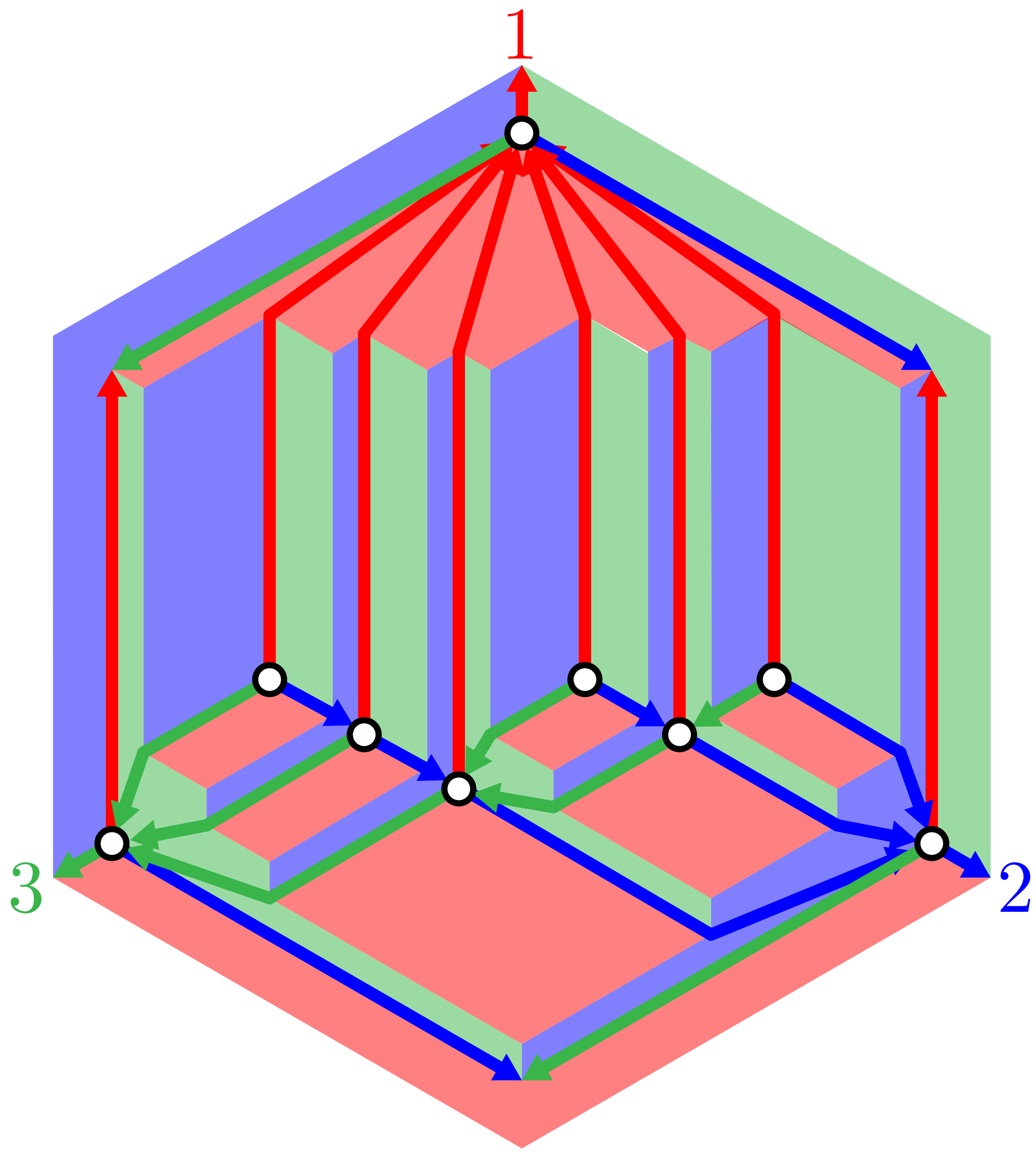




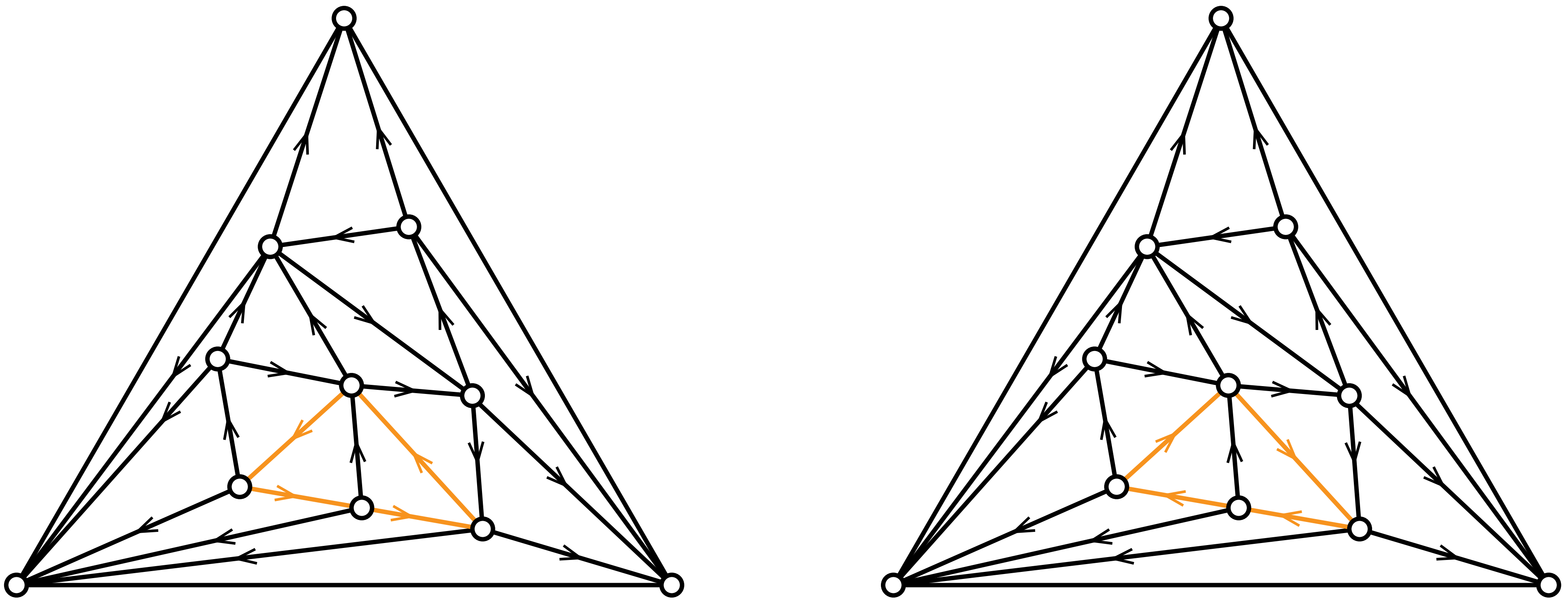
ALPHA-ORIENTATIONS

α-ORIENTATION

DEF. $G = (V, E)$ a graph, $\alpha : V \to \mathbb{N}$.
α-orientation = edge orientation of G such that any vertex v has $\alpha(v)$ outgoing edges.

remark: α-orientation do not always exists,
even when $\sum_{v \in V} \alpha(v) = |E|$ and $\alpha(v) \leq \deg(v)$ for all $v \in V$.

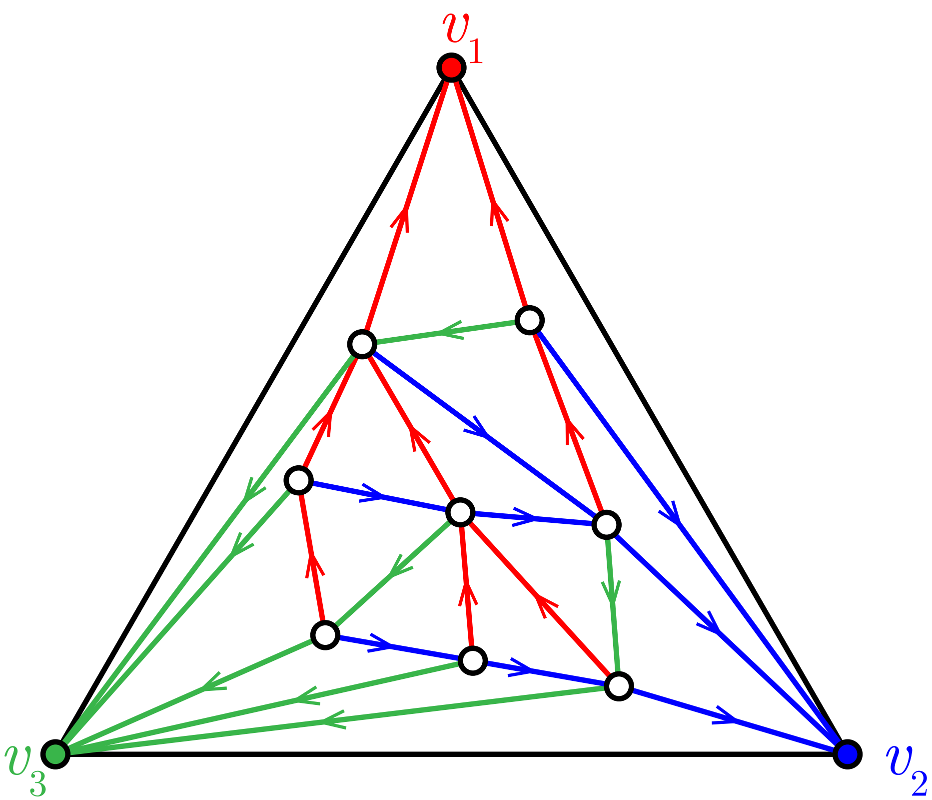
PROP. Reversing an oriented cycle in an α-orientation yields another α-orientation.

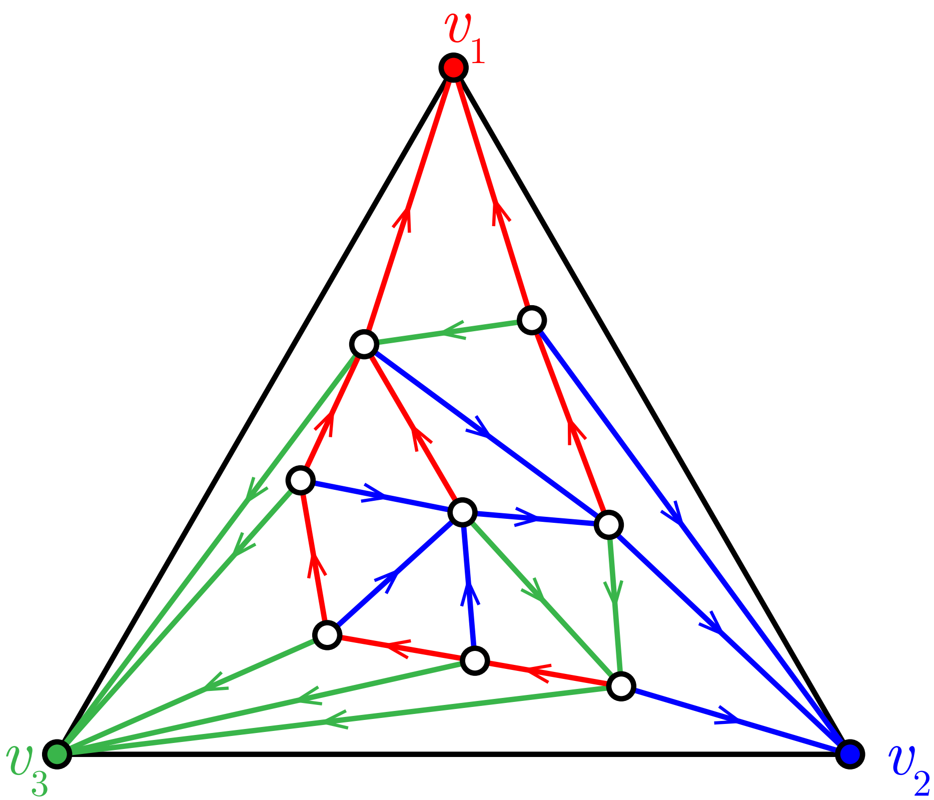


3-ORIENTATIONS IN TRIANGULATIONS

DEF. M = triangulated planar map with external vertices v_1, v_2, v_3, and edges e_1, e_2, e_3
3-orientation = α-orientation of $M \setminus \{e_1, e_2, e_3\}$,
where $\alpha(v) = 3$ except $\alpha(v_1) = \alpha(v_2) = \alpha(v_3) = 0$.

THM. For a triangulated triangulated map M, there is a bijection
3-orientations of $M \longleftrightarrow$ Schnyder woods of M.





