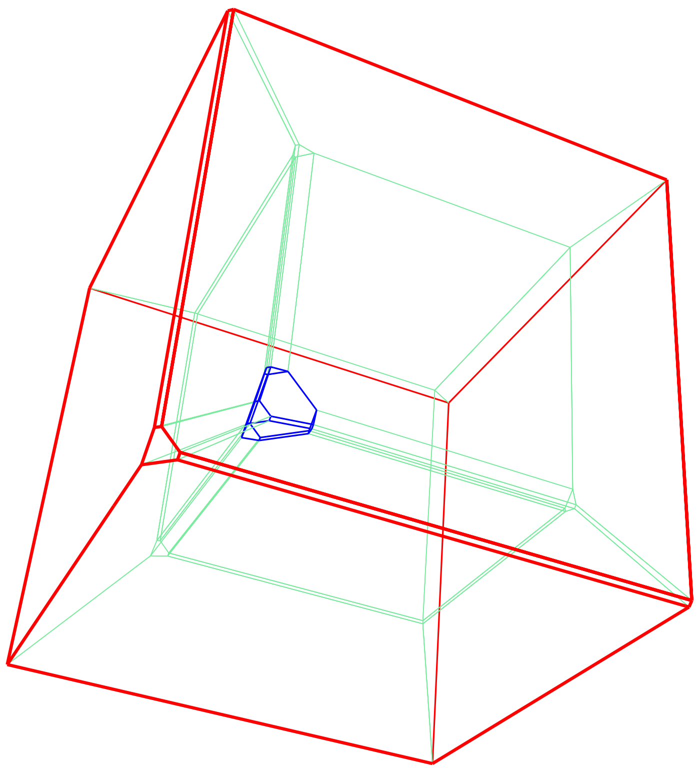
Polytopes



V. PILAUD

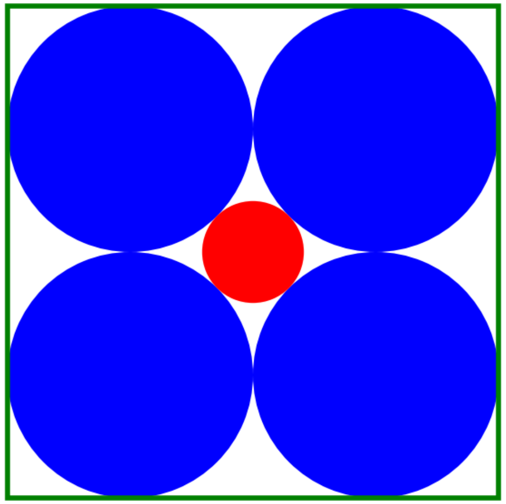
MPRI 2-38-1. Algorithms and combinatorics for geometric graphs

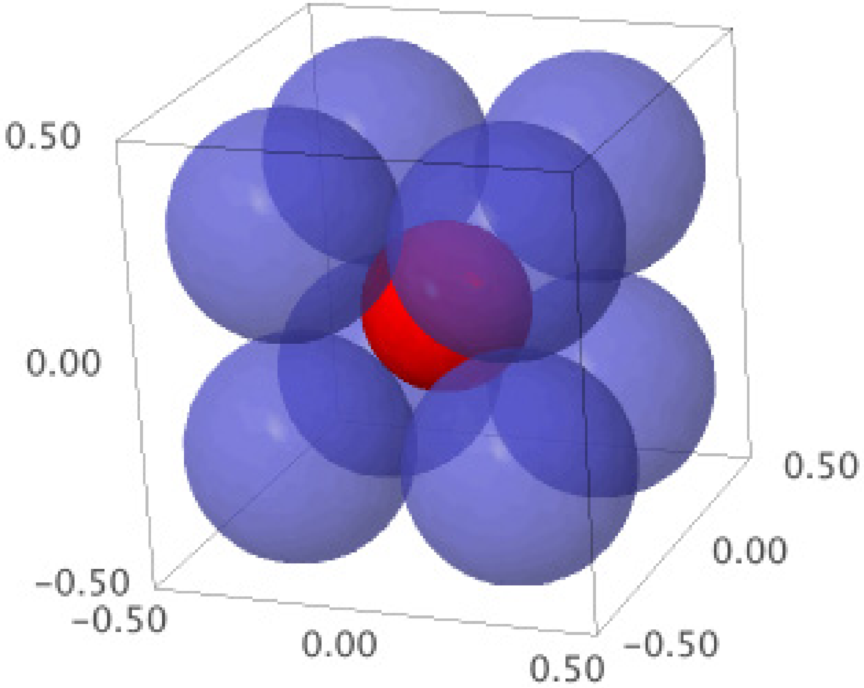
Fridays September 30th & October 7th, 2022

slides available at: http://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/MPRI-2-38-1-VP-3.pdf

Course notes available at: https://www.lix.polytechnique.fr/~pilaud/enseignement/MPRI/notesCoursMPRI21.pdf

BOULES DE PETANQUE & COCHONET

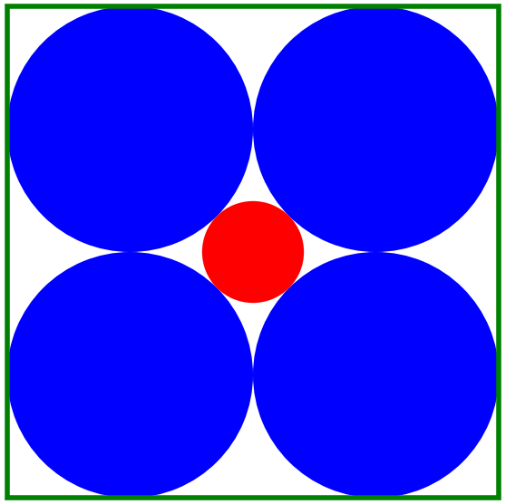


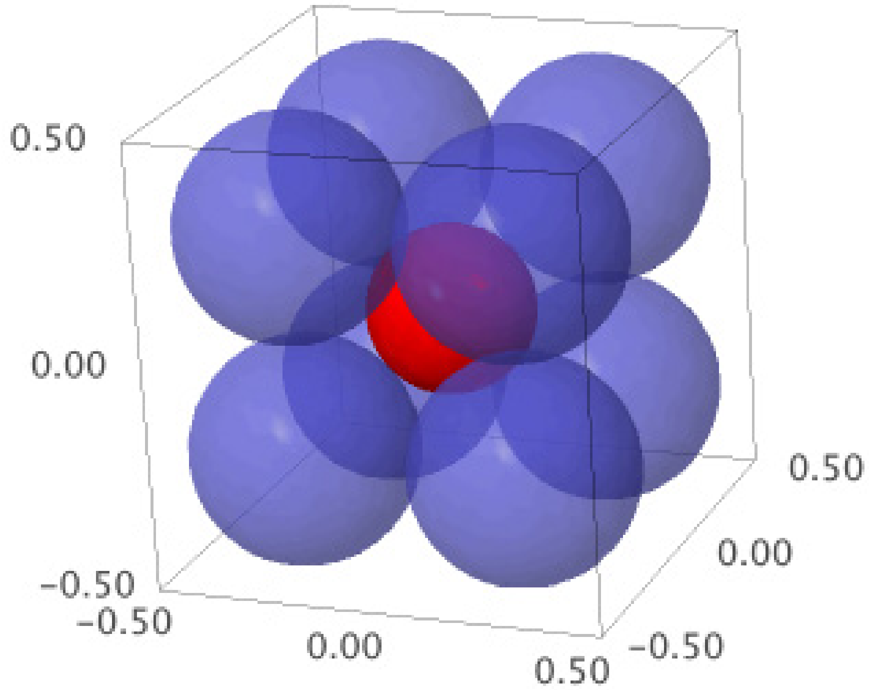


DEF. Pétanque = ... long story ... played with balls (blue) and a cochonet (red).

QU. What is the diameter of the cochonet ? and in dimension d? and in dimension 10?

COCHONET PARADOX

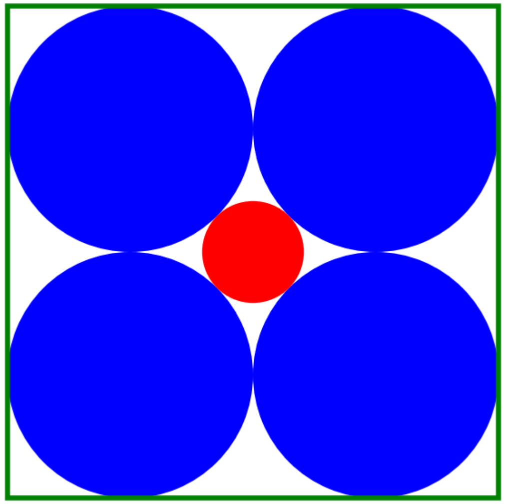


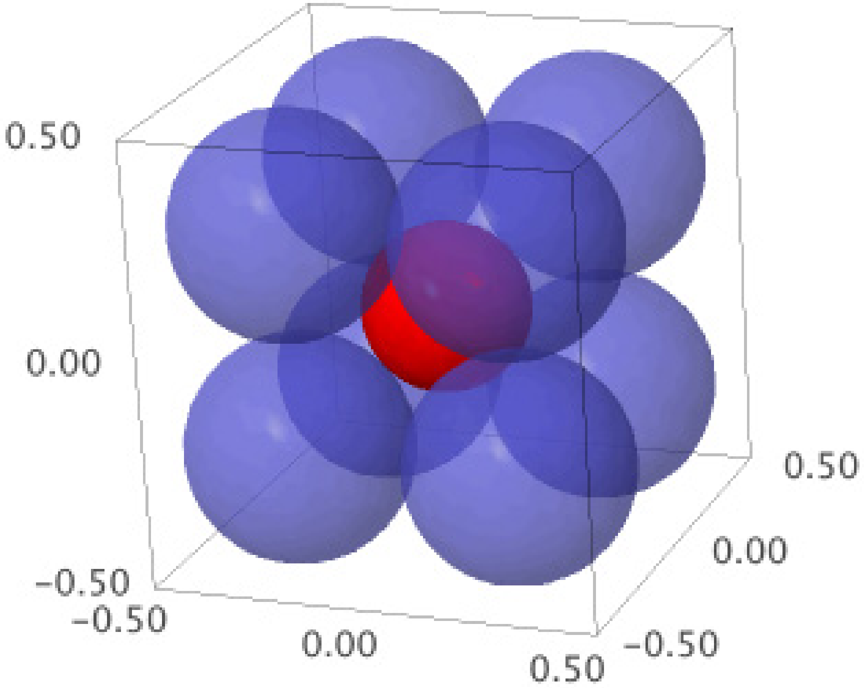


dimension d	1	2	3	...	9	10	11	...
diameter $= (\sqrt{d}-1)/2$	0	0.207	0.366	...	1	1.08	1.16	...
volume $= \dfrac{\left(\Gamma(1/2)\cdot(\sqrt{d}-1)/4\right)^d}{\Gamma(d/2+1)}$	0	0.0337	0.0257	...	0.00644	0.00543	0.00463	...

REM. In dimension ≥ 10, the cochonet is out of the box!!

COCHONET PARADOX





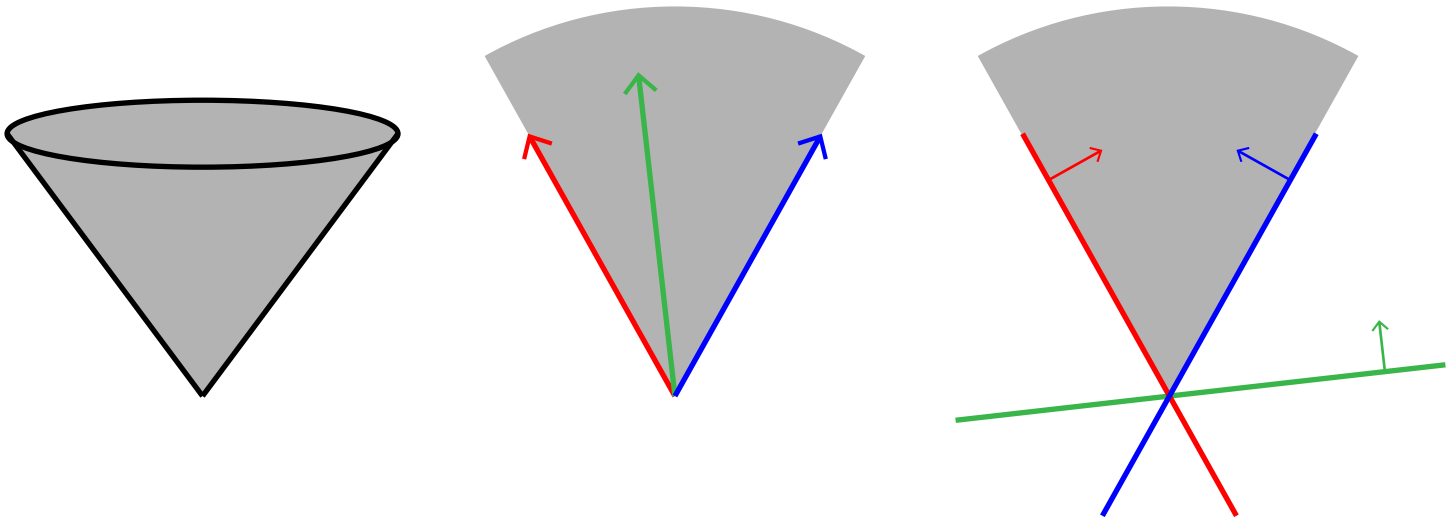
In high dimension, intuition is wrong, computations are correct.

POLYHEDRAL CONES

CONES

DEF. $\mathbb{C} \subseteq \mathbb{R}^n$ convex cone $\iff$ $\mu \boldsymbol{u} + \nu \boldsymbol{v} \in \mathbb{C}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}$ and $\mu, \nu \in \mathbb{R}_{\geq 0}$.

DEF. dimension of $\mathbb{C}$ = dimension of its linear span.



DEF. $\mathcal{V}$-cone = convex cone generated by finitely many vectors

$$= \left\{ \sum_{\boldsymbol{u} \in \boldsymbol{U}} \mu_{\boldsymbol{u}} \boldsymbol{u} \;\middle|\; \mu_{\boldsymbol{u}} \geq 0 \text{ for all } \boldsymbol{u} \in \boldsymbol{U} \right\} \text{ for some finite } \boldsymbol{U}.$$

DEF. $\mathcal{H}$-cone = intersection of finitely many linear halfspaces

$$= \left\{ \boldsymbol{u} \in \mathbb{R}^n \;\middle|\; \langle\, \boldsymbol{u} \mid \boldsymbol{v} \,\rangle \leq 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{V} \right\} \text{ for some finite } \boldsymbol{V}.$$

$\mathcal{V}$-CONES VS $\mathcal{H}$-CONES

THM. (Minkowski-Weyl for cones) $\mathcal{V}$-cone $\iff$ $\mathcal{H}$-cone.

remark: different proofs are possible.

Classical algorithmic proof = Fourier-Motzkin elimination procedure (projections on coordinate hyperplanes).

Here, induction + polarity...

$\mathcal{V}$-CONES VS $\mathcal{H}$-CONES

THM. (Minkowski-Weyl for cones) $\mathcal{V}$-cone $\iff$ $\mathcal{H}$-cone.

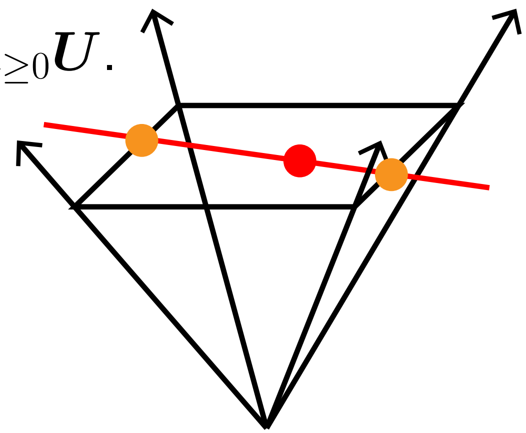
proof: $\mathcal{H}$-cone $\Longrightarrow$ $\mathcal{V}$-cone by induction on the dimension.

Consider an $\mathcal{H}$-cone $\mathbb{C} = \left\{ \boldsymbol{u} \in \mathbb{R}^n \;\middle|\; \langle\, \boldsymbol{u} \mid \boldsymbol{v} \,\rangle \le 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{V} \right\}$.

It is clearly a $\mathcal{V}$-cone if $\dim(\mathbb{C}) = 0$ or if $\boldsymbol{V}$ does not contain two independent vectors.
Otherwise, there exist $\boldsymbol{v}, \boldsymbol{v}'$ in $\boldsymbol{V}$ and $\boldsymbol{w} \in \mathbb{R}^n$ st $\langle\, \boldsymbol{w} \mid \boldsymbol{v} \,\rangle \le 0$ and $\langle\, \boldsymbol{w} \mid \boldsymbol{v}' \,\rangle \ge 0$
(consider $\boldsymbol{w} = \langle\, \boldsymbol{v} \mid \boldsymbol{v}' \,\rangle \boldsymbol{v} + \langle\, \boldsymbol{v}' \mid \boldsymbol{v}' \,\rangle \boldsymbol{v} - \langle\, \boldsymbol{v} \mid \boldsymbol{v}' \,\rangle \boldsymbol{v}' - \langle\, \boldsymbol{v} \mid \boldsymbol{v} \,\rangle \boldsymbol{v}'$)

For $\boldsymbol{v} \in \boldsymbol{V}$, define $\mathbb{C}_{\boldsymbol{v}} = \mathbb{C} \cap \boldsymbol{v}^{\perp}$.
By induction, the $\mathcal{H}$-cone $\mathbb{C}_{\boldsymbol{v}}$ is the $\mathcal{V}$-cone generated by some finite set $\boldsymbol{U}_{\boldsymbol{v}}$.
We claim that the $\mathcal{H}$-cone $\mathbb{C}$ is the $\mathcal{V}$-cone generated by the finite set $\boldsymbol{U} = \bigcup_{\boldsymbol{v} \in \boldsymbol{V}} \boldsymbol{U}_{\boldsymbol{v}}$.

Let $\boldsymbol{u} \in \mathbb{C}$.
If $\boldsymbol{u}$ is on the boundary of $\mathbb{C}$, it belongs to some $\mathbb{C}_{\boldsymbol{v}} = \mathbb{R}_{\ge 0} \boldsymbol{U}_{\boldsymbol{v}} \subseteq \mathbb{R}_{\ge 0} \boldsymbol{U}$.
Otherwise, $(\boldsymbol{u} + \mathbb{R}\boldsymbol{w}) \cap \mathbb{C}$ is a segment $[\boldsymbol{u}^+, \boldsymbol{u}^-]$.
There is $\boldsymbol{v}^+, \boldsymbol{v}^- \in \boldsymbol{V}$ st $\boldsymbol{u}^+ \in \mathbb{C}_{\boldsymbol{v}^+}$ and $\boldsymbol{u}^- \in \mathbb{C}_{\boldsymbol{v}^-}$.
Thus $\boldsymbol{u} \in \mathbb{R}_{\ge 0}\{\boldsymbol{u}^+, \boldsymbol{u}^-\} \subseteq \mathbb{R}_{\ge 0}(\boldsymbol{U}_{\boldsymbol{v}^+} \cup \boldsymbol{U}_{\boldsymbol{v}^-}) \subseteq \mathbb{R}_{\ge 0} \boldsymbol{U}$.



$\mathcal{V}$-CONES VS $\mathcal{H}$-CONES

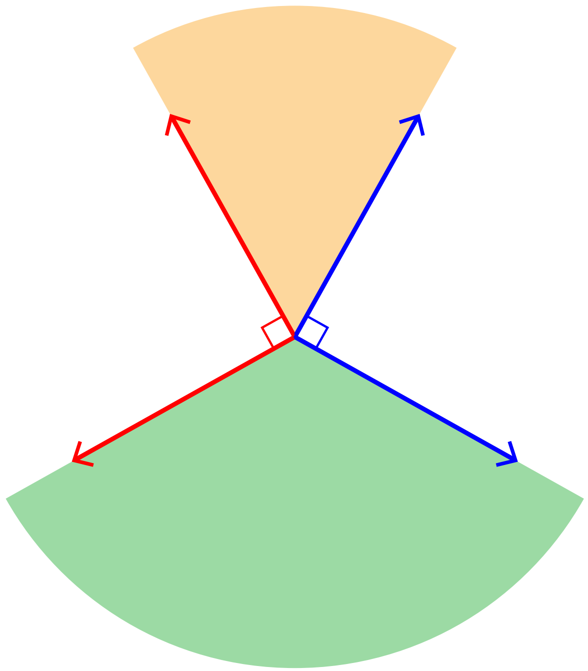
THM. (Minkowski-Weyl for cones) $\mathcal{V}$-cone $\iff$ $\mathcal{H}$-cone.

proof: $\mathcal{V}$-cone $\Longrightarrow$ $\mathcal{H}$-cone by polarity.

DEF. linear polar $\mathbb{U}^\circ = \{\boldsymbol{v} \in \mathbb{R}^n \mid \langle\, \boldsymbol{u} \mid \boldsymbol{v}\,\rangle \le 0 \text{ for all } \boldsymbol{u} \in \mathbb{U}\}$.

PROP. $\mathbb{U}^\circ$ is a closed convex cone. If $\mathbb{U}$ is convex and closed, then $(\mathbb{U}^\circ)^\circ = \mathbb{U}$.

PROP. The polar of a $\mathcal{V}$-cone is an $\mathcal{H}$-cone.



$\mathcal{V}$-CONES VS $\mathcal{H}$-CONES

THM. (Minkowski-Weyl for cones) $\mathcal{V}$-cone $\iff$ $\mathcal{H}$-cone.

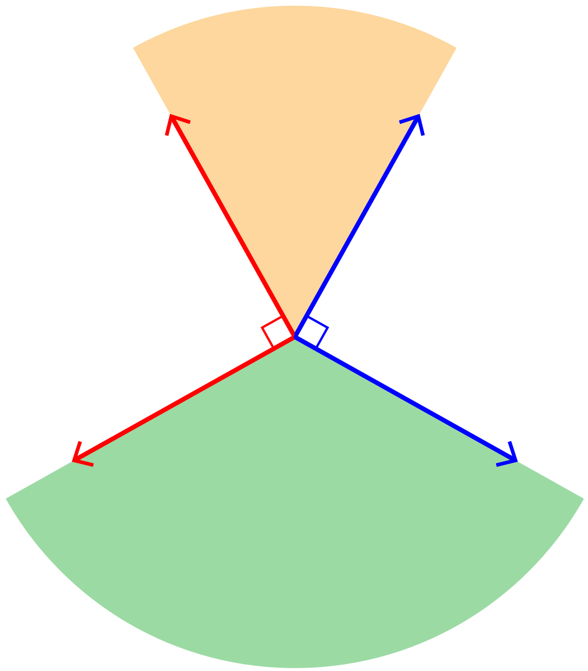
proof: $\mathcal{V}$-cone $\Longrightarrow$ $\mathcal{H}$-cone by polarity.

Consider an $\mathcal{V}$-cone $\mathbb{C}$.
Its polar $\mathbb{C}^\circ$ is an $\mathcal{H}$-cone, thus a $\mathcal{V}$-cone according to the first part of the proof.
Therefore, $\mathbb{C} = (\mathbb{C}^\circ)^\circ$ is an $\mathcal{H}$-cone.

DEF. linear polar $\mathbb{U}^\circ = \{\boldsymbol{v} \in \mathbb{R}^n \mid \langle \boldsymbol{u} \mid \boldsymbol{v} \rangle \leq 0 \text{ for all } \boldsymbol{u} \in \mathbb{U}\}$.

PROP. $\mathbb{U}^\circ$ is a closed convex cone. If $\mathbb{U}$ is convex and closed, then $(\mathbb{U}^\circ)^\circ = \mathbb{U}$.

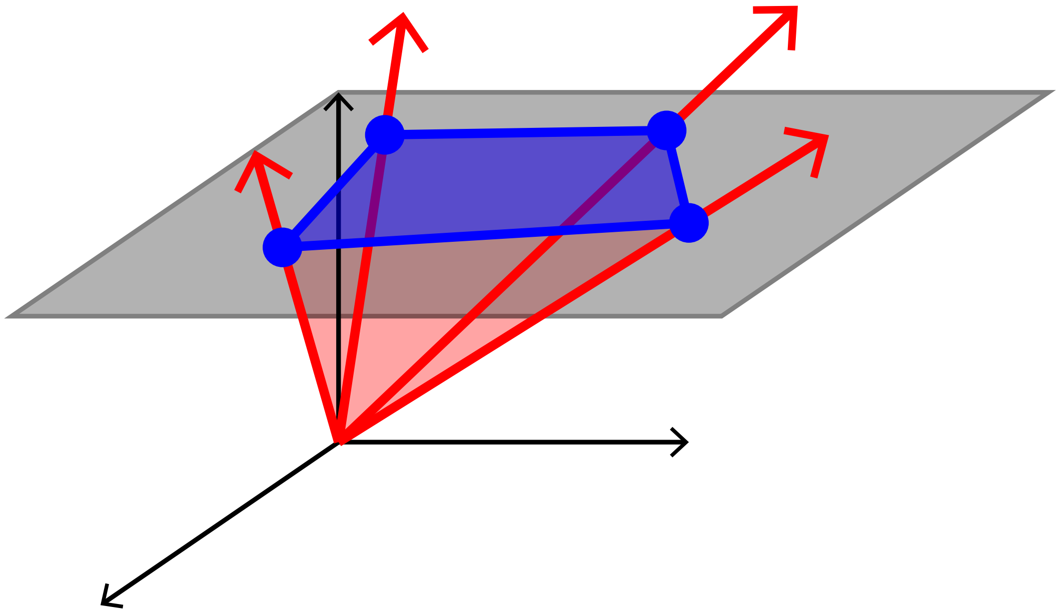
PROP. The polar of a $\mathcal{V}$-cone is an $\mathcal{H}$-cone.



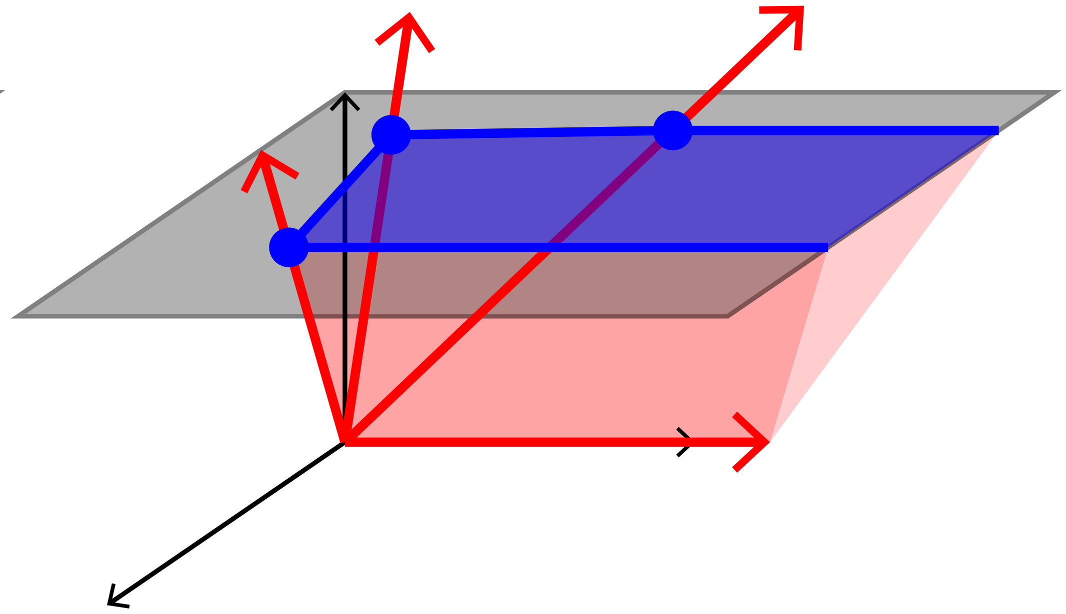
INTERSECTING A CONE BY A HYPERPLANE

DEF. polyhedral cone $= \mathcal{V}$-cone $= \mathcal{H}$-cone.

