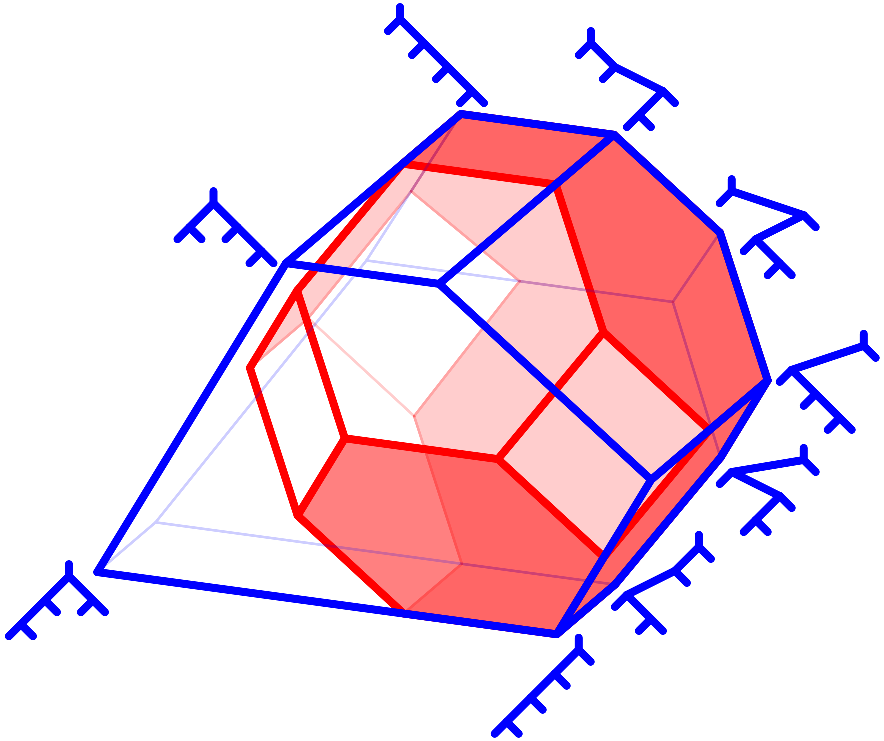
Permutahedra & Associahedra



V. PILAUD

MPRI 2-38-1. Algorithms and combinatorics for geometric graphs

Friday October 14th, 2022

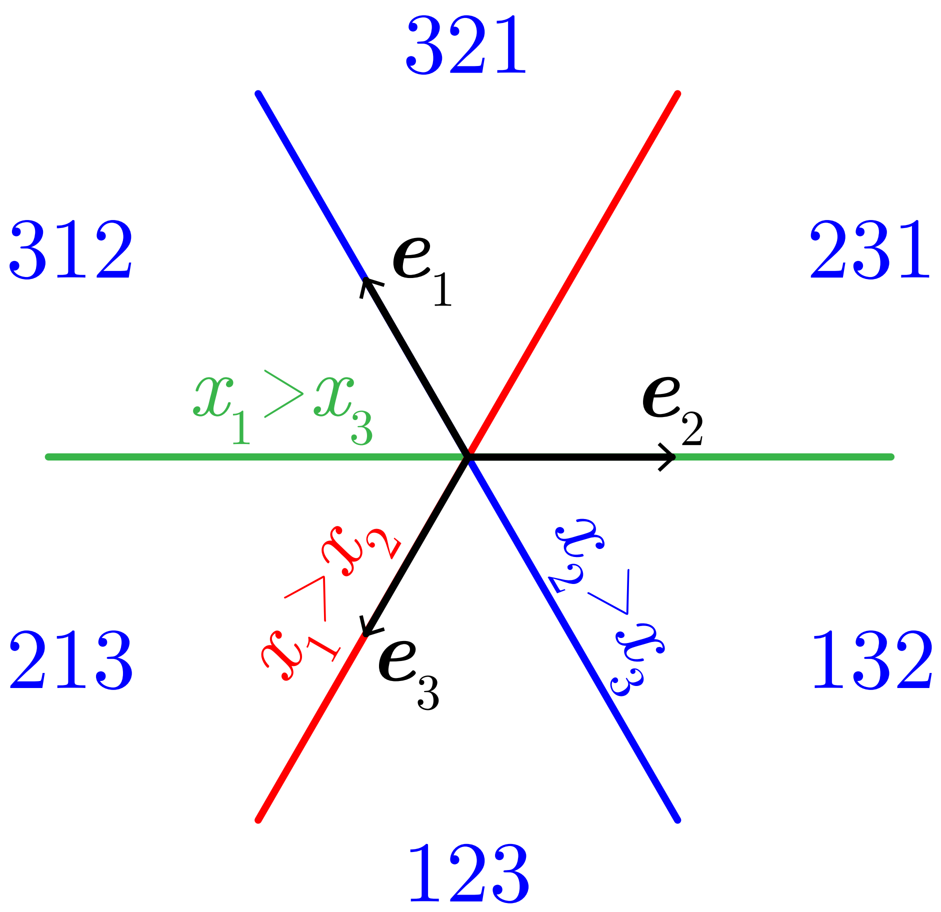
slides available at: http://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/MPRI-2-38-1-VP-5.pdf

Course notes available at: https://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/notesCoursMPRI20.pdf

PERMUTAHEDRA

BRAID FAN

braid fan $\mathcal{F}(n)$ = fan defined by hyperplanes $\{\boldsymbol{x} \in \mathbb{R}^n \mid x_i = x_j\}$ for $1 \le i < j \le n$



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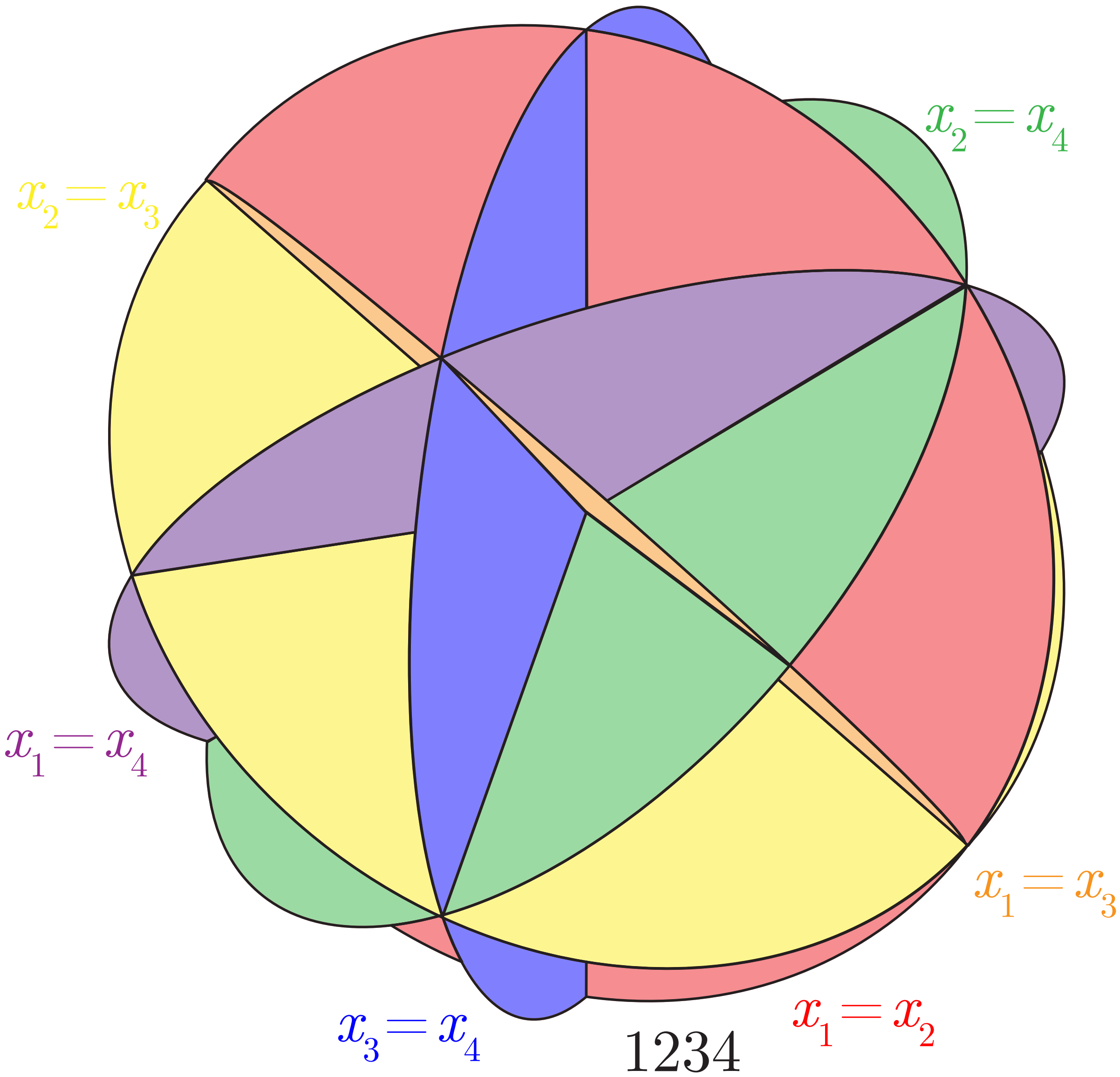

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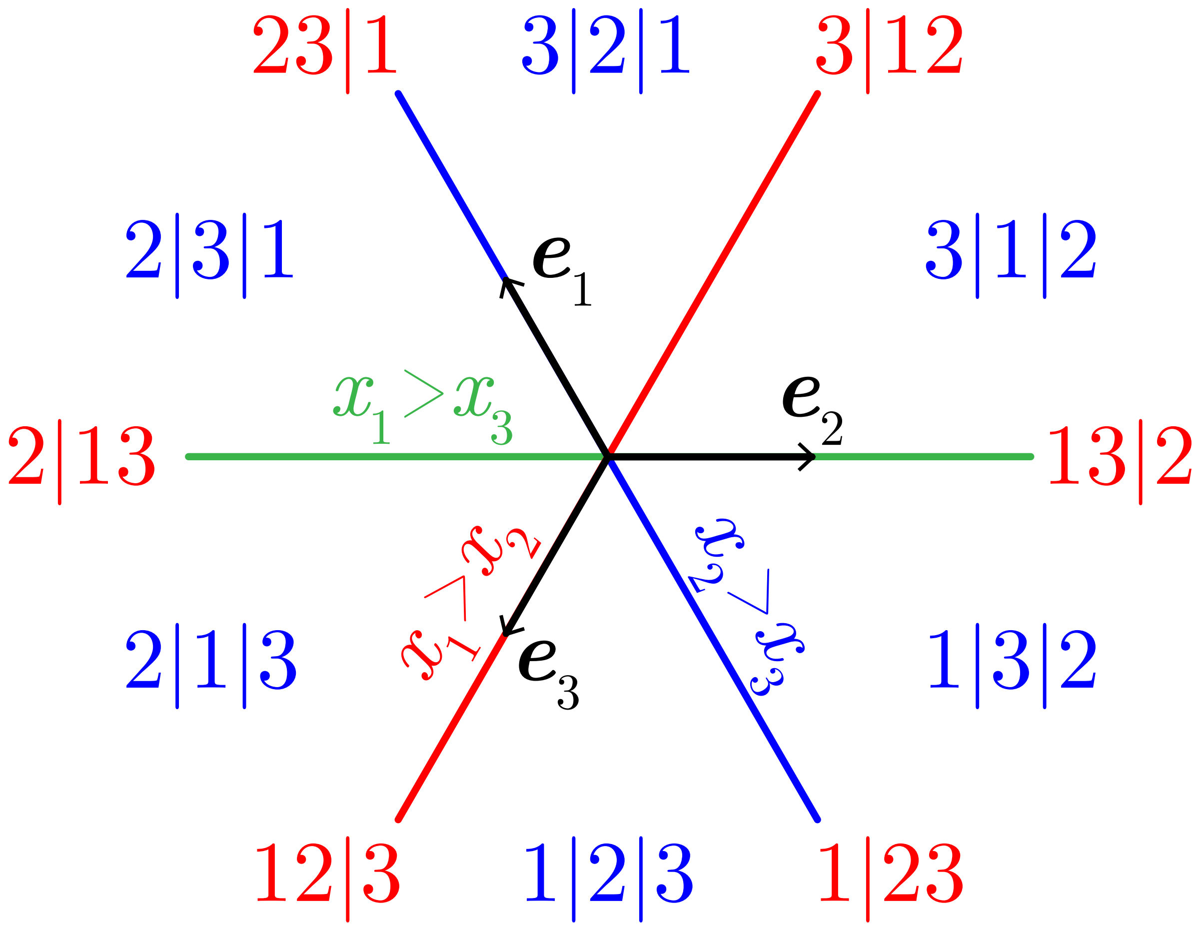

regions $\longleftrightarrow$ permutations of $[n]$

rays $\longleftrightarrow$ proper subsets of $[n]$

cones $\longleftrightarrow$ ordered partitions of $[n]$

$i \le_\mu j \iff i$ before j in μ

$C(\mu) = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \le x_j$ for $i \le_\mu j\}$