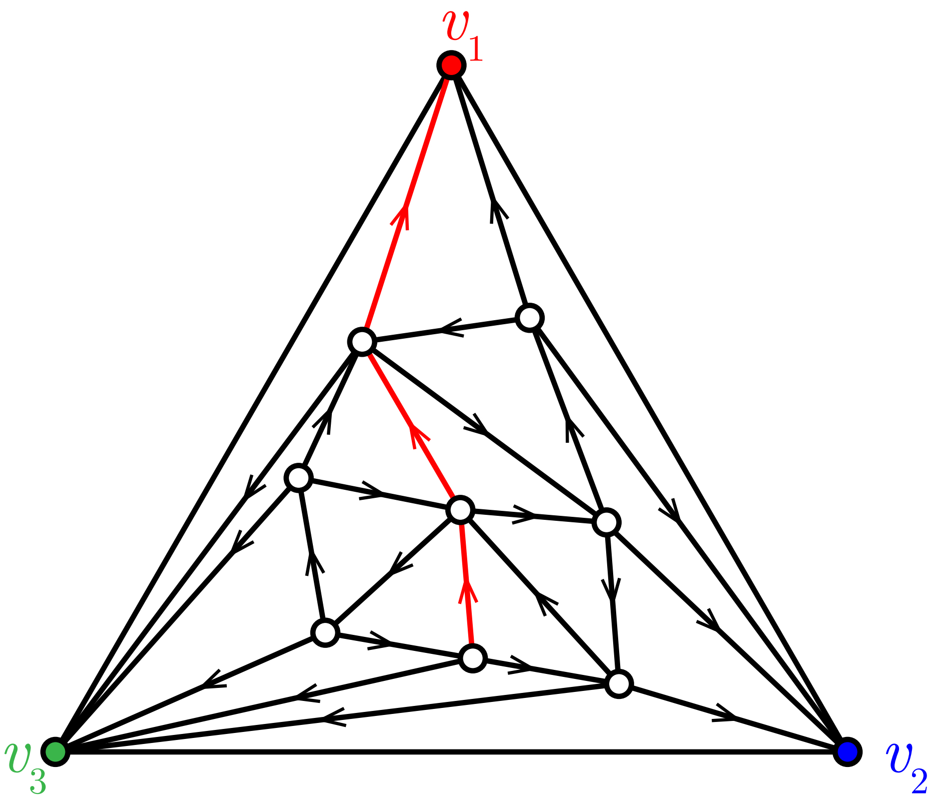
3-ORIENTATIONS IN TRIANGULATIONS

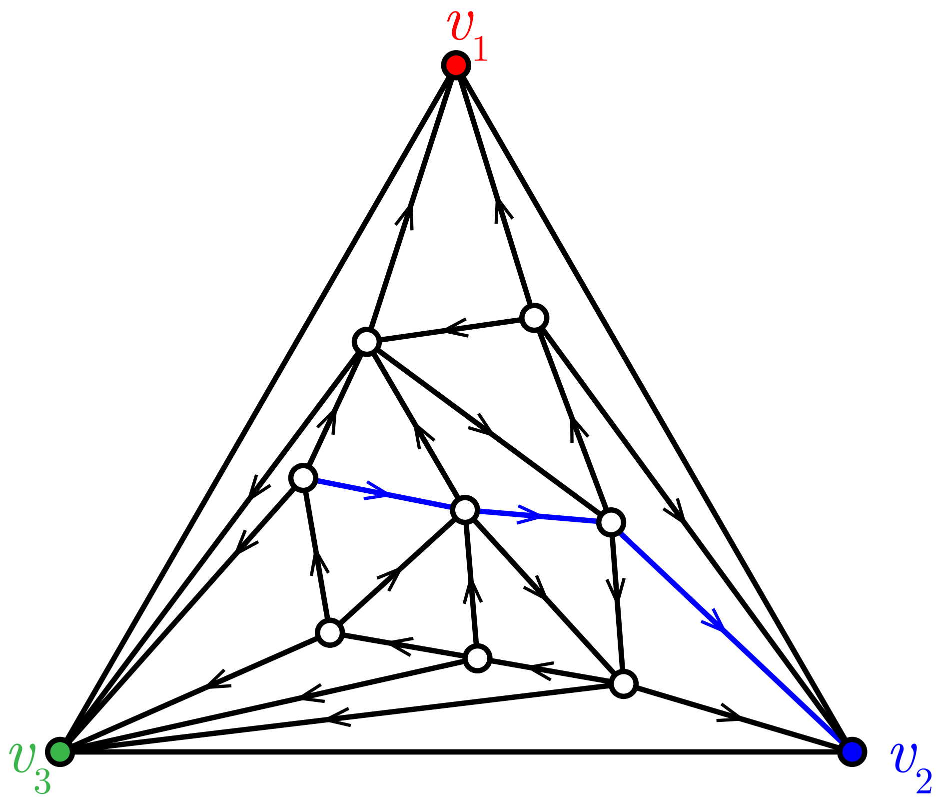
THM. For a triangulated triangulated map M, there is a bijection

$$3\text{-orientations of } M \longleftrightarrow \text{Schnyder woods of } M.$$

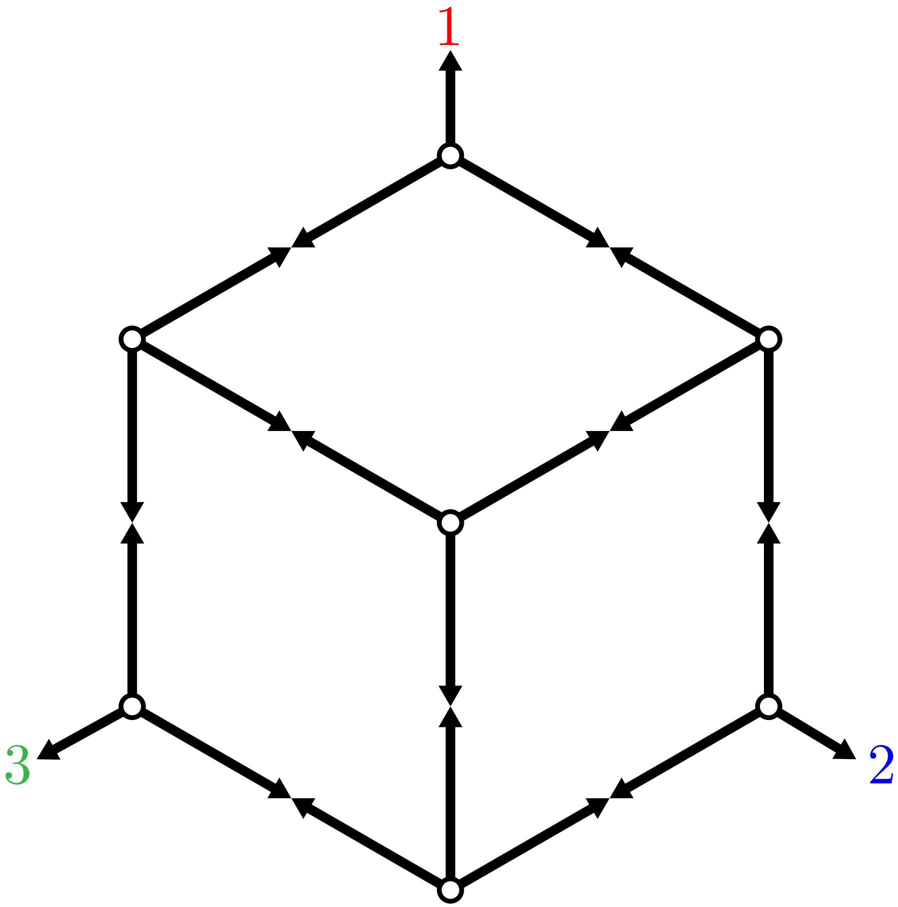
proof idea:

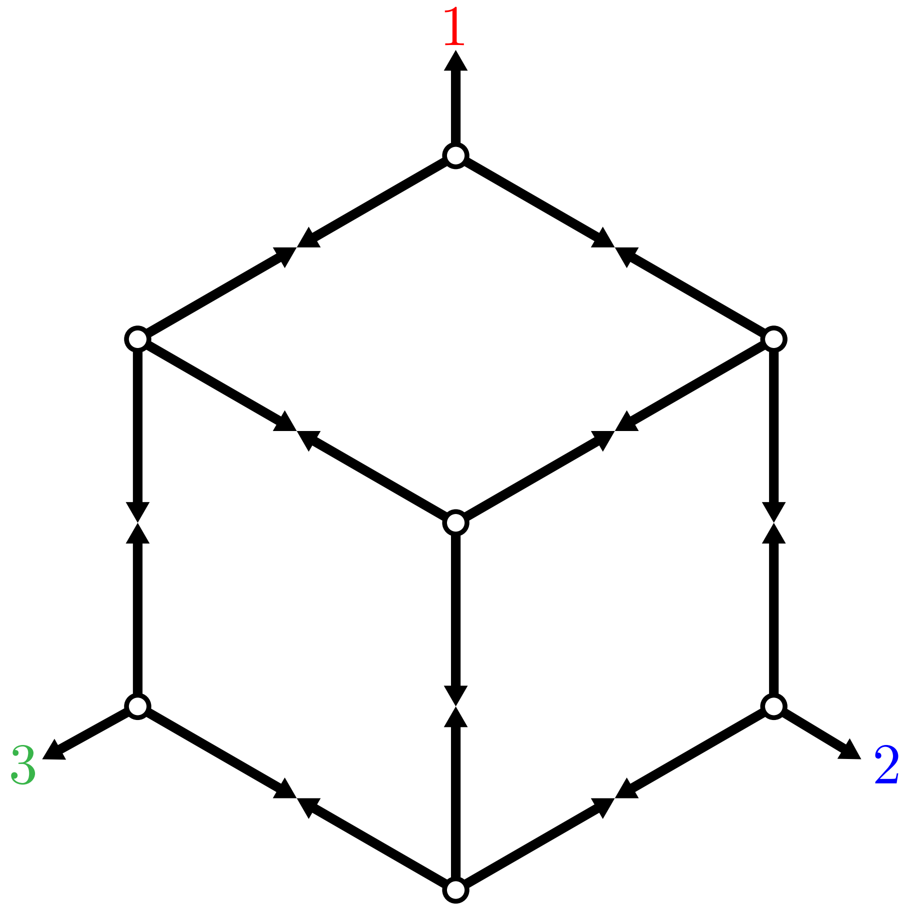
- A Schnyder woods clearly gives a 3-orientation.
- Conversely, consider the central paths in a 3-orientation and prove that they never self-intersect, nor intersect twice.





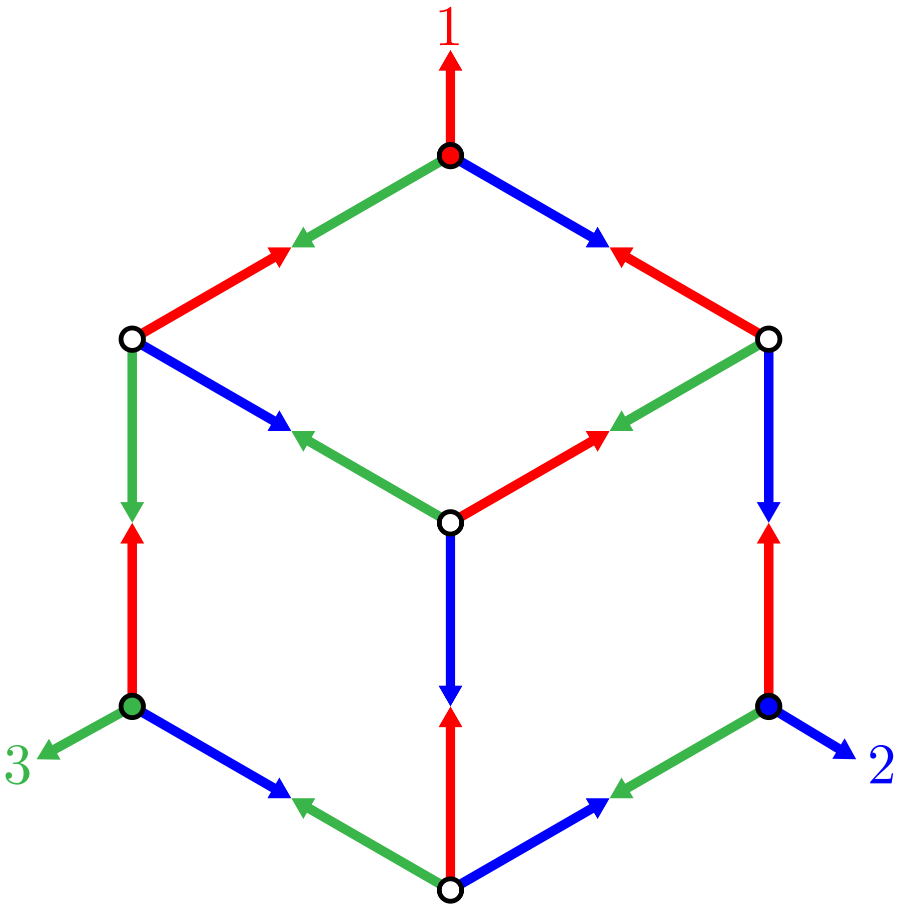
BEYOND TRIANGULATIONS

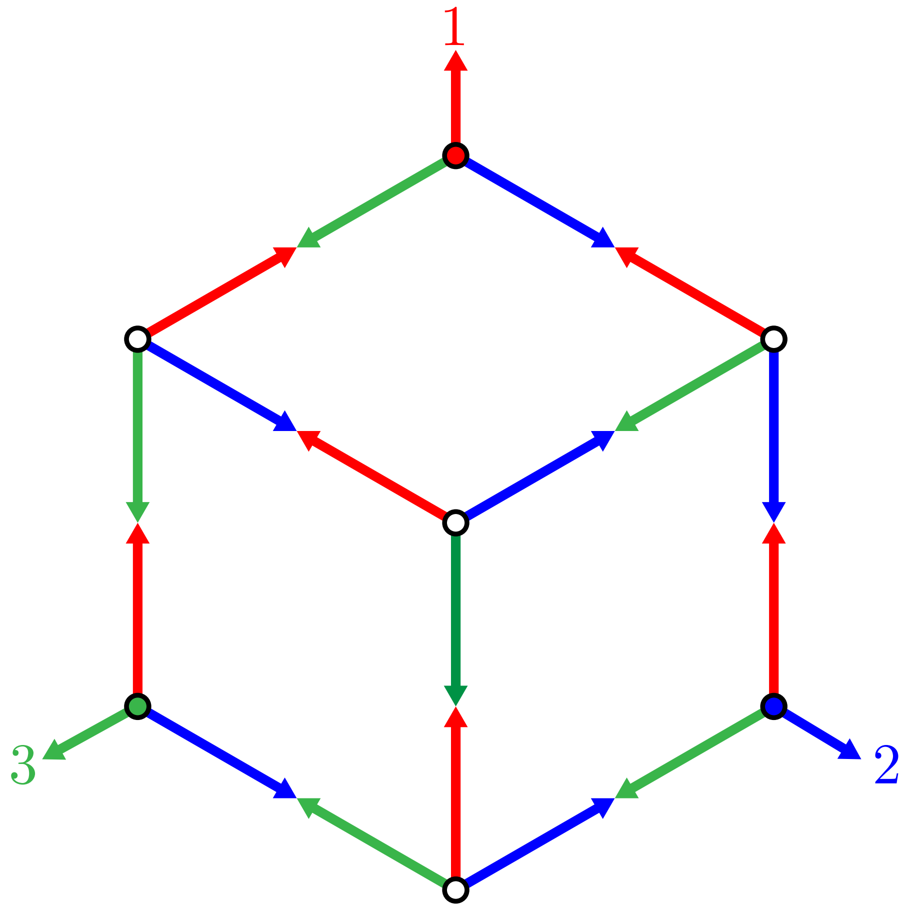




remark: for an arbitrary planar map, there are more Schnyder woods than 3-orientations...