DEF. polyhedron $=$ intersection of a polyhedral cone by an affine hyperplane.



bounded
$=$ polytope



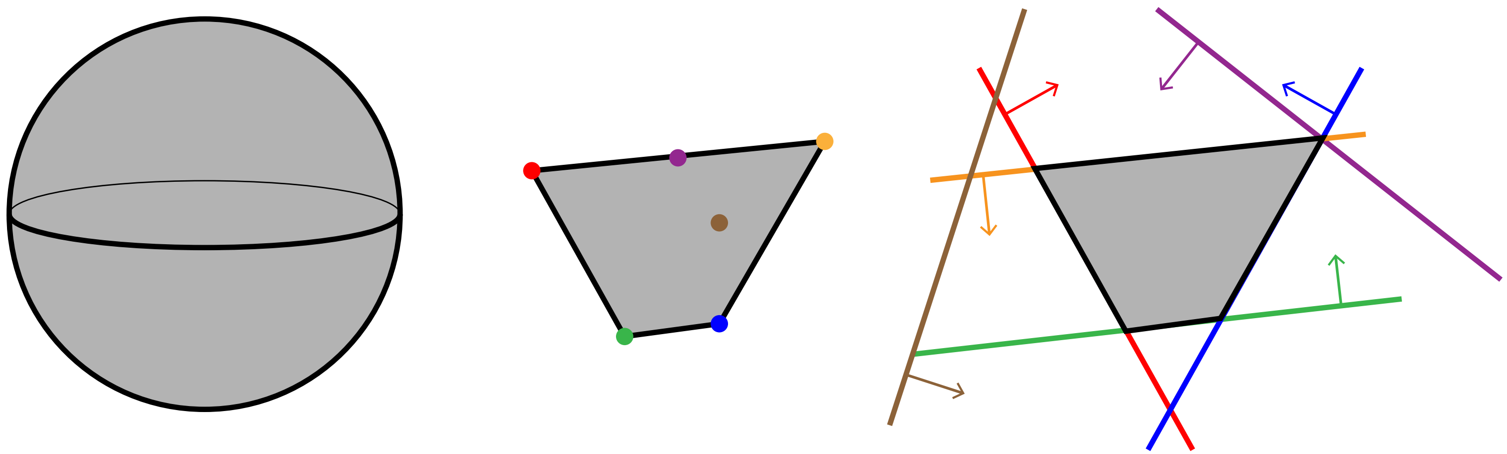
undounded
$=$ polytope $+$ recession cone

POLYTOPES

POLYTOPES

DEF. $\mathbb{P} \subseteq \mathbb{R}^n$ convex $\iff$ $\mu\boldsymbol{x}+\nu\boldsymbol{y} \in \mathbb{P}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{P}$ and $\mu, \nu \in \mathbb{R}_{\geq 0}$ with $\mu+\nu = 1$.

DEF. dimension of $\mathbb{P}$ = dimension of its affine span.



DEF. $\mathcal{V}$-polytope = convex hull of finite point set in $\mathbb{R}^n$

$$= \Big\{ \sum_{\boldsymbol{x}\in\boldsymbol{X}} \mu_{\boldsymbol{x}}\boldsymbol{x} \;\Big|\; \sum_{\boldsymbol{x}\in\boldsymbol{X}} \mu_{\boldsymbol{x}} = 1 \text{ and } \mu_{\boldsymbol{x}} \geq 0 \text{ for all } \boldsymbol{x} \in \boldsymbol{X} \Big\} \text{ for a finite } \boldsymbol{X}.$$

DEF. $\mathcal{H}$-polytope = bounded intersection of finitely many affine halfspaces of $\mathbb{R}^n$

$$= \big\{ \boldsymbol{x} \in \mathbb{R}^n \;\big|\; \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \leq c_{\boldsymbol{y}} \text{ for all } \boldsymbol{y} \in \boldsymbol{Y} \big\} \text{ for a finite } \boldsymbol{Y}.$$

$\mathcal{V}$-POLYTOPES VS $\mathcal{H}$-POLYTOPES

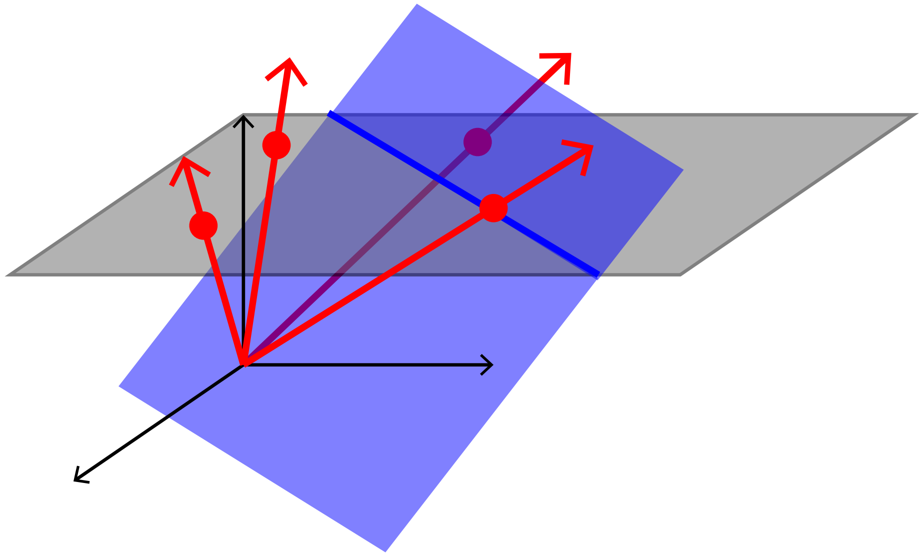
THM. (Minkowski-Weyl for polytopes) $\mathcal{V}$-polytope $\Longleftrightarrow$ $\mathcal{H}$-polytope.

proof: embed the affine space $\mathbb{R}^n$ into the linear space $\mathbb{R}^{n+1}$.

$$\boldsymbol{x} \qquad\qquad \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \le c_{\boldsymbol{y}}$$

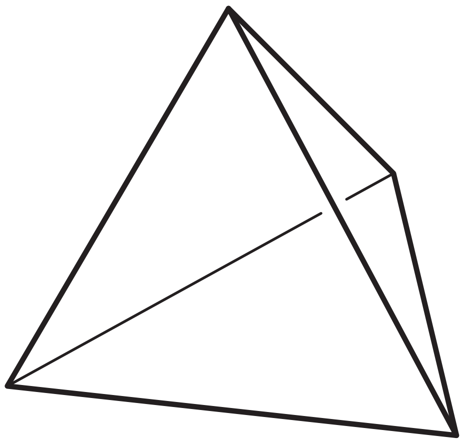
$$\updownarrow \qquad\qquad \updownarrow$$

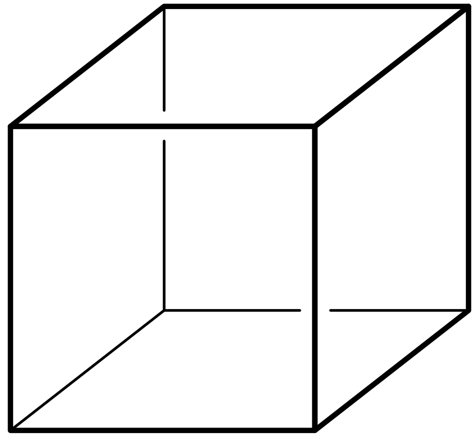
$$\begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} \qquad \left\langle \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} \,\middle|\, \begin{bmatrix} \boldsymbol{y} \\ -c_{\boldsymbol{y}} \end{bmatrix} \right\rangle \le 0$$

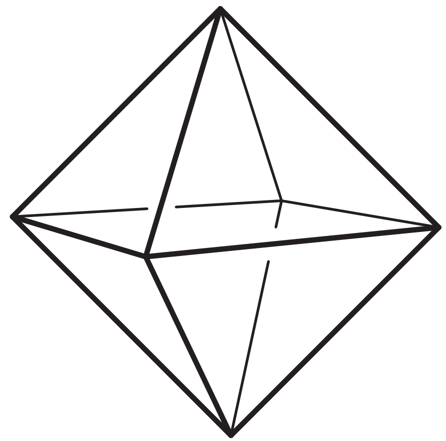


DEF. polytope $=$ $\mathcal{V}$-polytope $=$ $\mathcal{H}$-polytope.

CLASSICAL POLYTOPES







DEF. <u>d-simplex</u> = convex hull of $d+1$ affinely independent points.

<u>standard d-simplex</u> $\triangle_d = \operatorname{conv}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{d+1}\}$

$$= \left\{ \boldsymbol{x} \in \mathbb{R}^{d+1} \;\middle|\; \sum_{i \in [d+1]} x_i = 1 \text{ and } x_i \ge 0 \text{ for all } i \in [d+1] \right\}.$$

DEF. <u>d-cube</u> $\square_d = \operatorname{conv}(\{\pm 1\}^d) = \left\{ \boldsymbol{x} \in \mathbb{R}^d \;\middle|\; -1 \le x_i \le 1 \text{ for all } i \in [d] \right\}.$

DEF. <u>d-cross-pol.</u> $\lozenge_d = \operatorname{conv}\{\pm \boldsymbol{e}_i \mid i \in [d]\} = \left\{ \boldsymbol{x} \in \mathbb{R}^d \;\middle|\; \sum_{i \in [d]} \varepsilon_i x_i \le 1 \text{ for all } \varepsilon \in \{\pm 1\}^d \right\}.$

AFFINE POLARITY

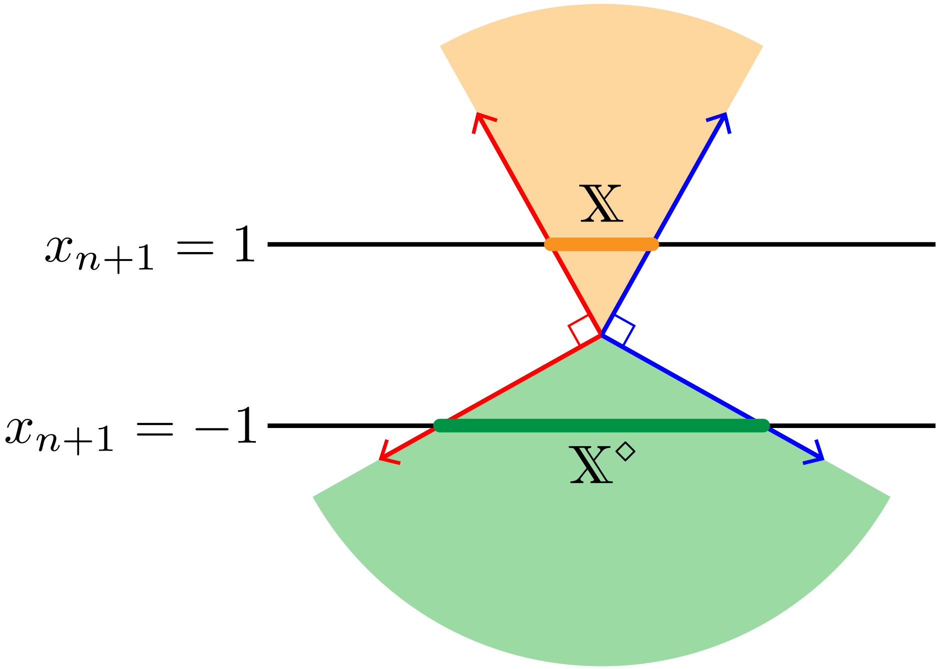
DEF. linear polar $\mathbb{U}^\circ = \left\{ \boldsymbol{v} \in \mathbb{R}^{n+1} \;\middle|\; \langle\, \boldsymbol{u} \mid \boldsymbol{v} \,\rangle \le 0 \text{ for all } \boldsymbol{u} \in \mathbb{U} \right\}$.

DEF. affine polar $\mathbb{X}^\diamond = \{ \boldsymbol{y} \in \mathbb{R}^n \mid \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \le 1 \text{ for all } \boldsymbol{x} \in \mathbb{X} \}$.

$$\langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \le 1$$

$$\updownarrow$$

$$\left\langle \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} \;\middle|\; \begin{bmatrix} \boldsymbol{y} \\ -1 \end{bmatrix} \right\rangle \le 0$$

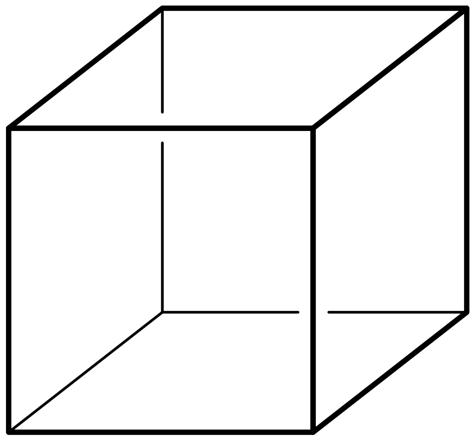


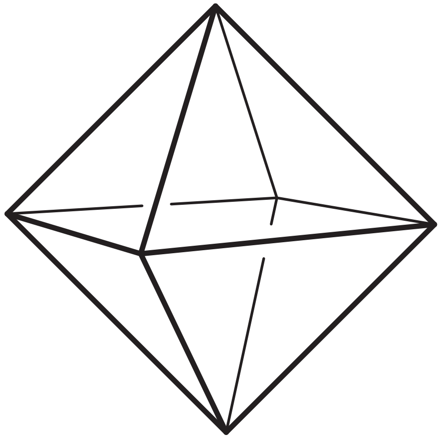
PROP. $\mathbb{X}^\diamond$ is closed and convex, and bounded iff $\mathbf{0} \in \operatorname{int}(\mathbb{X})$. If $\mathbb{X}$ is closed, convex and contains $\mathbf{0}$, then $(\mathbb{X}^\diamond)^\diamond = \mathbb{X}$.

POLAR POLYTOPE

DEF. affine polar $\mathbb{X}^\diamond = \{\boldsymbol{y} \in \mathbb{R}^n \mid \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \le 1 \text{ for all } \boldsymbol{x} \in \mathbb{X}\}$.

PROP. Assume $\mathbf{0} \in \operatorname{int}(\mathbb{P})$.
If $\mathbb{P} = \operatorname{conv}(\boldsymbol{X}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \le 1 \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}\}$,
then $\mathbb{P}^\diamond = \operatorname{conv}(\boldsymbol{Y}) = \{\boldsymbol{y} \in \mathbb{R}^n \mid \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \le 1 \text{ for all } \boldsymbol{x} \in \boldsymbol{X}\}$.





EXM. d-cube $\square_d = \operatorname{conv}(\{\pm 1\}^d) = \left\{\boldsymbol{x} \in \mathbb{R}^d \;\middle|\; -1 \le x_i \le 1 \text{ for all } i \in [d]\right\}$.

d-cross-pol. $\lozenge_d = \operatorname{conv}\{\pm \boldsymbol{e}_i \mid i \in [d]\} = \left\{\boldsymbol{x} \in \mathbb{R}^d \;\middle|\; \sum_{i \in [d]} \varepsilon_i x_i \le 1 \text{ for all } \varepsilon \in \{\pm 1\}^d\right\}$.

EXM: MATCHING POLYTOPES

DEF. $G = (V, E)$ graph.
matching on G = subset of E with at most one edge incident to each vertex.
matching polytope $\mathbb{M}(G)$ = convex hull of the characteristic vectors $\chi_M \in \mathbb{R}^E$ of all matchings M on G.

QU. Consider the polytope $\mathbb{N}(G)$ defined by

$$x_e \geq 0 \quad \text{for all } e \in E, \qquad \text{and} \qquad \sum_{e \ni v} x_e \leq 1 \quad \text{for all } v \in V.$$

- Show that $\mathbb{M}(G) \subseteq \mathbb{N}(G)$.
- Give an example where this inclusion is strict.
- Show that $\mathbb{M}(G) = \mathbb{N}(G)$ when G is bipartite.

EXM: MATCHING POLYTOPES

DEF. $G = (V, E)$ graph.
matching on G = subset of E with at most one edge incident to each vertex.
matching polytope $\mathbb{M}(G)$ = convex hull of the characteristic vectors $\chi_M \in \mathbb{R}^E$ of all matchings M on G.

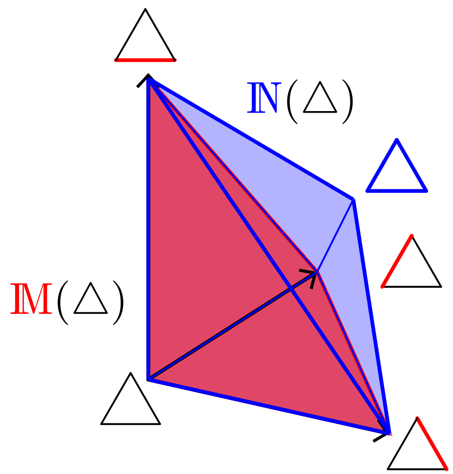
PROP. The matching polytope $\mathbb{M}(G)$ is contained in the polytope $\mathbb{N}(G)$ defined by

$$x_e \geq 0 \quad \text{for all } e \in E, \qquad \text{and} \qquad \sum_{e \ni v} x_e \leq 1 \quad \text{for all } v \in V,$$

and $\mathbb{M}(G) = \mathbb{N}(G)$ when G is bipartite.

proof: $\mathbb{M}(G) \subseteq \mathbb{N}(G)$ as $(\chi_M)_e \geq 0$ and $\sum_{e \ni v} (\chi_M)_e \leq 1$ (at most one edge per vertex).
Strict inclusion in general:

$\mathbb{M}(\triangle) = \operatorname{conv}\{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$
$\mathbb{N}(\triangle) = \operatorname{conv}\{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, (\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3)/2\}$



EXM: MATCHING POLYTOPES

DEF. $G = (V, E)$ graph.
matching on G = subset of E with at most one edge incident to each vertex.
matching polytope $\mathbb{M}(G)$ = convex hull of the characteristic vectors $\chi_M \in \mathbb{R}^E$ of all matchings M on G.

PROP. The matching polytope $\mathbb{M}(G)$ is contained in the polytope $\mathbb{N}(G)$ defined by

$$x_e \geq 0 \quad \text{for all } e \in E, \qquad \text{and} \qquad \sum_{e \ni v} x_e \leq 1 \quad \text{for all } v \in V,$$

and $\mathbb{M}(G) = \mathbb{N}(G)$ when G is bipartite.

proof: $\mathbb{M}(G) \subseteq \mathbb{N}(G)$ as $(\chi_M)_e \geq 0$ and $\sum_{e \ni v} (\chi_M)_e \leq 1$ (at most one edge per vertex).
Assume now that G is bipartite, so that all its cycles are even.
For $\boldsymbol{x} \in \mathbb{N}(G)$, let $U(\boldsymbol{x}) = \{e \in E \mid 0 < \boldsymbol{x}_e < 1\}$.
If $U(\boldsymbol{x}) \neq \varnothing$, it contains a cycle $C = e_1, \dots, e_{2p}$, which is even since G is bipartite.
Let $\lambda = \min \{\boldsymbol{x}_e \mid e \in C\} \cup \{1 - \boldsymbol{x}_e \mid e \in C\}$.
Then $\boldsymbol{x}$ is in the middle of $\boldsymbol{x} + \lambda\chi_C$ and $\boldsymbol{x} - \lambda\chi_C$, which both belong to $\mathbb{N}(G)$.
Therefore, all vertices of $\mathbb{N}(G)$ belong to $\{0, 1\}^E$, and thus $\mathbb{M}(G) = \mathbb{N}(G)$.

OPERATIONS ON POLYTOPES

CARTESIAN PRODUCT

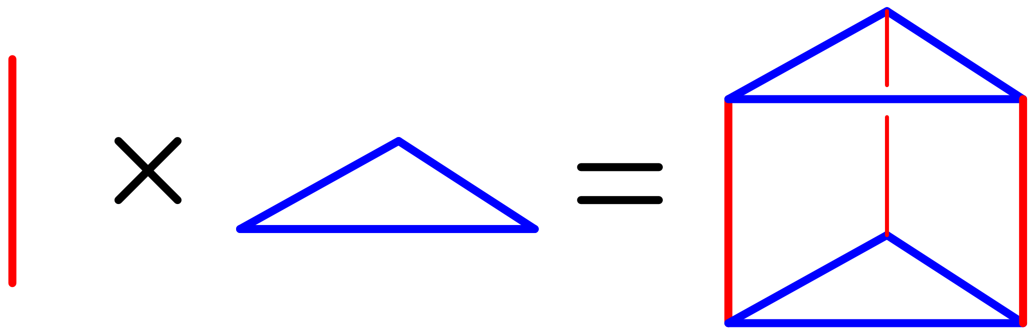
DEF. $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{X}' \subseteq \mathbb{R}^{n'}$.
Cartesian product $\mathbb{X} \times \mathbb{X}' = \{(\boldsymbol{x}, \boldsymbol{x'}) \mid \boldsymbol{x} \in \mathbb{X} \text{ and } \boldsymbol{x'} \in \mathbb{X}'\} \subseteq \mathbb{R}^{n+n'}$.

PROP. The Cartesian product $\mathbb{P} \times \mathbb{P}'$ of two polytopes $\mathbb{P}$ and $\mathbb{P}'$ is a polytope. Moreover

$$\begin{aligned} \mathbb{P} \times \mathbb{P}' &= \operatorname{conv}(\boldsymbol{X} \times \boldsymbol{X'}) \\ &= \left\{ (\boldsymbol{x}, \boldsymbol{x'}) \in \mathbb{R}^{n+n'} \;\middle|\; \begin{array}{l} \langle\, (\boldsymbol{x}, \boldsymbol{x'}) \mid (\boldsymbol{y}, \boldsymbol{0}) \,\rangle \le c_{\boldsymbol{y}} \text{ for all } \boldsymbol{y} \in \boldsymbol{Y} \\ \langle\, (\boldsymbol{x}, \boldsymbol{x'}) \mid (\boldsymbol{0}, \boldsymbol{y'}) \,\rangle \le c_{\boldsymbol{y'}} \text{ for all } \boldsymbol{y'} \in \boldsymbol{Y'} \end{array} \right\} \end{aligned}$$

where $\mathbb{P} = \operatorname{conv}(\boldsymbol{X}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \le c_{\boldsymbol{y}} \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}\}$.
and $\mathbb{P}' = \operatorname{conv}(\boldsymbol{X'}) = \left\{\boldsymbol{x'} \in \mathbb{R}^{n'} \;\middle|\; \langle\, \boldsymbol{x'} \mid \boldsymbol{y'} \,\rangle \le c_{\boldsymbol{y'}} \text{ for all } \boldsymbol{y'} \in \boldsymbol{Y'}\right\}$.

exm:
cube: $\square_d = [-1, 1]^d$
prism: $\mathbb{P}\mathrm{rism}(\mathbb{P}) = [-1, 1] \times \mathbb{P}$



DIRECT SUM

DEF. $\mathbb{P} \subset \mathbb{R}^n$ and $\mathbb{P}' \subset \mathbb{R}^{n'}$ two polytopes with $\mathbf{0} \in \operatorname{int} \mathbb{P}$ and $\mathbf{0} \in \operatorname{int} \mathbb{P}'$.
direct sum $\mathbb{P} \oplus \mathbb{P}' = \operatorname{conv}\big(\{(\boldsymbol{x}, \mathbf{0}) \mid \boldsymbol{x} \in \mathbb{P}\} \cup \{(\mathbf{0}, \boldsymbol{x}') \mid \boldsymbol{x}' \in \mathbb{P}'\} \big) \subset \mathbb{R}^{n+n'}$

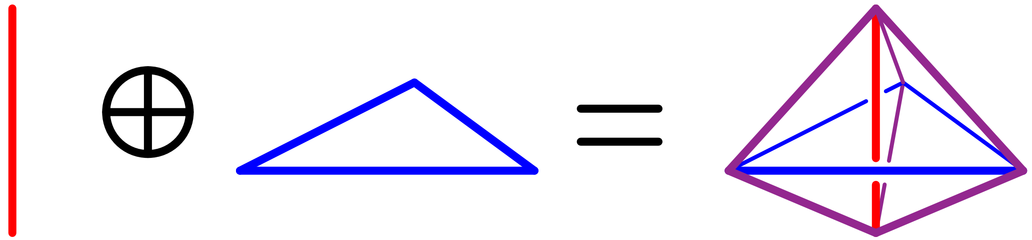
PROP. $\mathbb{P} \oplus \mathbb{P}' = \operatorname{conv}\big(\{(\boldsymbol{x}, \mathbf{0}) \mid \boldsymbol{x} \in \boldsymbol{X}\} \cup \{(\mathbf{0}, \boldsymbol{x}') \mid \boldsymbol{x}' \in \boldsymbol{X}'\} \big)$
$= \big\{ (\boldsymbol{x}, \boldsymbol{x}') \in \mathbb{R}^{n+n'} \;\big|\; \langle\, (\boldsymbol{x}, \boldsymbol{x}') \mid (\boldsymbol{y}, \boldsymbol{y}') \,\rangle \le 1 \text{ for all } \boldsymbol{y} \in \boldsymbol{Y} \text{ and } \boldsymbol{y}' \in \boldsymbol{Y}' \big\}$

where $\mathbb{P} = \operatorname{conv}(\boldsymbol{X}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle \le 1 \text{ for all } \boldsymbol{y} \in \boldsymbol{Y}\}$.
and $\mathbb{P}' = \operatorname{conv}(\boldsymbol{X}') = \big\{\boldsymbol{x}' \in \mathbb{R}^{n'} \;\big|\; \langle\, \boldsymbol{x}' \mid \boldsymbol{y}' \,\rangle \le 1 \text{ for all } \boldsymbol{y}' \in \boldsymbol{Y}'\big\}$.

exm:

cross-poly.: $\lozenge_d = [-1, 1] \oplus \cdots \oplus [-1, 1]$

bipyramid: $\mathbb{B}\mathrm{ipyr}(\mathbb{P}) = [-1, 1] \oplus \mathbb{P}$



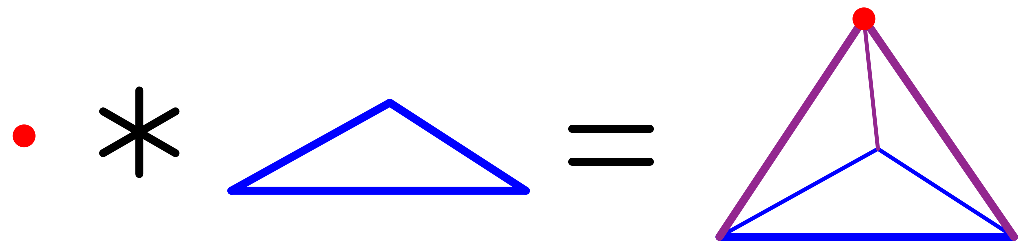
PROP. $(\mathbb{P} \oplus \mathbb{P}')^{\diamond} = \mathbb{P}^{\diamond} \times \mathbb{P}'^{\diamond}$.