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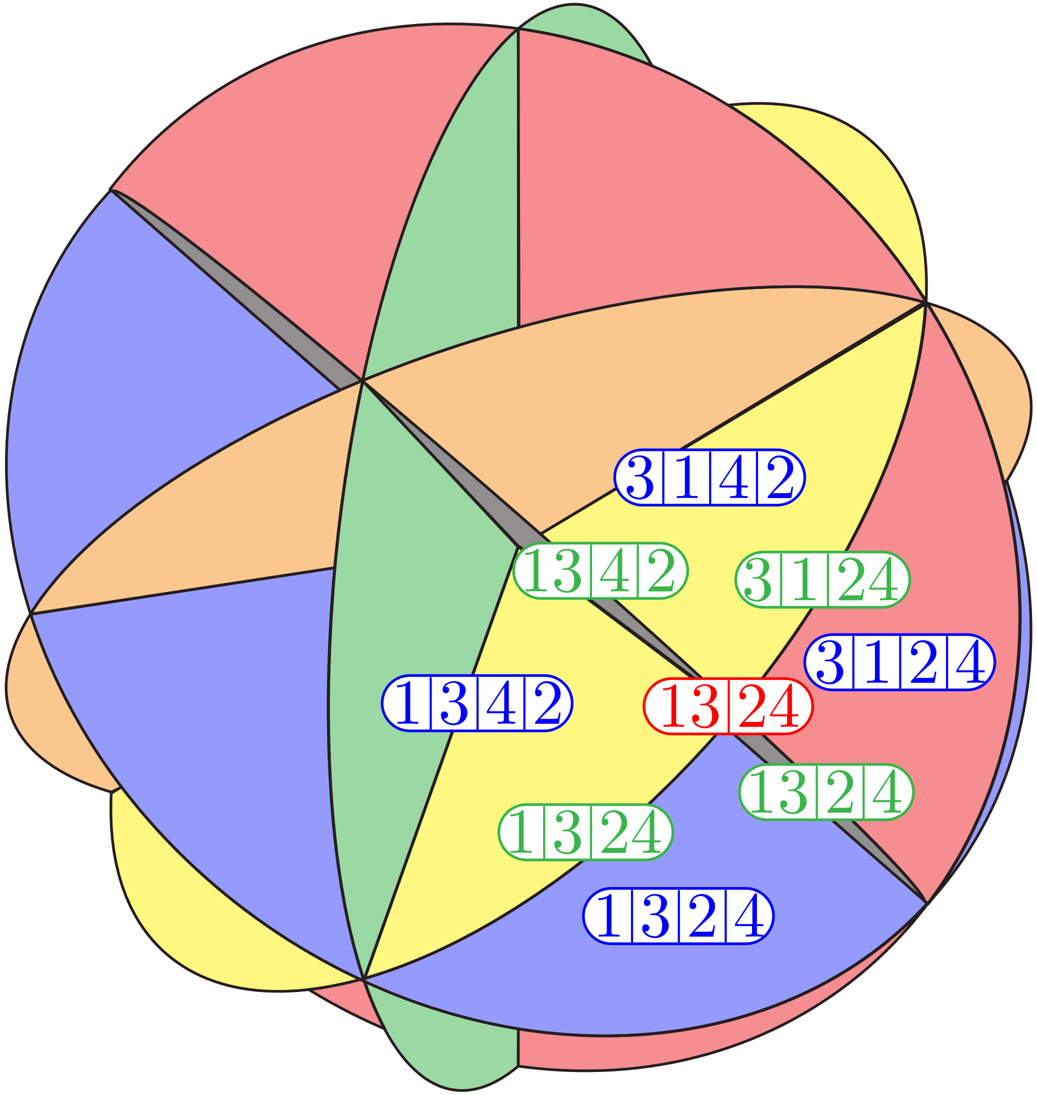

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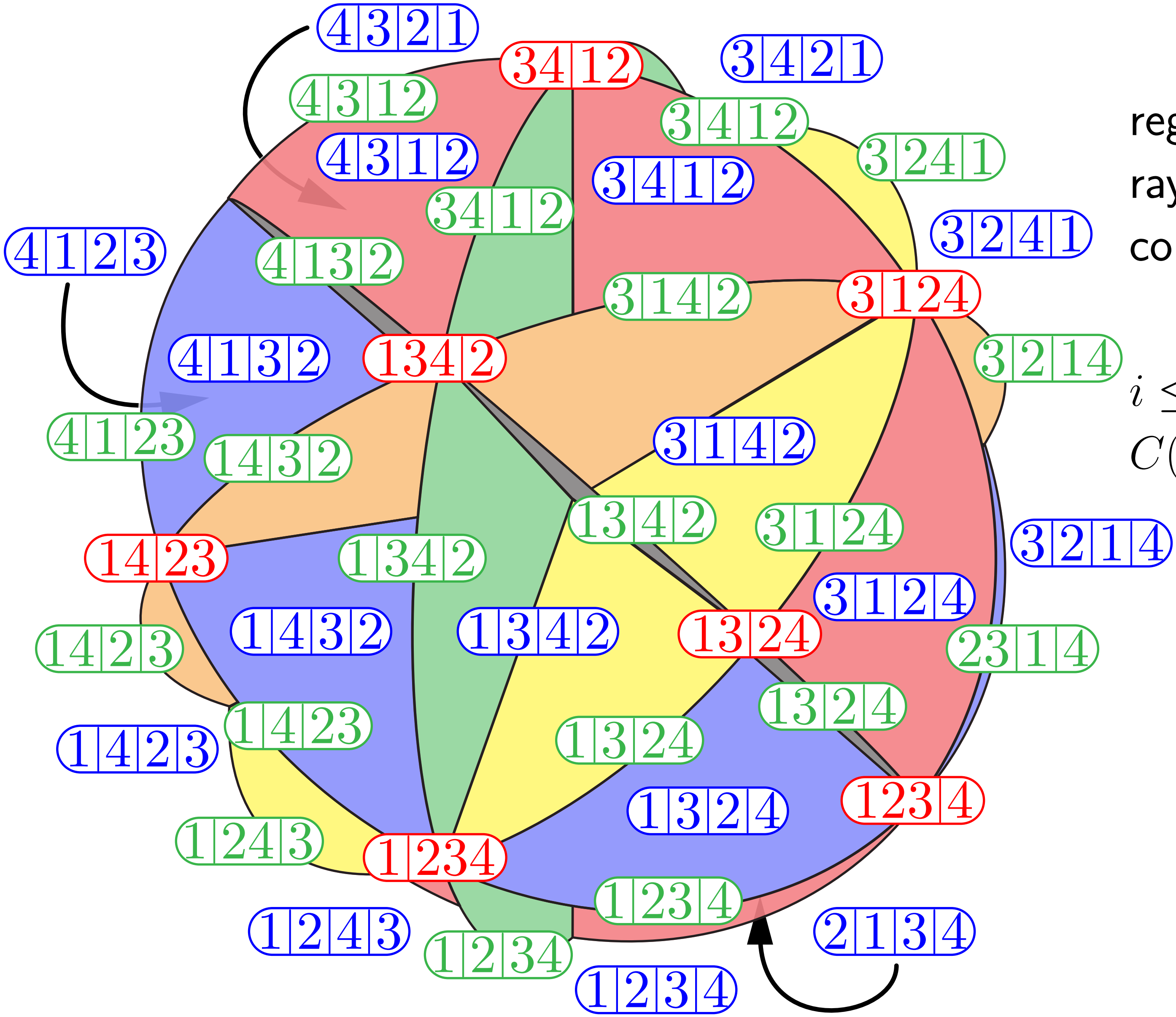

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NUMBER OF ORDERED PARTITIONS

QU. Show that

- Ordered partitions of $[n]$ into k parts are in bijection with surjections from $[n]$ to $[k]$.
- The number of surjections from A to B, with $|A| \geq |B|$ is given by

$$\sum_{p=0}^{|B|} (-1)^p \binom{|B|}{p} (|B| - p)^{|A|}.$$

(Apply the inclusion-exclusion formula to the sets $X_b := \{f : A \to B \mid b \notin f(A)\}$ for $b \in B$ to compute the number of non-surjective applications from A to B).

NUMBER OF ORDERED PARTITIONS

PROP. The number of ordered partitions of $[n]$ into k parts is $\sum_{p=0}^{k}(-1)^p\binom{k}{p}(k-p)^n$.

proof: For finite sets A and B, we have

$$\{f : A \to B \mid f(A) \neq B\} = \bigcup_{b\in B}\{f : A \to B \mid b \notin f(A)\}.$$

Thus by inclusion-exclusion principle

$$|\{f : A \to B \mid f(A) \neq B\}| = \sum_{\emptyset\neq C\subseteq B}(-1)^{|C|+1}\left|\bigcap_{c\in C}\{f : A \to B \mid c \notin f(A)\}\right|$$

$$= \sum_{\emptyset\neq C\subseteq B}(-1)^{|C|+1}(|B|-|C|)^{|A|} = \sum_{p=1}^{|B|}(-1)^{p+1}\binom{|B|}{p}(|B|-p)^{|A|}.$$

Thus

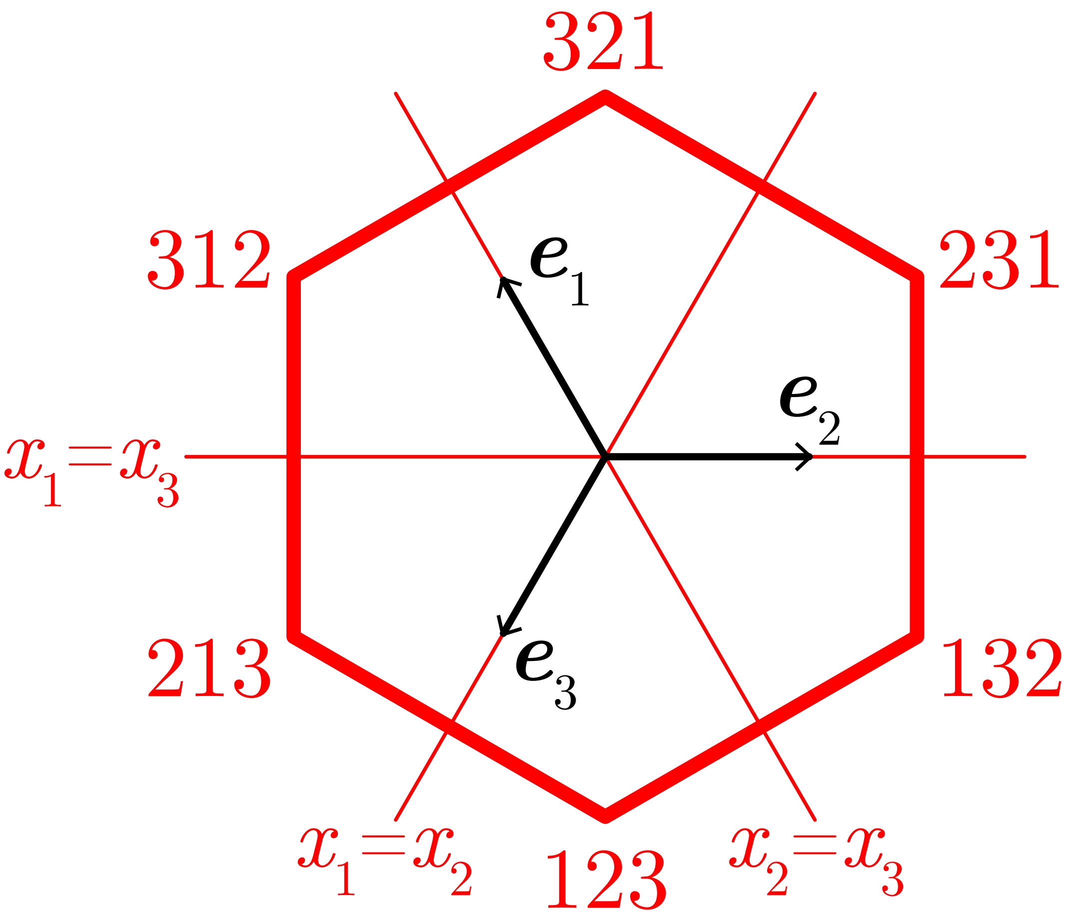
$$|\{f : A \to B \mid f(A) = B\}| = \sum_{p=0}^{|B|}(-1)^{p}\binom{|B|}{p}(|B|-p)^{|A|}.$$

Ordered partitions of $[n]$ into k parts are in bijection with surjections from $[n]$ to $[k]$. (the parts of the partition are the fibers of the surjection)