BEYOND TRIANGULATIONS





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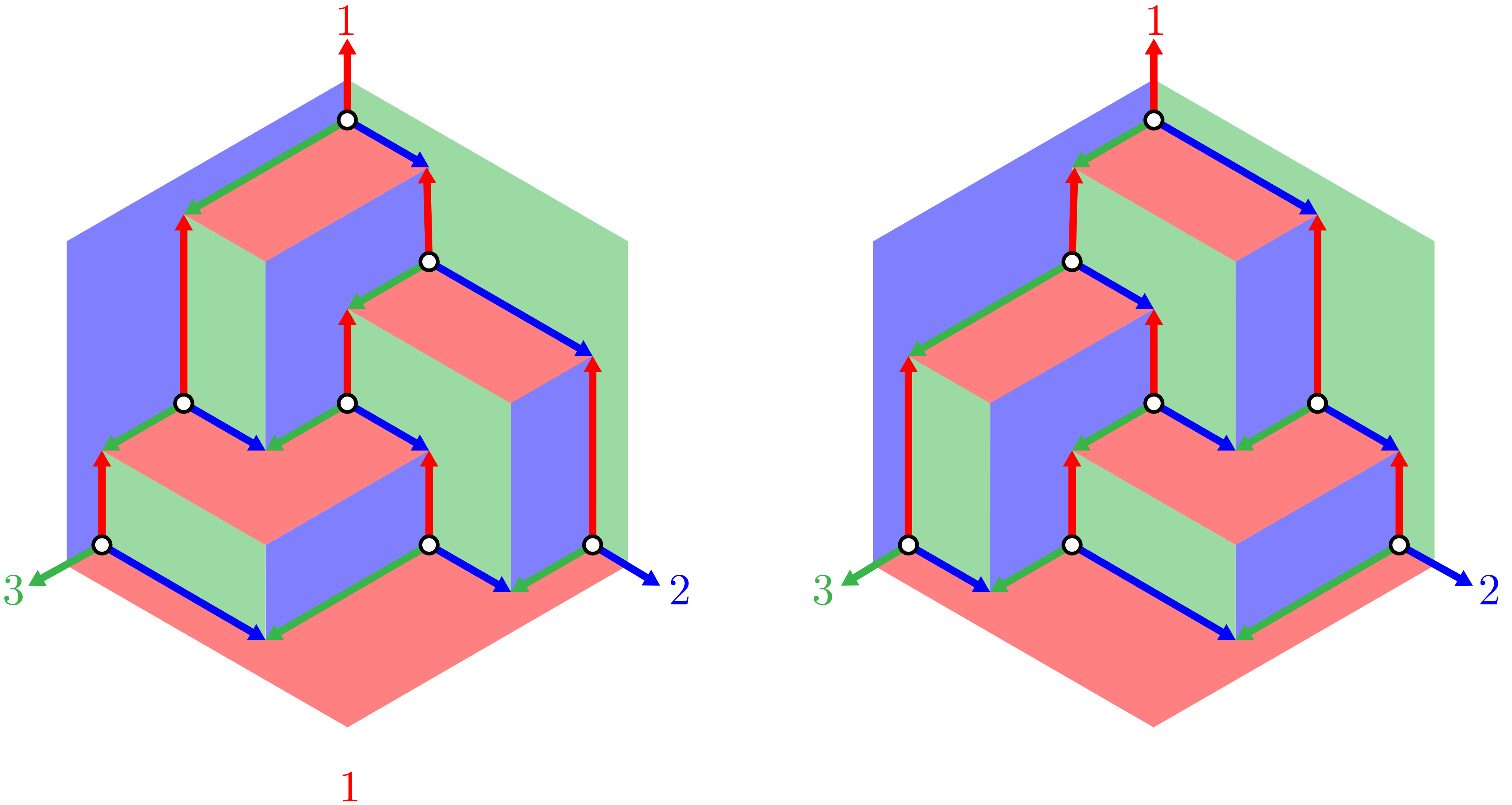
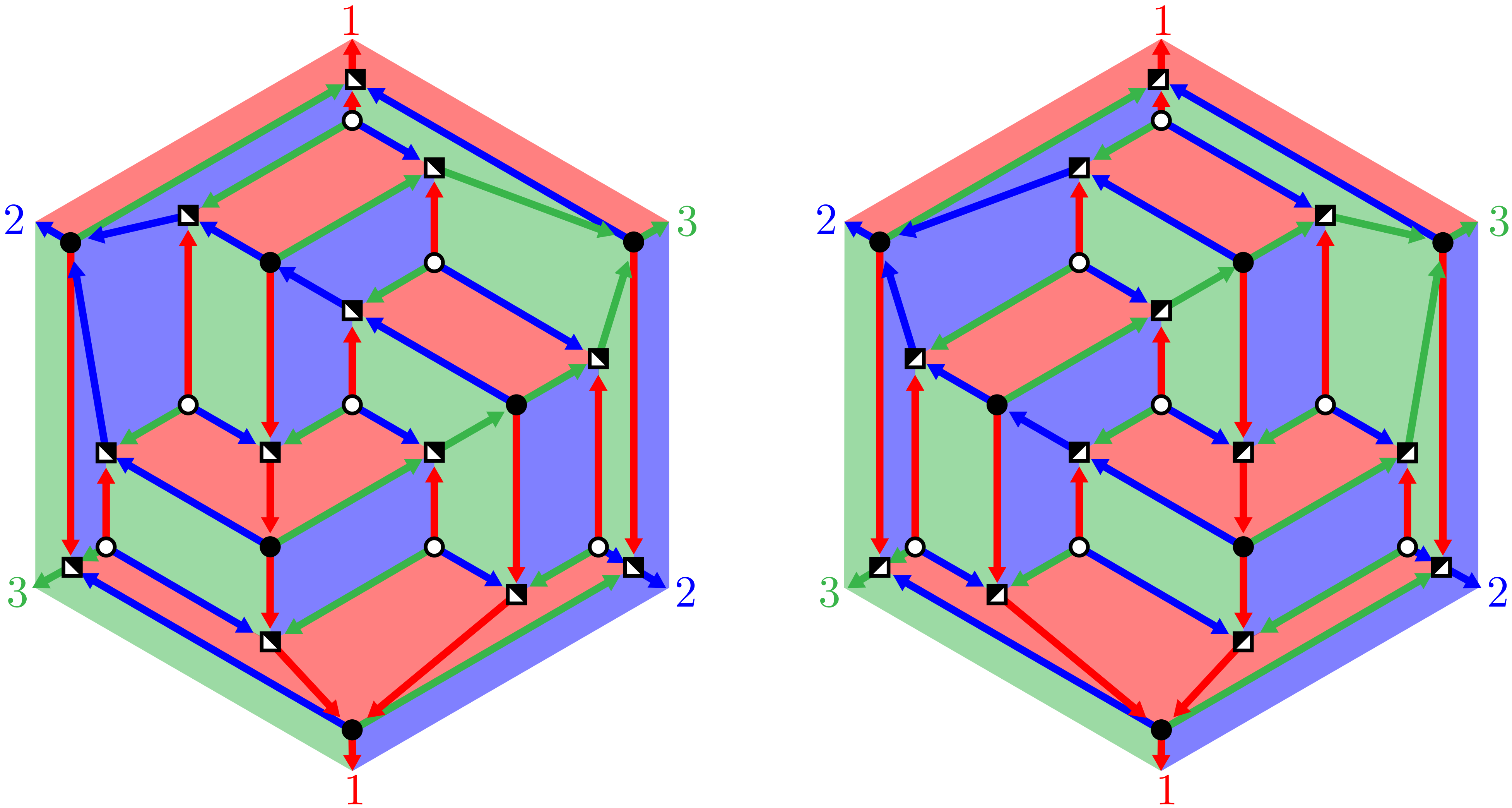


remark: for an arbitrary planar map, there are more Schnyder woods than 3-orientations...

... but distinct Schnyder woods yield geodesic embeddings on distinct orthogonal surfaces...

BEYOND TRIANGULATIONS

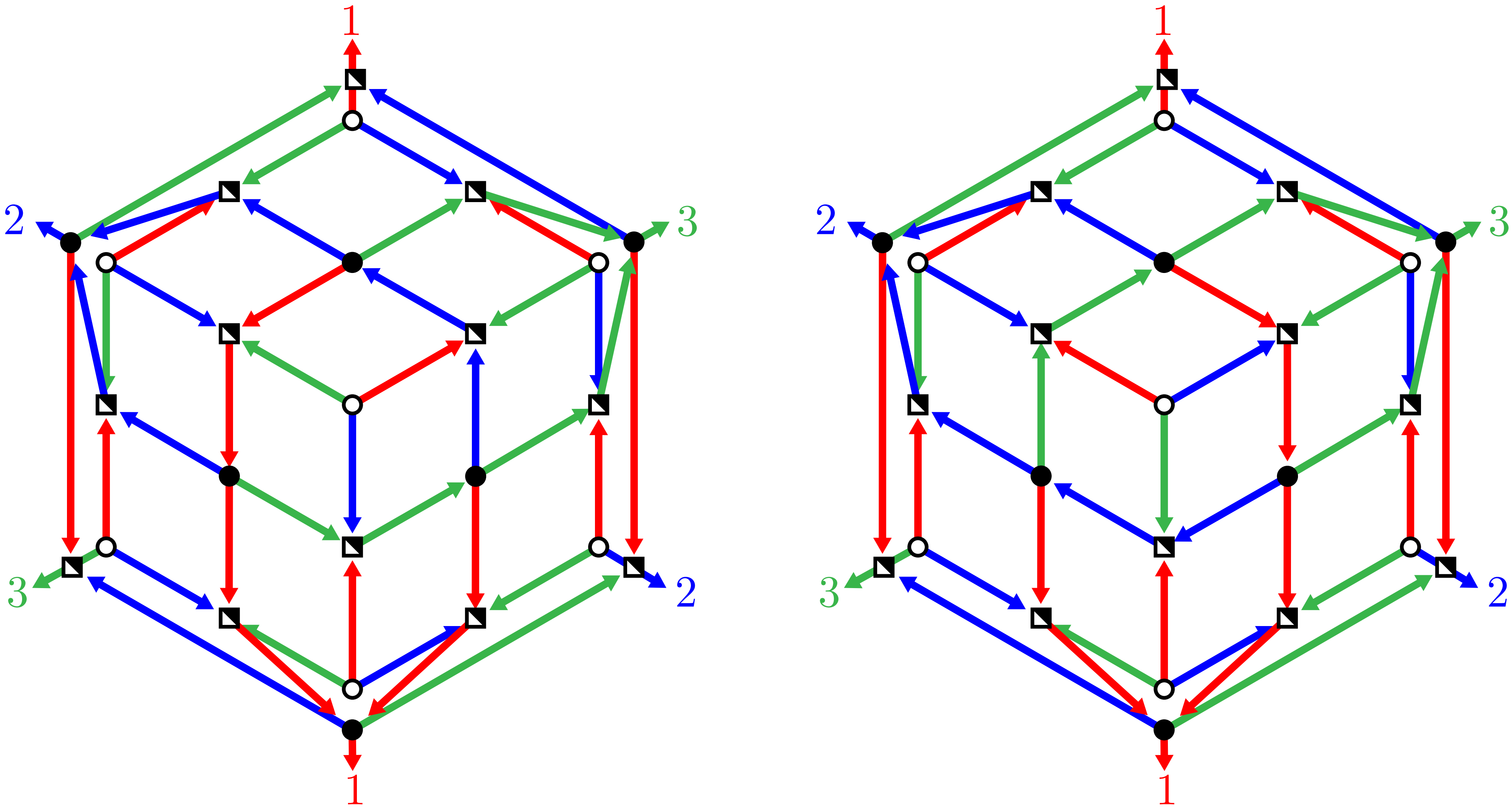


remark: for an arbitrary planar map, there are more Schnyder woods than 3-orientations...

... but distinct Schnyder woods yield geodesic embeddings on distinct orthogonal surfaces...

... with distinct orientations for the suspended duals...

BEYOND TRIANGULATIONS

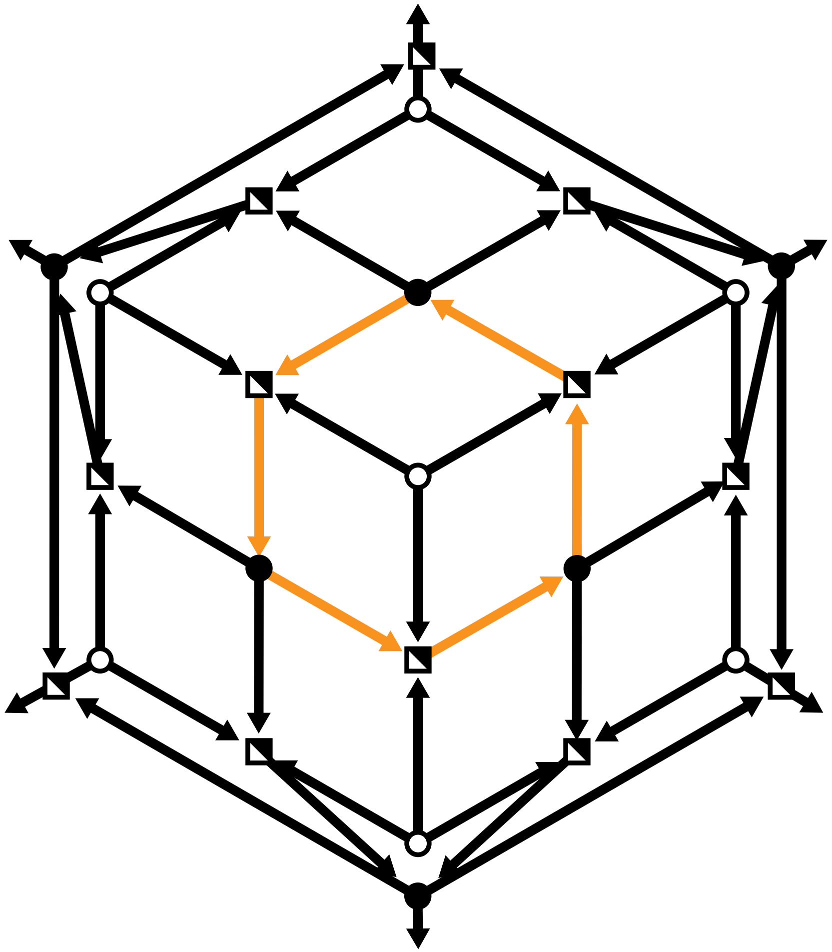
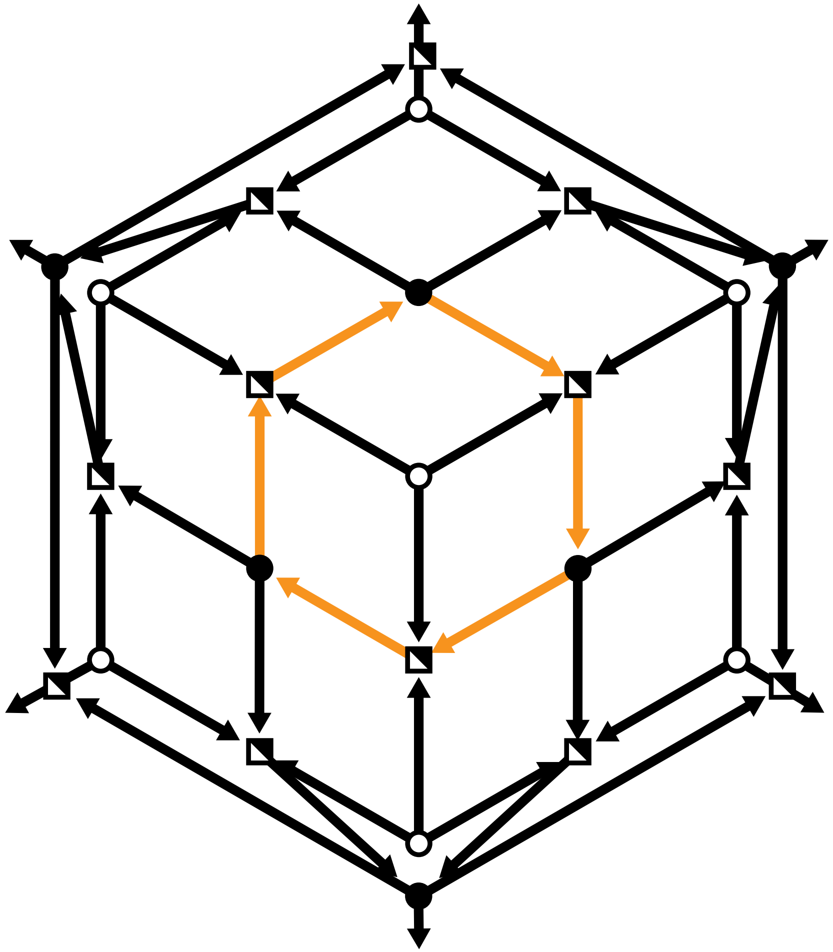


THM. For a 3-connected planar map M, there is a bijection

α-orientations of the primal-dual $\widetilde{M}$ $\longleftrightarrow$ Schnyder woods of M

where $\alpha(\circ) = \alpha(\bullet) = 3$ while $\alpha(\square) = 1$.