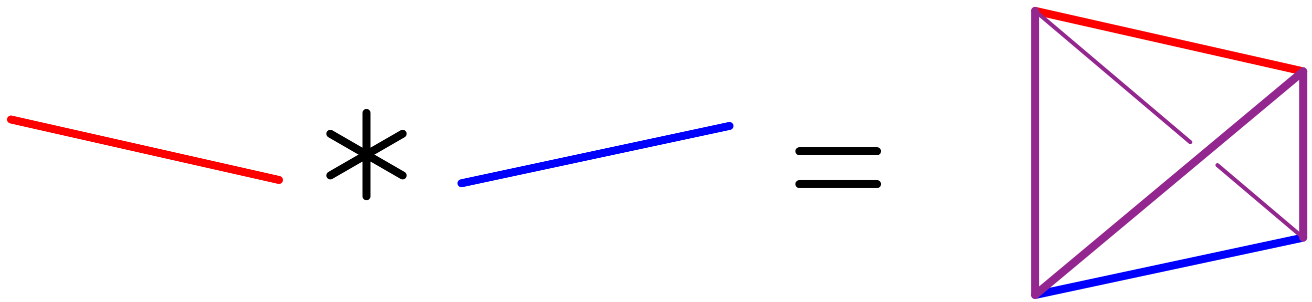
JOIN

DEF. $\mathbb{P} \subset \mathbb{R}^n$ and $\mathbb{P}' \subset \mathbb{R}^{n'}$ two polytopes.
join $\mathbb{P} * \mathbb{P}'$ = convex hull of $\mathbb{P}$ and $\mathbb{P}'$ in independent affine subspaces

$$= \operatorname{conv}\Big(\{(\boldsymbol{x}, \mathbf{0}, 1) \mid \boldsymbol{x} \in \mathbb{P}\} \cup \{(\mathbf{0}, \boldsymbol{x'}, -1) \mid \boldsymbol{x'} \in \mathbb{P}'\}\Big) \subset \mathbb{R}^{n+n'+1}$$

exm:
simplex: $\triangle_d = \triangle_i * \triangle_{d-i}$
pyramid: $\mathbb{P}\mathrm{yr}(\mathbb{P})$ = point $* \mathbb{P}$
k-fold pyramid: $\mathbb{P}\mathrm{yr}_k(\mathbb{P}) = \triangle_{k-1} * \mathbb{P}$

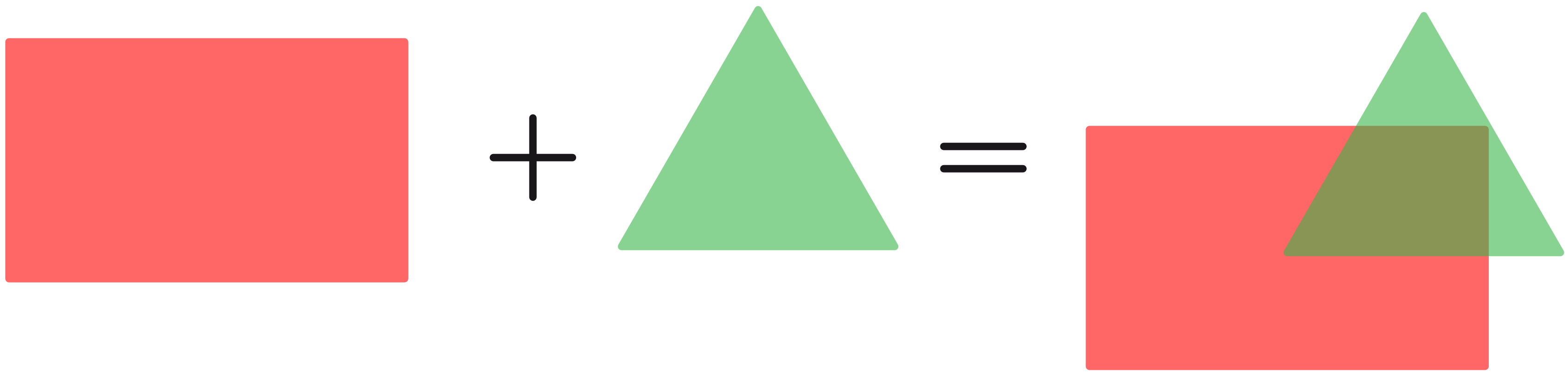




MINKOWSKI SUM

DEF. $\mathbb{X}, \mathbb{X}' \subseteq \mathbb{R}^n$ (same space!).
Minkowski sum $\mathbb{X} + \mathbb{X}' = \{\boldsymbol{x} + \boldsymbol{x'} \mid \boldsymbol{x} \in \mathbb{X} \text{ and } \boldsymbol{x'} \in \mathbb{X}'\} \subseteq \mathbb{R}^n$.

PROP. The Minkowski sum $\mathbb{P} + \mathbb{P}'$ of two polytopes $\mathbb{P}$ and $\mathbb{P}'$ is a polytope.



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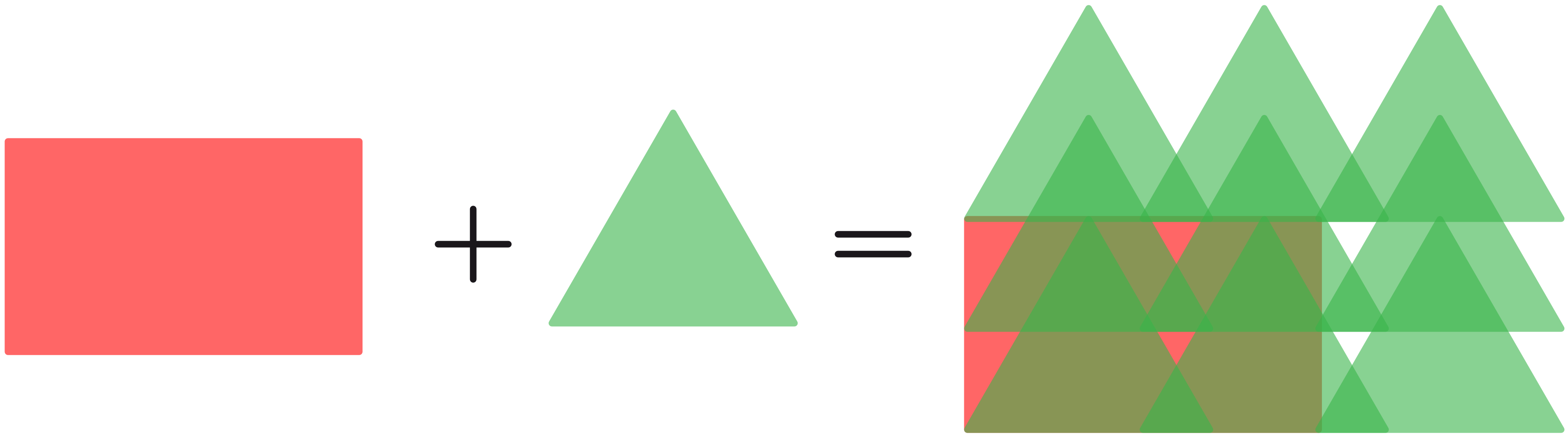

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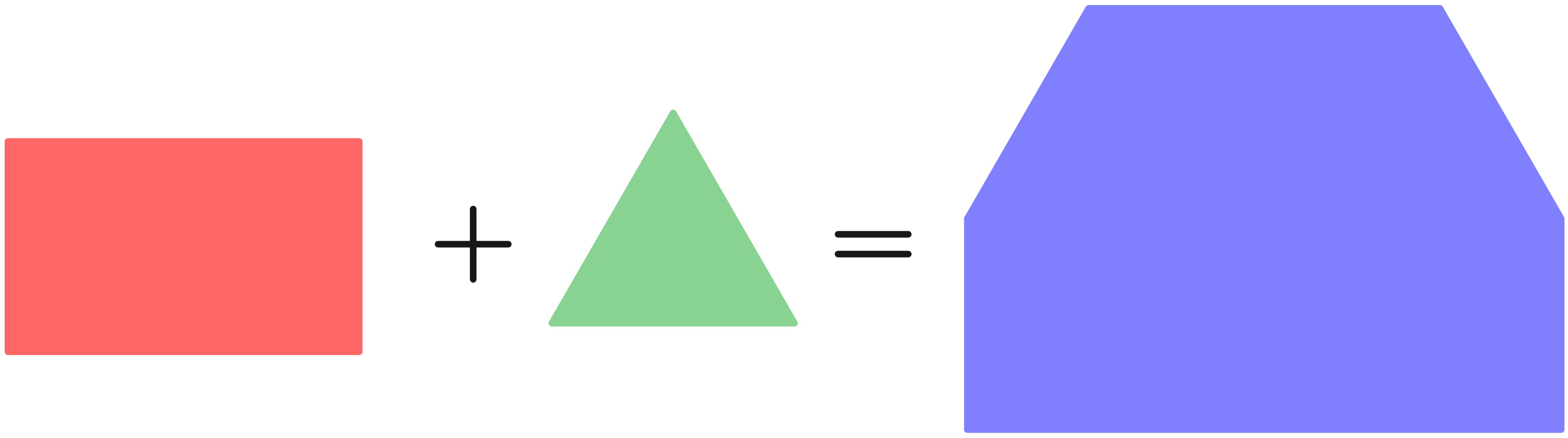


MINKOWSKI SUM

DEF. $\mathbb{X}, \mathbb{X}' \subseteq \mathbb{R}^n$ (same space!).
Minkowski sum $\mathbb{X} + \mathbb{X}' = \{\boldsymbol{x} + \boldsymbol{x'} \mid \boldsymbol{x} \in \mathbb{X} \text{ and } \boldsymbol{x'} \in \mathbb{X}'\} \subseteq \mathbb{R}^n$.

PROP. The Minkowski sum $\mathbb{P} + \mathbb{P}'$ is the image of the Cartesian product $\mathbb{P} \times \mathbb{P}'$ under the affine projection $(\boldsymbol{x}, \boldsymbol{x'}) \longmapsto \boldsymbol{x} + \boldsymbol{x'}$.

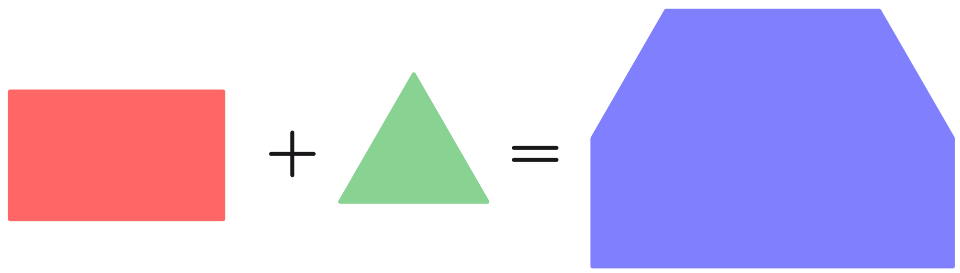
MINKOWSKI SUM

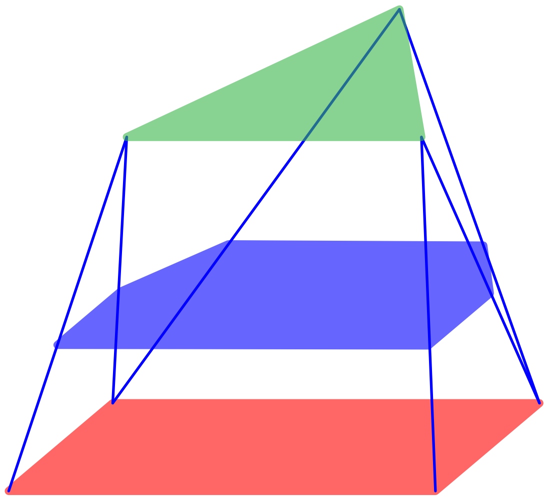
DEF. $\mathbb{X}, \mathbb{X}' \subseteq \mathbb{R}^n$ (same space!).
Minkowski sum $\mathbb{X} + \mathbb{X}' = \{\boldsymbol{x} + \boldsymbol{x}' \mid \boldsymbol{x} \in \mathbb{X} \text{ and } \boldsymbol{x}' \in \mathbb{X}'\} \subseteq \mathbb{R}^n$.

PROP. For any $-1 \le \lambda \le 1$, the section of the Cayley polytope

$$\mathbb{C}\mathrm{ay}(\mathbb{P}, \mathbb{P}') = \mathrm{conv}\Big(\{(\boldsymbol{x}, -1) \mid \boldsymbol{x} \in \mathbb{P}\} \cup \{(\boldsymbol{x}', 1) \mid \boldsymbol{x}' \in \mathbb{P}'\}\Big) \subset \mathbb{R}^{n+1}$$

by the hyperplane $\big\{\boldsymbol{x} \in \mathbb{R}^{n+1} \;\big|\; x_{n+1} = \lambda\big\}$ is the Minkowski sum $\dfrac{1-\lambda}{2} \cdot \mathbb{P} + \dfrac{1+\lambda}{2} \cdot \mathbb{P}'$.

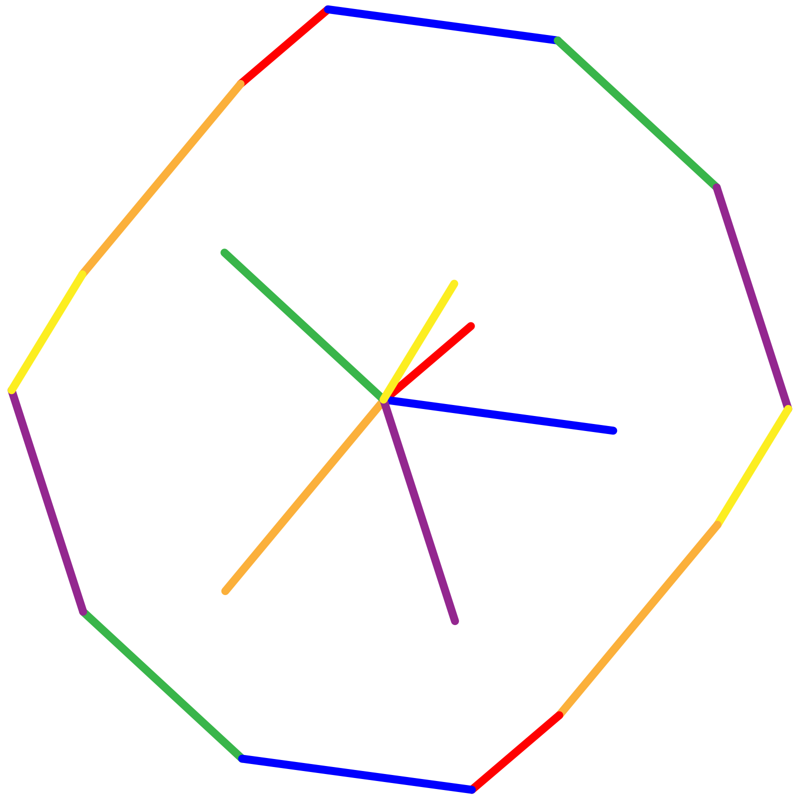


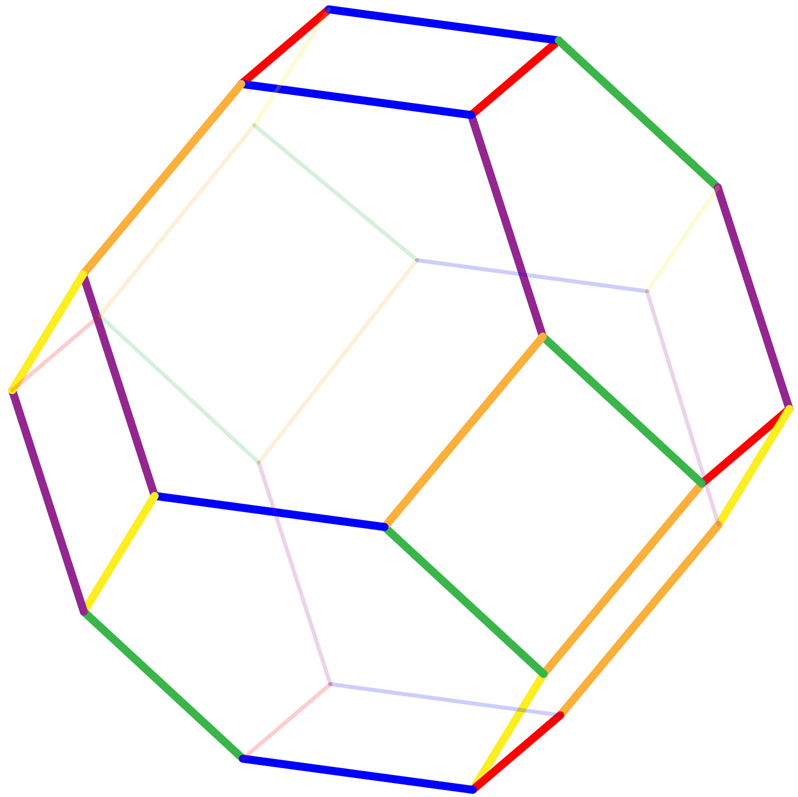


ZONOTOPE

DEF. $\mathbb{X}, \mathbb{X}' \subseteq \mathbb{R}^n$ (same space!).
Minkowski sum $\mathbb{X} + \mathbb{X}' = \{\boldsymbol{x} + \boldsymbol{x}' \mid \boldsymbol{x} \in \mathbb{X} \text{ and } \boldsymbol{x}' \in \mathbb{X}'\} \subseteq \mathbb{R}^n$.

DEF. zonotope = Minkowki sum of segments
= projection of a cube $\square_d$





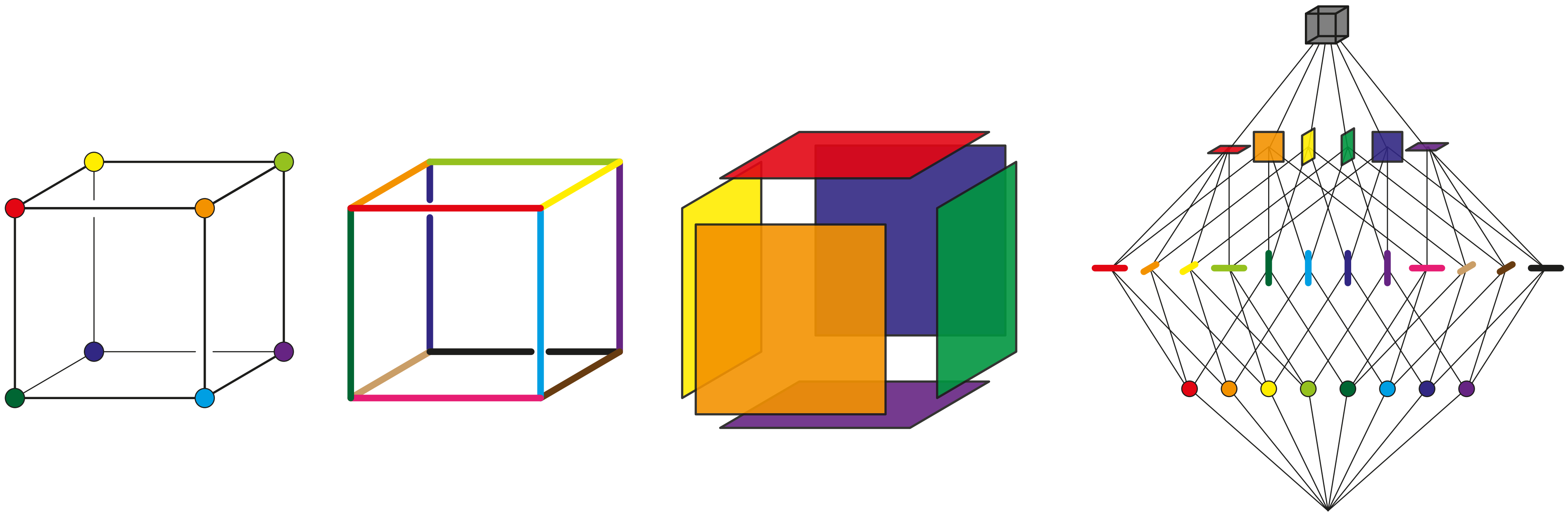
FACES

FACES

DEF. face of a polytope $\mathbb{P}$ =

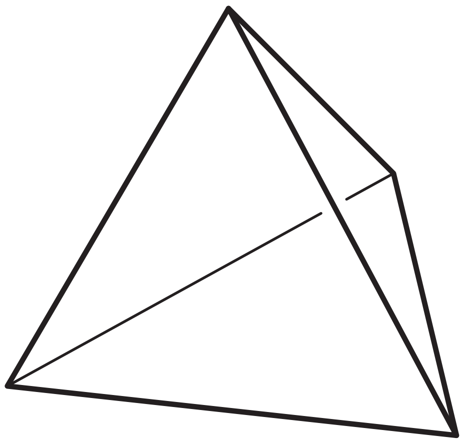
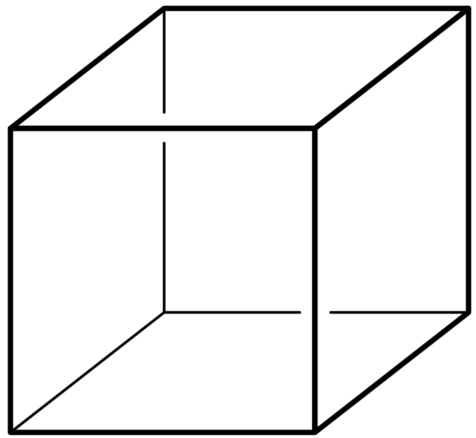
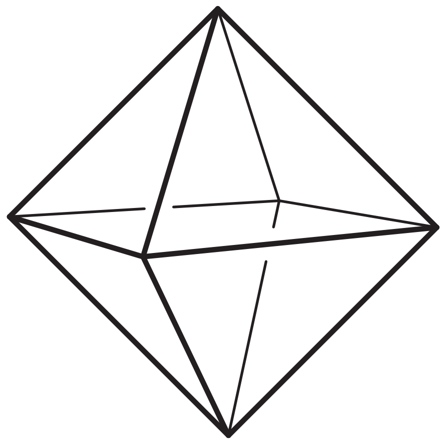
- either the polytope $\mathbb{P}$ itself,
- or the intersection of $\mathbb{P}$ with a supporting hyperplane of $\mathbb{P}$,
- or the empty set.

NOT. $\mathcal{F}(\mathbb{P}) = \{\text{faces of } \mathbb{P}\}$ and $\mathcal{F}_k(\mathbb{P}) = \{k\text{-dimensional faces of } \mathbb{P}\}$.



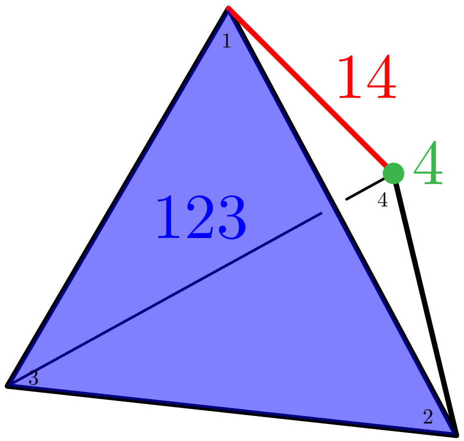
vertices $= \mathcal{F}_0(\mathbb{P})$ edges $= \mathcal{F}_1(\mathbb{P})$ ridges $= \mathcal{F}_{d-2}(\mathbb{P})$ facets $= \mathcal{F}_{d-1}(\mathbb{P})$

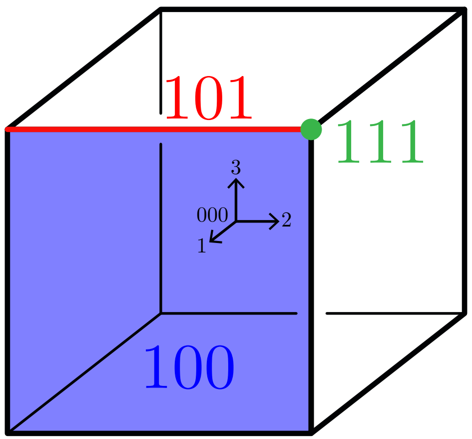
EXM: FACES OF CLASSICAL POLYTOPES

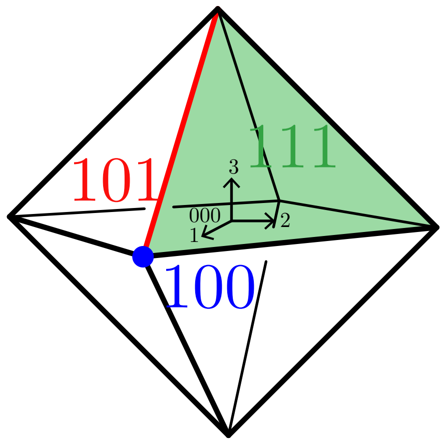
  

QU. Describe the faces of the d-simplex $\triangle_d$, the d-cube $\square_d$ and the d-cross-polytope $\Diamond_d$.

EXM: FACES OF CLASSICAL POLYTOPES







PROP. The faces of the d-simplex $\triangle_d$, the d-cube $\square_d$ and the d-cross-polytope $\lozenge_d$ are:

- d-simplex $\triangle_d$:

  subset I of $[d+1]$ $\quad\longleftrightarrow\quad$ face $\triangle_I = \operatorname{conv}\{\boldsymbol{e}_i \mid i \in I\}$.

- d-cube $\square_d$: the empty face $\varnothing$ and

  word w in $\{-1,0,1\}^d$ $\longleftrightarrow$ face $\square_w = \{\boldsymbol{x} \in \square_d \mid w_i(x_i - w_i) = 0$ for all $i \in [d]\}$.

- d-cross-polytope $\lozenge_d$: the d-cross-polytope $\lozenge_d$ itself and

  word w in $\{-1,0,1\}^d$ $\quad\longleftrightarrow\quad$ face $\triangle_w = \operatorname{conv}\{w_i\boldsymbol{e}_i \mid i \in [d]$ st $w_i \neq 0\}$.

FACE PROPERTIES

PROP. For a polytope $\mathbb{P}$,

- $\mathbb{P} = \mathrm{conv}(\mathcal{F}_0(\mathbb{P}))$ (a polytope is the convex hull of its vertices),
- $\mathbb{P} = \mathrm{conv}(\boldsymbol{X}) \Longrightarrow \mathcal{F}_0(\mathbb{P}) \subseteq \boldsymbol{X}$ (all vertices of a polytope are extreme).

PROP. For a face $\mathbb{F}$ of a polytope $\mathbb{P}$,

- $\mathbb{F}$ is a polytope,
- $\mathcal{F}_0(\mathbb{F}) = \mathcal{F}_0(\mathbb{P}) \cap \mathbb{F}$,
- $\mathcal{F}(\mathbb{F}) = \{\mathbb{G} \in \mathcal{F}(\mathbb{P}) \mid \mathbb{G} \subseteq \mathbb{F}\} \subseteq \mathcal{F}(\mathbb{P})$.

PROP. $\mathcal{F}(\mathbb{P})$ is stable by intersection: $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{P}) \Longrightarrow \mathbb{F} \cap \mathbb{G} \in \mathcal{F}(\mathbb{P})$.

proof ideas: separation theorems, finding a suitable supporting hyperplane, ...