Permutahedron $\mathbb{P}\mathsf{erm}(n) = \mathrm{conv}\left\{(\tau^{-1}(1), \dots, \tau^{-1}(n)) \;\middle|\; \tau \in \mathfrak{S}_n\right\}$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n]} \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$$

$$= \mathbb{1} + \sum_{1 \leq i < j \leq n} [\boldsymbol{e}_i, \boldsymbol{e}_j]$$



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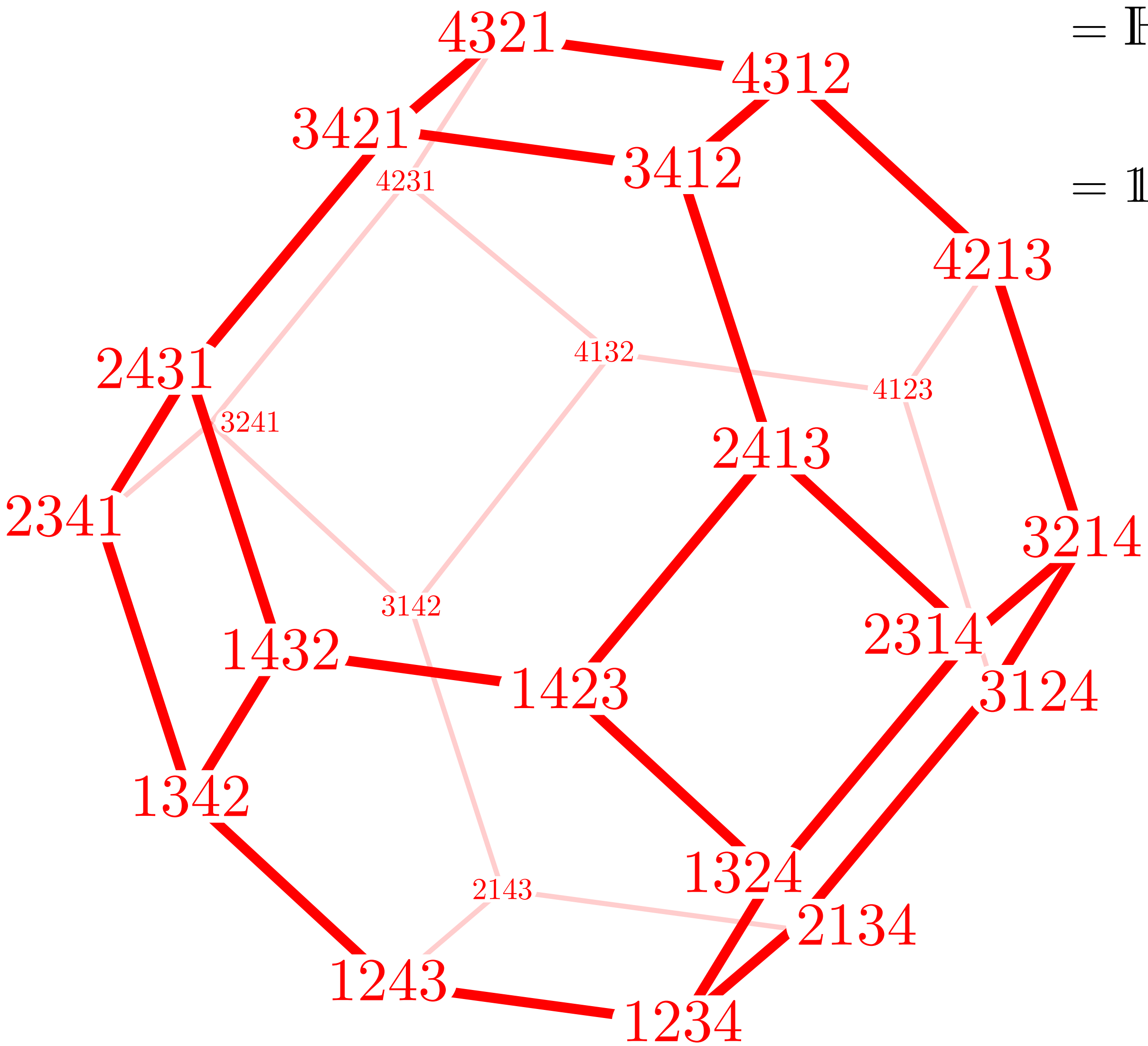

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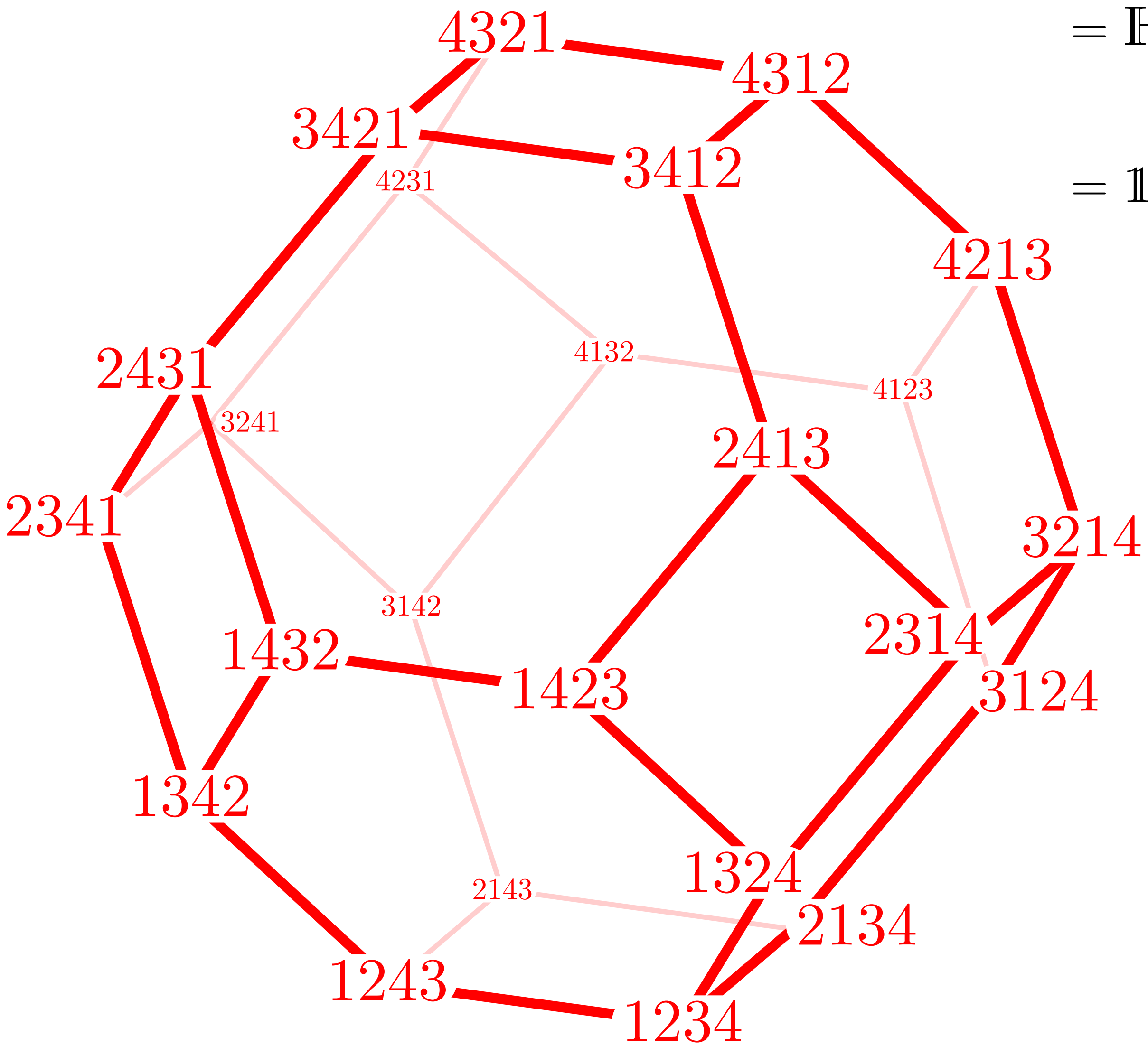
normal fan of $\mathbb{P}\mathrm{erm}(n)$ = braid fan $\mathcal{F}(n)$



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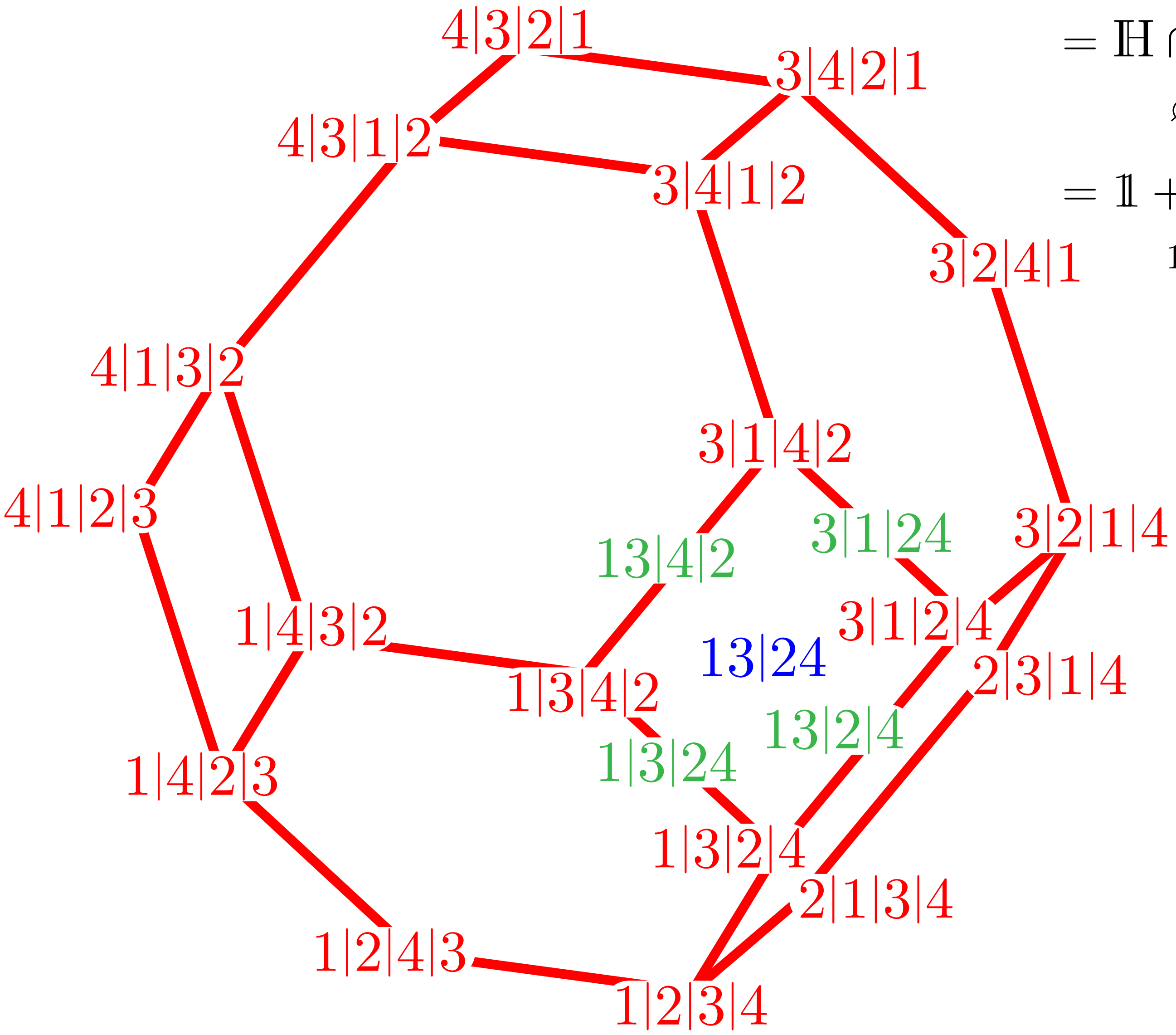