BEYOND TRIANGULATIONS

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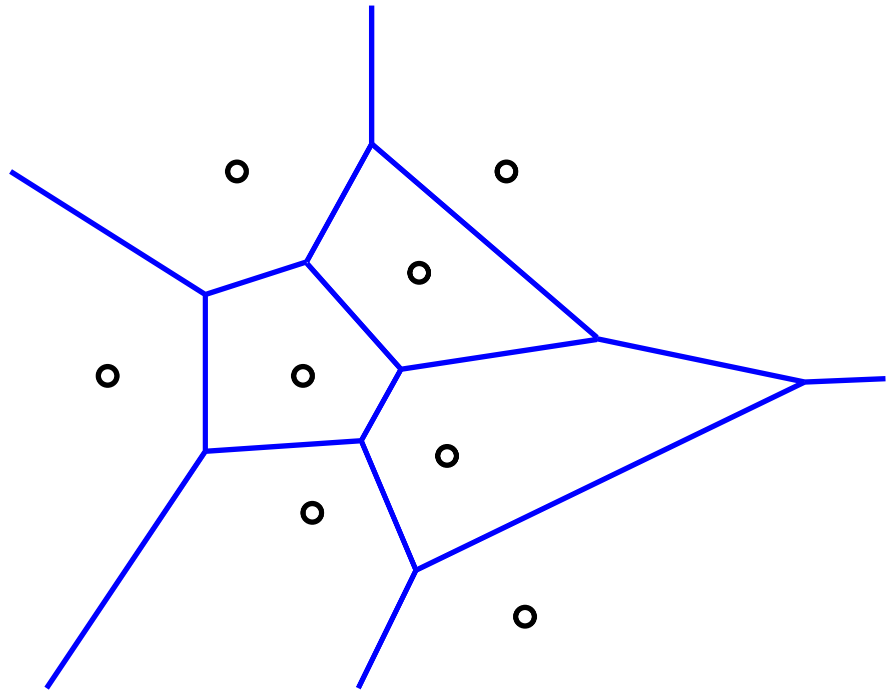
TD-DELAUNAY TRIANGULATIONS

VORONOI DIAGRAM

DEF. $\boldsymbol{P}$ = set of sites in $\mathbb{R}^n$.

Voronoi region $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P}) = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \|\boldsymbol{x} - \boldsymbol{p}\| \leq \|\boldsymbol{x} - \boldsymbol{q}\| \text{ for all } \boldsymbol{q} \in \boldsymbol{P}\}$.

Voronoi diagram $\mathrm{Vor}(\boldsymbol{P})$ = partition of $\mathbb{R}^n$ formed by $\mathrm{Vor}(\boldsymbol{p}, \boldsymbol{P})$ for $\boldsymbol{p} \in \boldsymbol{P}$.



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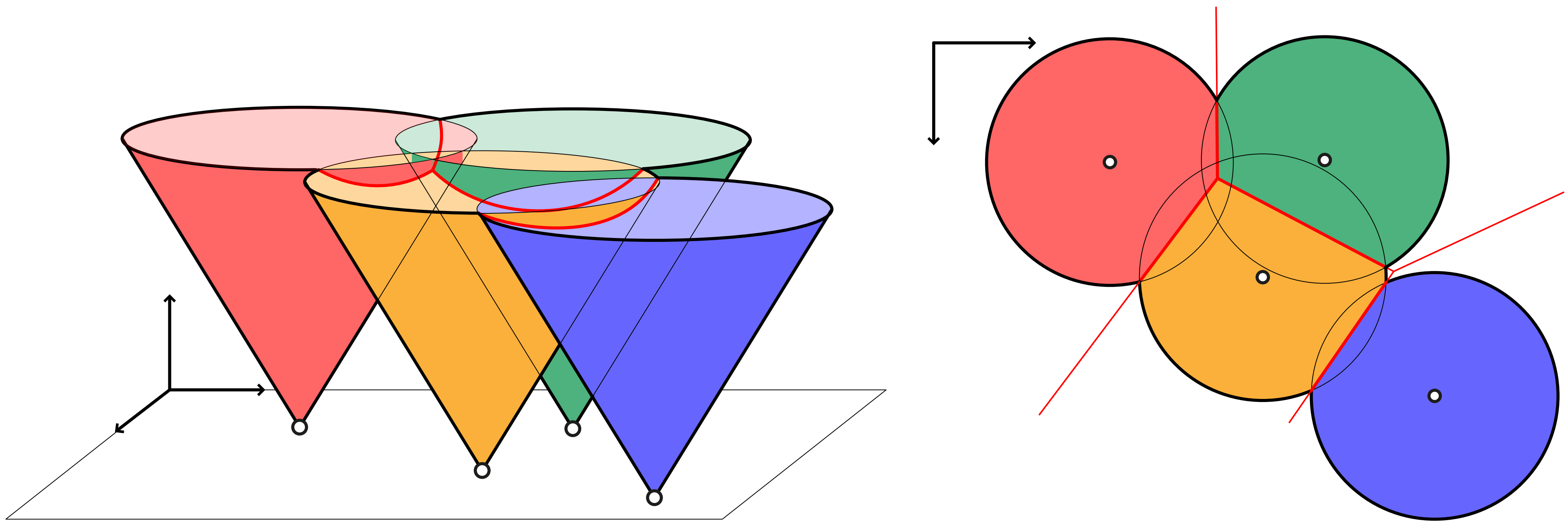
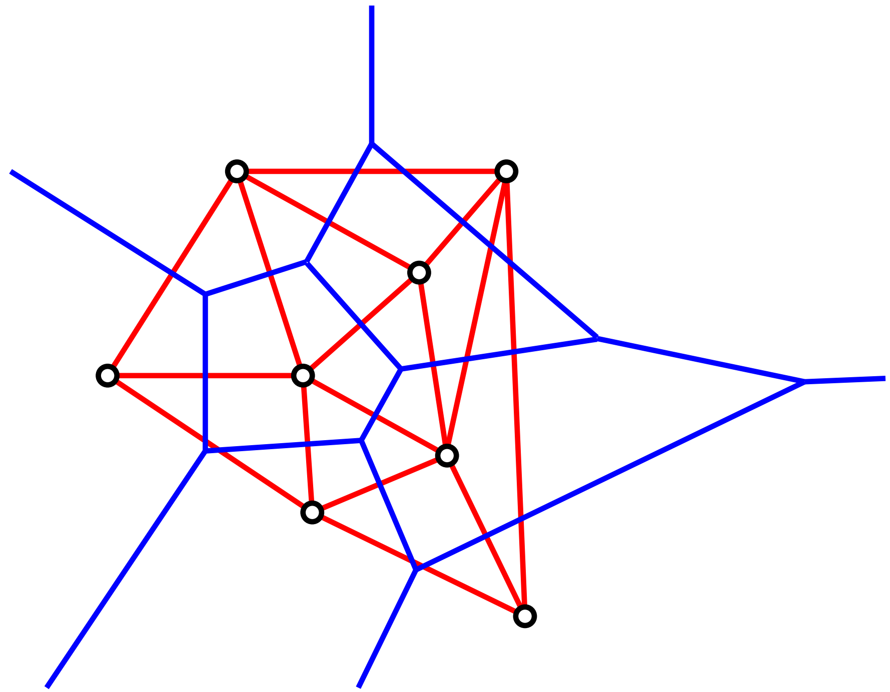


DELAUNAY COMPLEX

DEF. $\boldsymbol{P}$ = set of sites in $\mathbb{R}^n$.

Voronoi region $\mathrm{Vor}(\boldsymbol{p},\boldsymbol{P}) = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \|\boldsymbol{x}-\boldsymbol{p}\| \le \|\boldsymbol{x}-\boldsymbol{q}\| \text{ for all } \boldsymbol{q} \in \boldsymbol{P}\}$.

Voronoi diagram $\mathrm{Vor}(\boldsymbol{P})$ = partition of $\mathbb{R}^n$ formed by $\mathrm{Vor}(\boldsymbol{p},\boldsymbol{P})$ for $\boldsymbol{p} \in \boldsymbol{P}$.



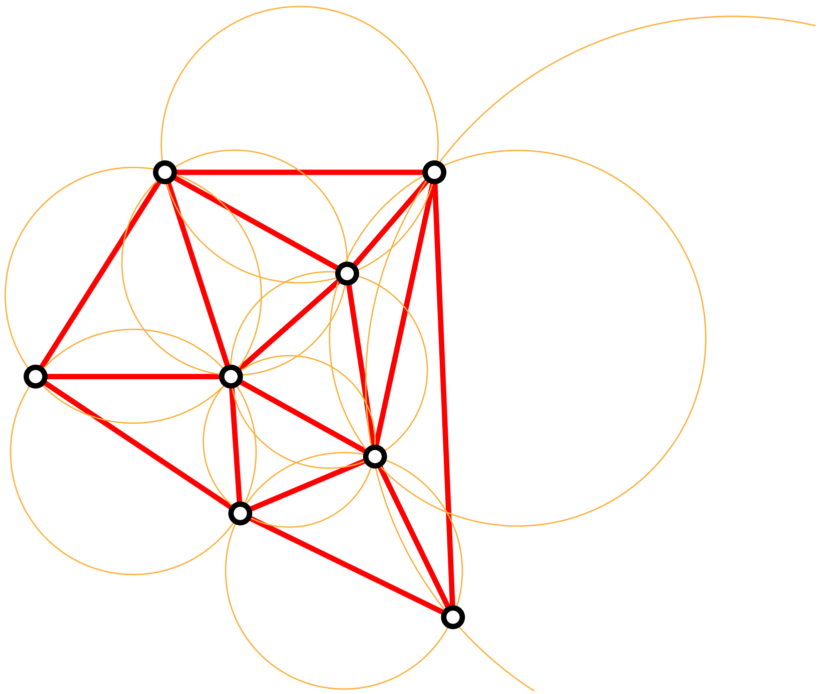
DEF. Delaunay complex $\mathrm{Del}(\boldsymbol{P})$ = intersection complex of $\mathrm{Vor}(\boldsymbol{P})$

$$\mathrm{Del}(\boldsymbol{P}) = \left\{ \mathrm{conv}(\boldsymbol{X}) \;\middle|\; \boldsymbol{X} \subseteq \boldsymbol{P} \text{ and } \bigcap_{\boldsymbol{p}\in\boldsymbol{X}} \mathrm{Vor}(\boldsymbol{p},\boldsymbol{P}) \neq \varnothing \right\}.$$

EMPTY CIRCLES

PROP. For any three points $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ of $\boldsymbol{P}$,

- $\boldsymbol{pq}$ is an edge of $\mathrm{Del}(\boldsymbol{P}) \iff$ there is an empty circle passing through $\boldsymbol{p}$ and $\boldsymbol{q}$,
- $\boldsymbol{pqr}$ is a triangle of $\mathrm{Del}(\boldsymbol{P}) \iff$ the circumcircle of $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ is empty.



proof idea: consider the circle centered at the intersection of the Voronoi regions and passing through the Voronoi sites.

EMPTY CIRCLES

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- $\boldsymbol{pq}$ is an edge of $\mathrm{Del}(\boldsymbol{P}) \iff$ there is an empty circle passing through $\boldsymbol{p}$ and $\boldsymbol{q}$,
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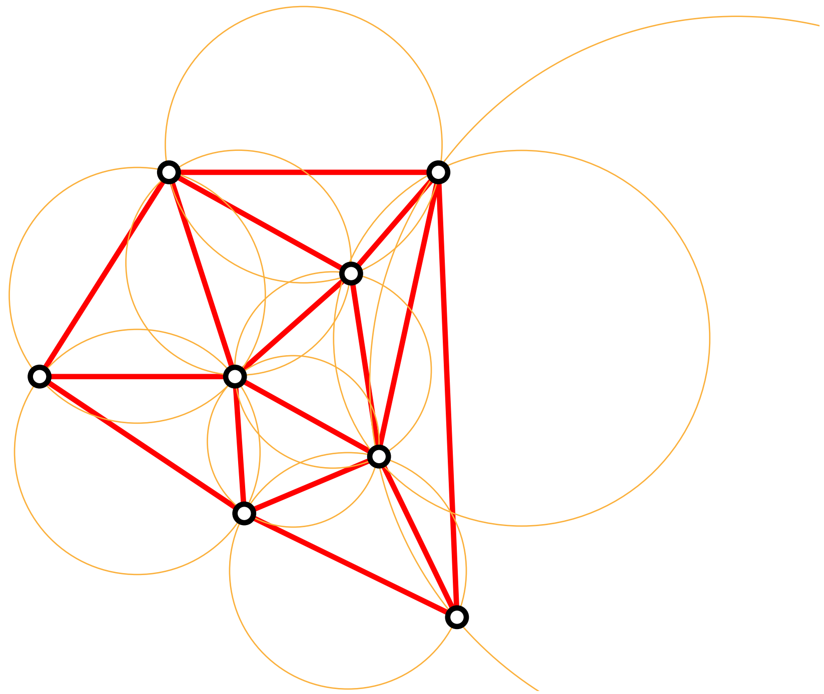
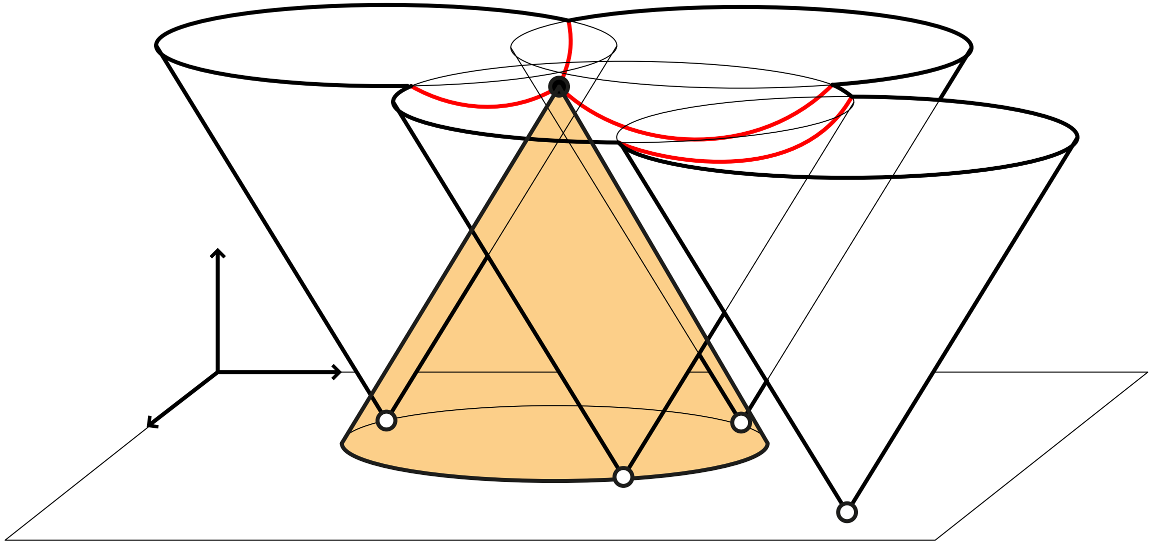