LATTICE

DEF. lattice = partially ordered set $(\mathcal{L}, \leq)$ where any subset $\mathcal{X} \subseteq \mathcal{L}$ admits

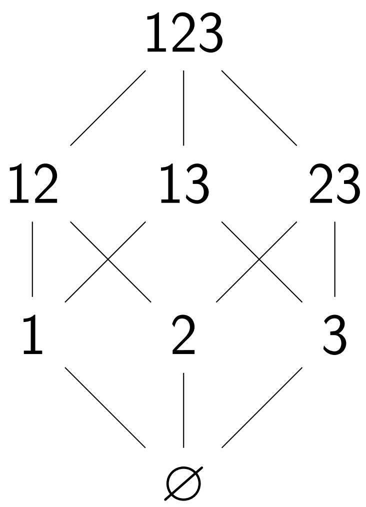
- a meet $\bigwedge \mathcal{X}$ = greatest lower bound

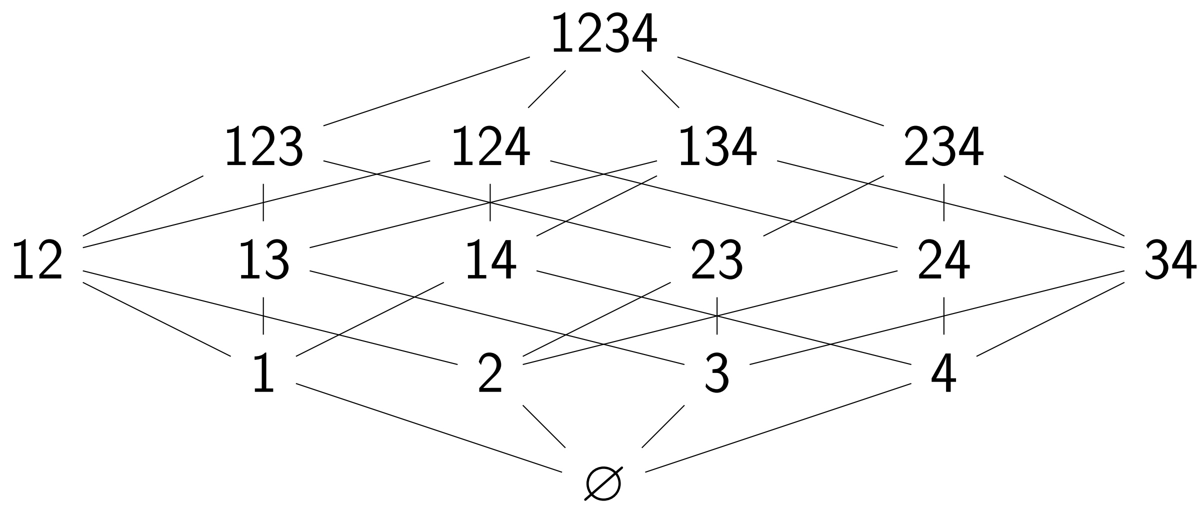
  $\bigwedge \mathcal{X} \leq X$ for all $X \in \mathcal{X}$ and $Y \leq X$ for all $X \in \mathcal{X}$ implies $Y \leq \bigwedge \mathcal{X}$.

- a join $\bigvee \mathcal{X}$ = least upper bound

  $X \leq \bigwedge \mathcal{X}$ for all $X \in \mathcal{X}$ and $X \leq Y$ for all $X \in \mathcal{X}$ implies $\bigwedge \mathcal{X} \leq Y$.

EXM. boolean lattice $\mathcal{B}(Y)$ = subsets of Y ordered by inclusion



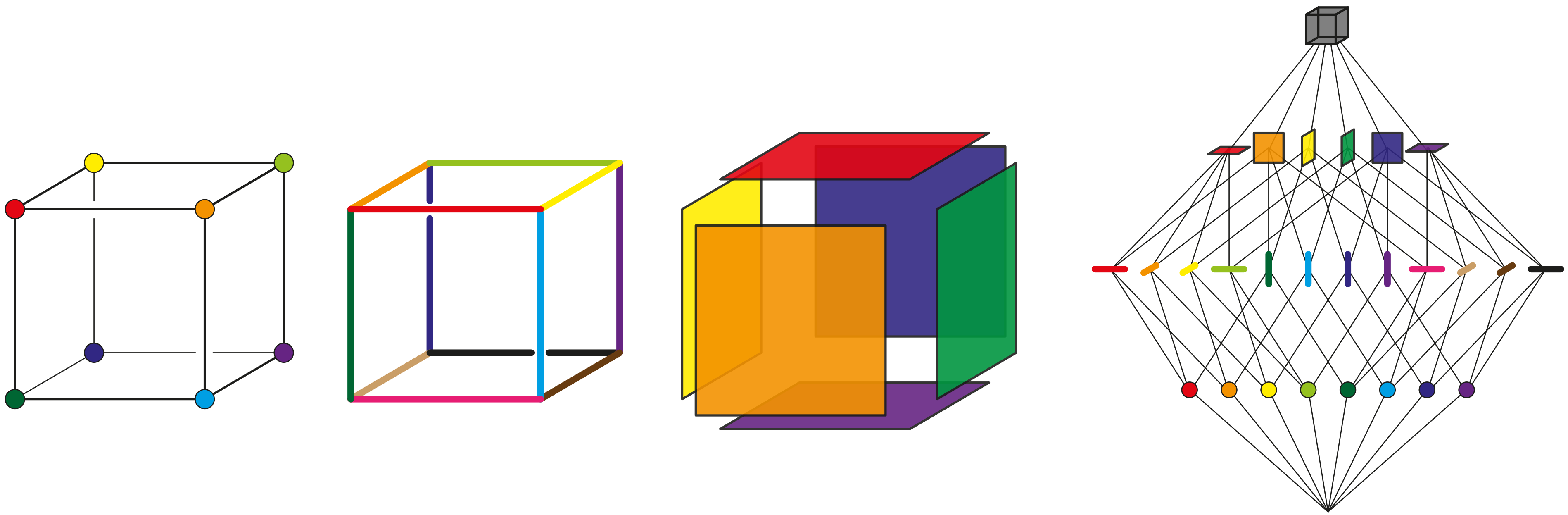


$$\bigwedge \mathcal{X} = \bigcap_{X \in \mathcal{X}} X \quad \text{and} \quad \bigvee \mathcal{X} = \bigcup_{X \in \mathcal{X}} X.$$

FACE LATTICE

PROP. The inclusion poset $\mathcal{F}(\mathbb{P})$ of faces of $\mathbb{P}$

- is a graded lattice (with rank function $\mathrm{rank}(\mathbb{F}) = \dim(\mathbb{F}) + 1$),
- is atomic (every face is the join of its vertices) and coatomic (every face is the meet of the facets containing it),
- every interval of $\mathcal{F}(\mathbb{P})$ is the face lattice of a polytope,
- has the diamond property (every interval of rank 2 has 4 elements).



EXM: FACE LATTICES OF SIMPLICES

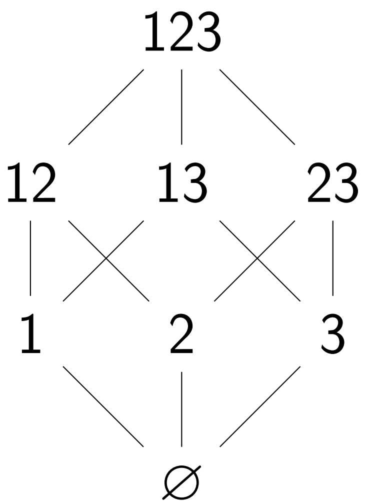
QU. Draw the face lattice of a 2- and 3-dimensional simplices. What is this lattice?

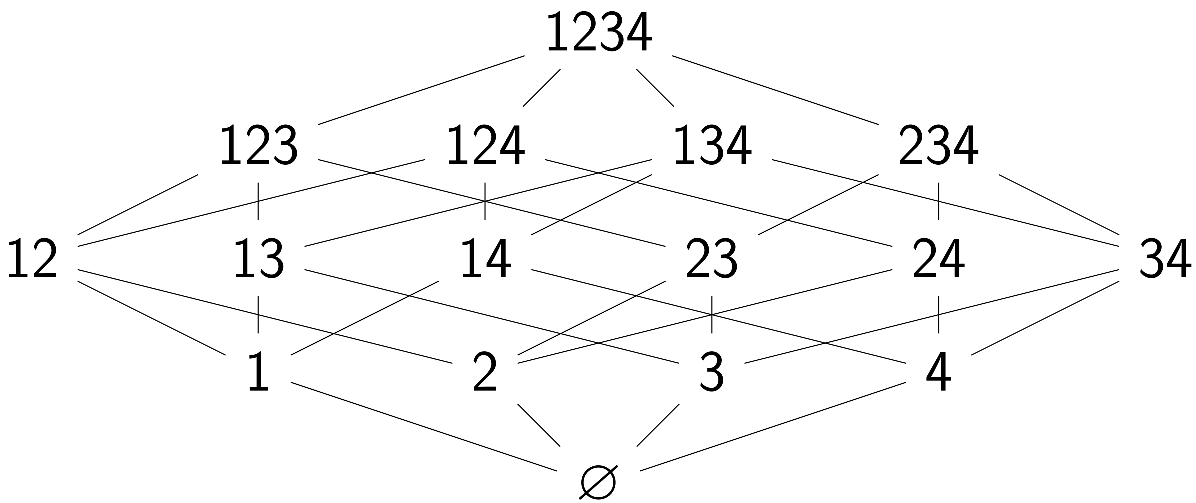
EXM: FACE LATTICES OF SIMPLICES

remark:

- any subset $I \subseteq [d+1]$ corresponds to a face $\triangle_I = \operatorname{conv}\{\boldsymbol{e}_i \mid i \in I\}$ of $\triangle_d$,
- $I \subseteq J \iff \triangle_I \subseteq \triangle_J$.

The face lattice of $\triangle_d$ is thus the boolean lattice on subsets of $[d+1]$:



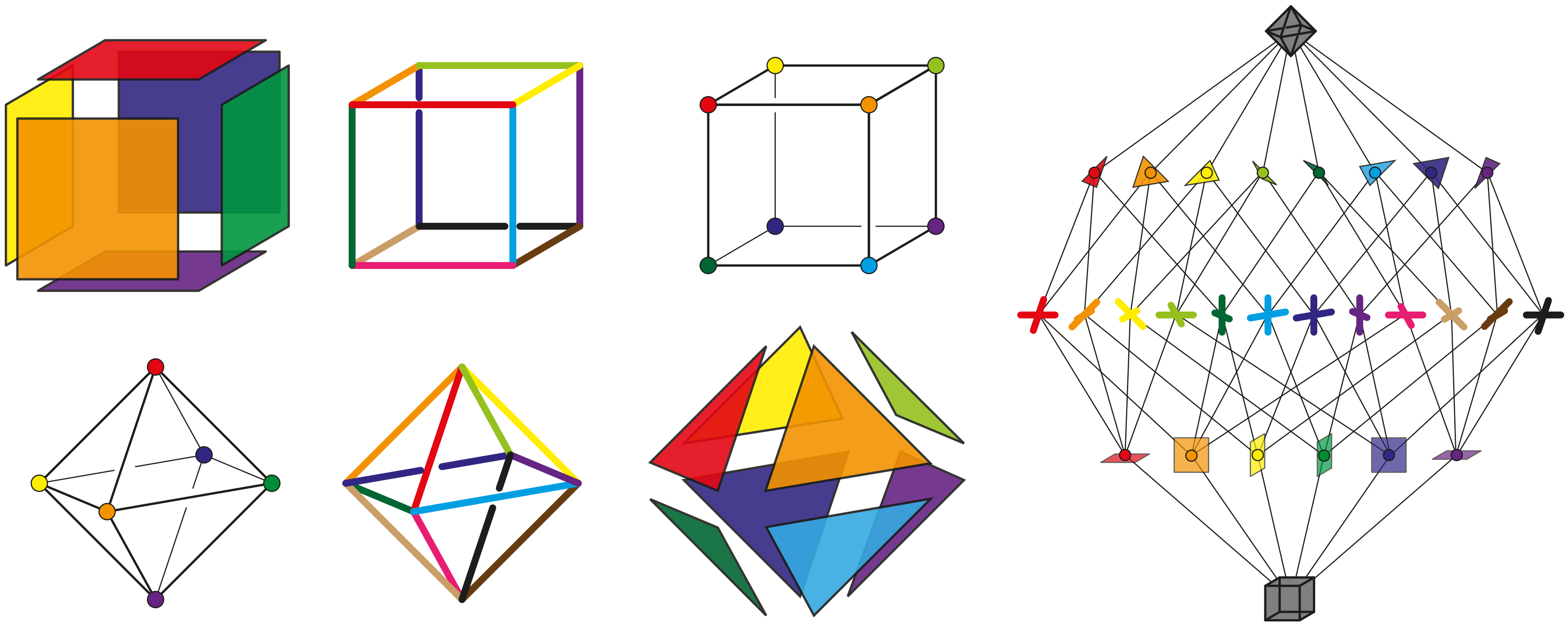


POLARITY AND FACES

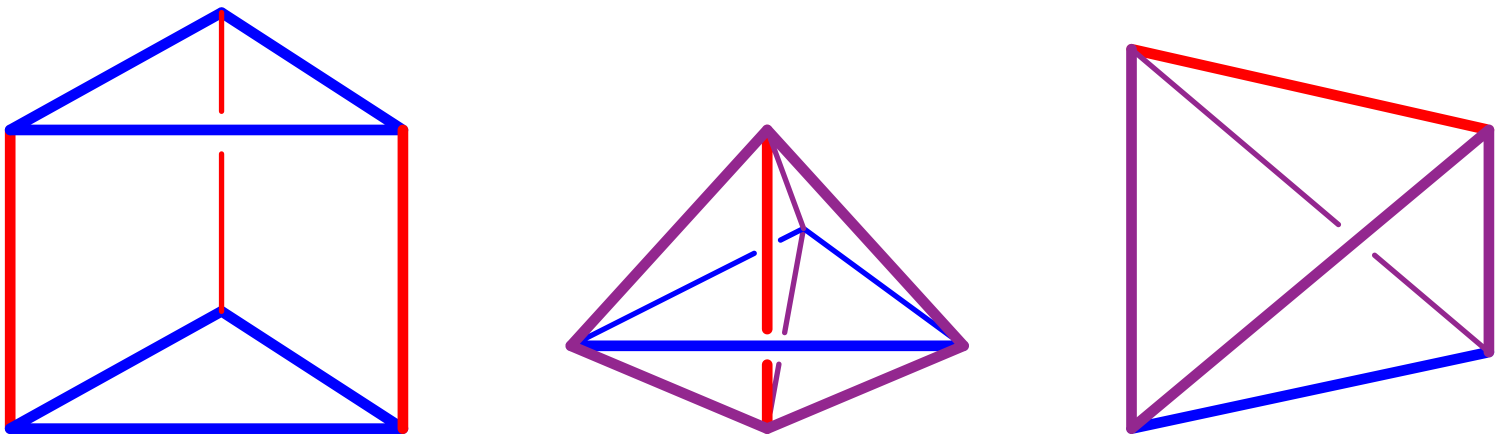
Assume $\mathbf{0} \in \mathrm{int}(\mathbb{P})$.

DEF. A face $\mathbb{F}$ of $\mathbb{P}$ defines a polar face $\mathbb{F}^\diamond = \{\boldsymbol{y} \in \mathbb{P}^\diamond \mid \langle\, \boldsymbol{x} \mid \boldsymbol{y} \,\rangle = 1 \text{ for all } \boldsymbol{x} \in \mathbb{F}\}$.

PROP. The map $\mathbb{F} \longmapsto \mathbb{F}^\diamond$ is a lattice anti-isomorphism $\mathcal{F}(\mathbb{P}) \longrightarrow \mathcal{F}(\mathbb{P}^\diamond)$.



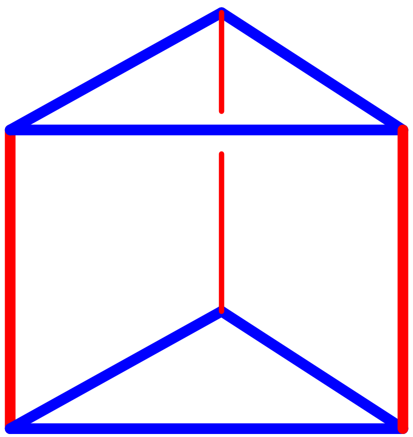
OPERATIONS AND FACES



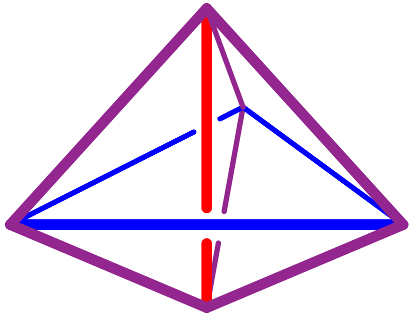
$\mathbb{P} \times \mathbb{P}'$ $\quad$ $\mathbb{P} \oplus \mathbb{P}'$ $\quad$ $\mathbb{P} * \mathbb{P}'$

QU. Describe the faces of the Cartesian product $\mathbb{P} \times \mathbb{P}'$, the direct sum $\mathbb{P} \oplus \mathbb{P}'$ and the join $\mathbb{P} * \mathbb{P}'$ in terms of that of $\mathbb{P}$ and $\mathbb{P}'$.
What can you say about the faces of the Minkowski sum $\mathbb{P} + \mathbb{P}'$?

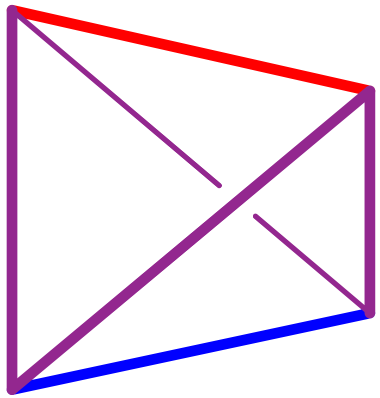
OPERATIONS AND FACES



$\mathbb{P} \times \mathbb{P}'$



$\mathbb{P} \oplus \mathbb{P}'$

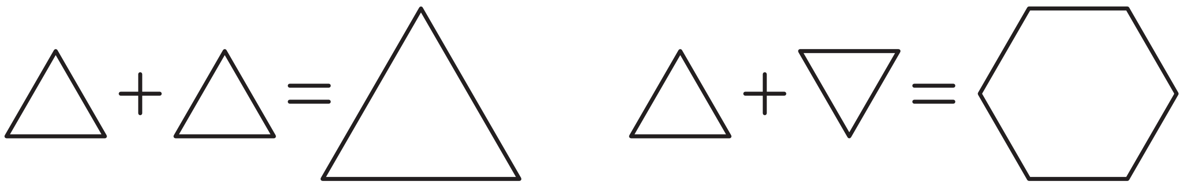


$\mathbb{P} * \mathbb{P}'$

PROP. Define $\mathcal{F}_\star(\mathbb{P}) = \mathcal{F}(\mathbb{P}) \smallsetminus \{\varnothing\}$ and $\mathcal{F}^\star(\mathbb{P}) = \mathcal{F}(\mathbb{P}) \smallsetminus \{\mathbb{P}\}$. Then

$$\mathcal{F}_\star(\mathbb{P} \times \mathbb{P}') = \{\mathbb{F} \times \mathbb{F}' \mid \mathbb{F} \in \mathcal{F}_\star(\mathbb{P}) \text{ and } \mathbb{F}' \in \mathcal{F}_\star(\mathbb{P}')\}$$
$$\mathcal{F}^\star(\mathbb{P} \oplus \mathbb{P}') = \{\mathbb{F} * \mathbb{F}' \mid \mathbb{F} \in \mathcal{F}^\star(\mathbb{P}) \text{ and } \mathbb{F}' \in \mathcal{F}^\star(\mathbb{P}')\}$$
$$\mathcal{F}(\mathbb{P} * \mathbb{P}') = \{\mathbb{F} * \mathbb{F}' \mid \mathbb{F} \in \mathcal{F}(\mathbb{P}) \text{ and } \mathbb{F}' \in \mathcal{F}(\mathbb{P}')\}$$

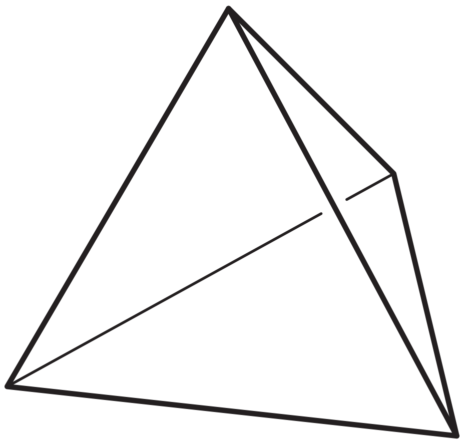
remark: the combinatorial structure of $\mathbb{P} + \mathbb{P}'$ depends on the geometry of $\mathbb{P}$ and $\mathbb{P}'$.



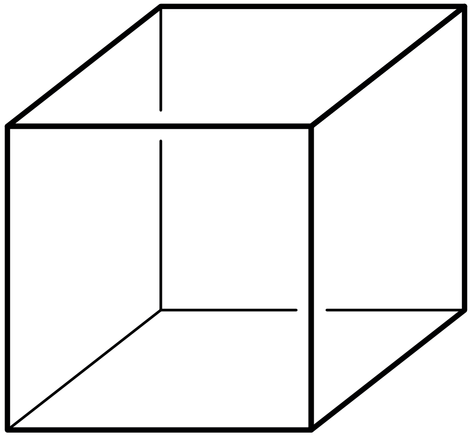
SIMPLE OR SIMPLICIAL POLYTOPES

DEF. A d-polytope $\mathbb{P}$ is

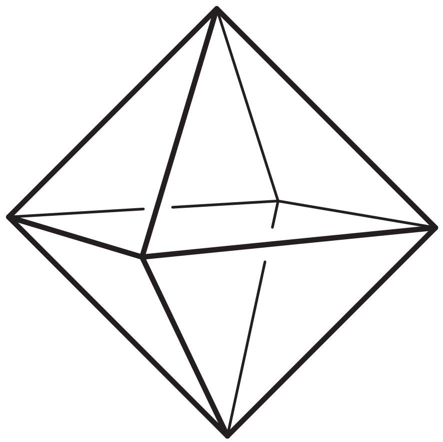
- simplicial if its vertices are in general position,
- simple if its facets are in general position.



simple and simplicial



simple but not simplicial



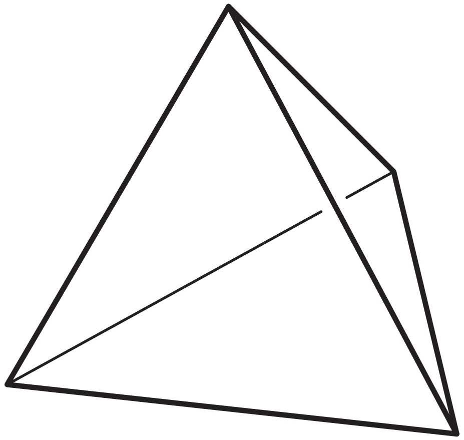
not simple but simplicial

PROP. $\mathbb{P}$ is simple $\iff$ $\mathbb{P}^\diamond$ is simplicial.

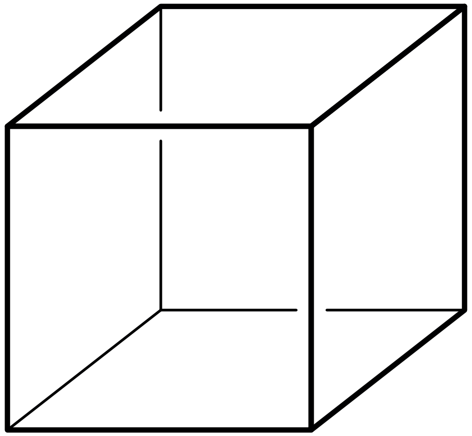
SIMPLE OR SIMPLICIAL POLYTOPES

DEF. A d-polytope $\mathbb{P}$ is

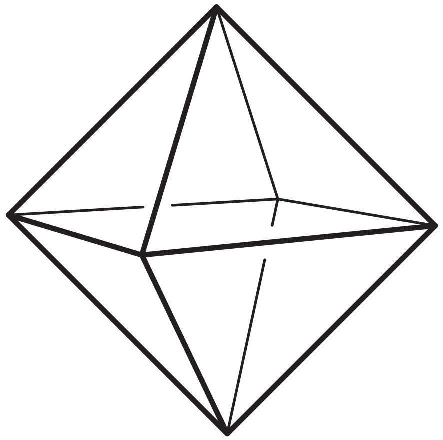
- simplicial if each facet is a simplex contains d vertices (ie. is a simplex),
- simple if each vertex is contained in d edges (or equiv. in d facets).



simple and simplicial



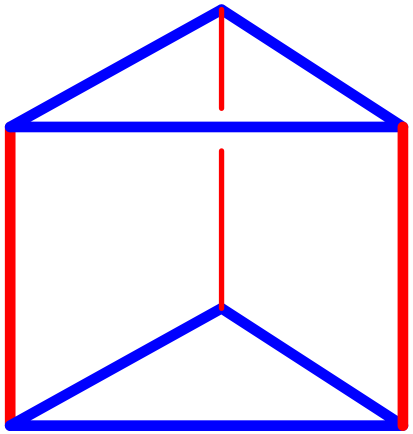
simple but not simplicial



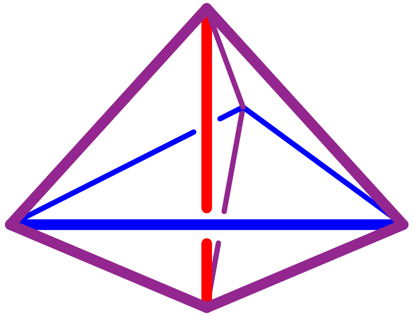
not simple but simplicial

PROP. $\mathbb{P}$ is simple $\iff$ $\mathbb{P}^\diamond$ is simplicial.

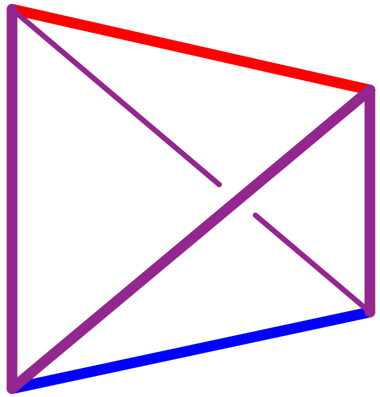
SIMPLE OR SIMPLICIAL POLYTOPE OPERATIONS



$\mathbb{P} \times \mathbb{P}'$



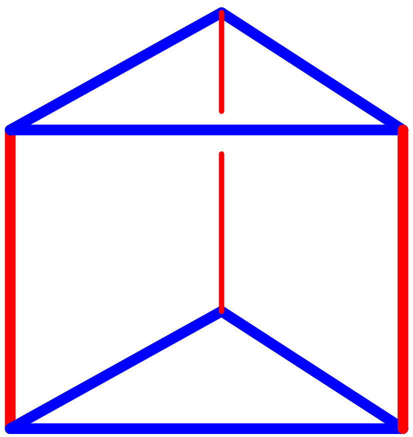
$\mathbb{P} \oplus \mathbb{P}'$



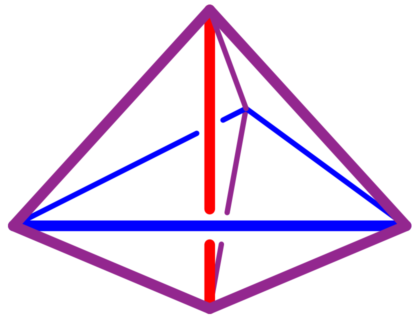
$\mathbb{P} * \mathbb{P}'$

QU. When is $\mathbb{P} \times \mathbb{P}'$ (resp. $\mathbb{P} \oplus \mathbb{P}'$, resp. $\mathbb{P} * \mathbb{P}'$) simple or simplicial?