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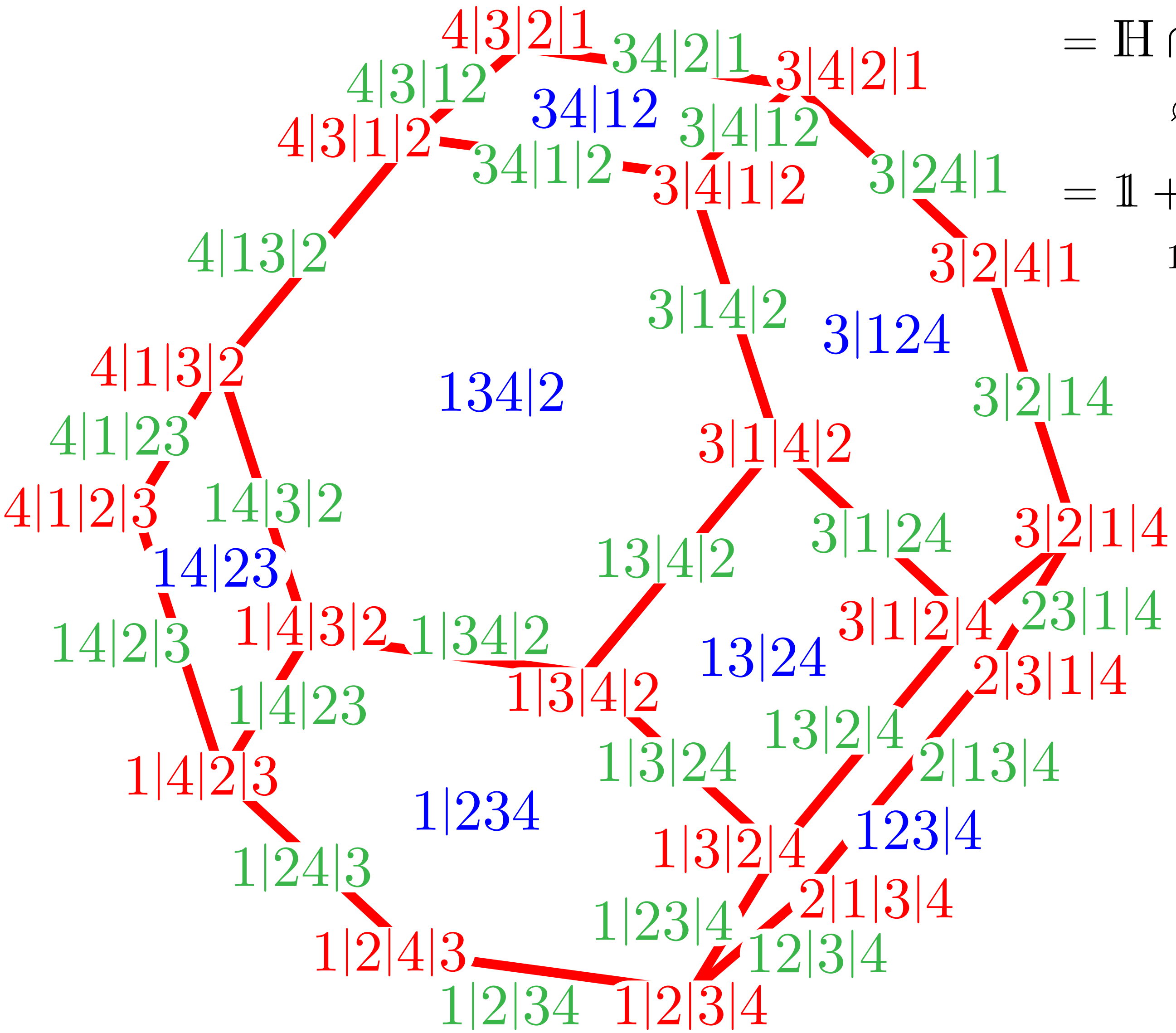

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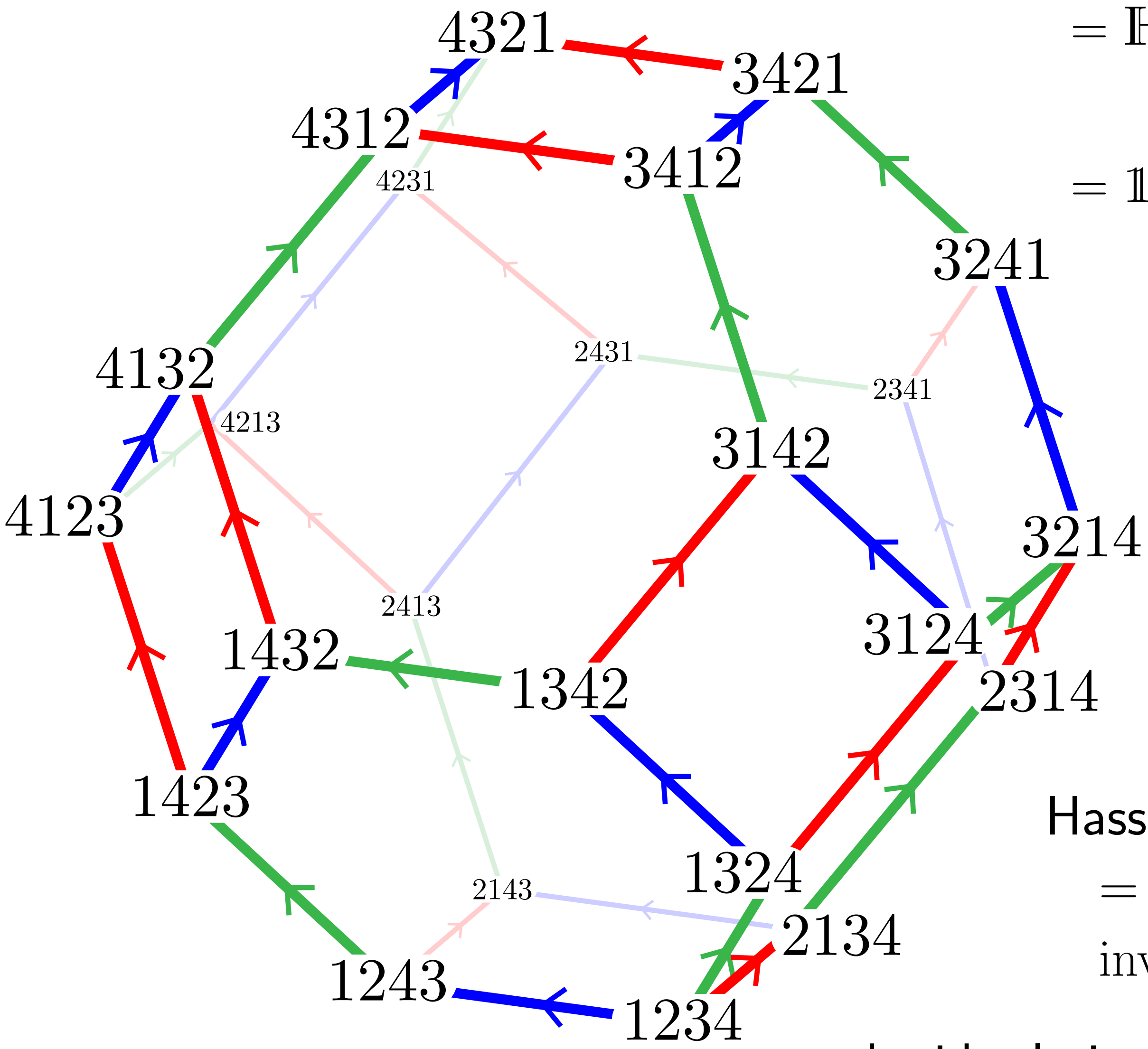

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Cayley graph of <ins>simple transpositions</ins>

transpositions $\tau_i = (i \;\; i+1)$

Hasse diagram of the <ins>weak order</ins>

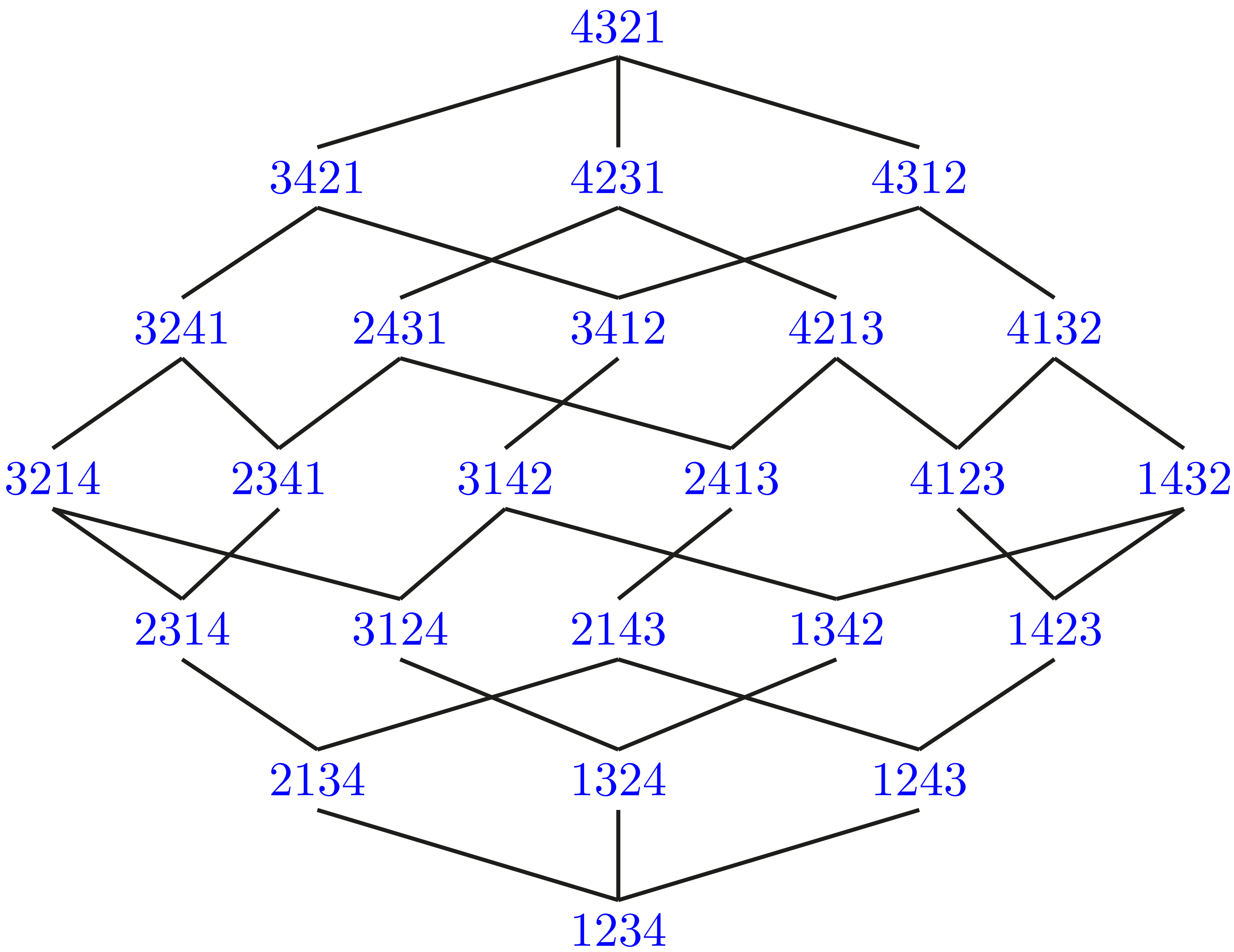
= inclusion of inversion sets

$\mathrm{inv}(\sigma) = \{(\sigma_i, \sigma_j) \mid i < j \text{ and } \sigma_i > \sigma_j\}$

<ins>braid relations</ins> $\quad \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \qquad \tau_i \tau_j = \tau_j \tau_i$

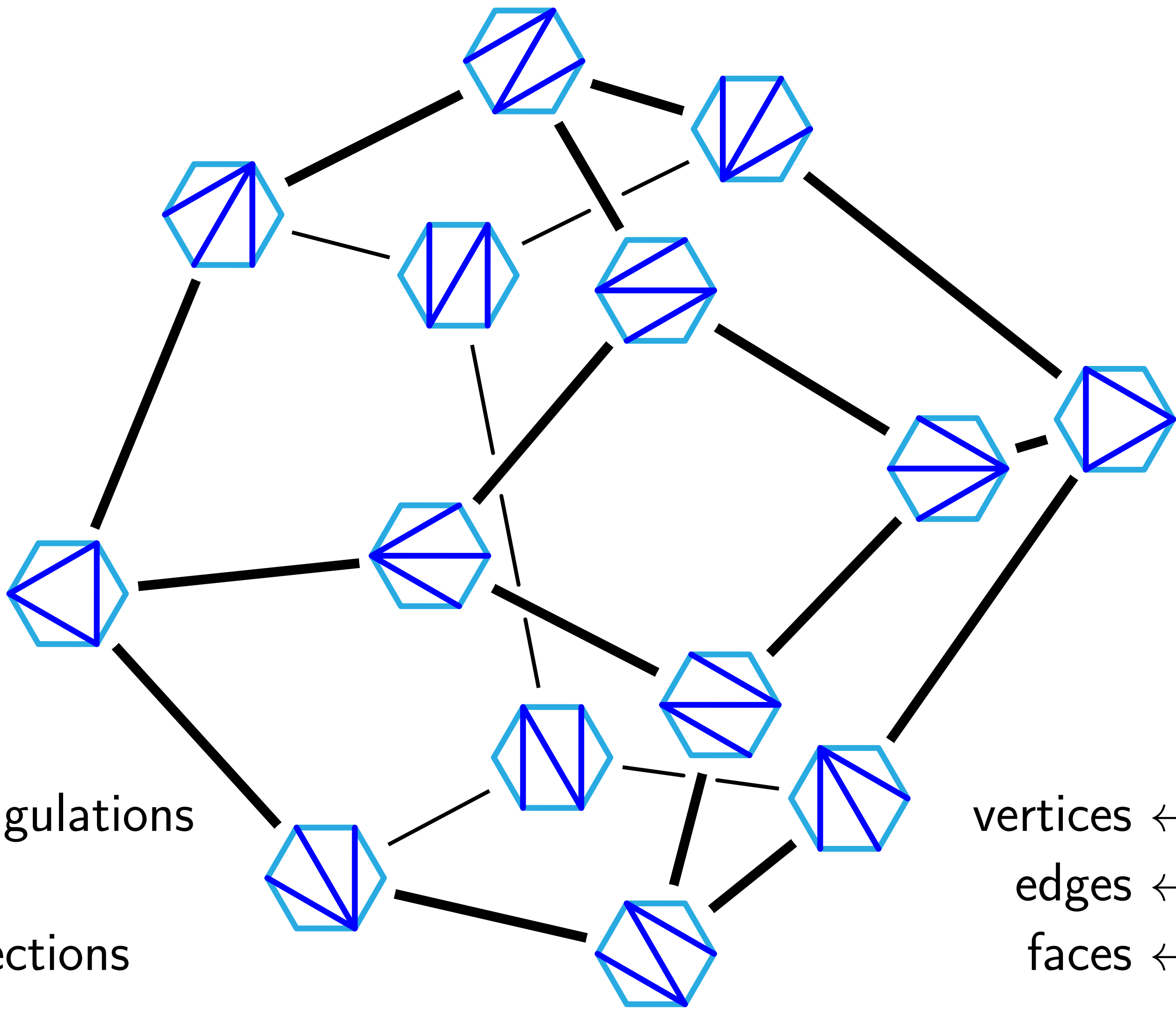
WEAK ORDER

DEF. weak order $=$ inclusion of inversion sets $\mathrm{inv}(\sigma) = \{(\sigma_i, \sigma_j) \mid i < j \text{ and } \sigma_i > \sigma_j\}$



ASSOCIAHEDRA

associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion.