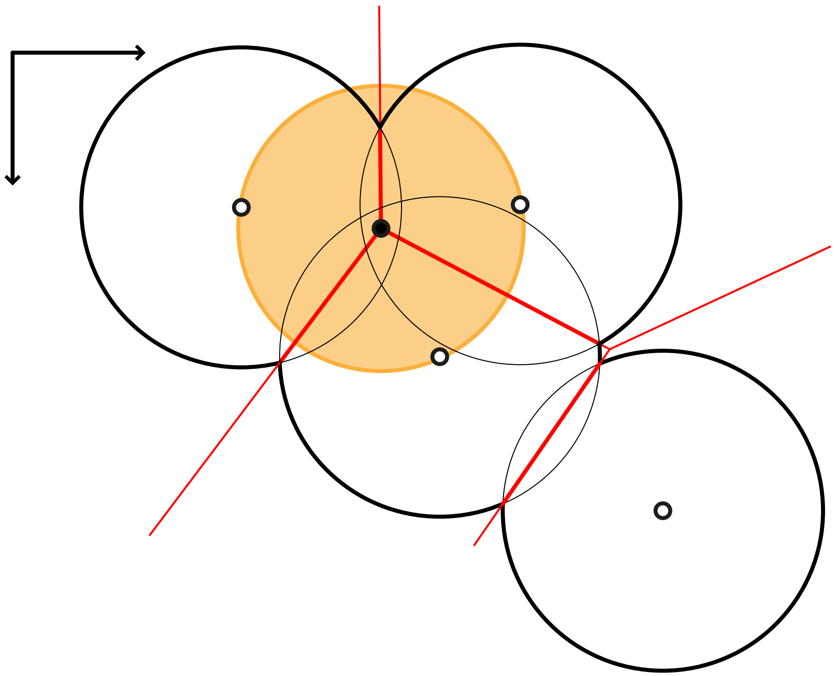
CORO. In two adjacent triangles of a Delaunay triangulation, the sum of the two opposite angles is at most π.

EMPTY CIRCLES

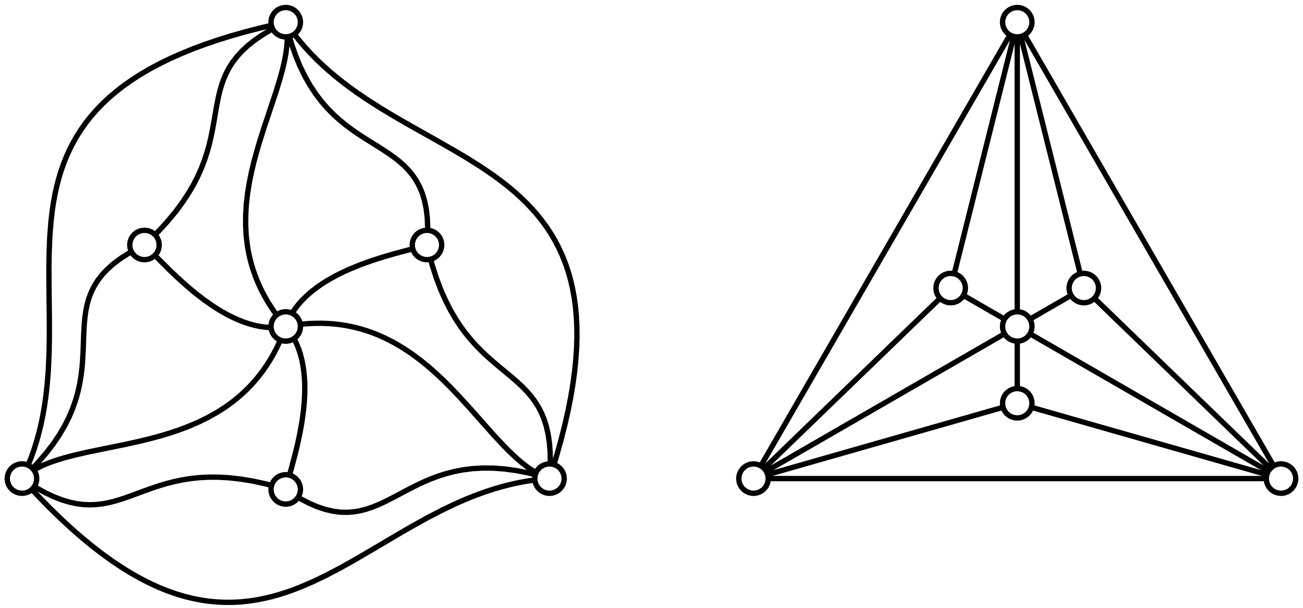
PROP. For any three points $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ of $\boldsymbol{P}$,

- $\boldsymbol{pq}$ is an edge of $\mathrm{Del}(\boldsymbol{P}) \iff$ there is an empty circle passing through $\boldsymbol{p}$ and $\boldsymbol{q}$,
- $\boldsymbol{pqr}$ is a triangle of $\mathrm{Del}(\boldsymbol{P}) \iff$ the circumcircle of $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ is an empty circle.



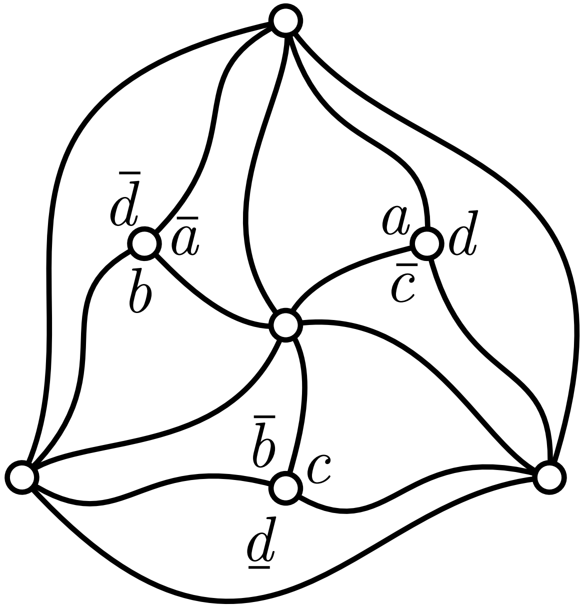


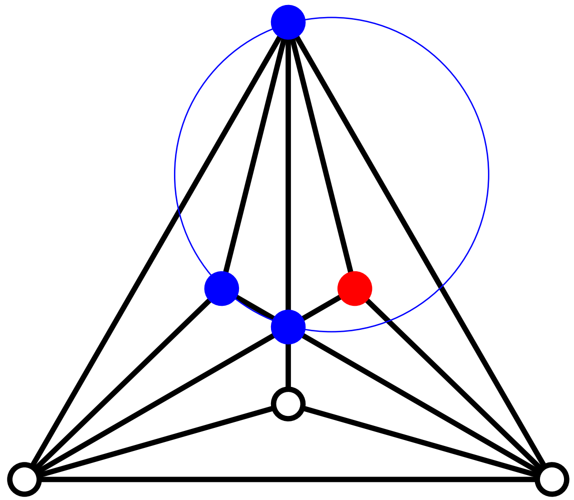
QU. Consider the stacked triangulation



Does this realization look Delaunay? Can you provide a Delaunay realization?

REM. The stacked triangulation





has no Delaunay realization.

proof: In a Delaunay realization of this stacked triangulation, we would have

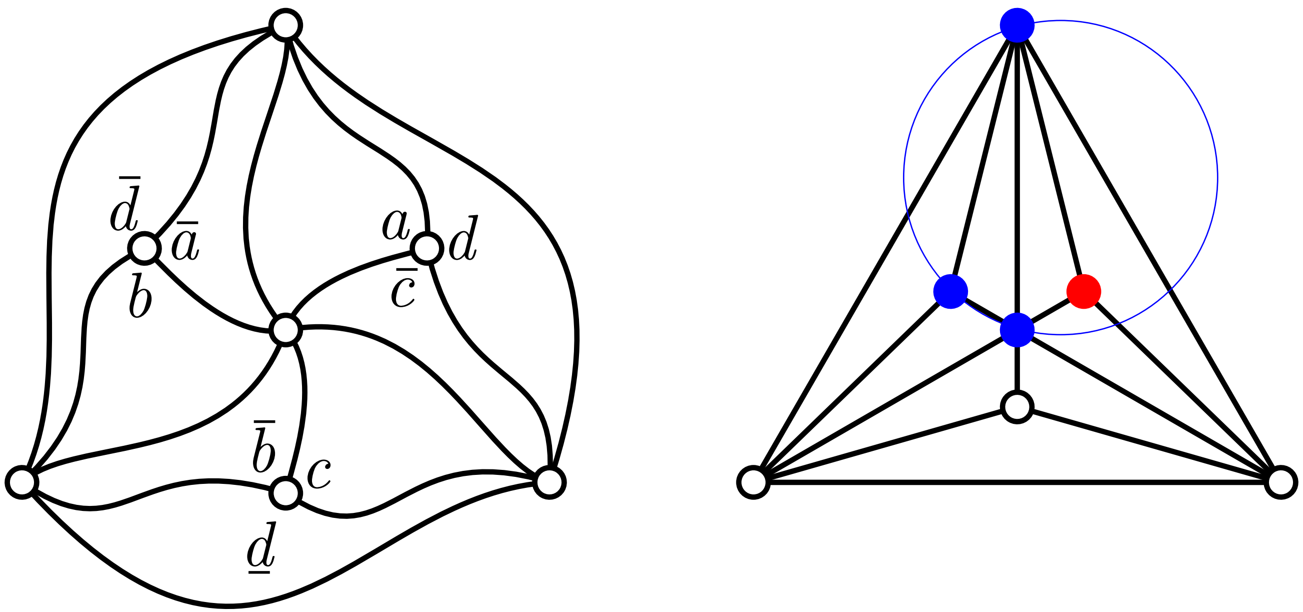
$$a + \overline{a} < \pi, \qquad b + \overline{b} < \pi, \qquad c + \overline{c} < \pi,$$

$$\text{and} \qquad a + \overline{c} + d = \overline{a} + b + \overline{d} = \overline{b} + c + \underline{d} = 2\pi.$$

Thus $d + \overline{d} + \underline{d} > 3\pi$ and at least one of d, $\overline{d}$ and $\underline{d}$ is larger than π, a contradiction.

EXM: STACKED TRIANGULATIONS

REM. The stacked triangulation



has no Delaunay realization.

THM. A stacked triangulation admits a Delaunay realization if and only if its construction tree has no ternary node after deletion of all its leaves.

proof ideas:

- one direction follows from the example above,
- for the opposite direction, find an explicit construction (see Exercise 113 course notes).

QUASI-METRICS

DEF. quasi-metric on Q = function $\delta : Q^2 \to \mathbb{R}_{\geq 0}$ st:

- separability: $\delta(p,q) = 0 \iff p = q$,
- triangular inequality: $\delta(p,q) + \delta(q,r) \geq \delta(p,r)$.

DEF. $P \subseteq Q$ a set of sites of Q.

δ-Voronoi region $\mathrm{Vor}_\delta(p,P) = \{r \in Q \mid \delta(p,r) \leq \delta(q,r)$ for all $q \in P\}$.

δ-Voronoi diagram $\mathrm{Vor}_\delta(P)$ = partition of Q formed by $\mathrm{Vor}_\delta(p,P)$ for $p \in P$.

DEF. δ-Delaunay complex $\mathrm{Del}_\delta(P)$ = intersection complex of $\mathrm{Vor}_\delta(P)$

$$\mathrm{Del}_\delta(P) = \left\{ X \subseteq P \;\middle|\; \bigcap_{p \in X} \mathrm{Vor}_\delta(p,P) \neq \varnothing \right\} \subseteq 2^P.$$

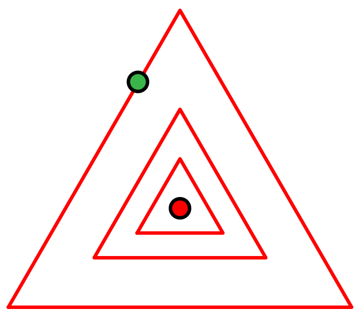
TRIANGULAR DISTANCE

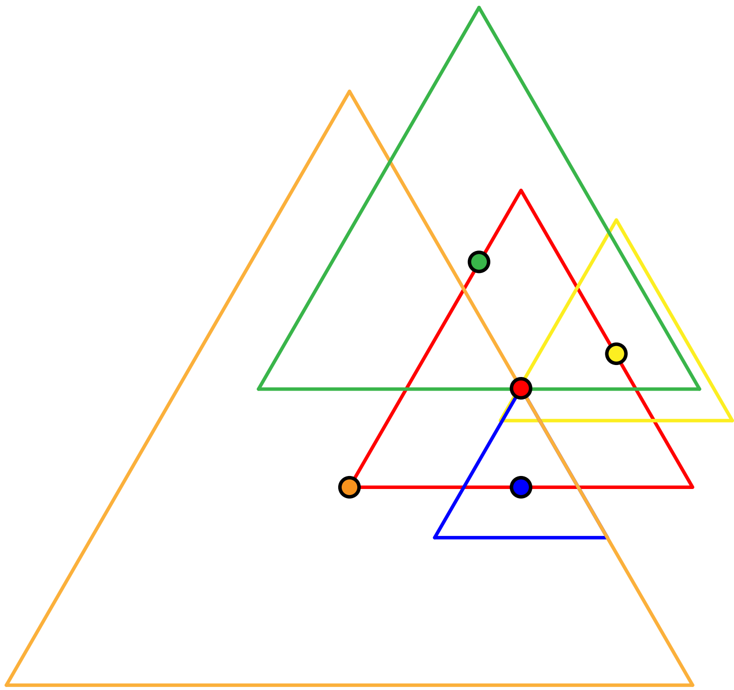
Fix $c \in \mathbb{R}_{\geq 0}$, and consider the hyperplane $\boldsymbol{H} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; x_1 + x_2 + x_3 = c \right\}$.
and its standard equilateral triangle $\triangle = \mathrm{conv}(c\boldsymbol{e}_1, c\boldsymbol{e}_2, c\boldsymbol{e}_3)$

DEF. triangular distance between $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{H}$ =

$$\mathrm{TD}(\boldsymbol{x}, \boldsymbol{y}) = \min \left\{ \lambda \in \mathbb{R}_{\geq 0} \;\middle|\; \boldsymbol{x} \in \boldsymbol{y} + \lambda(\triangle - c\mathbb{1}/3) \right\}.$$

remark: intuitively, $\mathrm{TD}(\boldsymbol{x}, \boldsymbol{y})$ is obtained by dilating a standard equilateral triangle $\triangle$ centered at $\boldsymbol{y}$ until it reaches $\boldsymbol{x}$.