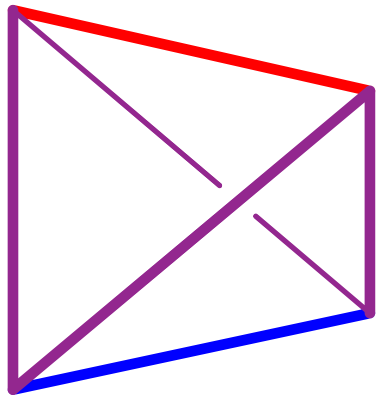
SIMPLE OR SIMPLICIAL POLYTOPE OPERATIONS



$\mathbb{P} \times \mathbb{P}'$



$\mathbb{P} \oplus \mathbb{P}'$



$\mathbb{P} * \mathbb{P}'$

PROP. $\mathbb{P}$ and $\mathbb{P}'$ simple $\iff$ $\mathbb{P} \times \mathbb{P}'$ simple

$\mathbb{P}$ and $\mathbb{P}'$ simplicial $\iff$ $\mathbb{P} \oplus \mathbb{P}'$ simplicial

$\mathbb{P}$ and $\mathbb{P}'$ simplices $\iff$ $\mathbb{P} * \mathbb{P}'$ simple (or simplicial)

SIMPLE AND SIMPLICIAL POLYTOPES

QU. Show that a simple and simplicial polytope is a polygon or a simplex.

SIMPLE AND SIMPLICIAL POLYTOPES

PROP. A simple and simplicial polytope is a polygon or a simplex.

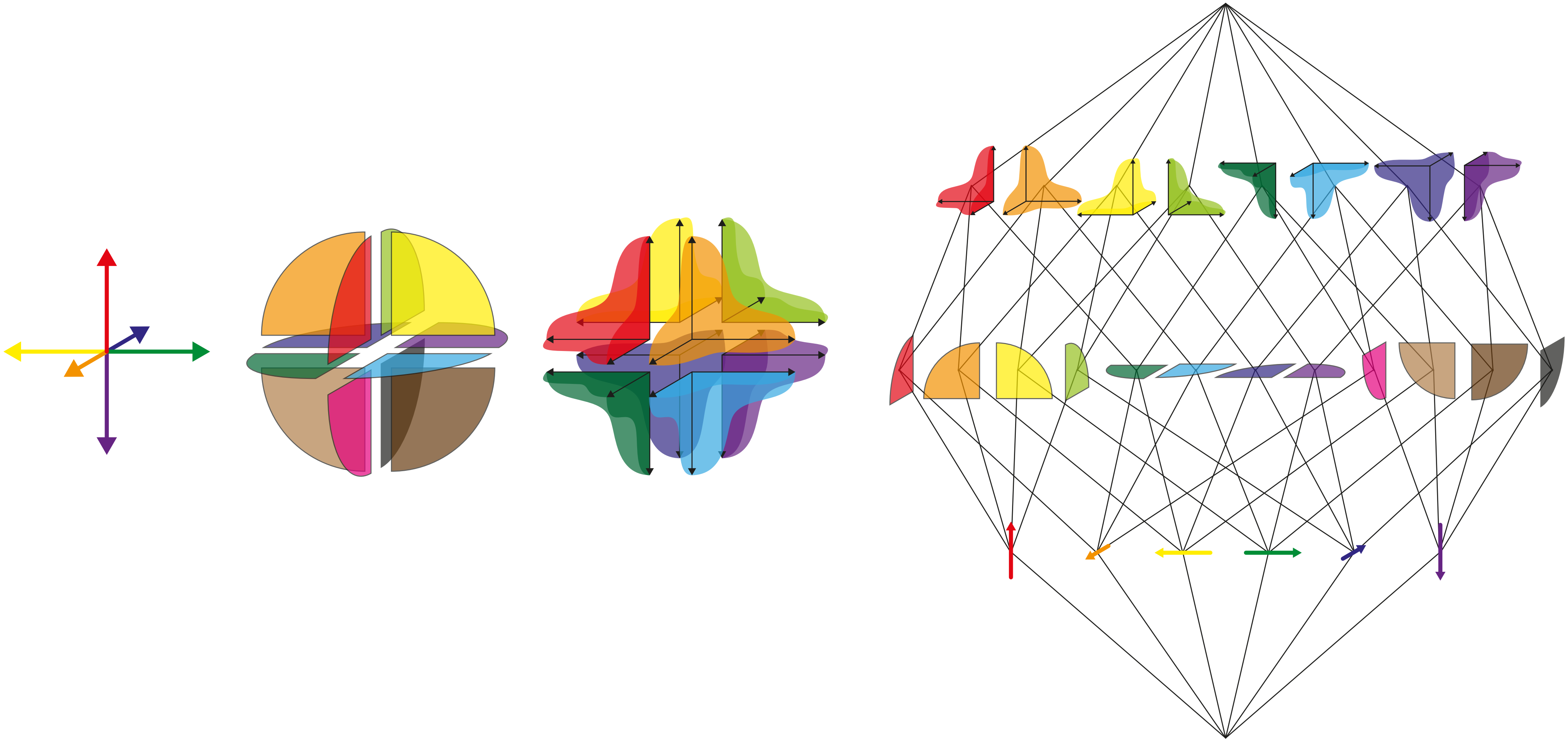
proof: Assume $\mathbb{P}$ is a simple and simplicial d-polytope with $d \geq 3$. Pick a vertex $\boldsymbol{v}_0$ of $\mathbb{P}$.
Since $\mathbb{P}$ is simplicial, $\boldsymbol{v}_0$ has d neighbors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$.
For $k \in [d]$, $\{\boldsymbol{v}_i \mid i \neq k\}$ is contained in a facet ($\mathbb{P}$ simple) and forms a facet ($\mathbb{P}$ simplicial).
Thus $\boldsymbol{v}_k$ incident to $\boldsymbol{v}_i$ for $i \neq k$, and $\{\boldsymbol{v}_i \mid i \in [d]\}$ forms a facet ($\mathbb{P}$ simple and simplicial).
Thus $\mathbb{P}$ is a simplex.

FANS

FAN

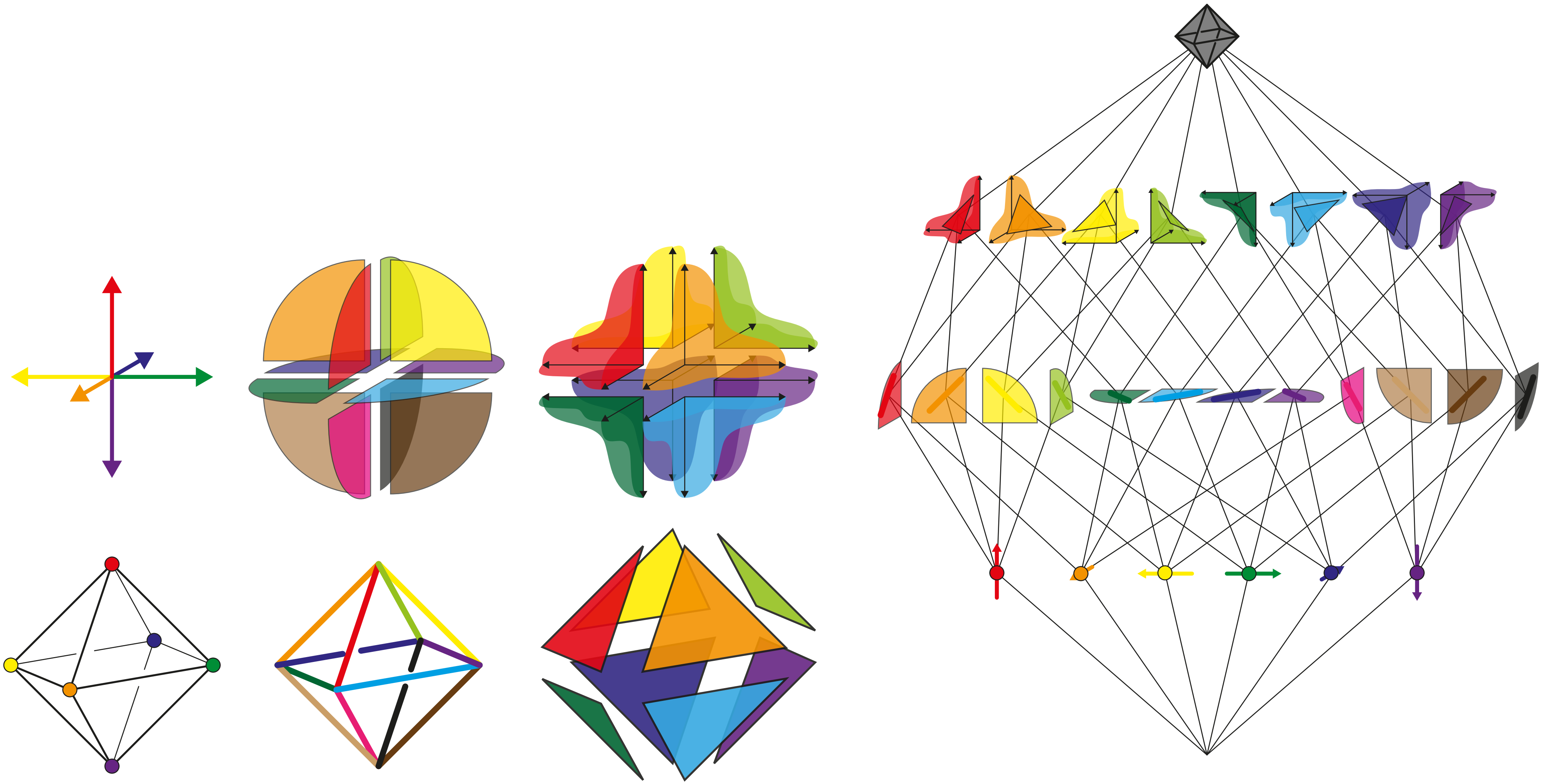
DEF. fan $\mathcal{F}$ = collection of polyhedral cones st

- closed by faces: if $\mathbb{C} \in \mathcal{F}$ and $\mathbb{C}'$ is a face of $\mathbb{C}$, then $\mathbb{C}' \in \mathcal{F}$,
- intersecting properly: if $\mathbb{C}, \mathbb{C}' \in \mathcal{F}$, the intersection $\mathbb{C} \cap \mathbb{C}'$ is a face of $\mathbb{C}$ and $\mathbb{C}'$.



FAN

DEF. $\mathbb{P}$ polytope with $\mathbf{0} \in \mathrm{int}(\mathbb{P})$. $\mathbb{F}$ face of $\mathbb{P}$.
face cone of $\mathbb{F}$ = cone $\mathbb{R}_{\geq 0}\mathbb{F}$ generated by $\mathbb{F}$.
face fan of $\mathbb{P}$ = collection of face cones of all faces of $\mathbb{P}$.

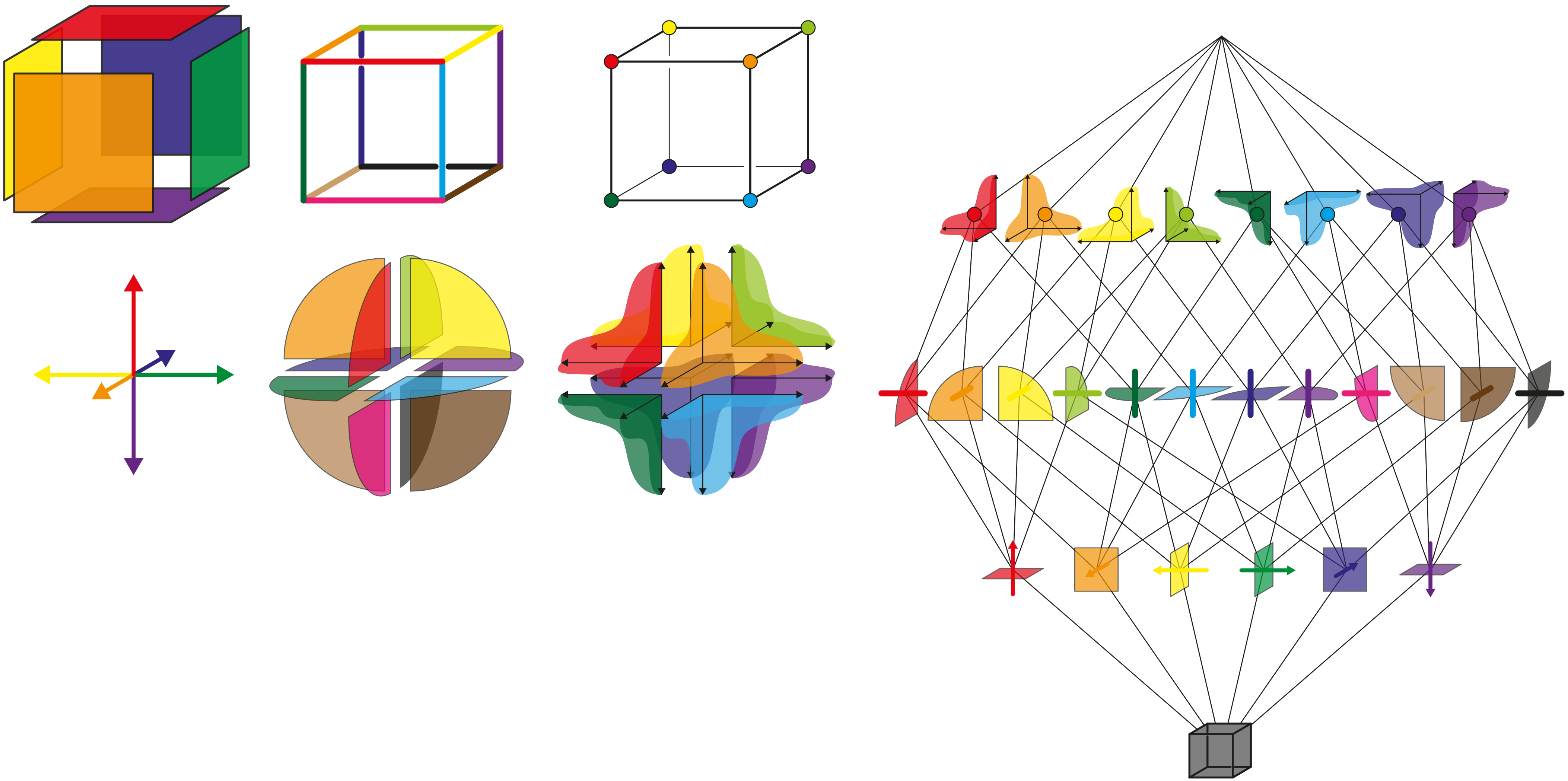


FAN

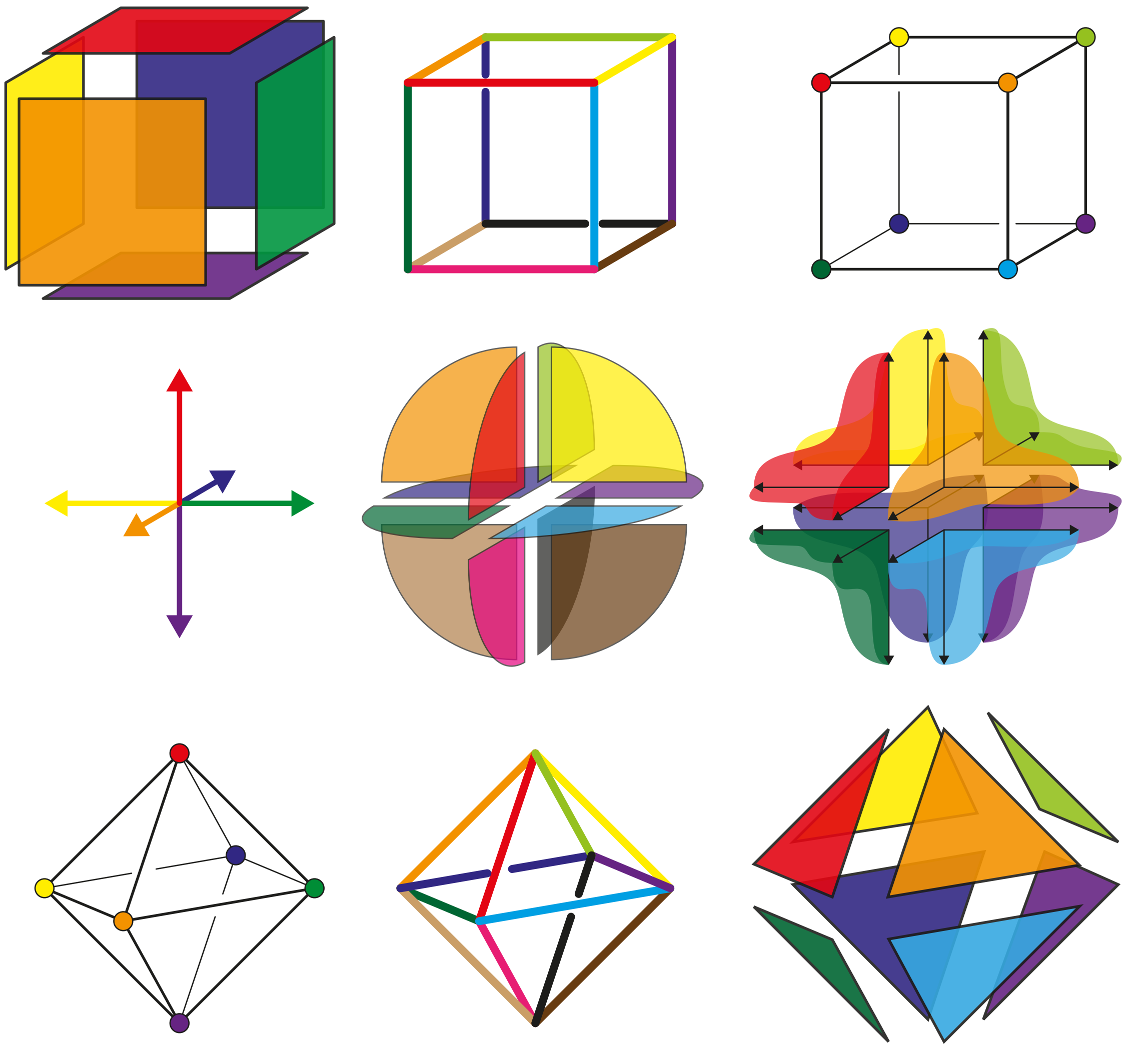
DEF. $\mathbb{P}$ polytope. $\mathbb{F}$ face of $\mathbb{P}$.

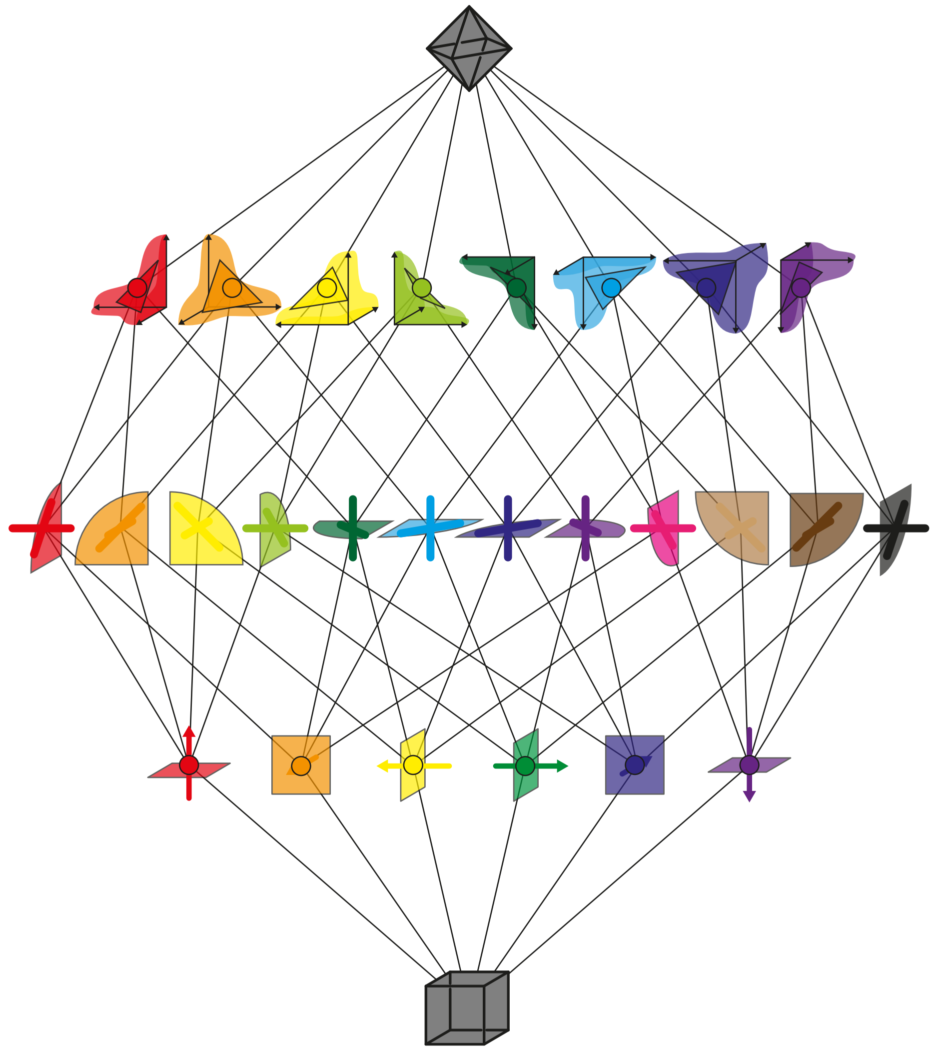
normal cone of $\mathbb{F}$ = cone generated by outer normal vectors to facets of $\mathbb{P}$ containing $\mathbb{F}$.

normal fan of $\mathbb{P}$ = collection of normal cones of all faces of $\mathbb{P}$.



PROP. If $\mathbf{0} \in \operatorname{int}(\mathbb{P})$, then the face fan of $\mathbb{P}$ coincides with the normal fan of $\mathbb{P}^{\diamond}$.

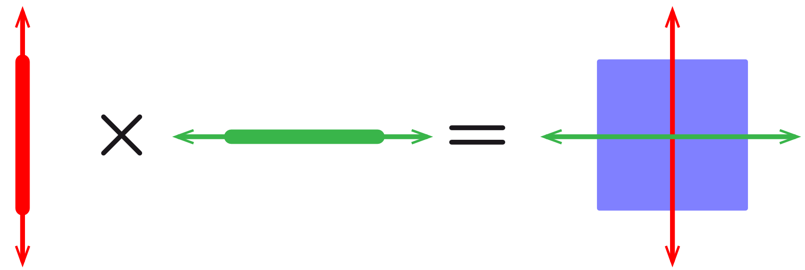




NORMAL FANS AND POLYTOPE OPERATIONS

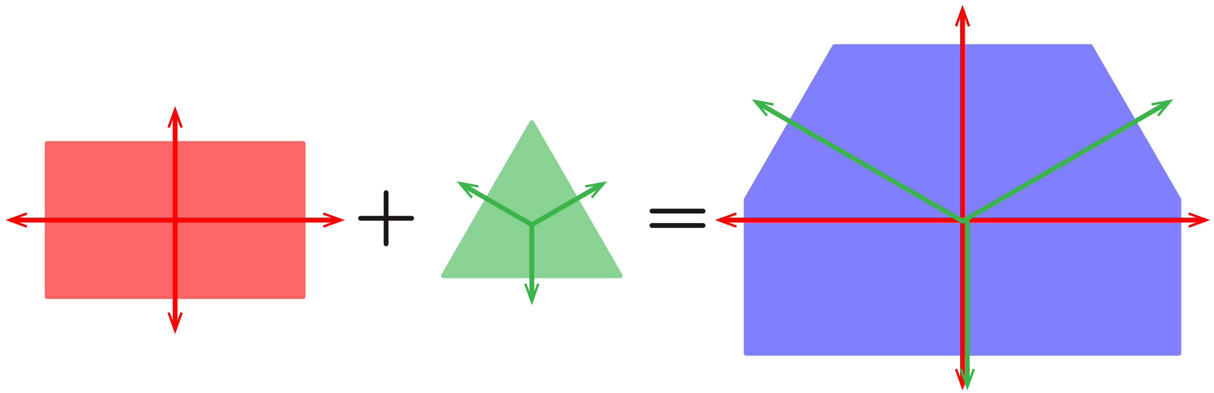
DEF. direct sum $\mathcal{F} \oplus \mathcal{F}' = \{\mathbb{C} \times \mathbb{C}' \mid \mathbb{C} \in \mathcal{F} \text{ and } \mathbb{C}' \in \mathcal{F}'\}$

PROP. normal fan of $\mathbb{P} \times \mathbb{P}'$ = direct sum of normal fans of $\mathbb{P}$ and $\mathbb{P}'$.



DEF. common refinement $\mathcal{F} \wedge \mathcal{F}' = \{\mathbb{C} \cap \mathbb{C}' \mid \mathbb{C} \in \mathcal{F} \text{ and } \mathbb{C}' \in \mathcal{F}'\}$

PROP. normal fan of $\mathbb{P} + \mathbb{P}'$ = common refinement of normal fans of $\mathbb{P}$ and $\mathbb{P}'$.