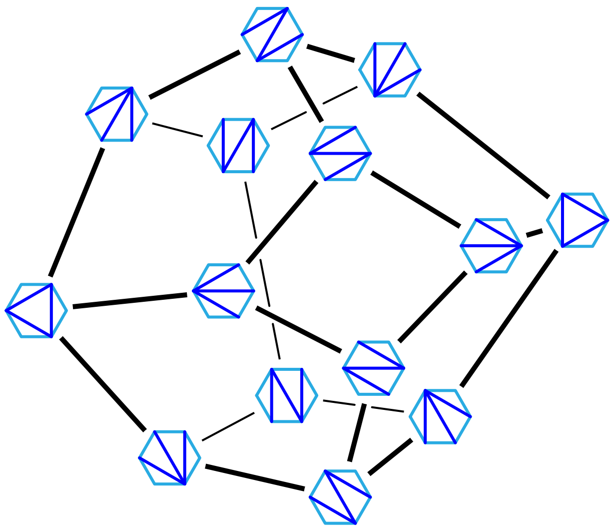


vertices $\leftrightarrow$ triangulations
edges $\leftrightarrow$ flips
faces $\leftrightarrow$ dissections

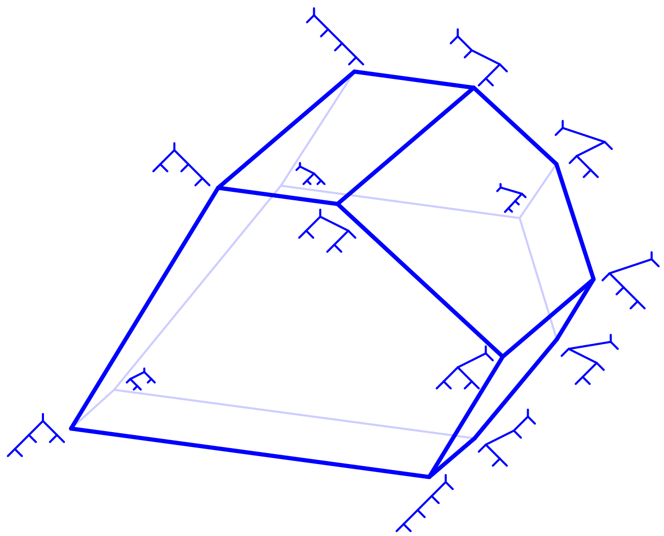
vertices $\leftrightarrow$ binary trees
edges $\leftrightarrow$ rotations
faces $\leftrightarrow$ Schröder trees

THREE FAMILIES OF ASSOCIAHEDRA

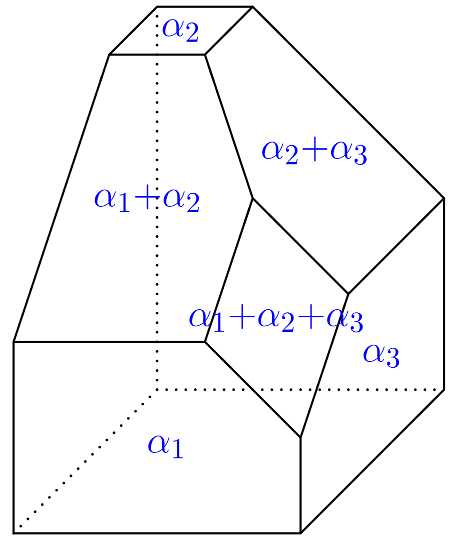
SECONDARY POLYTOPE



LODAY'S ASSOCIAHEDRON



CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

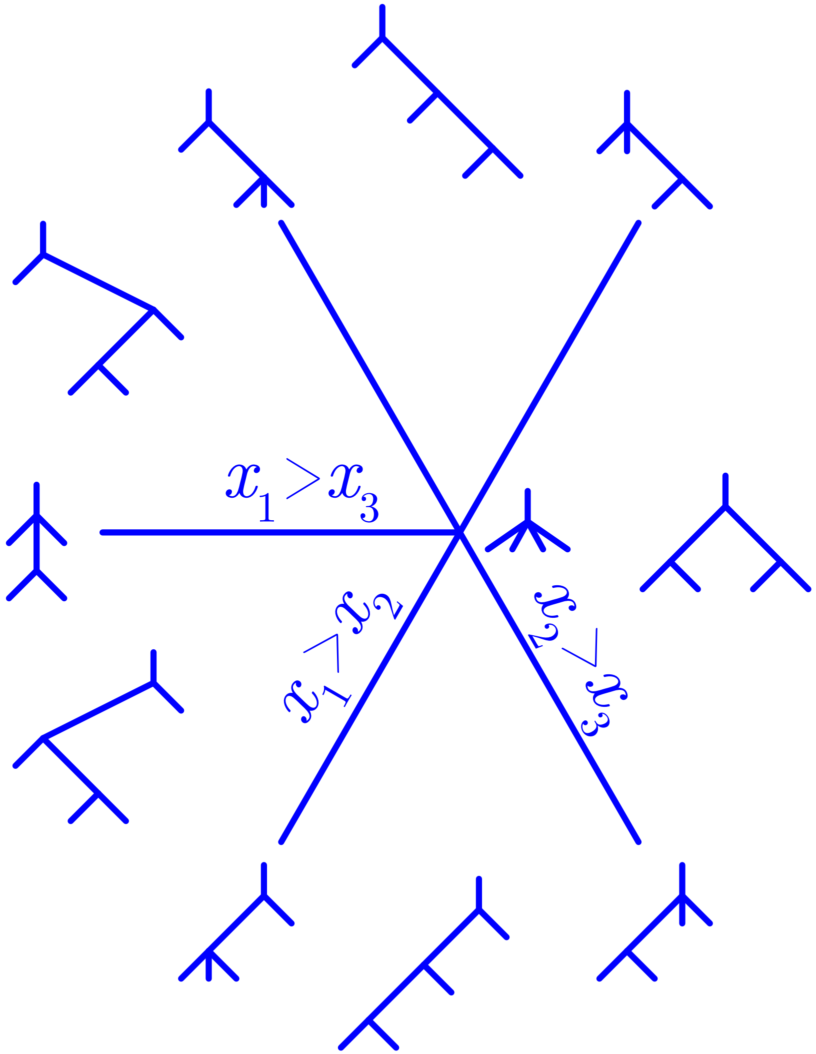


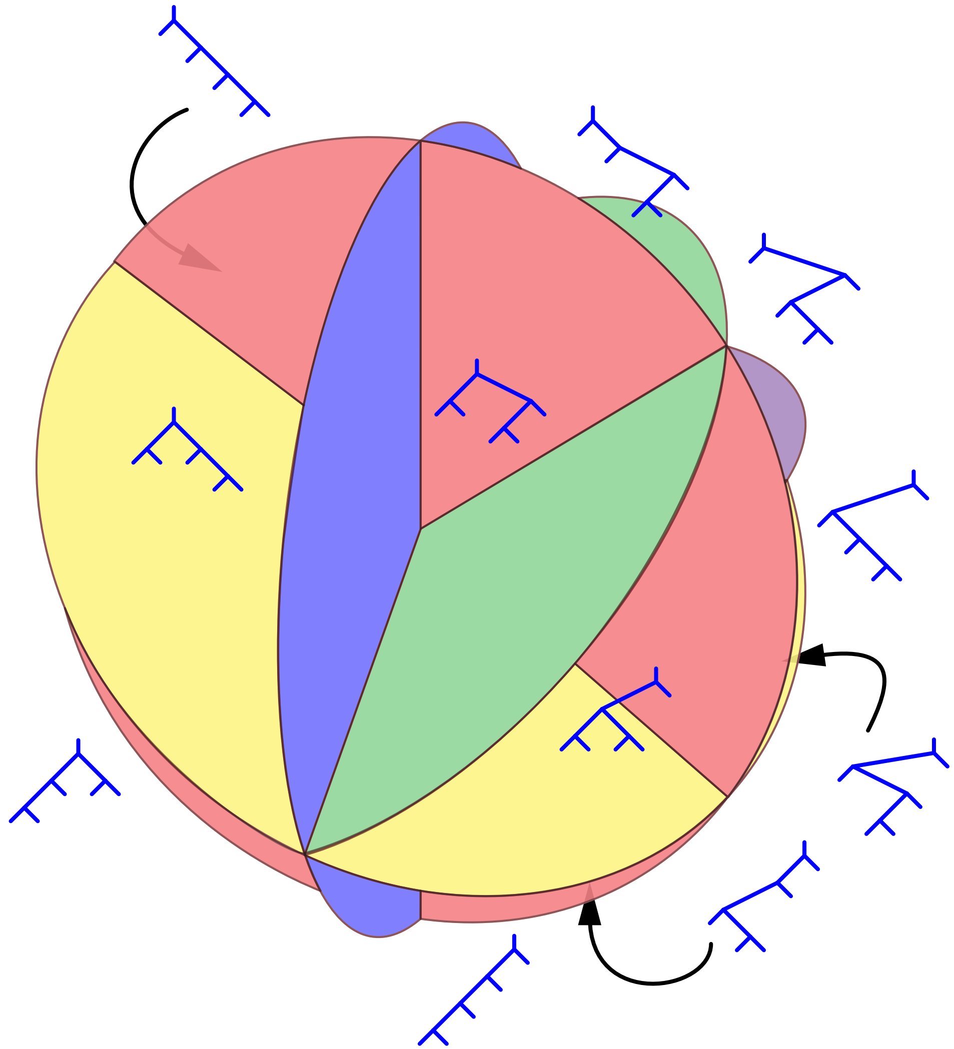
SYLVESTER FAN

DEF. binary tree T = tree where each internal node has exactly 2 children.
Schröder tree S = tree where each internal node has at least 2 children.
inorder labeling = label left subtree, then angle, then right subtree.
$i \le_S j \iff$ there is a path from i to j in S.

SYLVESTER FAN

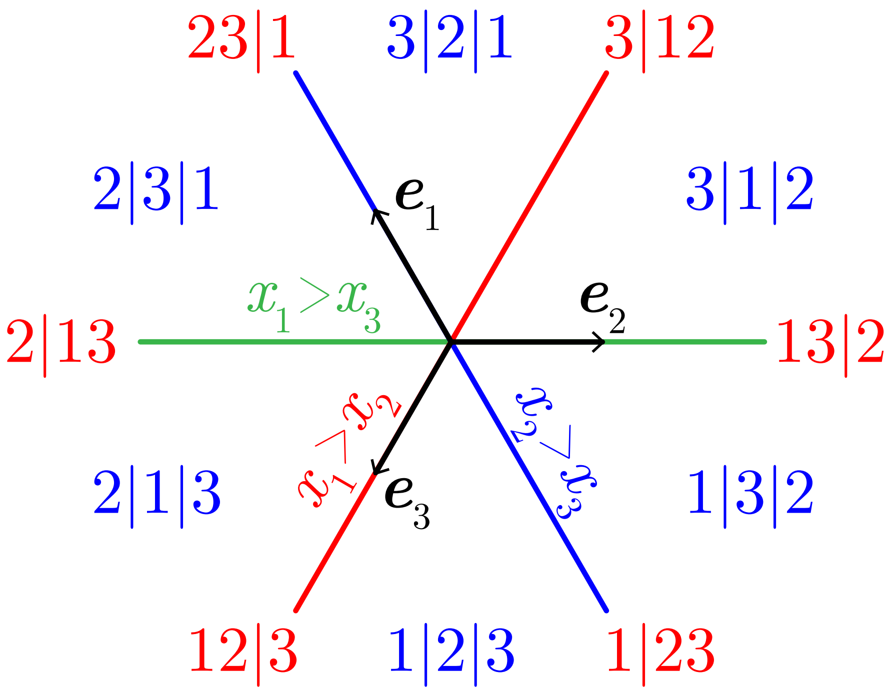
DEF. incidence cone $C(S) = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \leq x_j \text{ for } i \leq_S j\}$.
sylvester fan $= \{C(S) \mid S$ Schröder tree with $n+1$ leaves$\}$.