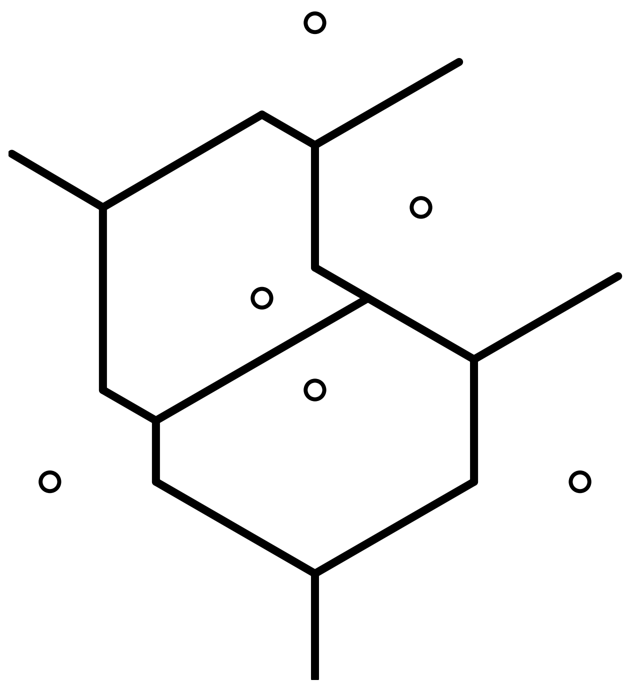


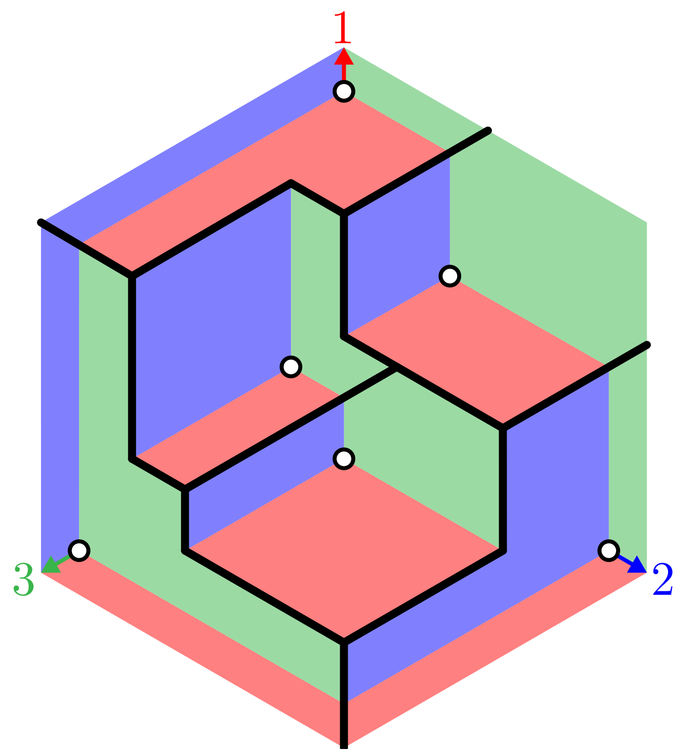


remark: TD is a quasi-distance, but is not symmetric.

GEODESIC EMBEDDINGS VS TD-DELAUNAY REALIZATIONS

PROP. Given a Schnyder wood W on a planar map M, the region vectors of the vertices of M with respect to W define a point-set whose TD-Delaunay triangulation is isomorphic to M.





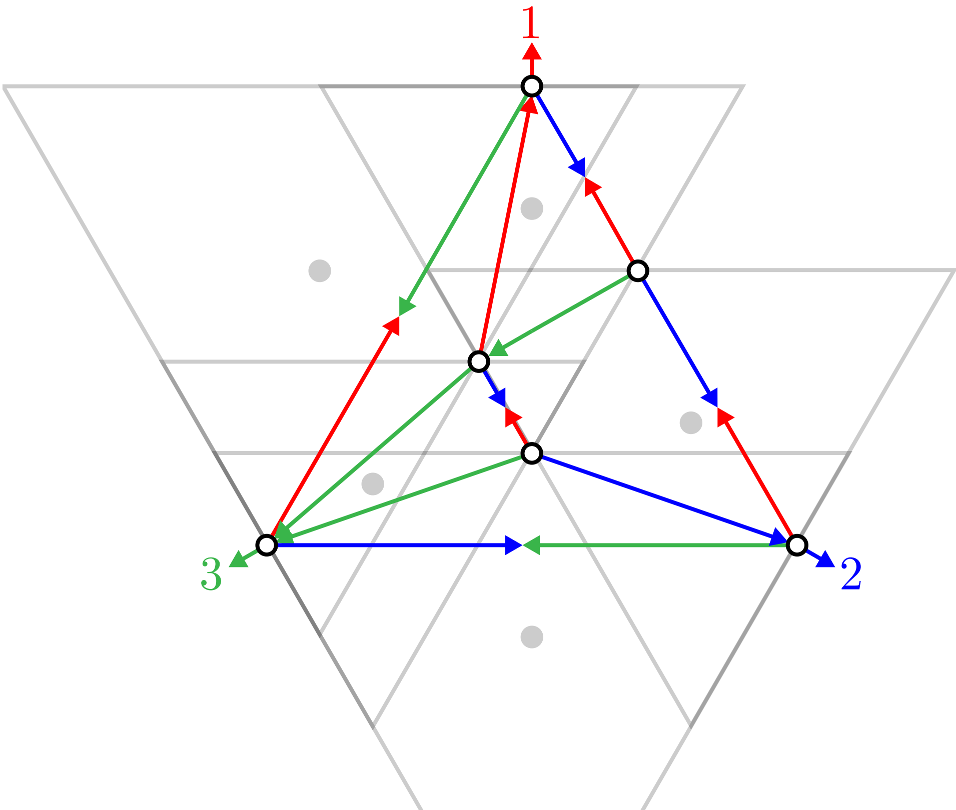
CORO. Any 3-connected planar graph admits a TD-Delaunay realization.

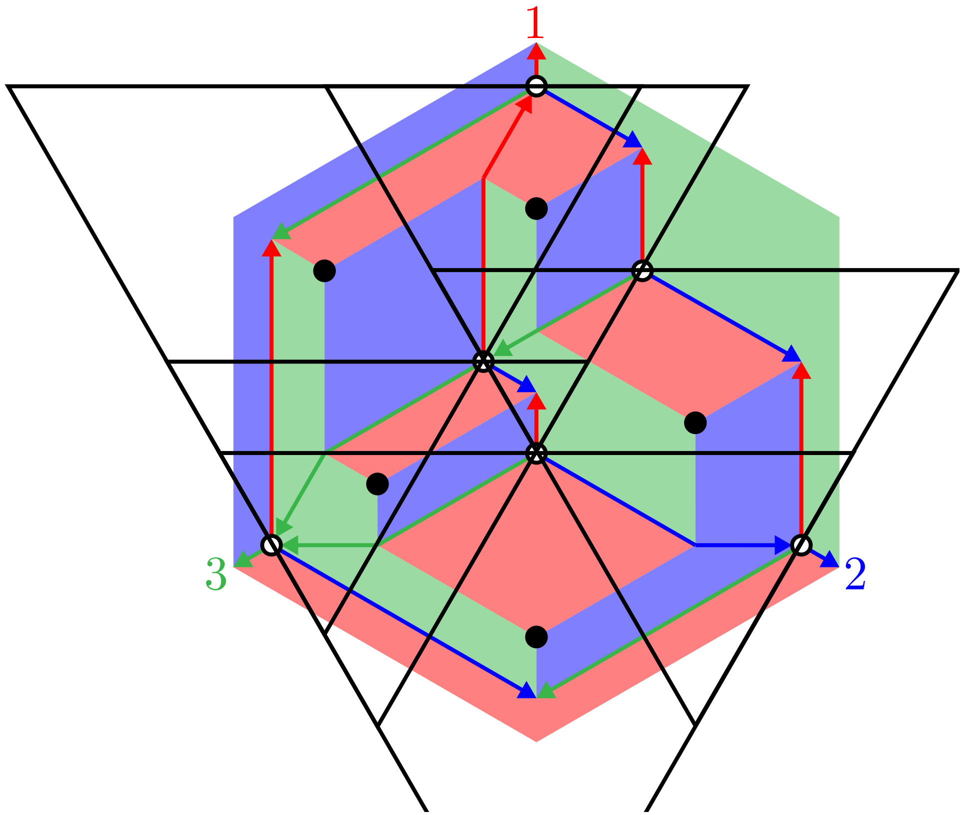
EMPTY REVERSED EQUILATERAL TRIANGLES

anti-standard equilateral triangle $\nabla = -\triangle$

PROP. For any points $\boldsymbol{p}$, $\boldsymbol{q}$ of $\boldsymbol{P}$ and any $\boldsymbol{Q} \subseteq \boldsymbol{P}$,

- $\boldsymbol{pq}$ is an edge of $\mathrm{Del}_{\mathrm{TD}}(\boldsymbol{P}) \iff$ there is an empty ∇ passing through $\boldsymbol{p}$ and $\boldsymbol{q}$,
- $\boldsymbol{Q}$ belongs to a face of $\mathrm{Del}_{\mathrm{TD}}(\boldsymbol{P}) \iff$ the circumscribed ∇ of $\boldsymbol{Q}$ is empty.





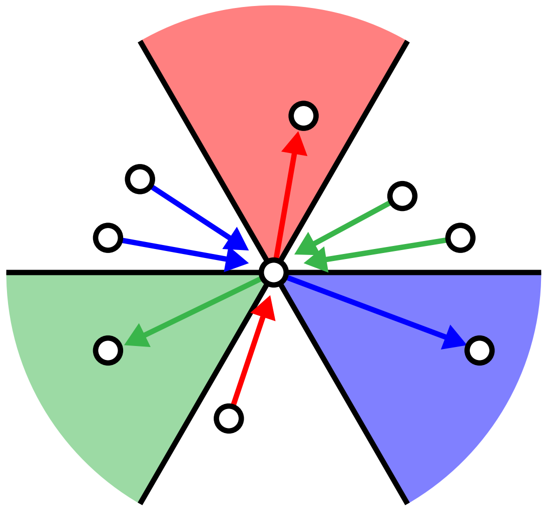
EMPTY REVERSED EQUILATERAL TRIANGLES

anti-standard equilateral triangle $\triangledown = -\triangle$

PROP. For any points $\boldsymbol{p}$, $\boldsymbol{q}$ of $\boldsymbol{P}$ and any $\boldsymbol{Q} \subseteq \boldsymbol{P}$,

- $\boldsymbol{pq}$ is an edge of $\mathrm{Del}_{\mathrm{TD}}(\boldsymbol{P}) \iff$ there is an empty $\triangledown$ passing through $\boldsymbol{p}$ and $\boldsymbol{q}$,
- $\boldsymbol{Q}$ belongs to a face of $\mathrm{Del}_{\mathrm{TD}}(\boldsymbol{P}) \iff$ the circumscribed $\triangledown$ of $\boldsymbol{Q}$ is empty.

PROP. In a TD-triangulation, the edges around a vertex look geometrically like



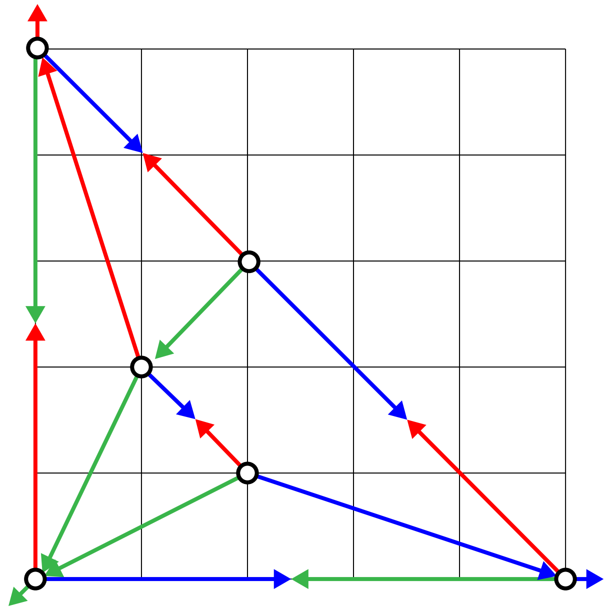
In particular, the paths $P_1(v)$, $P_1(v)$ and $P_1(v)$ stay in the red, blue and green angles.

EXISTENCE OF SCHNYDER WOODS

EXISTENCE

THM. Any 3-connected planar map admits a Schnyder wood.

CORO. Any 3-connected planar map with f faces admits a straight line embedding with vertices located on a $(f-1) \times (f-1)$ grid.



remark: Original proofs of Schnyder (for triangulations) and Felsner (for maps) based on edge contractions (difficult since contractions do not preserve 3-connectedness).
Here, proof for triangulations based on canonical orderings (a similar proof for arbitrary 3-connected planar maps is possible but more difficult).

CANONICAL ORDERING

M = triangulated planar map (except the external face)

DEF. canonical ordering of M = order on the vertices $v_1, \ldots, v_n$ such that for all $k \geq 3$, the submap M_k of M induced by $\{v_1, \ldots, v_k\}$ satisfies:

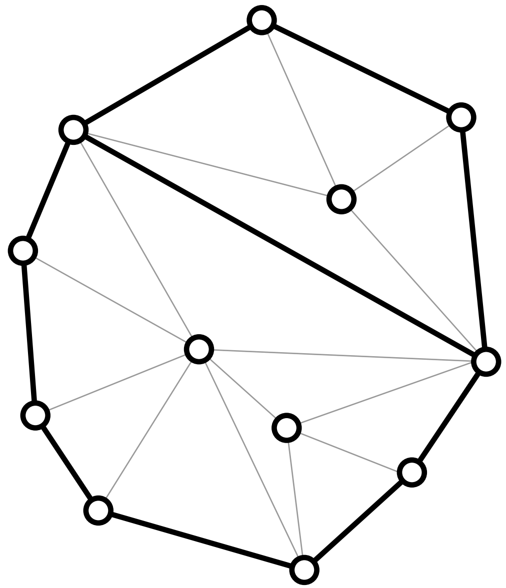
- M_k is connected and its boundary is a simple cycle,
- M_k is triangulated,
- v_{k+1} is in the outer face of M_k.