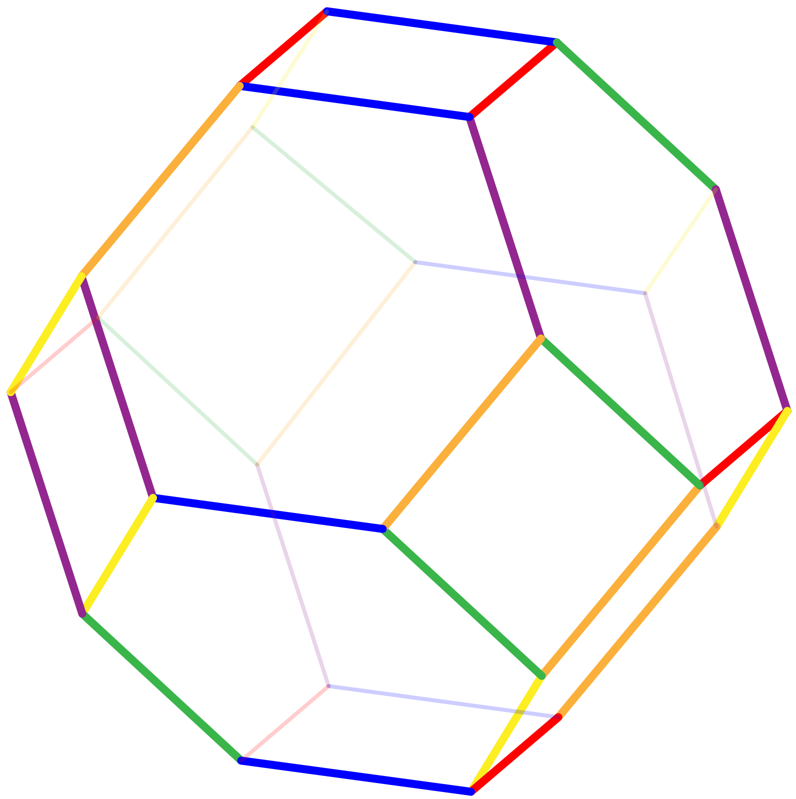


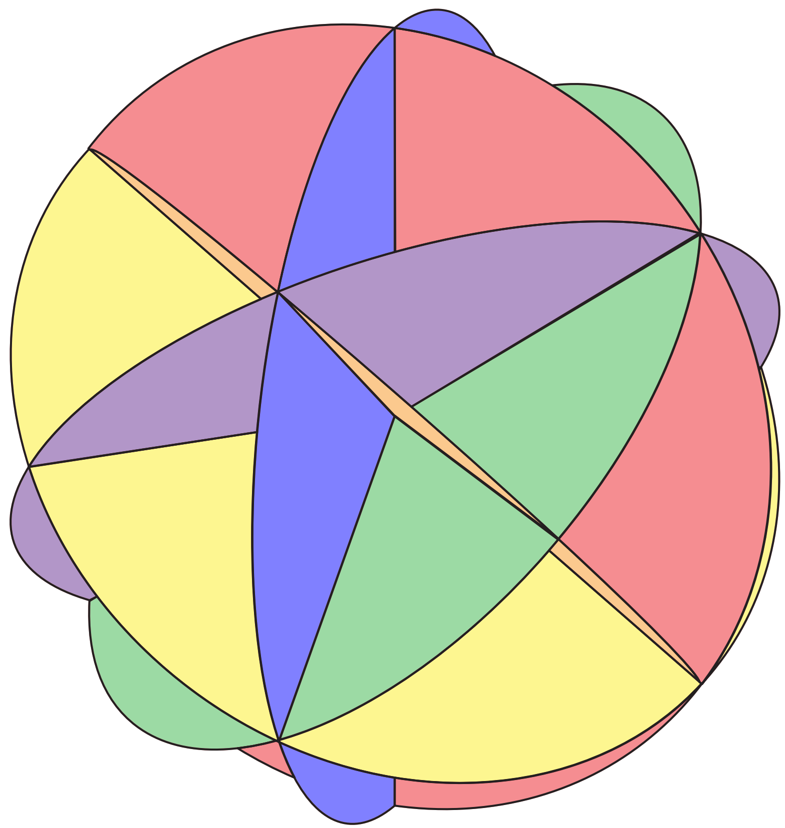
NORMAL FANS OF ZONOTOPES

DEF. common refinement $\mathcal{F} \wedge \mathcal{F}' = \{\mathbb{C} \cap \mathbb{C}' \mid \mathbb{C} \in \mathcal{F} \text{ and } \mathbb{C}' \in \mathcal{F}'\}$

PROP. normal fan of $\mathbb{P} + \mathbb{P}'$ = common refinement of normal fans of $\mathbb{P}$ and $\mathbb{P}'$.

PROP. normal fans of zonotopes $\iff$ fans defined by hyperplane arrangements.



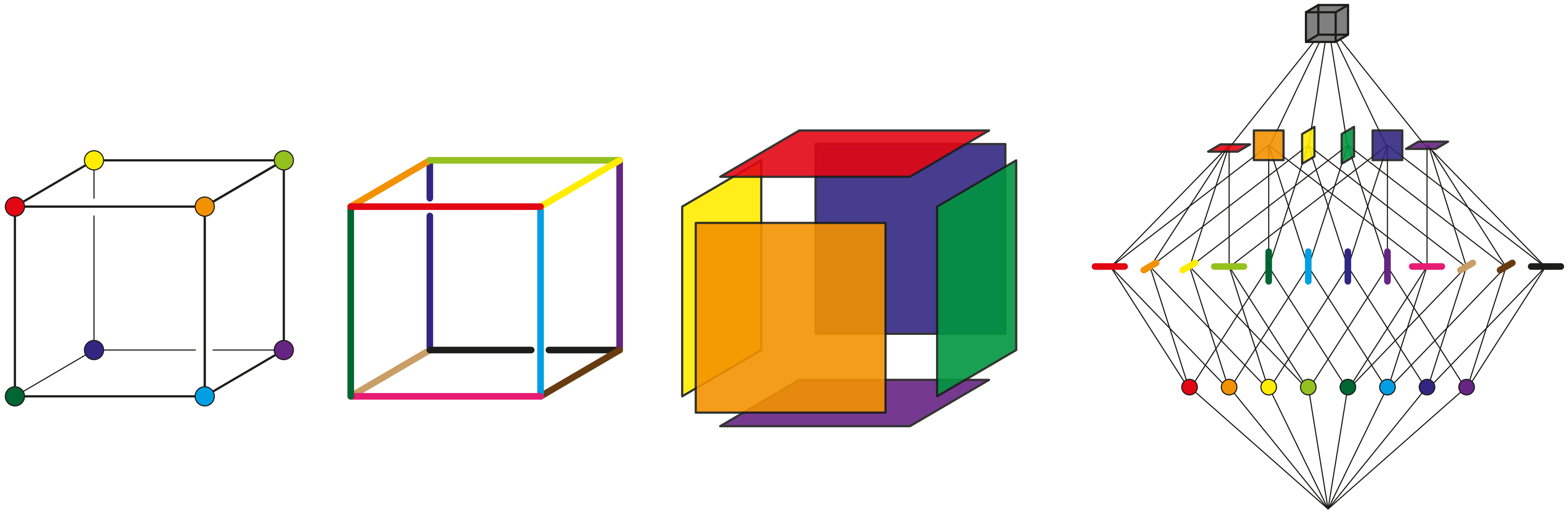


F-VECTOR & EULER RELATION

F-VECTOR & F-POLYNOMIAL

DEF. For a d-polytope $\mathbb{P}$,

- $f_i(\mathbb{P})$ = number of i-faces of $\mathbb{P}$,
- f-vector $f(\mathbb{P}) = \big(f_0(\mathbb{P}), \dots, f_d(\mathbb{P})\big)$,
- f-polynomial $f(\mathbb{P}, x) = \sum_{i=0}^{d} f_i(\mathbb{P})\, x^i$.



$$f(\square_3) = 8 + 12\,x + 6\,x^2 + x^3$$

In fact, for the exercises below, it is convenient to define

$$F(\mathbb{P},x)=\sum_{i=-1}^{d} f_i(\mathbb{P})\,x^{i+1}$$

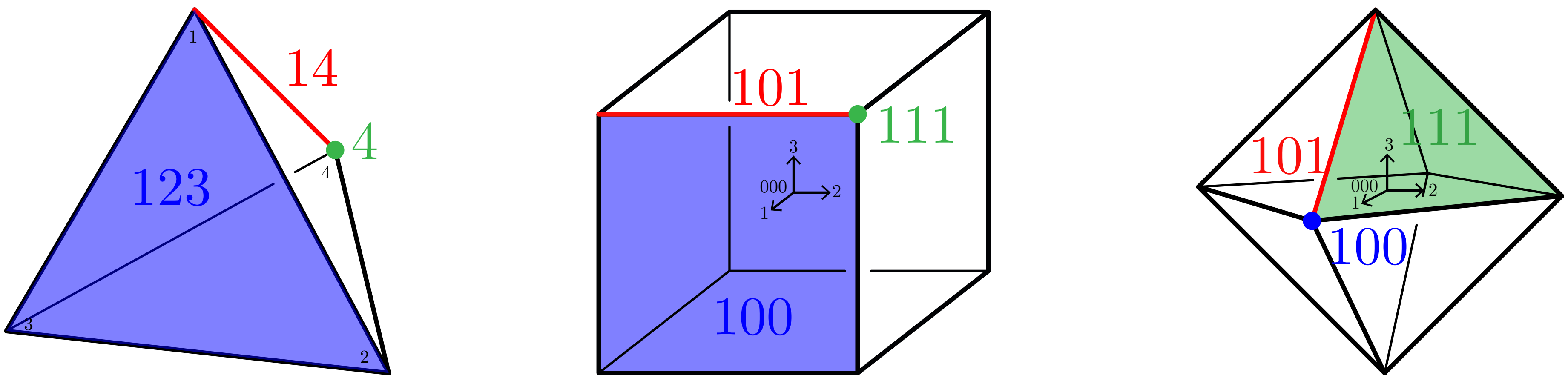
and to consider

$$f(\mathbb{P},x)=\sum_{i=0}^{d} f_i(\mathbb{P})\,x^{i}=\frac{F(\mathbb{P},x)-1}{x}$$

and

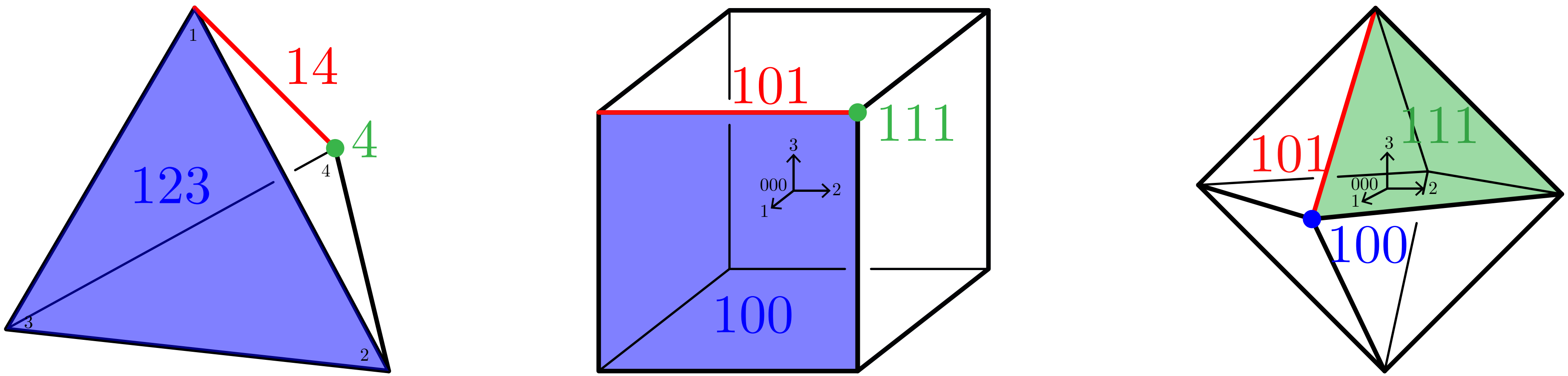
$$\bar{f}(\mathbb{P},x)=\sum_{i=-1}^{d-1} f_i(\mathbb{P})\,x^{i+1}=F(\mathbb{P},x)-x^{d+1}$$

EXM: F-VECTOR OF CLASSICAL POLYTOPES



QU. Compute the f-vectors and F-polynomials of the d-simplex $\triangle_d$, the d-cube $\square_d$ and the d-cross-polytope $\lozenge_d$.

EXM: F-VECTOR OF CLASSICAL POLYTOPES



PROP. The f-vectors and F-polynomials of the d-simplex $\triangle_d$, the d-cube $\square_d$ and the d-cross-polytope $\lozenge_d$ are given by

$$f_i(\triangle_d) = \binom{d+1}{i+1} \qquad f_i(\square_d) = \binom{d}{i} 2^{d-i} \qquad f_i(\lozenge_d) = \binom{d}{i+1} 2^{i+1}$$

$$F(\triangle_d, x) = (x+1)^{d+1} \qquad F(\square_d, x) = 1 + x(x+2)^d \qquad F(\lozenge_d, x) = x^{d+1} + (2x+1)^d$$

REM. In other words,

$$F(\triangle_d, x) = (x+1)^{d+1} \qquad f(\square_d, x) = (x+2)^d \qquad \bar{f}(\lozenge_d, x) = (2x+1)^d$$

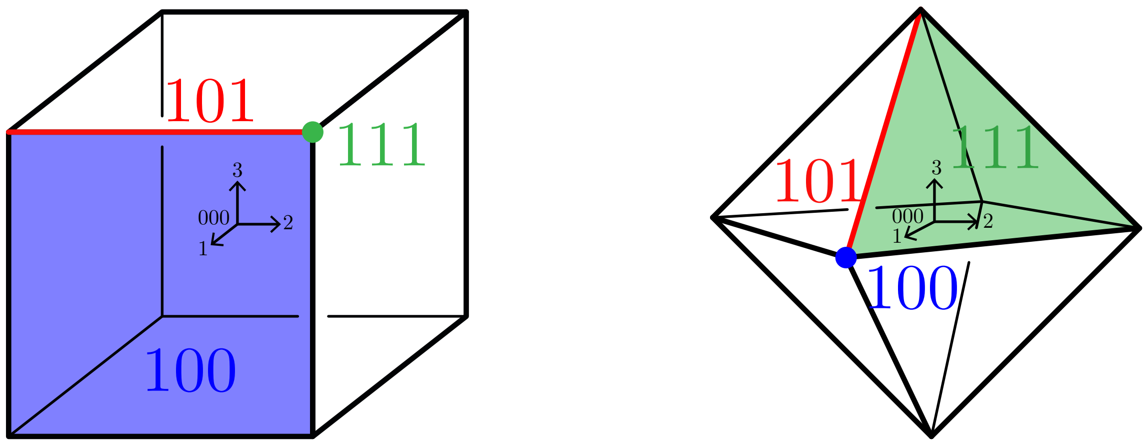
EXM: F-VECTOR & POLARITY

QU. Relate $F(\mathbb{P}, x)$ to $F(\mathbb{P}^\diamond, x)$.

EXM: F-VECTOR & POLARITY

PROP. $F(\mathbb{P}, x) = x^{d+1} F(\mathbb{P}^\diamond, 1/x)$

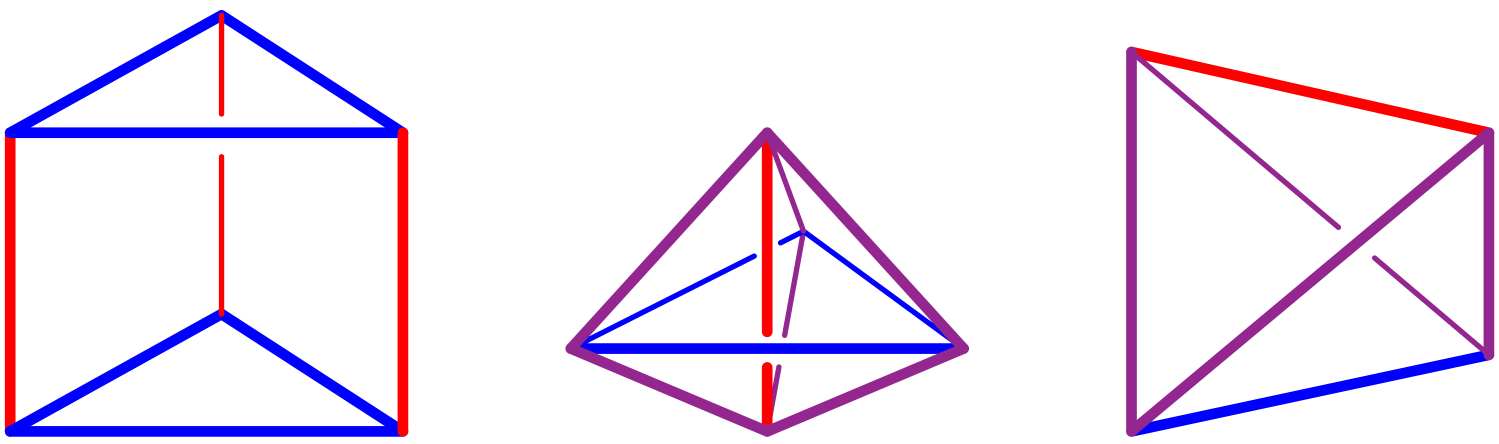
proof: $\mathbb{F} \longmapsto \mathbb{F}^\diamond$ anti-isomorphism, thus $f_i(\mathbb{P}) = f_{d-i-1}(\mathbb{P}^\diamond)$, thus $F_i(\mathbb{P}) = F_{d+1-i}(\mathbb{P}^\diamond)$.



remark: sanity check on classical polytopes

$$F(\square_d, x) = 1 + x(x+2)^d \qquad F(\Diamond_d, x) = x^{d+1} + (2x+1)^d \qquad F(\triangle_d, x) = (x+1)^{d+1}$$

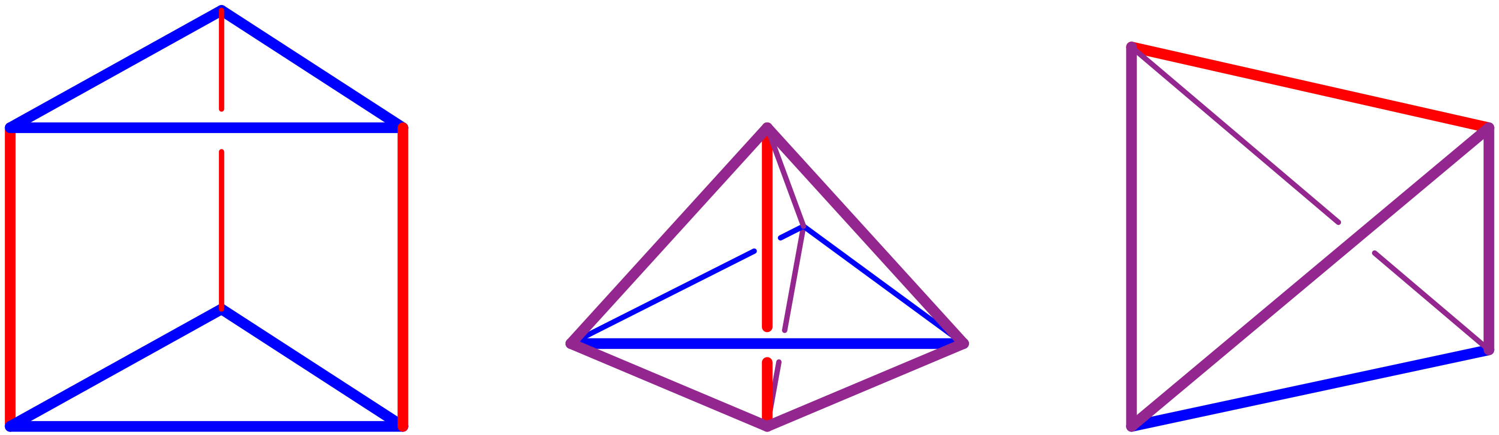
EXM: F-VECTORS & POLYTOPE OPERATIONS



$\mathbb{P} \times \mathbb{P}'$ $\quad$ $\mathbb{P} \oplus \mathbb{P}'$ $\quad$ $\mathbb{P} * \mathbb{P}'$

QU. Express the f-vectors of the Cartesian product $\mathbb{P} \times \mathbb{P}'$, the direct sum $\mathbb{P} \oplus \mathbb{P}'$ and the join $\mathbb{P} * \mathbb{P}'$ in terms of that of $\mathbb{P}$ and $\mathbb{P}'$.

EXM: F-VECTORS & POLYTOPE OPERATIONS



$\mathbb{P} \times \mathbb{P}'$ $\qquad$ $\mathbb{P} \oplus \mathbb{P}'$ $\qquad$ $\mathbb{P} * \mathbb{P}'$

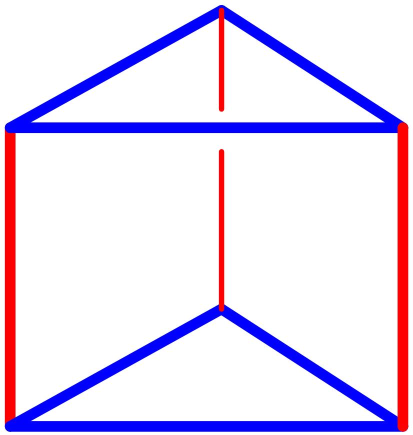
PROP. The f-vectors and f-polynomials of the Cartesian product $\mathbb{P} \times \mathbb{P}'$, the direct sum $\mathbb{P} \oplus \mathbb{P}'$ and the join $\mathbb{P} * \mathbb{P}'$ are given by

$$f_i(\mathbb{P} \times \mathbb{P}') = \sum_{j+j'=i} f_j(\mathbb{P}) \cdot f_{j'}(\mathbb{P}') \qquad f(\mathbb{P} \times \mathbb{P}', x) = f(\mathbb{P}, x) \cdot f(\mathbb{P}', x)$$

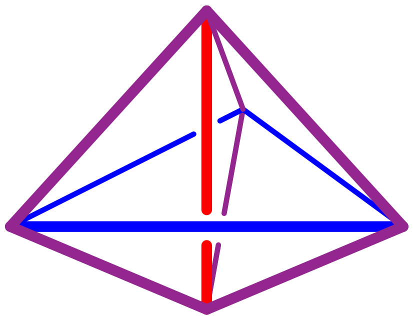
$$f_i(\mathbb{P} \oplus \mathbb{P}') = \sum_{\substack{j<d,\ j'<d' \\ j+j'=i-1}} f_j(\mathbb{P}) \cdot f_{j'}(\mathbb{P}') \qquad \bar{f}(\mathbb{P} \oplus \mathbb{P}', x) = \bar{f}(\mathbb{P}, x) \cdot \bar{f}(\mathbb{P}', x)$$

$$f_i(\mathbb{P} * \mathbb{P}') = \sum_{j+j'=i-1} f_j(\mathbb{P}) \cdot f_{j'}(\mathbb{P}') \qquad F(\mathbb{P} * \mathbb{P}', x) = F(\mathbb{P}, x) \cdot F(\mathbb{P}', x)$$

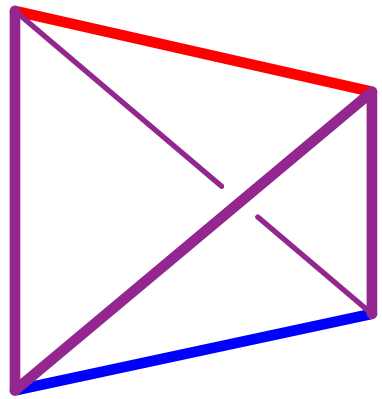
EXM: F-VECTORS & POLYTOPE OPERATIONS



$\mathbb{P} \times \mathbb{P}'$



$\mathbb{P} \oplus \mathbb{P}'$



$\mathbb{P} * \mathbb{P}'$

PROP. The f-vectors and f-polynomials of the Cartesian product $\mathbb{P} \times \mathbb{P}'$, the direct sum $\mathbb{P} \oplus \mathbb{P}'$ and the join $\mathbb{P} * \mathbb{P}'$ are given by

$$f(\mathbb{P} \times \mathbb{P}', x) = f(\mathbb{P}, x) \cdot f(\mathbb{P}', x)$$
$$\bar{f}(\mathbb{P} \oplus \mathbb{P}', x) = \bar{f}(\mathbb{P}, x) \cdot \bar{f}(\mathbb{P}', x)$$
$$F(\mathbb{P} * \mathbb{P}', x) = F(\mathbb{P}, x) \cdot F(\mathbb{P}', x)$$

remark: sanity check on classical polytopes

$$f(\square_d, x) = (x+2)^d \qquad \bar{f}(\Diamond_d, x) = (2x+1)^d \qquad F(\triangle_d, x) = (x+1)^{d+1}$$

HANNER POLYTOPES

DEF. Hanner polytope $=$ either the segment $I = [-1, 1]$ or a Cartesian product or direct sum of Hanner polytopes.

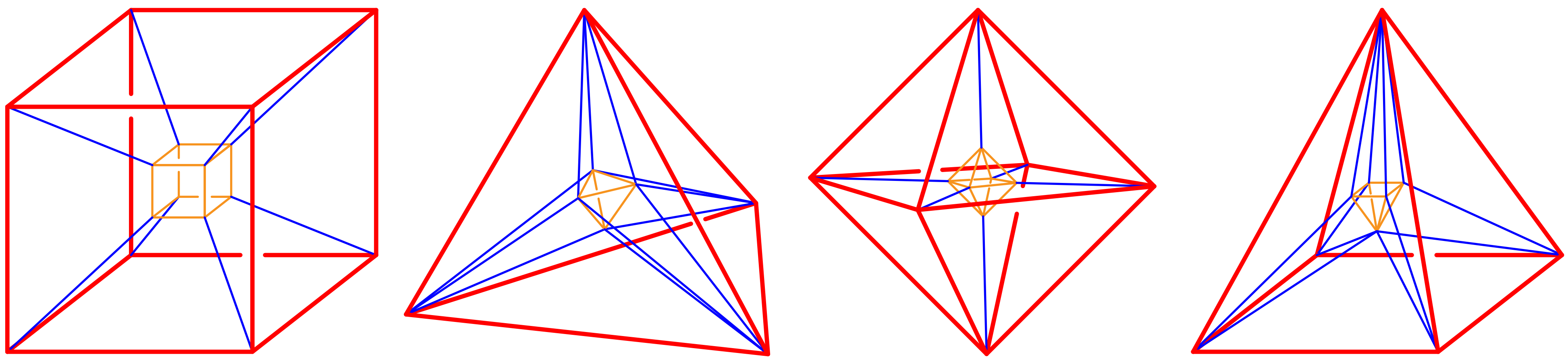
QU. What are the Hanner polytopes of dimension 1, 2, 3, 4?
Are all Hanner polytopes prisms or bipyramid?

HANNER POLYTOPES

DEF. Hanner polytope $=$ either the segment $I = [-1, 1]$ or a Cartesian product or direct sum of Hanner polytopes.

EXM. The small dimensional Hanner polytopes are:

- $d = 1$: interval I,
- $d = 2$: square $I \oplus I \sim I \times I$,
- $d = 3$: cube $I^{\times 3} := I \times I \times I$ and cross-polytope $I^{\oplus 3} := I \oplus I \oplus I$,
- $d = 4$: cube $I^{\times 4}$, cross-polytope $I^{\oplus 4}$, prism over an octahedron $I^{\oplus 3} \times I$ and bipyramid over a cube $I^{\times 3} \oplus I$.



(Schlegel diagrams...)

HANNER POLYTOPES

DEF. Hanner polytope $=$ either the segment $I = [-1, 1]$ or a Cartesian product or direct sum of Hanner polytopes.

EXM. The small dimensional Hanner polytopes are:

- $d = 1$: interval I,
- $d = 2$: square $I \oplus I \sim I \times I$,
- $d = 3$: cube $I^{\times 3} \coloneqq I \times I \times I$ and cross-polytope $I^{\oplus 3} \coloneqq I \oplus I \oplus I$,
- $d = 4$: cube $I^{\times 4}$, cross-polytope $I^{\oplus 4}$, prism over an octahedron $I^{\oplus 3} \times I$ and bipyramid over a cube $I^{\times 3} \oplus I$.

REM. The Hanner polytope $P \coloneqq (I \times I \times I) \oplus (I \times I \times I)$ cannot be

- a bipyramid: it has 16 vertices each of degree 11,
- a prism: it has 36 facets each of degree 8.

3^D CONJECTURE

DEF. Hanner polytope $=$ either the segment $I = [-1, 1]$ or a Cartesian product or direct sum of Hanner polytopes.