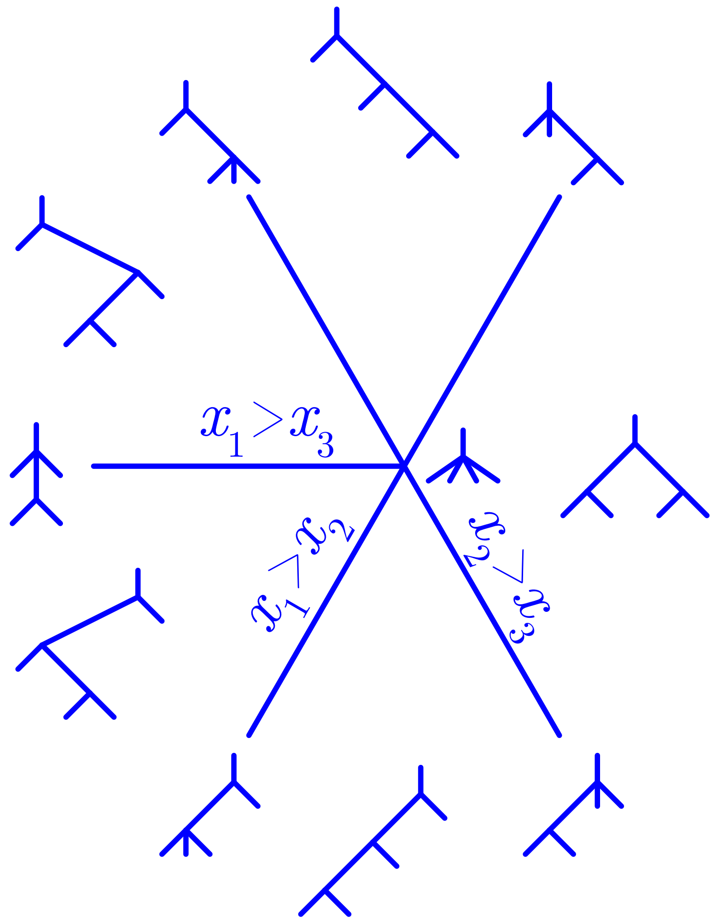




SYLVESTER FAN

PROP. $C(\mu) \subseteq C(S) \iff \mu$ extends $S \iff i \leq_S j \Rightarrow i \leq_\mu j$.





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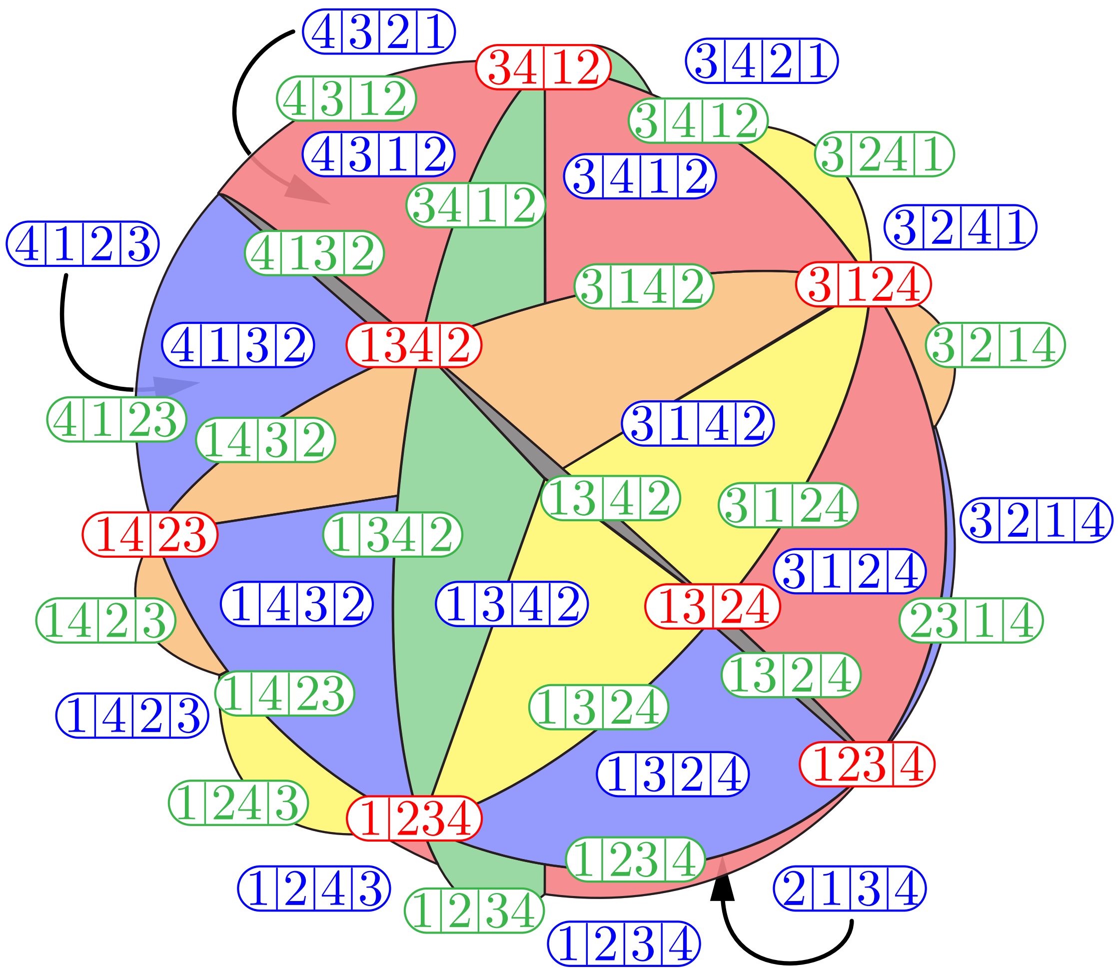
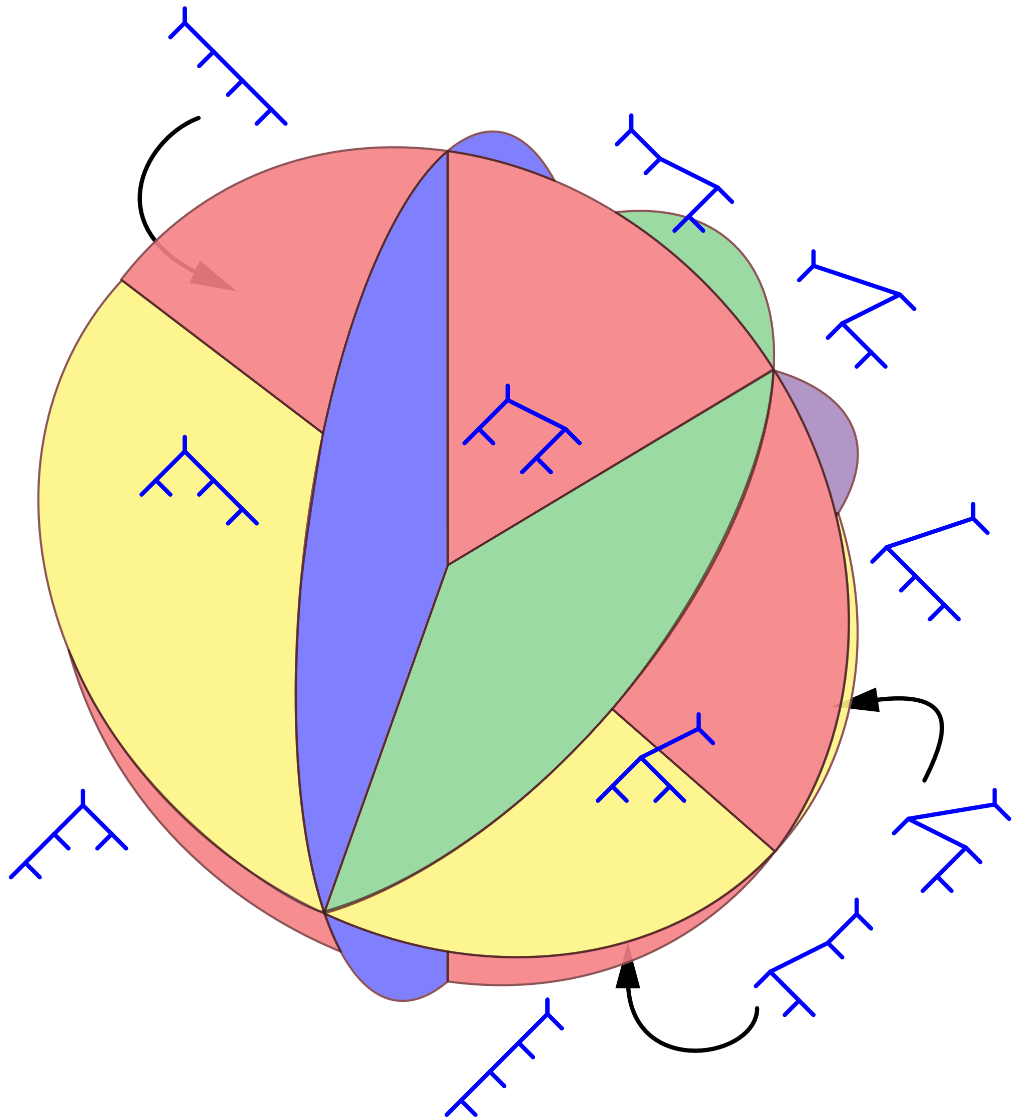
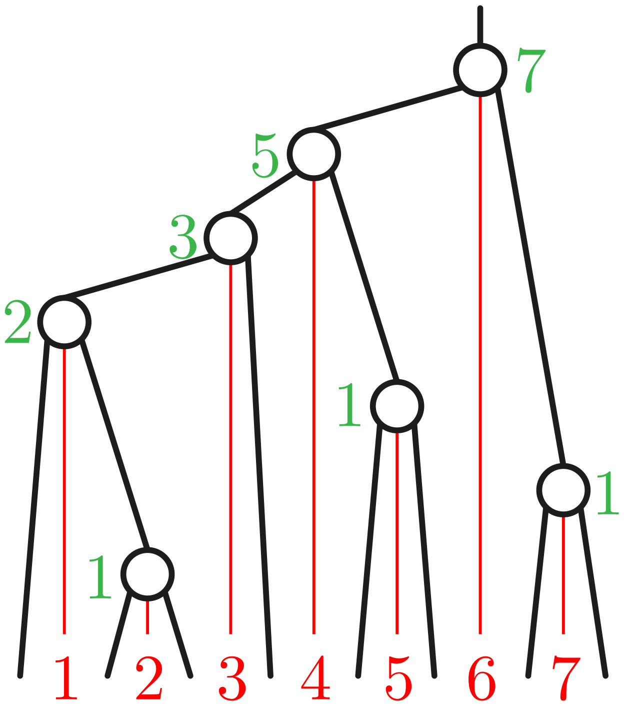




SYLVESTER FAN

QU. Prove that the number of linear extensions of a binary tree T is $n!/\prod_{i\in[n]} n_i$, where n = number of vertices and n_i = number of vertices in the subtree of node i.