PROP. Any triangulated map admits a canonical ordering.

proof idea: start from M and delete a vertex on the outer face incident to only two other vertices of the outer face.

Such a vertex exists since:

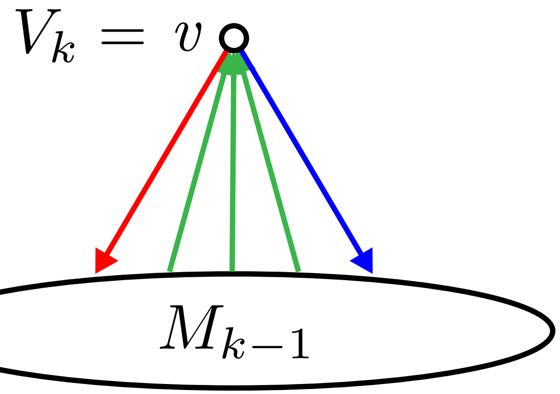
- either all vertices are valid,
- or there is a minimal length chord, separating at least a valid vertex.

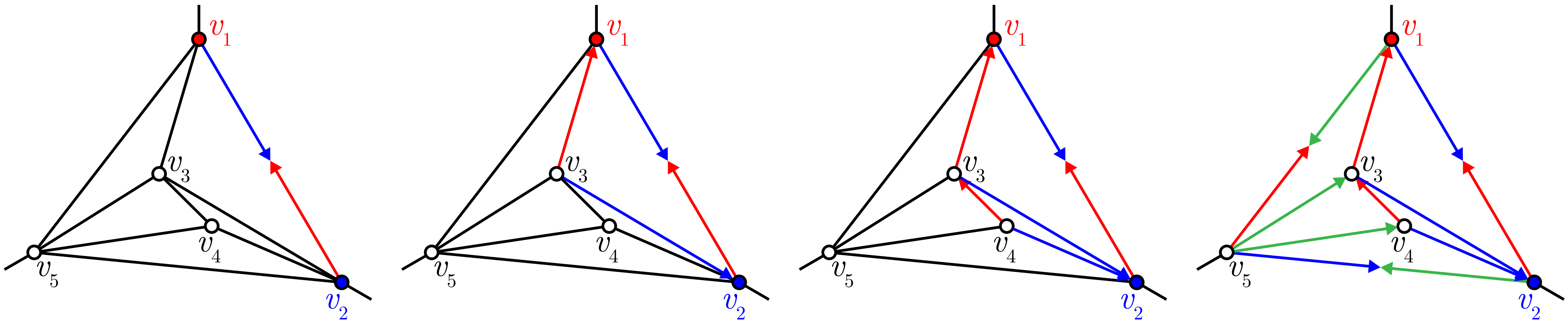


EXISTENCE FROM A CANONICAL ORDERING

PROP. Any triangulated map admits a canonical ordering.

PROP. A canonical ordering defines a Schnyder woods, using the local rule





THREE APPLICATIONS OF SCHNYDER WOODS

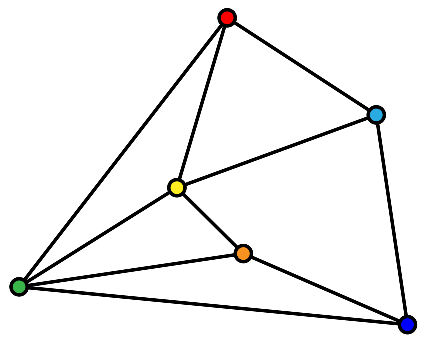
CONTACT REPRESENTATIONS

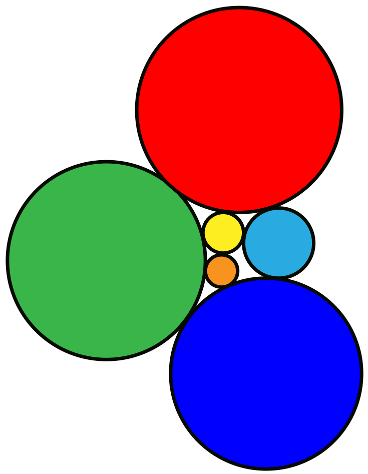
DEF. $\boldsymbol{X}$ = set of compact bodies whose interiors are pairwise disjoint.

contact graph of $\boldsymbol{X}$ = graph with

- vertices = bodies of $\boldsymbol{X}$
- edges = contacts between the bodies of $\boldsymbol{X}$.

contact representation of G = set $\boldsymbol{X}$ whose contact graph is isomorphic to G.



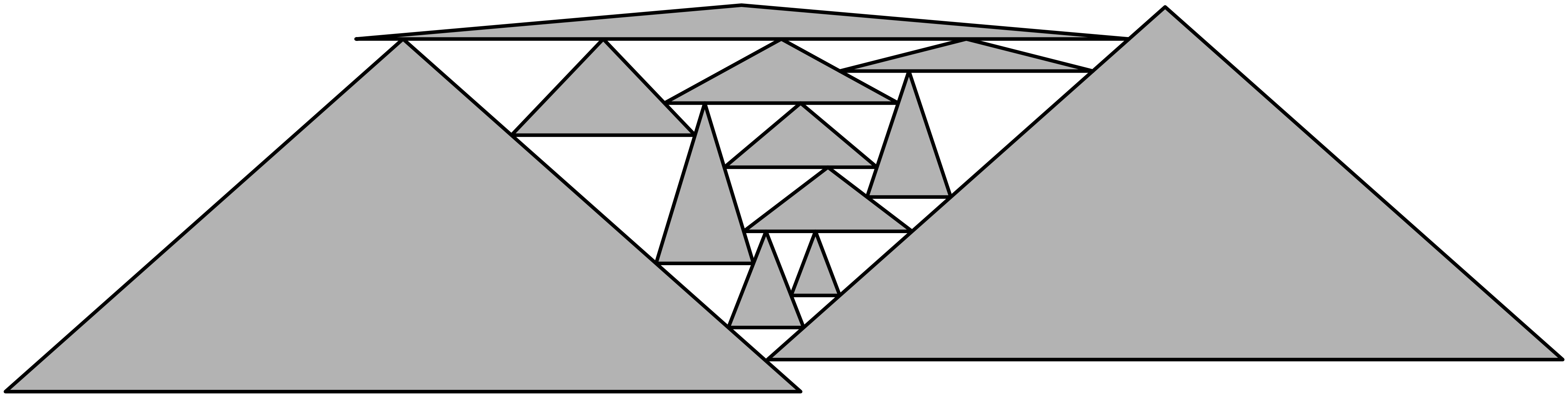


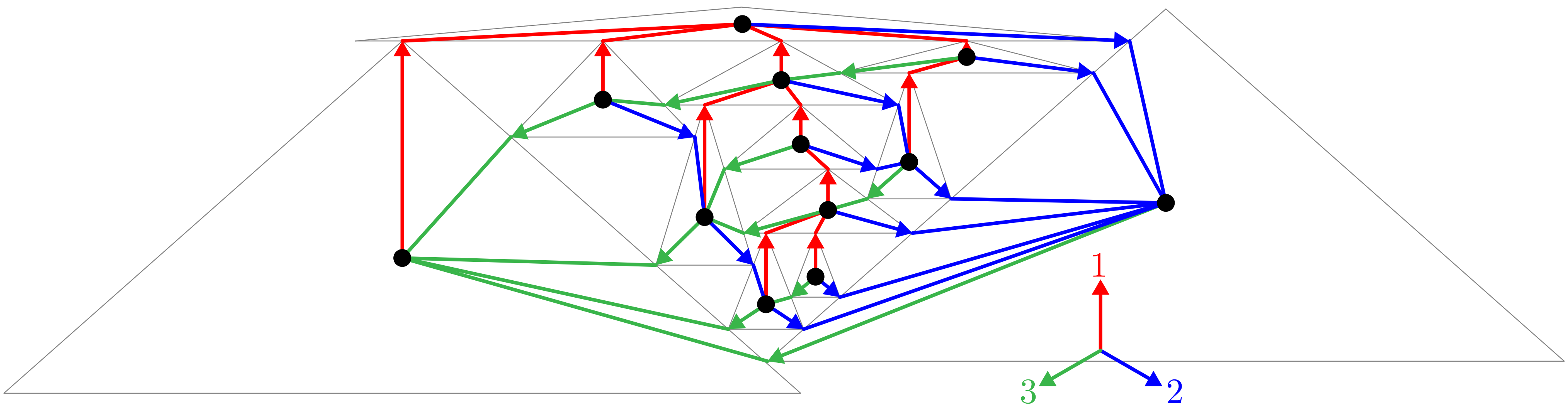
(img src: Wikipedia)

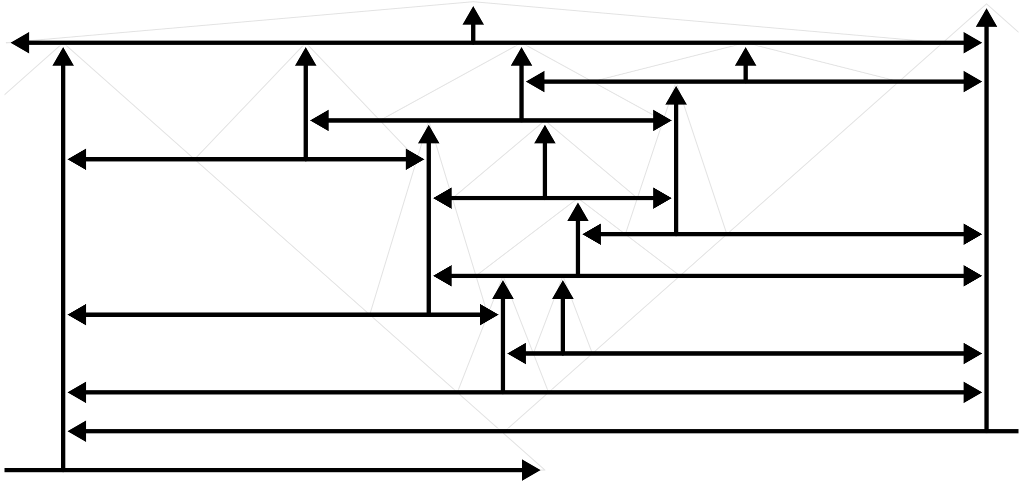
THM. (Circle packing) Any planar simple graph has a circle contact representation.

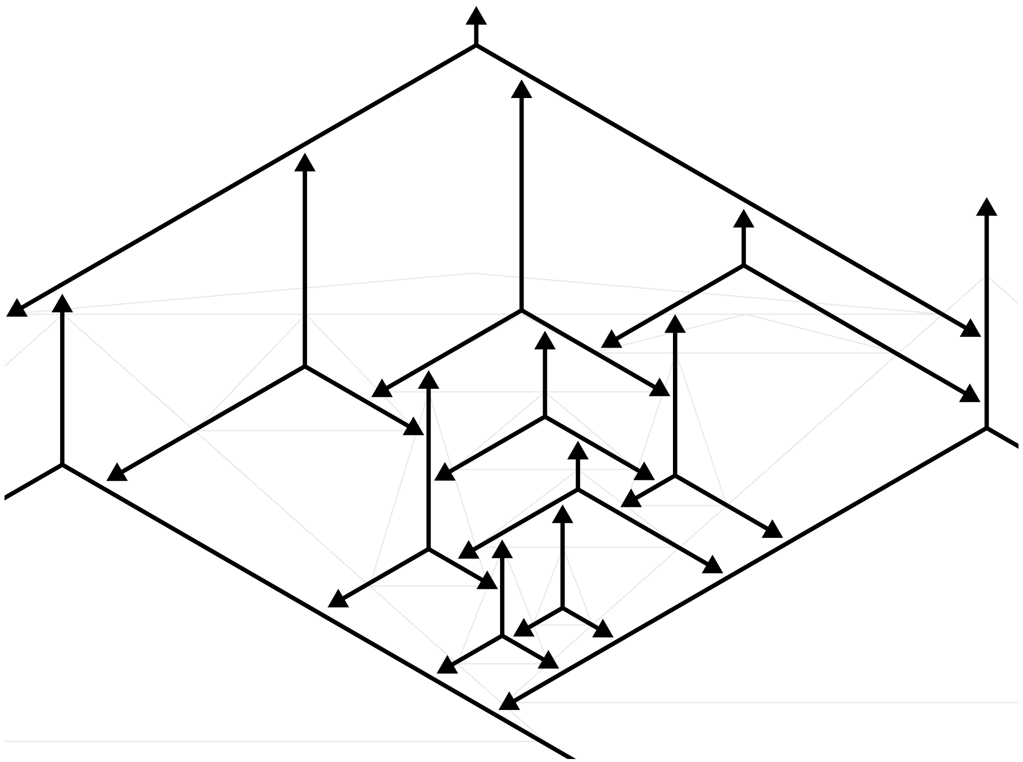
remark: in fact, the Koebe–Andreev–Thurston theorem says that this circle contact representation is unique up to Möbius transformations and reflections in lines.

TRIANGLE-CONTACT REPRESENTATIONS



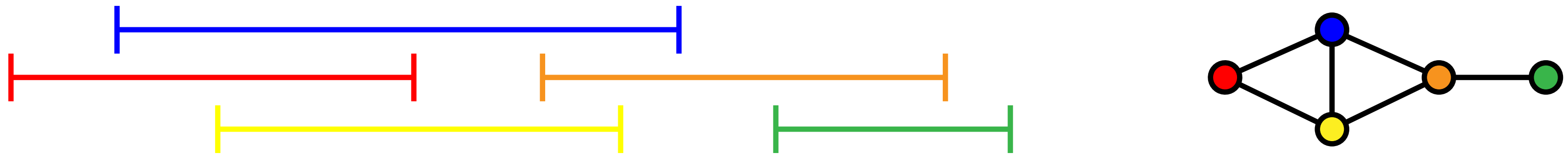






INTERVAL GRAPHS

DEF. interval graph = intersection graph of intervals.