PROP. For any d-dimensional Hanner polytope $\mathbb{H}$,

$$\sum_{i=0}^{d} f_i(\mathbb{H}) = 3^d.$$

proof: $\sum_{i=0}^{d} f_i(\mathbb{H}) = f(\mathbb{H}, 1) = \bar{f}(\mathbb{H}, 1)$ together with

$$f(\mathbb{P} \times \mathbb{P}', x) = f(\mathbb{P}, x) \cdot f(\mathbb{P}', x) \qquad \text{and} \qquad \bar{f}(\mathbb{P} \oplus \mathbb{P}', x) = \bar{f}(\mathbb{P}, x) \cdot \bar{f}(\mathbb{P}', x).$$

CONJ. (Kalai's 3^d conjecture) If $\mathbb{P}$ is centrally symmetric (meaning $\mathbb{P} = -\mathbb{P}$), then

$$\sum_{i=0}^{d} f_i(\mathbb{P}) \geq 3^d,$$

with equality if and only if $\mathbb{P}$ is a Hanner polytope.

EULER RELATION

DEF. Euler characteristic $\chi(\mathbb{P}) = \sum_{i=0}^{d} (-1)^i f_i(\mathbb{P}) = f(\mathbb{P}, -1)$.

PROP. For any polytope $\mathbb{P}$ and hyperplane $\mathbb{H}$,

$$\chi(\mathbb{P}) = \chi(\mathbb{P}^+) + \chi(\mathbb{P}^-) - \chi(\mathbb{P}^\circ).$$

where $\mathbb{P}^+ = \mathbb{P} \cap \mathbb{H}^+$, $\mathbb{P}^- = \mathbb{P} \cap \mathbb{H}^-$ and $\mathbb{P}^\circ = \mathbb{P} \cap \mathbb{H}$.

PROP. For any polytopes $\mathbb{P}, \mathbb{Q} \subset \mathbb{R}^n$ st $\mathbb{P} \cup \mathbb{Q}$ is a polytope,

$$\chi(\mathbb{P} \cup \mathbb{Q}) + \chi(\mathbb{P} \cap \mathbb{Q}) = \chi(\mathbb{P}) + \chi(\mathbb{Q}).$$

remark: These conditions define weak valuations and strong valuations.
For polytopes, any weak valuation is a strong valuation.
Exm: indicator function, volume, number of integer points, etc.

EULER RELATION

DEF. Euler characteristic $\chi(\mathbb{P}) = \sum_{i=0}^{d} (-1)^i f_i(\mathbb{P}) = f(\mathbb{P}, -1)$.

THM. (Euler relation) $\chi(\mathbb{P}) = f_0(\mathbb{P}) - f_1(\mathbb{P}) + \cdots + (-1)^d f_d(\mathbb{P}) = 1$.

proof: Induction on the dimension.

EULER RELATION

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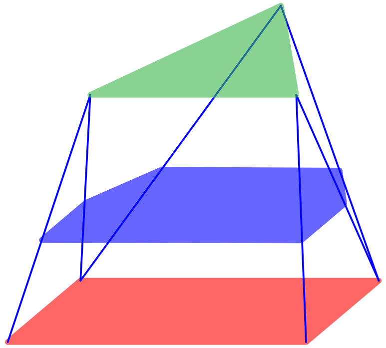
THM. (Euler relation) $\chi(\mathbb{P}) = f_0(\mathbb{P}) - f_1(\mathbb{P}) + \cdots + (-1)^d f_d(\mathbb{P}) = 1$.

proof: Induction on the dimension.

1. Observe first that it holds for Cayley polytopes (in particular for pyramids):



$$\chi(\mathbb{C}\mathrm{ay}(\mathbb{P}, \mathbb{R})) = \chi(\mathbb{P}) + \chi(\mathbb{R}) + (-1) \cdot \chi(\mathbb{Q})$$
$$= 1 + 1 - 1 = 1$$

EULER RELATION

DEF. Euler characteristic $\chi(\mathbb{P}) = \sum_{i=0}^{d} (-1)^i f_i(\mathbb{P}) = f(\mathbb{P}, -1)$.

THM. (Euler relation) $\chi(\mathbb{P}) = f_0(\mathbb{P}) - f_1(\mathbb{P}) + \dots + (-1)^d f_d(\mathbb{P}) = 1$.

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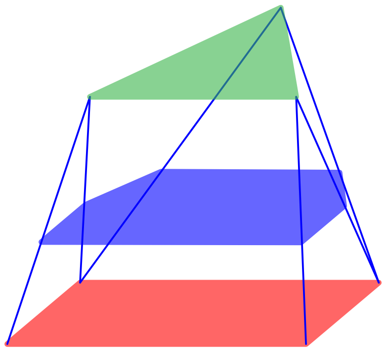
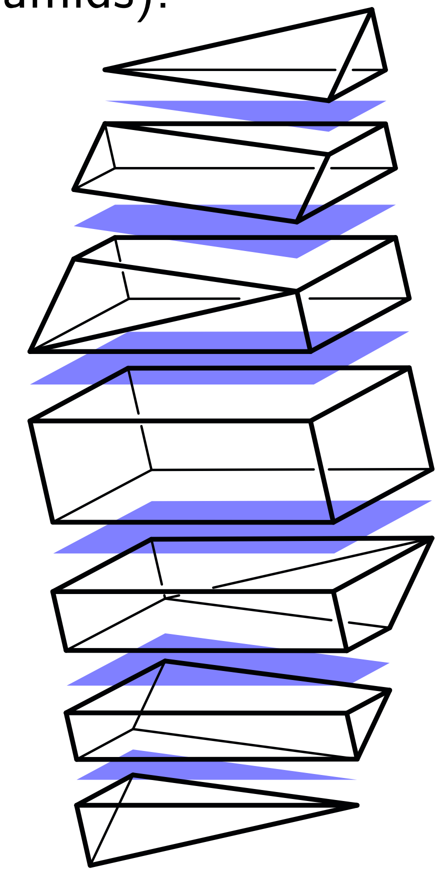
1. Observe first that it holds for Cayley polytopes (in particular for pyramids):

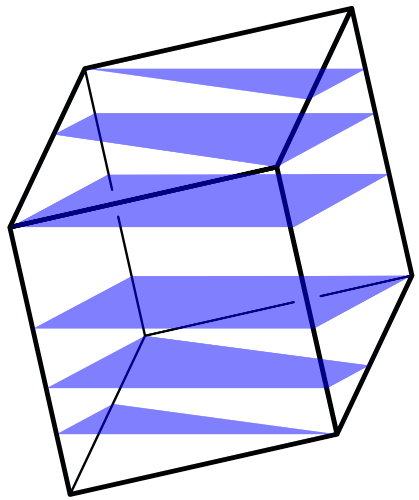


$$\begin{aligned}\chi(\mathbb{C}\mathrm{ay}(\mathbb{P}, \mathbb{R})) &= \chi(\mathbb{P}) + \chi(\mathbb{R}) + (-1) \cdot \chi(\mathbb{Q}) \\ &= 1 + 1 - 1 = 1\end{aligned}$$



2. Choose a Morse function ϕ,
slice the polytope $\mathbb{P}$ into Cayley polytopes,
and apply the valuation property:

$$\begin{aligned}\chi(\mathbb{P}) &= \chi(\mathbb{P}_0) - \chi(\mathbb{S}_1) + \dots - \chi(\mathbb{S}_k) + \chi(\mathbb{P}_k) \\ &= 1 - 1 + \dots - 1 + 1 = 1\end{aligned}$$



EULER RELATION

DEF. Euler characteristic $\chi(\mathbb{P}) = \sum_{i=0}^{d} (-1)^i f_i(\mathbb{P}) = f(\mathbb{P}, -1)$.

THM. (Euler relation) $\chi(\mathbb{P}) = f_0(\mathbb{P}) - f_1(\mathbb{P}) + \cdots + (-1)^d f_d(\mathbb{P}) = 1$.

PROP. Let $\mathbb{P}_{i,d} = \mathbb{P}\mathrm{yr}^{d-i}(\square_i)$ for $i \in [d]$. The f-vectors $f(\mathbb{P}_{i,d})$ are affinely independent.

proof: induction on the dimension d.

Affine dependance among f-vectors $\longleftrightarrow$ affine dependance among F-polynomials.

$\mathbb{P}_{i,d} = \square_i * \triangle_{d-i} \implies F(\mathbb{P}_{i,d}, x) = F(\square_i, x) \cdot F(\triangle_{d-i}, x) = (1 + x(x+2)^i) \cdot (x+1)^{d-i+1}$.

Assume $0 = \sum_{i=0}^{d} \lambda_i F(\mathbb{P}_{i,d}, x)$. Two cases:

- if $\lambda_d = 0$, then $0 = \sum_{i=0}^{d-1} \lambda_i F(\mathbb{P}_{i,d}, x) = (x+1) \cdot \sum_{i=0}^{d-1} \lambda_i F(\mathbb{P}_{i,d-1}, x)$ and induction.
- if $\lambda_d \neq 0$, then $F(\mathbb{P}_{d,d}, x) = -\sum_{i=0}^{d-1} \lambda_i/\lambda_d F(\mathbb{P}_{i,d}, x)$

$$(1 + x(x+2)^d) \cdot (x+1) = -(x+1)^2 \cdot \sum_{i=0}^{d-1} \lambda_i/\lambda_d (1 + x(x+2)^i) \cdot (x+1)^{d-i-1}$$

a contradiction since -1 is a simple root on the left and a double root on the right.

EULER RELATION

DEF. Euler characteristic $\chi(\mathbb{P}) = \sum_{i=0}^{d} (-1)^i f_i(\mathbb{P}) = f(\mathbb{P}, -1)$.

THM. (Euler relation) $\chi(\mathbb{P}) = f_0(\mathbb{P}) - f_1(\mathbb{P}) + \cdots + (-1)^d f_d(\mathbb{P}) = 1$.

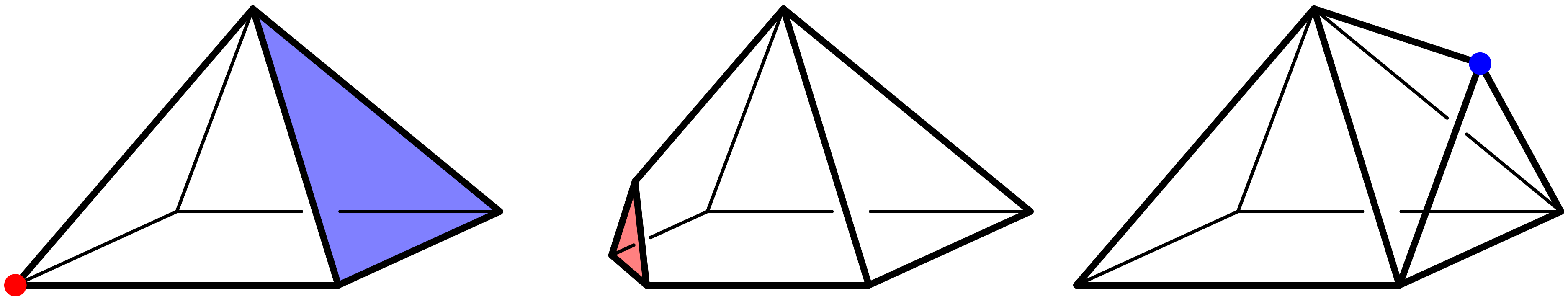
PROP. Let $\mathbb{P}_{i,d} = \mathrm{Pyr}^{d-i}(\square_i)$ for $i \in [d]$. The f-vectors $f(\mathbb{P}_{i,d})$ are affinely independent.

CORO. The Euler relation is the only relation among f-vectors of general polytopes.

F-VECTORS OF 3-POLYTOPES

QU. Describe the effect on the f-vector of the following (polar) operations:

- simple vertex truncation: cut a vertex whose vertex figure is a simplex,
- simplicial facet stacking: stack a vertex on a facet which is a simplex.



QU. What is the f-vector of a pyramid over a p-gon?

QU. Prove that the f-vectors of 3-polytopes are the integer vectors $(f_0, f_1, f_2, 1)$ st

$$f_0 - f_1 + f_2 = 2 \qquad f_0 \le 2f_2 - 4 \qquad \text{and} \qquad f_2 \le 2f_0 - 4.$$

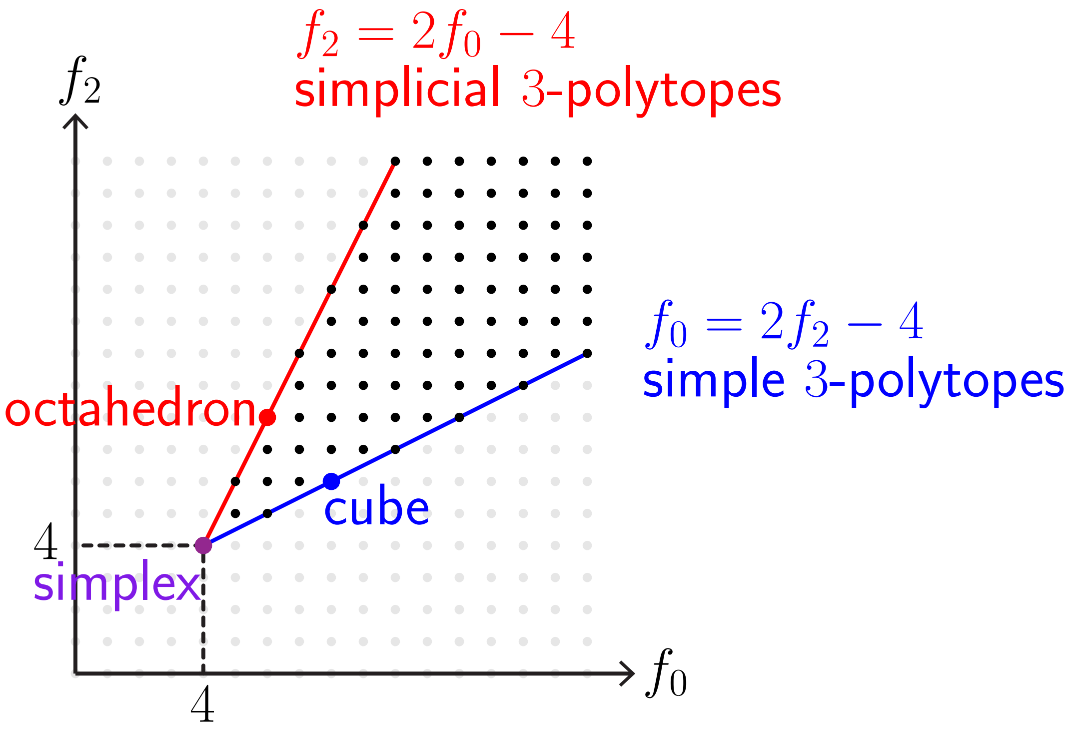
F-VECTORS OF 3-POLYTOPES

THM. The f-vectors of 3-polytopes are the integer vectors $(f_0, f_1, f_2, 1)$ st

$$f_0 - f_1 + f_2 = 2 \qquad f_0 \le 2f_2 - 4 \qquad \text{and} \qquad f_2 \le 2f_0 - 4.$$

proof: For one direction, combine the inequalities

- $f_0 - f_1 + f_2 = 2$ (Euler relation),
- $2f_1 \ge 3f_0$ (every vertex is contained in at least 3 edges, every edge contains 2 vertices),
- $2f_1 \ge 3f_2$ (every face contains at least 3 edges, every edge is contained in 2 faces).



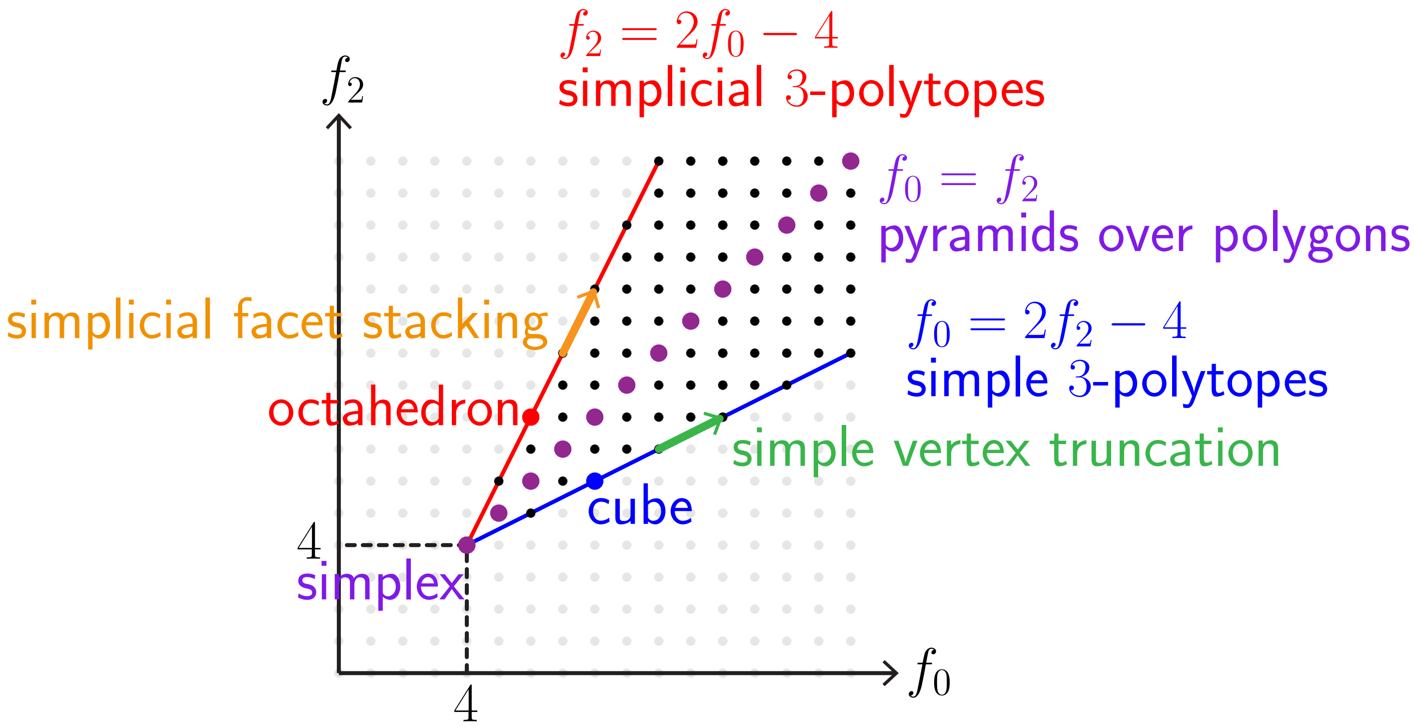
F-VECTORS OF 3-POLYTOPES

THM. The f-vectors of 3-polytopes are the integer vectors $(f_0, f_1, f_2, 1)$ st

$$f_0 - f_1 + f_2 = 2 \qquad f_0 \leq 2f_2 - 4 \qquad \text{and} \qquad f_2 \leq 2f_0 - 4.$$

proof: For the other direction, observe that

- the f-vector of a pyramid over a p-gon is $(p+1, 2p, p+1, 1)$,
- a simple vertex truncation adds $(2, 3, 1, 0)$ to the f-vector,
- a simplicial facet stacking adds $(1, 3, 2, 0)$ to the f-vector.



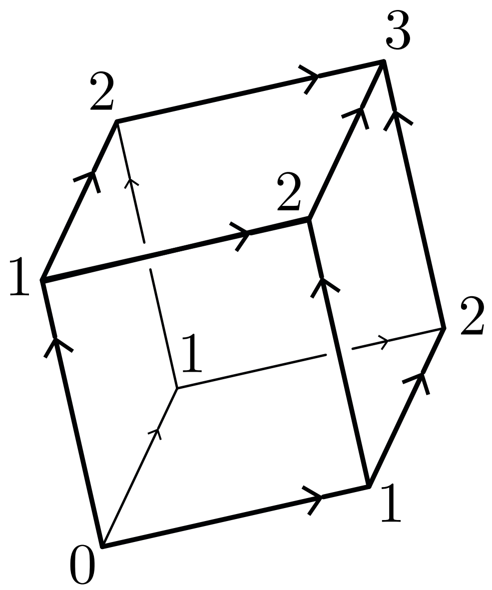
H-VECTOR & DEHN-SOMMERVILLE RELATIONS

H-VECTOR & H-POLYNOMIAL

DEF. A d-polytope is simple if each vertex is contained in d facets, or equiv. d edges.

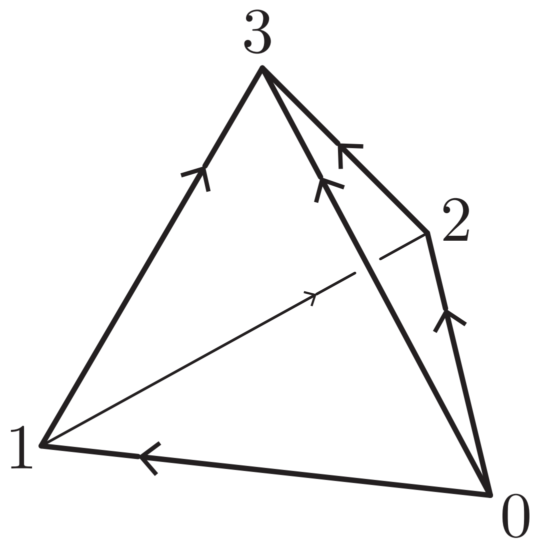
DEF. $\mathbb{P}$ = simple d-polytope,
ϕ = Morse function ($\phi(u) \neq \phi(v)$ for any edge (u, v) of $\mathbb{P}$)
Orient the edges of $\mathbb{P}$ according to ϕ and define

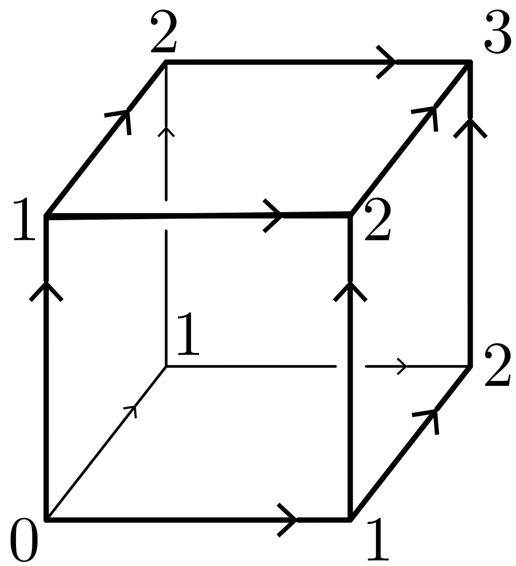
- $h_j(\mathbb{P})$ = number of vertices of $\mathbb{P}$ with indegree j,
- h-vector $h(\mathbb{P}) = \big(h_0(\mathbb{P}), \dots, h_d(\mathbb{P})\big)$,
- h-polynomial $h(\mathbb{P}, x) = \sum_{j=0}^{d} h_j(\mathbb{P})\, x^j$.



$$h(\square_3) = 1 + 3\,x + 3\,x^2 + x^3$$

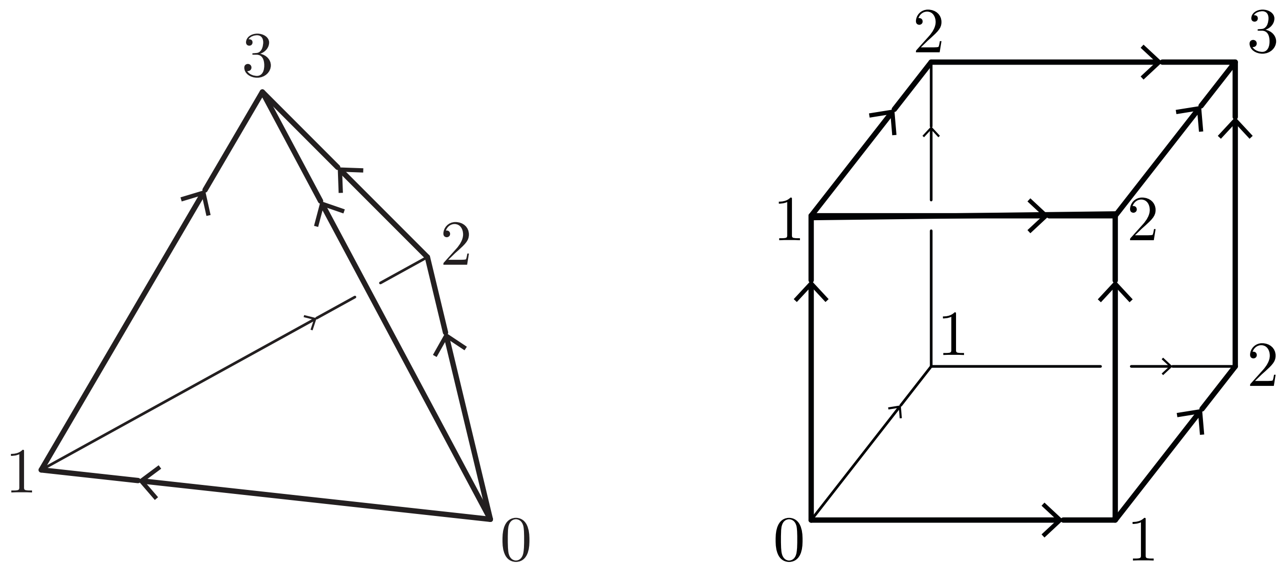
EXM: F-VECTOR OF CLASSICAL POLYTOPES





QU. Compute the h-vectors and h-polynomials of the d-simplex $\triangle_d$ and the d-cube $\square_d$.

EXM: F-VECTOR OF CLASSICAL POLYTOPES



PROP. The h-vectors and h-polynomials of the d-simplex $\triangle_d$ and the d-cube $\square_d$ are given by

$$h_j(\triangle_d) = 1 \qquad h_j(\square_d) = \binom{d}{j}$$

$$h(\triangle_d, x) = \sum_{j=0}^{d} x^j = \frac{x^{d+1} - 1}{x - 1} \qquad h(\square_d, x) = \sum_{j=0}^{d} \binom{d}{j} x^j = (x+1)^d$$

F-VECTOR VS H-VECTOR

THM. The f-vector and h-vector of any simple d-polytope $\mathbb{P}$ are related by

$$f_i(\mathbb{P}) = \sum_{j=0}^{d} \binom{j}{i} h_j(\mathbb{P}) \qquad \text{and} \qquad h_j(\mathbb{P}) = \sum_{i=0}^{d} (-1)^{i+j} \binom{i}{j} f_i(\mathbb{P})$$

and the f-polynomial and h-polynomial are related by

$$f(\mathbb{P}, x) = h(\mathbb{P}, x+1) \qquad \text{and} \qquad h(\mathbb{P}, x) = f(\mathbb{P}, x-1).$$

remark: sanity check on classical polytopes

$$f(\triangle_d, x) = \frac{(x+1)^{d+1} - 1}{x} = h(\triangle_d, x+1) \quad \text{and} \quad f(\square_d, x) = (x+2)^d = h(\square_d, x+1)$$

F-VECTOR VS H-VECTOR

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and the f-polynomial and h-polynomial are related by

$$f(\mathbb{P}, x) = h(\mathbb{P}, x+1) \qquad \text{and} \qquad h(\mathbb{P}, x) = f(\mathbb{P}, x-1).$$

proof: double counting the set $\mathcal{S}(i,\phi)$ of pairs $(\boldsymbol{v}, \mathbb{F})$ where $\mathbb{F}$ is an i-face of $\mathbb{P}$ and $\boldsymbol{v}$ is the ϕ-maximal vertex of $\mathbb{F}$:

$$f_i(\mathbb{P}) = \sum_{\mathbb{F} \in \mathcal{F}_i(\mathbb{P})} 1 = |\mathcal{S}(i,\phi)| = \sum_{\boldsymbol{v} \in \mathcal{F}_0(\mathbb{P})} \binom{\operatorname{indeg}(\boldsymbol{v})}{i} = \sum_{j=0}^{d} \binom{j}{i} h_j(\mathbb{P}).$$

This implies all other relations by the following lemma...