1235476	1253476	1523476	5123476
1235746	1253746	1523746	5123746
1237546	1257346	1527346	5127346
1273546	1275346	1572346	5172346
1723546	1725346	1752346	5712346
7123546	7125346	7152346	7512346

$$7!/(2 \cdot 1 \cdot 3 \cdot 5 \cdot 1 \cdot 7 \cdot 1) = 24$$

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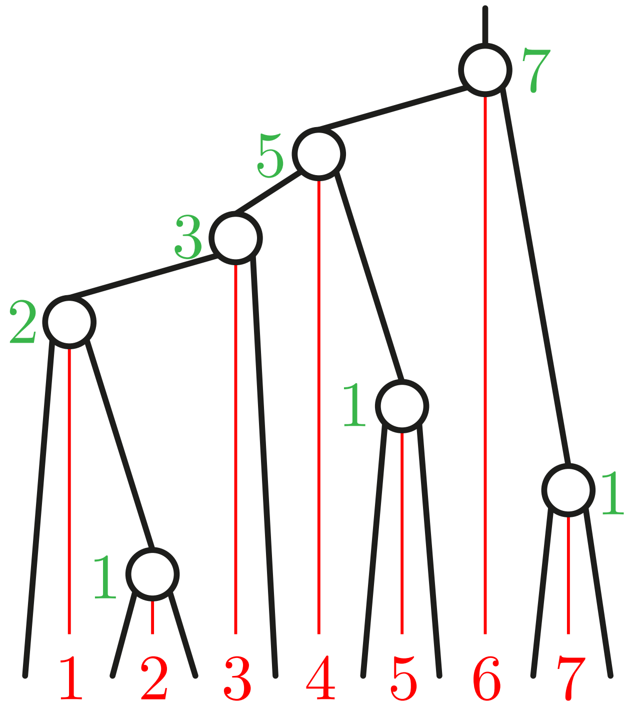

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$$7!/(2\cdot 1\cdot 3\cdot 5\cdot 1\cdot 7\cdot 1) = 24$$

proof: Induction. Let L and R denote the left and right subtrees of T, with ℓ and r nodes. Then the number $\phi(T)$ of linear extensions of T is

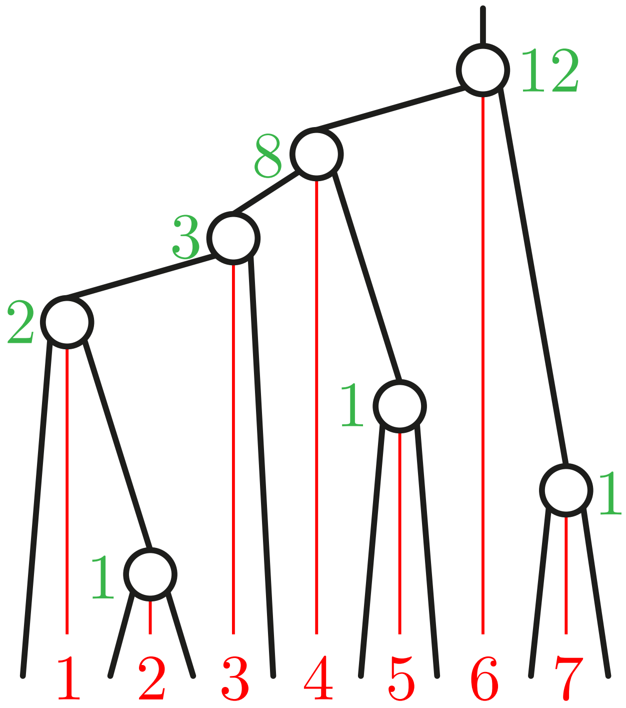
$$\phi(T) = \phi(L)\cdot\phi(R)\cdot\binom{\ell+r}{\ell} = \frac{\ell!}{\prod_{i\in L} n_i}\frac{r!}{\prod_{i\in R} n_i}\frac{(n-1)!}{\ell!r!} = \frac{n!}{\prod_{i\in T} n_i}$$

LODAY'S ASSOCIAHEDRON

DEF. Loday's associahedron

$$\mathbb{A}\mathsf{sso} := \operatorname{conv}\{\boldsymbol{L}(T) \mid T \text{ binary tree}\} = \mathbb{H} \cap \bigcap_{1\le i\le j\le n+1} \mathbb{H}(i,j)$$

$$\boldsymbol{L}(T) := \big[\ell(T,i)\cdot r(T,i)\big]_{i\in[n+1]} \qquad \mathbb{H}(i,j) := \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i\le k\le j} x_i \ge \binom{j-i+2}{2} \right\}$$



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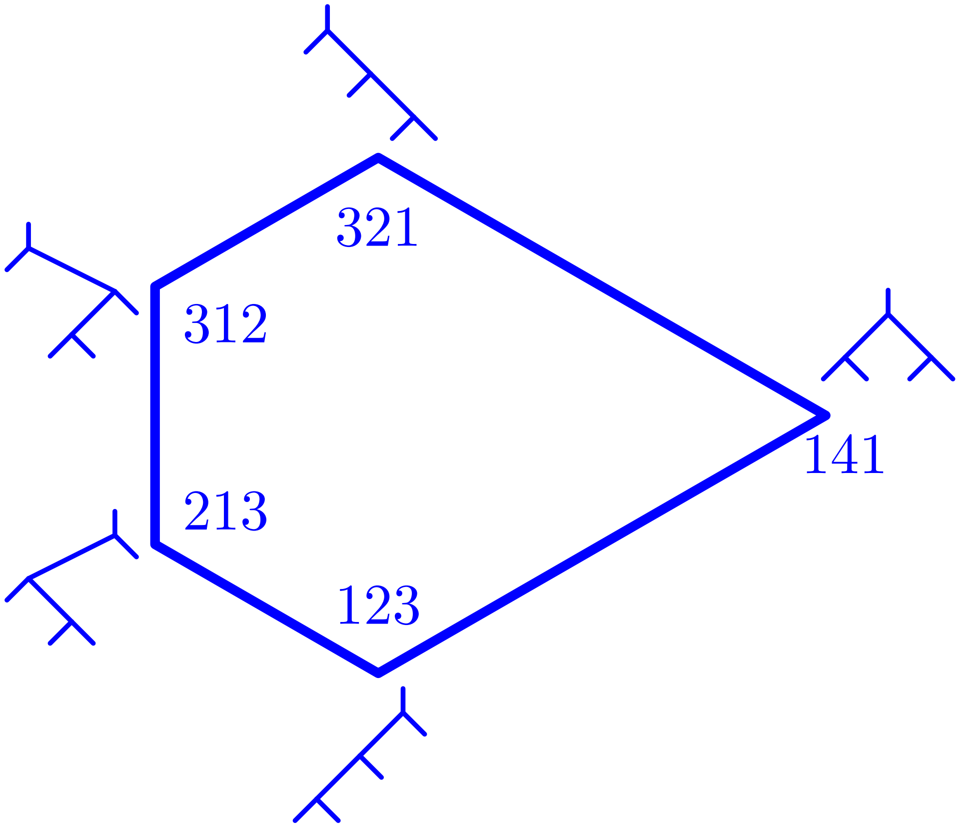