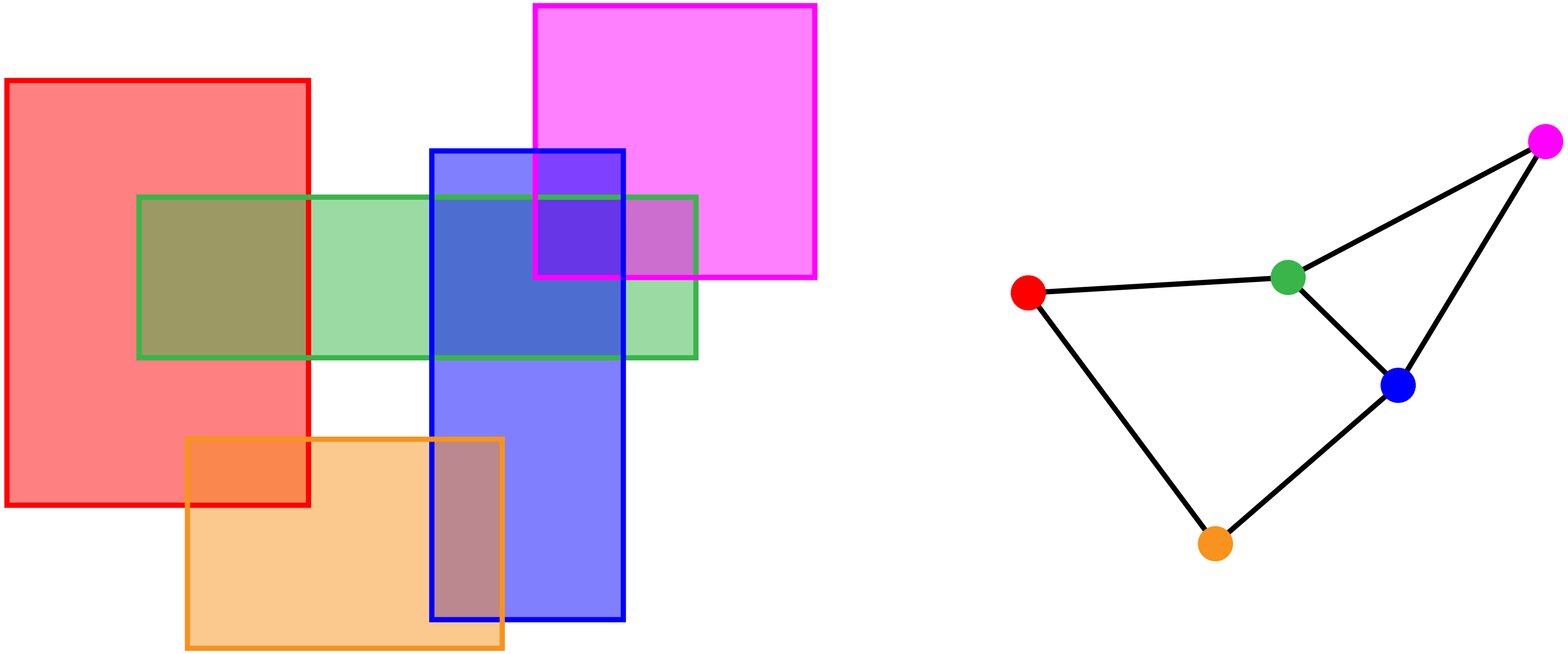


PROP. A graph $G = (V, E)$ is an interval graph if and only if

- all induced cycles are triangles,
- there is a partial order on V whose comparability graph is the complement of G.

BOXICITY

DEF. boxicity of G = smallest d such that there exists axis-parallel boxes in $\mathbb{R}^d$ whose intersection graph is isomorphic to G.

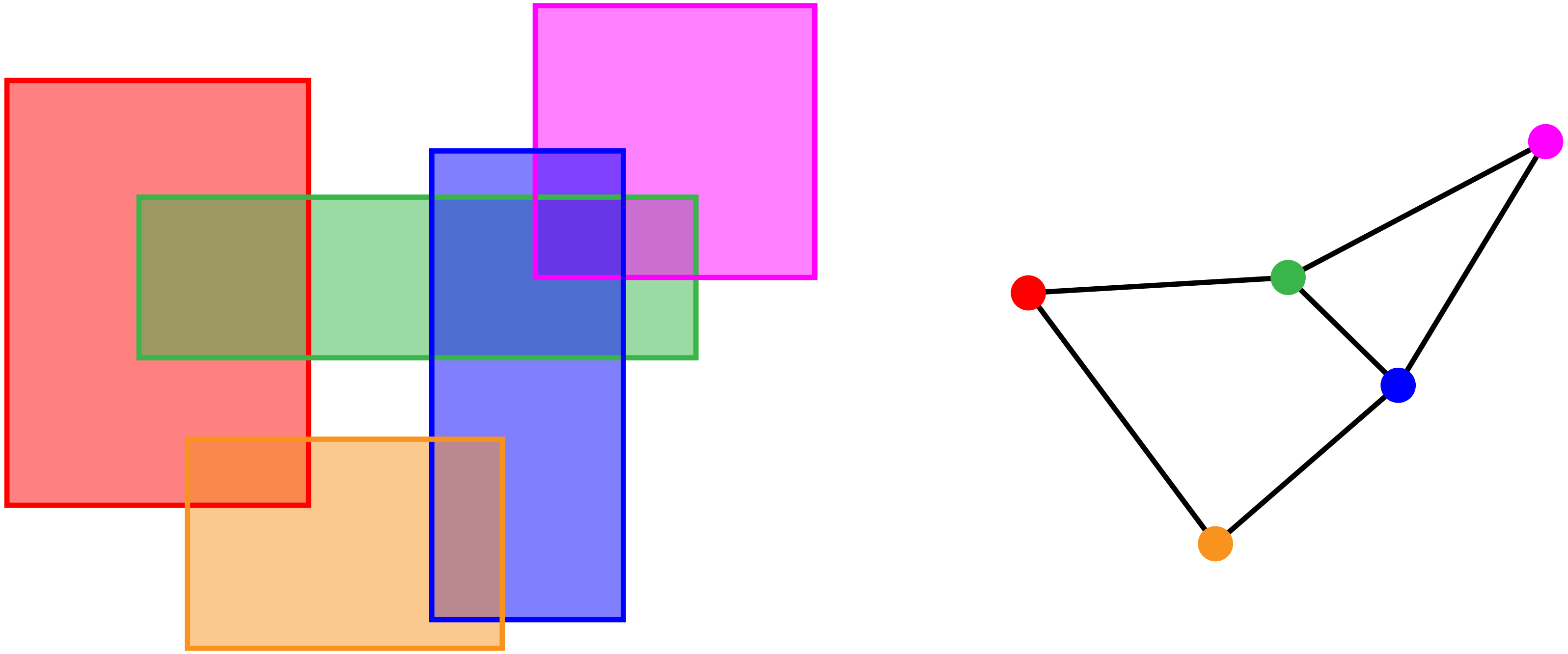


QU. What is the boxicity of

- a complete graph?
- a cycle of length at least 4?

BOXICITY

DEF. boxicity of G = smallest d such that there exists axis-parallel boxes in $\mathbb{R}^d$ whose intersection graph is isomorphic to G.



PROP. The boxicity of $G = (V, E)$ is the smallest d such that there exists d interval graphs $G_1 = (V, E_1), \ldots, G_d = (V, E_d)$ such that $E = E_1 \cap \cdots \cap E_d$.

PROP. The boxicity of $G = (V, E)$ is at most $|V|/2$.

BOXICITY

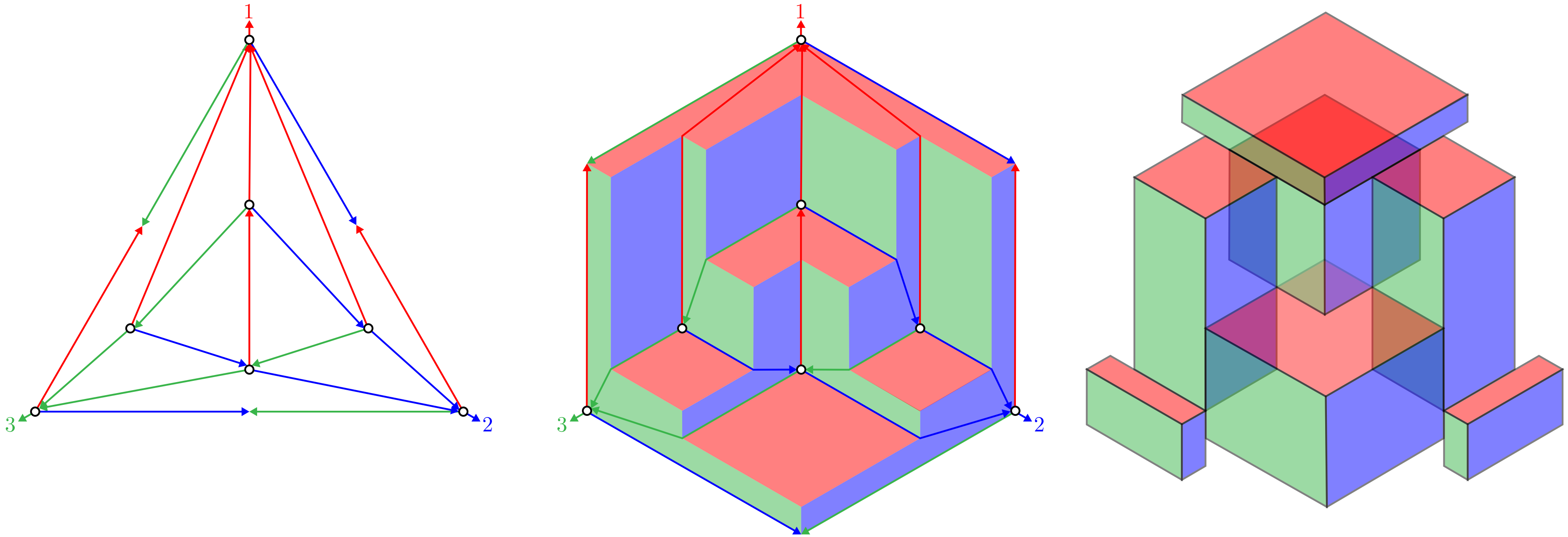
DEF. boxicity of G = smallest d such that there exists axis-parallel boxes in $\mathbb{R}^d$ whose intersection graph is isomorphic to G.

THM. Any planar graph has boxicity 3.

remark: initially proved by Thomassen with a different method.

proof idea:

- enough to consider triangulations,
- use Schnyder woods and geodesic embeddings.



GEOMETRIC SPANNERS

$G = (V, E)$ graph weighted by $\omega : E \to \mathbb{R}_{>0}$.
weight of a path $e_1, \ldots, e_k = \sum_{i \in [k]} \omega(e_i)$.
$d_G(u, v) =$ minimum weight of a path between u and v in G.

exm: $G = (V, E)$ geometric graph and $\omega(u, v) = |u - v|$.

DEF. t-spanner of G = subgraph H of G such that $d_H(u, v) \leq t \cdot d_G(u, v)$ for all $u, v \in V$.
stretch factor of H = smallest factor t such that H is a t-spanner of G.
geometric spanner = spanner of the complete geometric graph.

GEOMETRIC SPANNERS

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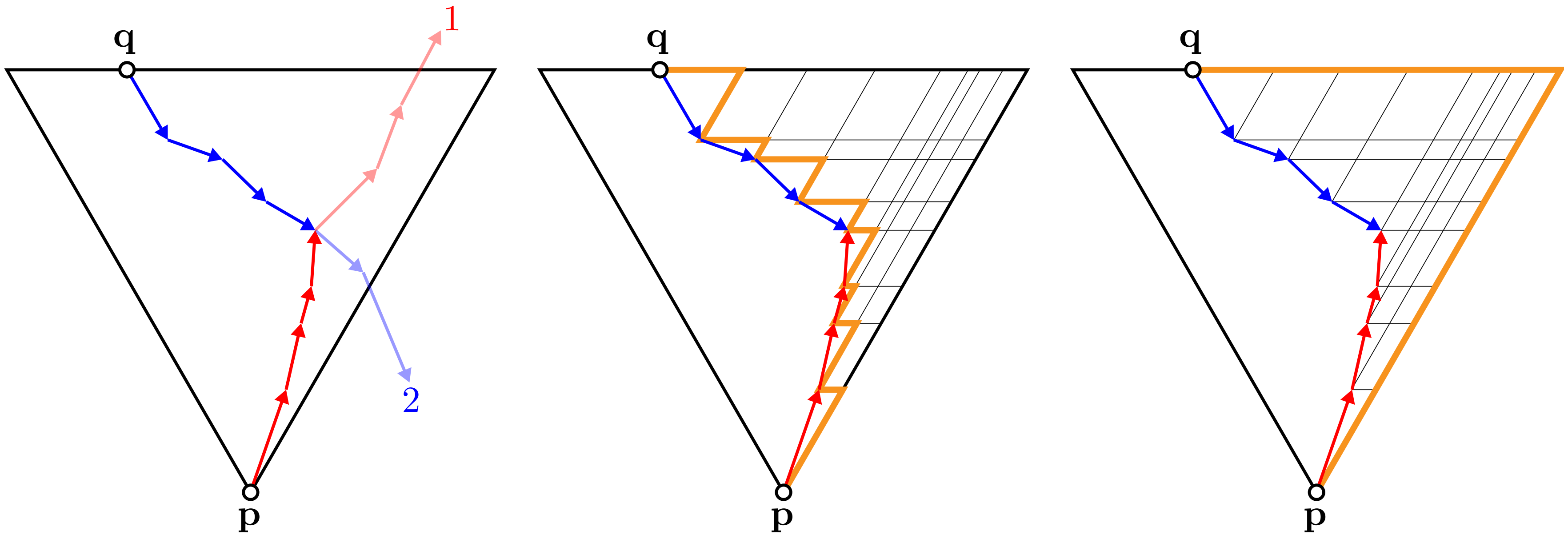
geometric spanner = spanner of the complete geometric graph.

THM.

- The complete geometric graph is a 1-spanner.
- The Delaunay triangulation is a t-spanner for $(\pi/2 <)\ 1.5846 < t < 1.998\ (< 2)$.
- The TD-Delaunay is a 2-spanner.

proof idea: for the TD-triangulation



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- The Delaunay triangulation is a t-spanner for $(\pi/2 <)\ 1.5846 < t < 1.998\ (< 2)$.
- The TD-Delaunay is a 2-spanner.

DEF. For $i \in [3]$ and $\boldsymbol{p} \in \boldsymbol{P}$, denote by

- parent$_i(\boldsymbol{p})$ = target of the unique outgoing edge of $\mathrm{Del}_{\mathrm{TD}}(\boldsymbol{P})$ colored by i.
- children$_i(\boldsymbol{p})$ = all points $\boldsymbol{q} \in \boldsymbol{P}$ such that $\boldsymbol{p} =$ parent$_i(\boldsymbol{q})$.
- closest$_i(\boldsymbol{p})$ = point of children$_i(\boldsymbol{p})$ closest to $\boldsymbol{p}$ for the triangular distance.
- first$_i(\boldsymbol{p})$ and last$_i(\boldsymbol{p})$ = first and last points of children$_i(\boldsymbol{p})$ clockwise around $\boldsymbol{p}$.

THM. (Bonichon, Gavoille, Hanusse, and Perkovic)
The subgraph of the TD-Delaunay triangulation $\mathrm{Del}_{\mathrm{TD}}(\boldsymbol{P})$ obtained by erasing at each vertex $\boldsymbol{p}$ all incoming arcs except the arcs first$_i(\boldsymbol{p})$, last$_i(\boldsymbol{p})$ and closest$_i(\boldsymbol{p})$ for $i \in [3]$ (if they exist) is a planar 6-spanner with degree at most 12.

SOME REFERENCES

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