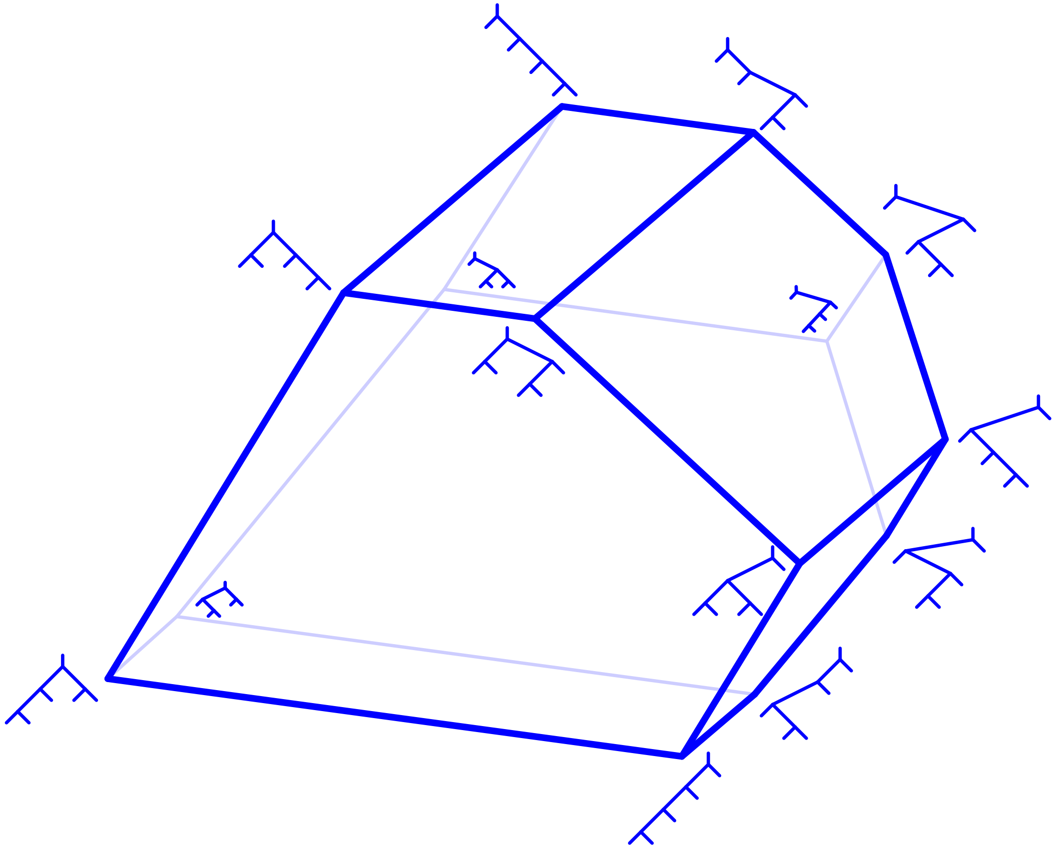

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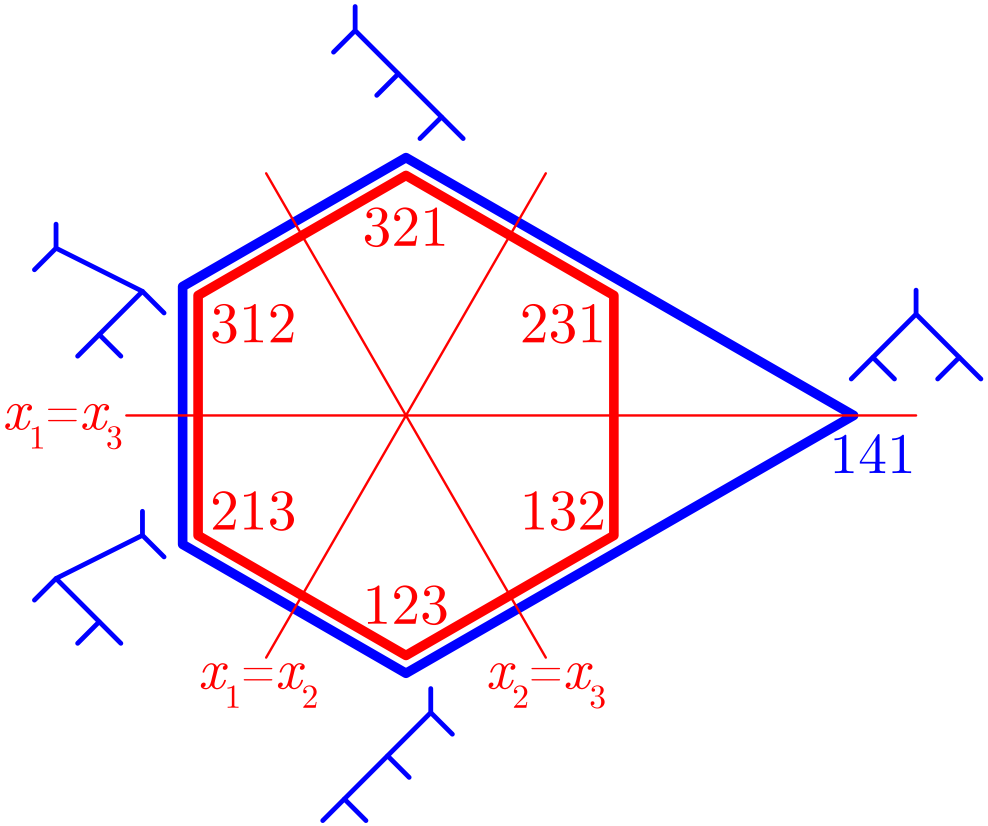
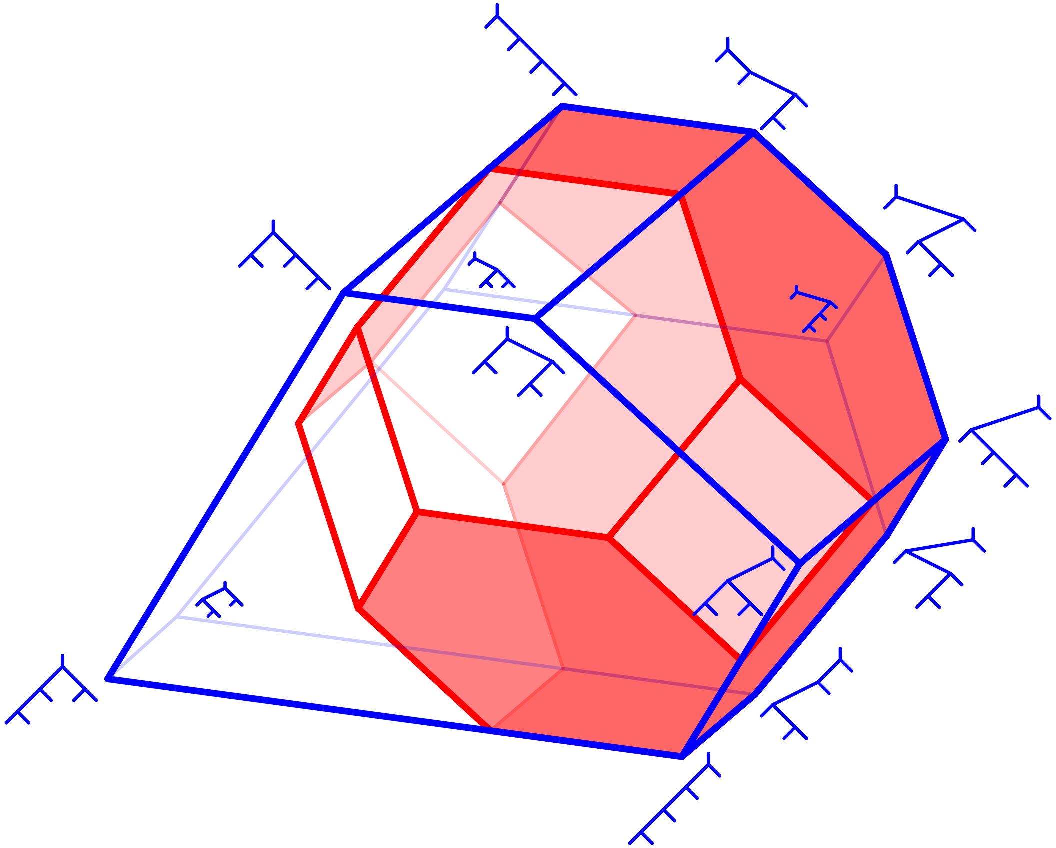
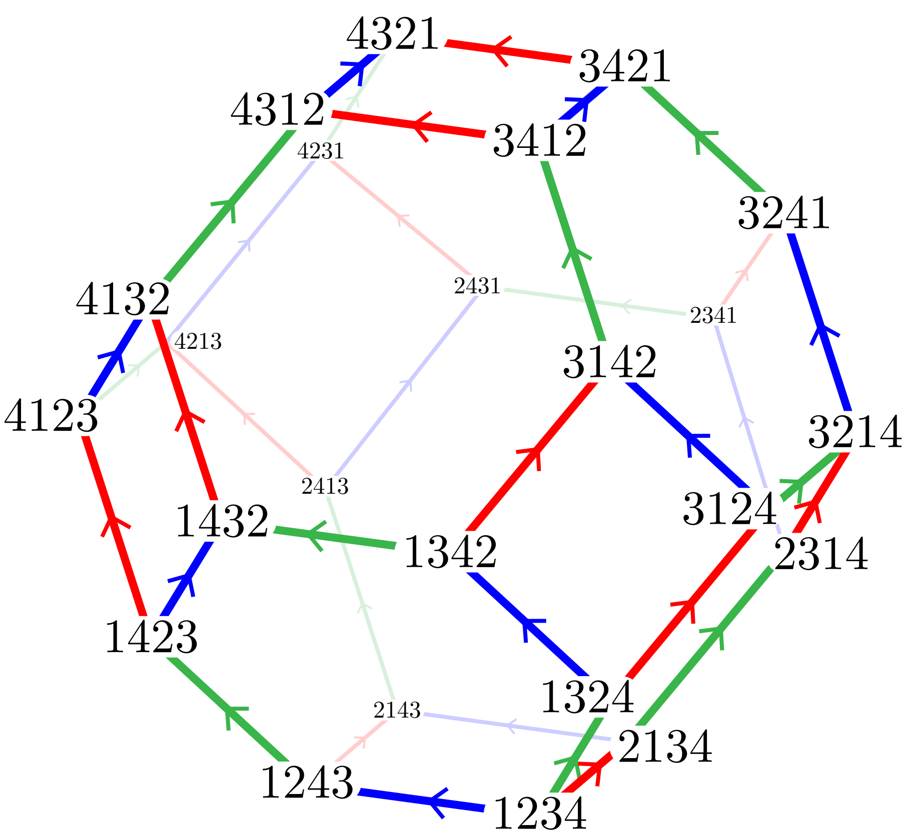
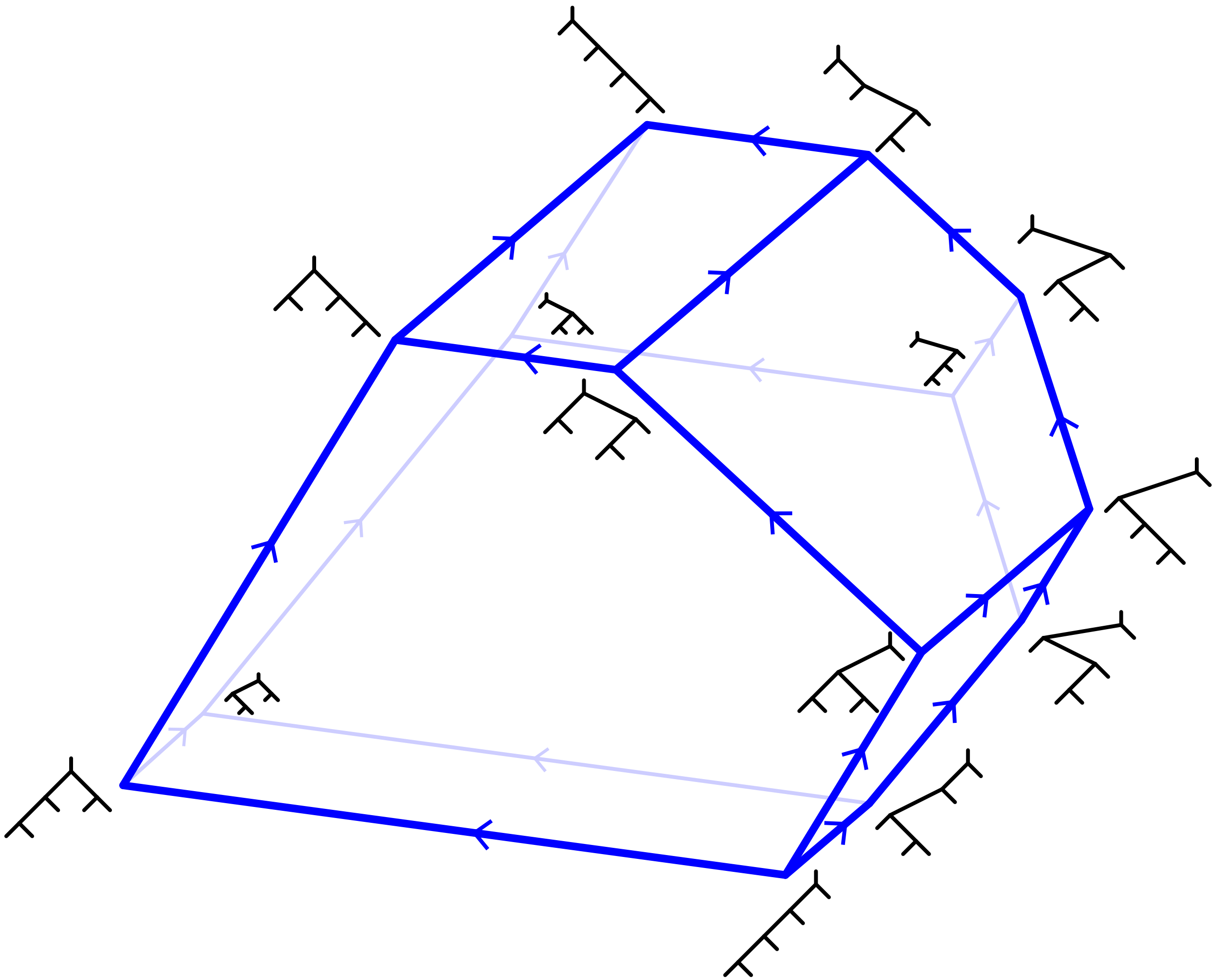




TAMARI LATTICE

DEF. Tamari lattice = right rotations on binary trees
= orientation of the graph of the associahedron
= quotient of the weak order by the sylvester congruence





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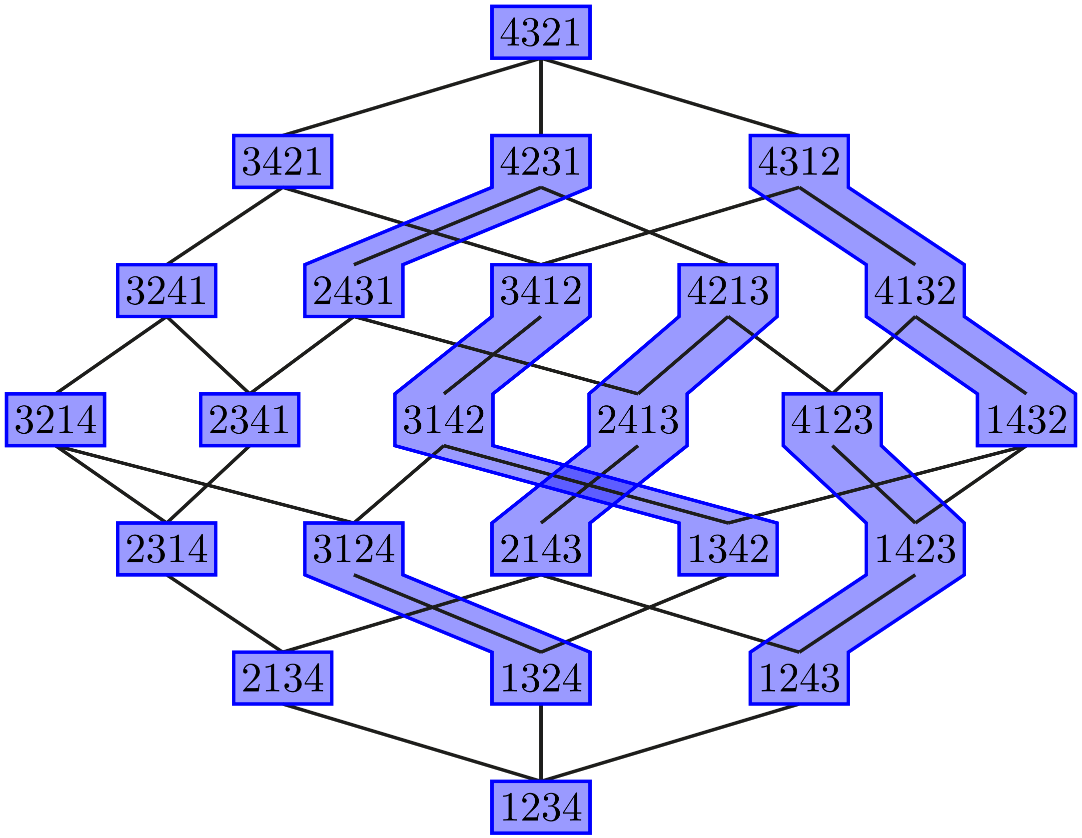

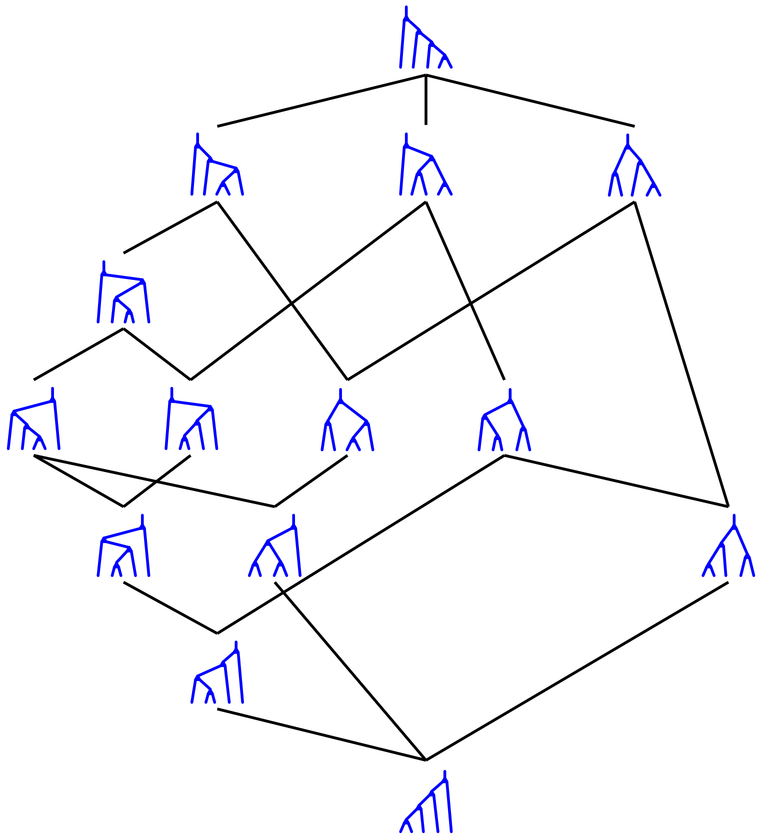

THANKS

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Contact me for internship ideas.