F-VECTOR VS H-VECTOR

QU. $$f_i = \sum_{j=0}^{d} \binom{j}{i} h_j \quad \Longleftrightarrow \quad f(x) = h(x+1) \quad \Longleftrightarrow \quad h_j = \sum_{i=0}^{d} (-1)^{i+j} \binom{i}{j} f_i.$$

F-VECTOR VS H-VECTOR

LEM. $$f_i = \sum_{j=0}^{d} \binom{j}{i} h_j \quad \Longleftrightarrow \quad f(x) = h(x+1) \quad \Longleftrightarrow \quad h_j = \sum_{i=0}^{d} (-1)^{i+j} \binom{i}{j} f_i.$$

proof:

$$f_i = \sum_{j=0}^{d} \binom{j}{i} h_j$$

$$\Updownarrow$$

$$\begin{aligned} h(x+1) &= \sum_{j=0}^{d} h_j (x+1)^j \\ &= \sum_{j=0}^{d} h_j \sum_{i=0}^{j} \binom{j}{i} x^i \\ &= \sum_{i=0}^{d} \left(\sum_{j=0}^{d} \binom{j}{i} h_j \right) x^i \\ &= \sum_{i=0}^{d} f_i x^i = f(x). \end{aligned}$$

$$h_j = \sum_{i=0}^{d} (-1)^{i+j} \binom{i}{j} f_i$$

$$\Updownarrow$$

$$\begin{aligned} f(x-1) &= \sum_{i=0}^{d} f_i (x-1)^i \\ &= \sum_{i=0}^{d} f_i \sum_{j=0}^{d} \binom{i}{j} (-1)^{i+j} x^j \\ &= \sum_{j=0}^{d} \left(\sum_{i=0}^{d} (-1)^{i+j} \binom{i}{j} f_i \right) x^j \\ &= \sum_{j=0}^{d} h_j x^j = h(x) \end{aligned}$$

DEHN-SOMMERVILLE RELATIONS

THM. (Dehn-Sommerville relations)
The h-vector of a simple d-polytope $\mathbb{P}$ is symmetric:

$$h_j(\mathbb{P}) = h_{d-j}(\mathbb{P}) \qquad \text{for all } 0 \leq j \leq d.$$

In terms of f-vectors,

$$\sum_{i=j}^{d} (-1)^{i+j} \binom{i}{j} f_i(\mathbb{P}) = \sum_{i=d-j}^{d} (-1)^{d+i-j} \binom{i}{d-j} f_i(\mathbb{P}) \qquad \text{for all } 0 \leq j \leq d.$$

proof: consider the Morse functions ϕ and $-\phi$ ...
A degree with ϕ-indegree j has $(-\phi)$-indegree $d-j$.

remark: for $j = 0$, $h_0(\mathbb{P}) = h_d(\mathbb{P})$ is the Euler relation.

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PROP. The f-vectors $f(\mathbb{C}\mathrm{yc}_{d,d+i})$ for $i \in [\lfloor d/2 \rfloor + 1]$ are affinely independent.

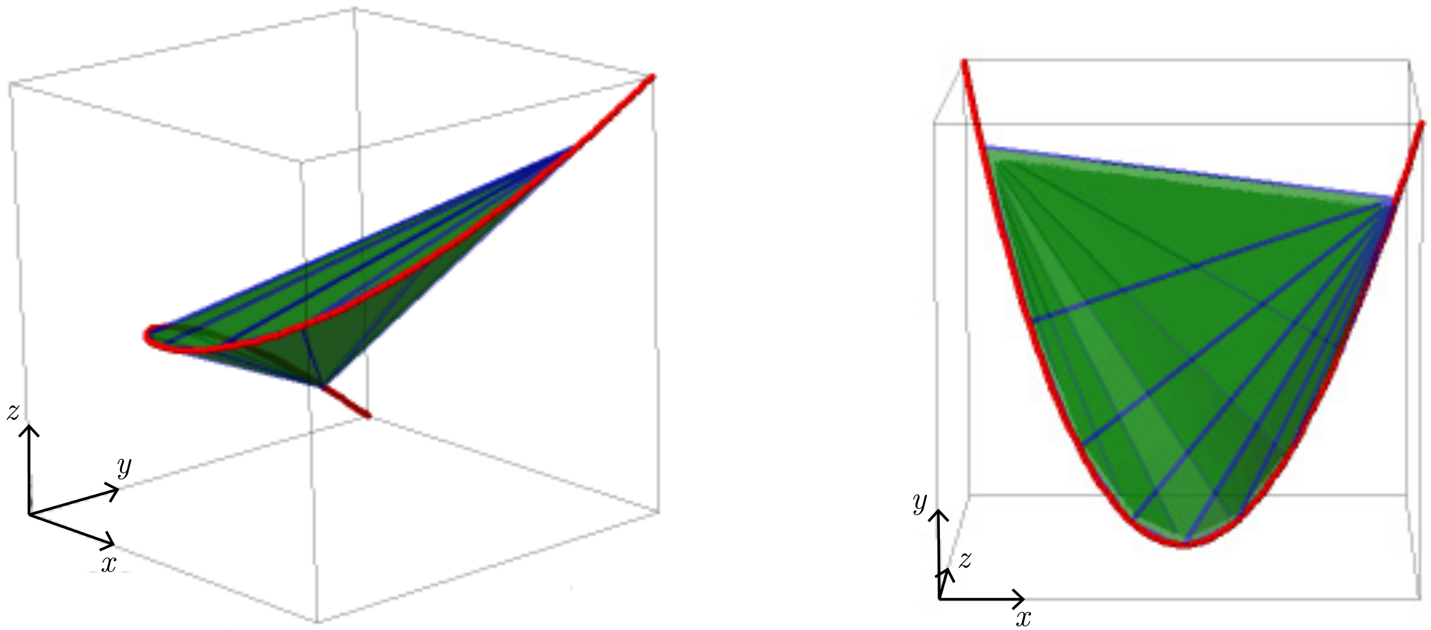
CORO. The Dehn-Sommerville relations are the only relations among f-vectors of simple polytopes.

MANY FACES: CYCLIC POLYTOPES

MOMENT CURVE & CYCLIC POLYTOPES

DEF. moment curve = curve parametrized by $\mu_d : t \mapsto (t, t^2, \dots, t^d) \in \mathbb{R}^d$.
cyclic polytope $\mathbb{C}\mathrm{yc}_d(n) = \mathrm{conv}\,\{\mu_d(t_i) \mid i \in [n]\}$ for arbitrary reals $t_1 < \dots < t_n$.

exm: two views of $\mathbb{C}\mathrm{yc}_3(9)$



remark: we will see later that the combinatorics of $\mathbb{C}\mathrm{yc}_d(n)$ is independent of $t_1 < \dots < t_n$.

CYCLIC POLYTOPES ARE NEIGHBORLY

DEF. moment curve = curve parametrized by $\mu_d : t \mapsto (t, t^2, \dots, t^d) \in \mathbb{R}^d$.
cyclic polytope $\mathbb{C}\mathrm{yc}_d(n) = \mathrm{conv}\,\{\mu_d(t_i) \mid i \in [n]\}$ for arbitrary reals $t_1 < \dots < t_n$.

THM. The cyclic polytope $\mathbb{C}\mathrm{yc}_d(n)$ is

- simplicial: all facets are simplices,
- neighborly: all j-subsets of vertices define a $(j-1)$-face of $\mathbb{C}\mathrm{yc}_d(n)$ for $j \le \lfloor d/2 \rfloor$.

proof: use polynomials!

- If $\mu_d(s_1), \dots, \mu_d(s_{d+1})$ belong to an affine hyperplane $\sum_{i\in[d]} \alpha_i\, x_i = -\alpha_0$, then $s_1, \dots, s_{d+1}$ are all roots of the polynomial $\sum_{i=0}^{d} \alpha_i\, t^i$. A contradiction.
- For $j \le \lfloor d/2 \rfloor$ and $s_1, \dots, s_j \in \{t_1, \dots, t_n\}$, the polynomial $\sum_{i=0}^{d} \alpha_i\, t^i = \prod_{i\in[j]} (t - s_i)^2$ is non-negative and vanishes on $s_1, \dots, s_j$. Thus the hyperplane $\sum_{i\in[d]} \alpha_i\, x_i = -\alpha_0$ supports a face of $\mathbb{C}\mathrm{yc}_d(n)$ with vertices $\mu_d(s_1), \dots, \mu_d(s_j)$.

H-VECTORS OF POLAR CYCLIC POLYTOPES

CORO. The polar of the cyclic polytope $\mathbb{C}\mathrm{yc}_d(n)^\diamond$ is simple and its h-vector is given by

$$h_j = \binom{n-d+j-1}{j} \text{ for } j \le \left\lfloor \frac{d}{2} \right\rfloor \quad \text{and} \quad h_j = \binom{n-j-1}{d-j} \text{ for } j > \left\lfloor \frac{d}{2} \right\rfloor.$$

proof: $\mathbb{C}\mathrm{yc}_d(n)$ is neighborly $\Longrightarrow f_i\big(\mathbb{C}\mathrm{yc}_d(n)\big) = \binom{n}{i}$ for $i \le \lfloor d/2 \rfloor$

$$\Longrightarrow f_i\big(\mathbb{C}\mathrm{yc}_d(n)^\diamond\big) = \binom{n}{d-i} \text{ for } i > \lfloor d/2 \rfloor.$$

Therefore

$$h_j\big(\mathbb{C}\mathrm{yc}_d(n)^\diamond\big) = \sum_{i=j}^{d} (-1)^{i+j} \binom{i}{j} \binom{n}{d-i} = \binom{n-j-1}{d-j}. \qquad \text{if } j > \left\lfloor \frac{d}{2} \right\rfloor \qquad (\star)$$

$$= h_{d-j}\big(\mathbb{C}\mathrm{yc}_d(n)^\diamond\big) = \binom{n-d+j-1}{j} \qquad \text{if } j \le \left\lfloor \frac{d}{2} \right\rfloor$$

For $(\star)$, check that

- it holds when $j = 0$ and $j = d$, and
- if it holds for (j, d) and $(j+1, d)$ then it holds for $(j+1, d+1)$.

UPPER BOUND THEOREM

THM. (Upper Bound Theorem, McMullen) For any d-polytope $\mathbb{P}$ with n vertices:

$$f_i(\mathbb{P}) \le f_i(\mathbb{C}\mathrm{yc}_d(n)).$$

remark:

- clear for $i \le \lfloor d/2 \rfloor$ since $f_i(\mathbb{C}\mathrm{yc}_d(n)) = \binom{n}{i+1}$,
- equivalent to polar version $f_i(\mathbb{P}) \le f_i(\mathbb{C}\mathrm{yc}_d(n)^\diamond)$ for any d-polytope $\mathbb{P}$ with n facets,
- enough to prove it for simplicial/simple polytopes,
- thus implied by h-vector version:

THM. (Upper Bound Theorem, McMullen) For any simple d-polytope $\mathbb{P}$ with n facets:

$$h_j(\mathbb{P}) \le \binom{n-d+j-1}{j} \text{ for } j \le \left\lfloor \frac{d}{2} \right\rfloor \quad \text{and} \quad h_j(\mathbb{P}) \le \binom{n-j-1}{d-j} \text{ for } j > \left\lfloor \frac{d}{2} \right\rfloor.$$

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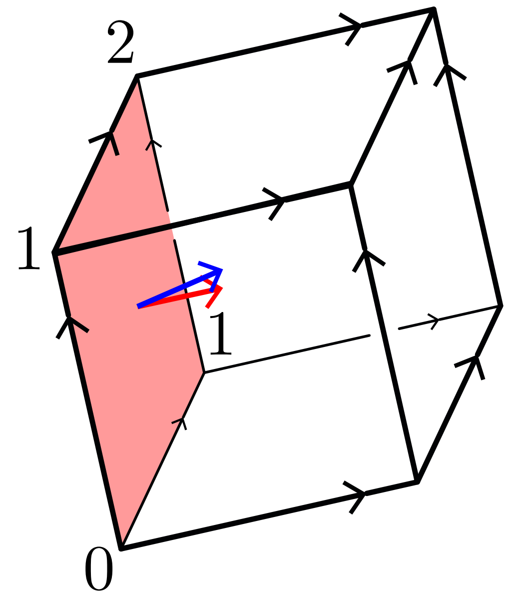
proof:

1. $h_i(\mathbb{F}) \leq h_i(\mathbb{P})$ for $\mathbb{F} \in \mathcal{F}_{d-1}(\mathbb{P})$
   ϕ obtained by perturbation of the inner normal of $\mathbb{F}$
   then $\operatorname{indeg}_{\mathbb{F}}(\boldsymbol{v}) = \operatorname{indeg}_{\mathbb{P}}(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathbb{F}$



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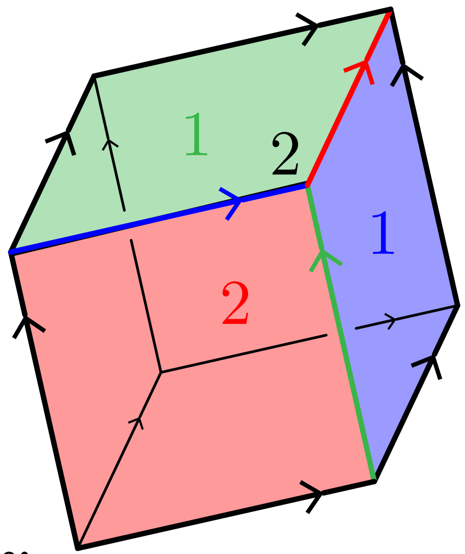
2. $\displaystyle\sum_{\mathbb{F} \in \mathcal{F}_{d-1}(\mathbb{P})} h_i(\mathbb{F}) = (d-i)\, h_i(\mathbb{P}) + (i+1)\, h_{i+1}(\mathbb{P})$
   Let $\boldsymbol{v} \in \mathbb{F}$, and e the edge of $\mathbb{P}$ st $\boldsymbol{v} \in e \not\subset \mathbb{F}$

   then $\mathrm{indeg}_{\mathbb{F}}(\boldsymbol{v}) = i \iff \begin{cases} \mathrm{indeg}_{\mathbb{P}}(\boldsymbol{v}) = i \text{ and } e \text{ leaving } \boldsymbol{v}, \text{ or} \\ \mathrm{indeg}_{\mathbb{P}}(\boldsymbol{v}) = i+1 \text{ and } e \text{ entering } \boldsymbol{v}. \end{cases}$



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proof:

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   ϕ obtained by perturbation of the inner normal of $\mathbb{F}$
   then $\mathrm{indeg}_{\mathbb{F}}(\boldsymbol{v}) = \mathrm{indeg}_{\mathbb{P}}(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathbb{F}$

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   Let $\boldsymbol{v} \in \mathbb{F}$, and e the edge of $\mathbb{P}$ st $\boldsymbol{v} \in e \not\subset \mathbb{F}$
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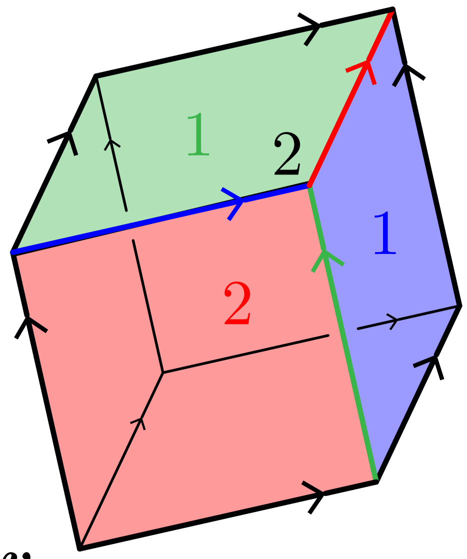


$1+2 \implies (d-i)\, h_i(\mathbb{P}) + (i+1)\, h_{i+1}(\mathbb{P}) \le n\, h_i(\mathbb{P}) \implies h_{i+1}(\mathbb{P}) \le \dfrac{n+d-i}{i+1} h_i(\mathbb{P}).$

and induction...

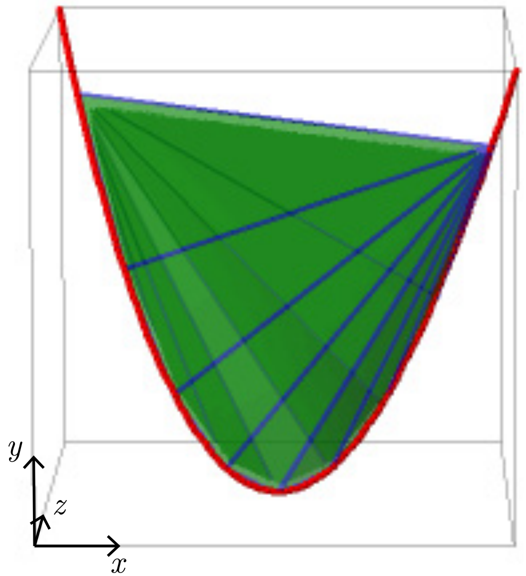
GALE'S EVENNESS CRITERION

DEF. For $I \subseteq [n] = \{1, \dots, n\}$, define

- block of I = intervals of I,
- even block of I = block of I of even size,
- internal block of I = block of I that does not contain 1 or n.

THM. (Gale's evenness criterion) For a d-subset I of $[n]$,
$\mathrm{conv}\,\{\mu_d(t_i) \mid i \in I\}$ is a facet of $\mathbb{C}\mathrm{yc}_d(n)$ $\iff$ all internal blocks of I are even.

exm: The facets $\mathbb{C}\mathrm{yc}_3(n)$ correspond to $\{i, i+1, n\}$ and $\{1, i+1, i+2\}$ for $i \in [n-2]$.



GALE'S EVENNESS CRITERION

DEF. For $I \subseteq [n] = \{1, \dots, n\}$, define

- block of I = maximal intervals of I,
- even block of I = block of I of even size,
- internal block of I = block of I that does not contain 1 or n.

THM. (Gale's evenness criterion) For a d-subset I of $[n]$,
$\operatorname{conv}\{\mu_d(t_i) \mid i \in I\}$ is a facet of $\mathbb{C}\mathrm{yc}_d(n)$ $\iff$ all internal blocks of I are even.

proof: For any $I = \{i_1, \dots, i_d\} \subseteq [n]$ and $k \in [n]$, the position of $\mu_d(t_k)$ with respect to the hyperplane $\mathbb{H}$ containing $\mu_d(t_{i_1}), \dots, \mu_d(t_{i_d})$ is given by the sign of the Vandermonde determinant

$$\det \begin{bmatrix} 1 & \dots & 1 & 1 \\ t_{i_1} & \dots & t_{i_d} & t_k \\ \vdots & \ddots & \vdots & \vdots \\ t_{i_1}^d & \dots & t_{i_d}^d & t_k^d \end{bmatrix} = \prod_{1 \le p < q \le d} (t_{i_q} - t_{i_p}) \prod_{1 \le p \le d} (t_k - t_{i_p}).$$

which is 0 if $k \in I$ and -1 to the parity of the number of $p \in [d]$ such that $i_p > k$. Therefore, all points $\mu_d(t_k)$ lie on the same side of $\mathbb{H}$ iff all internal blocks of I are even.

GALE'S EVENNESS CRITERION

THM. (Gale's evenness criterion) For a d-subset I of $[n]$,
$\mathrm{conv}\,\{\mu_d(t_i) \mid i \in I\}$ is a facet of $\mathbb{C}\mathrm{yc}_d(n)$ $\iff$ all internal blocks of I are even.

CORO. $\mathbb{C}\mathrm{yc}_d(n)$ is neighborly and independent of the choice of $t_1 < \cdots < t_n$.

proof:

- neighborly since for any $<\leq \lfloor d/2 \rfloor$, any j-subset can be completed into a d-subset satisfying Gale's evenness criterion (complete all odd blocks and add the remaining elements at the end).
- independent of the choice of $t_1 < \cdots < t_n$ since Gale's evenness criterion tells the vertices-facets incidences, which determine the whole combinatorics.

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CORO. $\mathbb{C}\mathrm{yc}_d(n)$ is neighborly and independent of the choice of $t_1 < \cdots < t_n$.

QU. Prove that $f_{d-1}\big(\mathbb{C}\mathrm{yc}_d(n)\big) = \binom{n - \lceil \frac{d}{2} \rceil}{\lfloor \frac{d}{2} \rfloor} + \binom{n - 1 - \lceil \frac{d-1}{2} \rceil}{\lfloor \frac{d-1}{2} \rfloor}$.

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CORO. $$f_{d-1}\big(\mathbb{C}\mathrm{yc}_d(n)\big) = \binom{n - \lceil \frac{d}{2} \rceil}{\lfloor \frac{d}{2} \rfloor} + \binom{n - 1 - \lceil \frac{d-1}{2} \rceil}{\lfloor \frac{d-1}{2} \rfloor}.$$

proof: number of $2k$-subsets of $[n]$ where all blocks are even $= \binom{n-k}{k}$

$$\circ \bullet \bullet \circ \bullet \bullet \bullet \bullet \circ \circ \bullet \bullet \qquad \longleftrightarrow \qquad \circ \bullet \circ \bullet \bullet \circ \circ \bullet$$

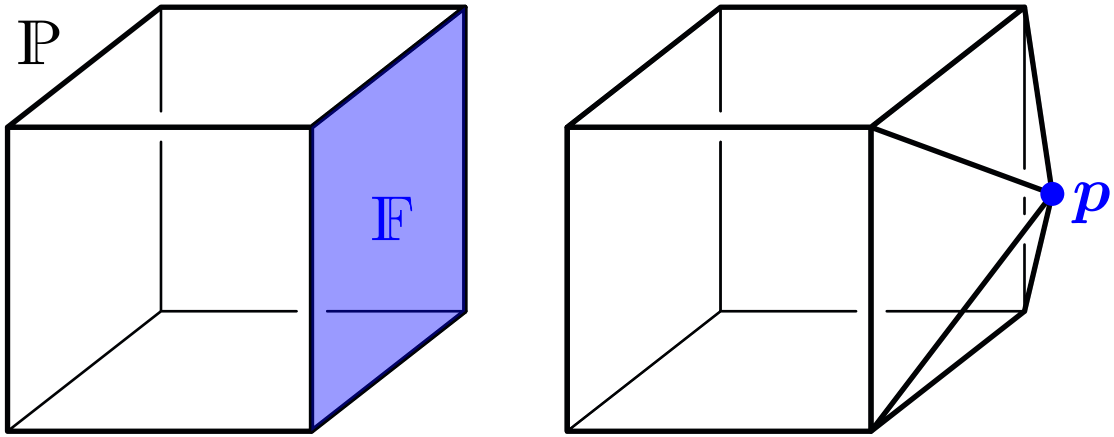
Then case analysis:

	1 in an odd block	otherwise
n in an odd block	d even $\binom{n-2-\frac{d-2}{2}}{\frac{d-2}{2}}$	d odd $\binom{n-1-\frac{d-1}{2}}{\frac{d-1}{2}}$
otherwise	d odd $\binom{n-1-\frac{d-1}{2}}{\frac{d-1}{2}}$	d even $\binom{n-\frac{d}{2}}{\frac{d}{2}}$

FEW FACES: STACKED POLYTOPES

STACKING OVER A FACET

DEF. stacking over a facet $\mathbb{F}$ of $\mathbb{P}$ = constructing $\mathrm{conv}(\mathbb{P} \cup \{\boldsymbol{p}\})$ where $\boldsymbol{p}$ is beyond $\mathbb{F}$ but beneath all other facets of $\mathbb{P}$.



QU. Express the f-vector of $\mathbb{P}' = \mathrm{conv}(\mathbb{P} \cup \{\boldsymbol{p}\})$ in terms of that of $\mathbb{P}$ and $\mathbb{F}$.

STACKING OVER A FACET

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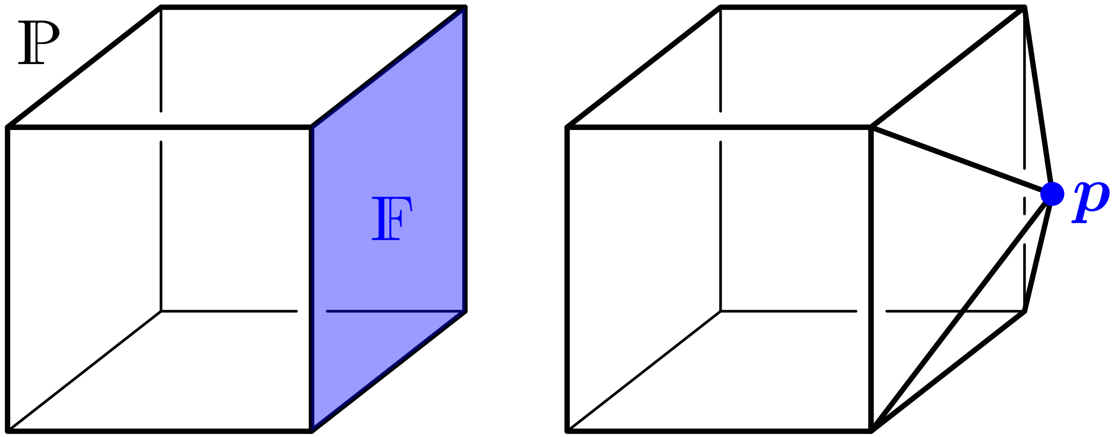
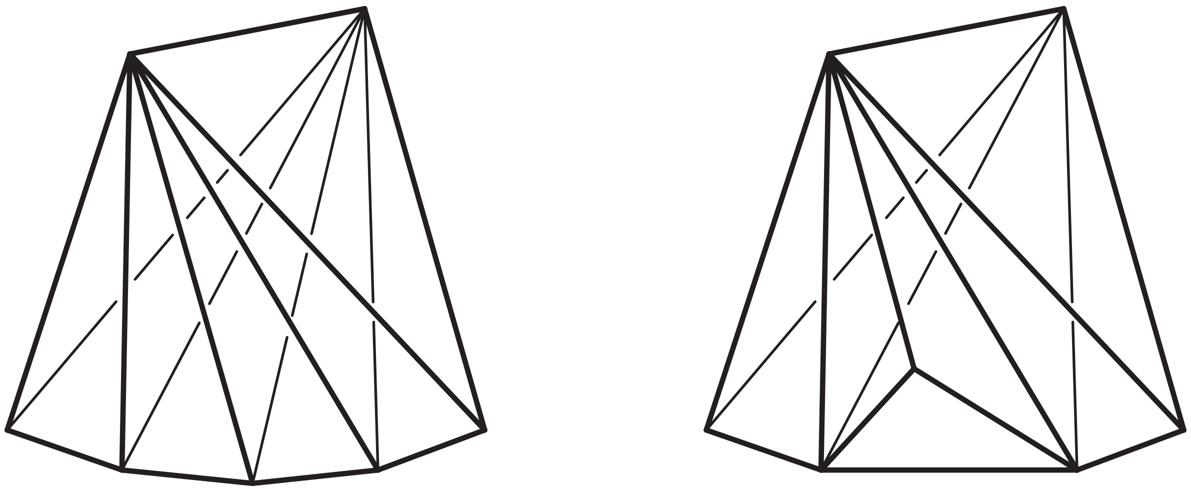


LEM. If $\mathbb{P}'$ is obtained from $\mathbb{P}$ by staking on $\mathbb{F}$, then

$$\begin{aligned} f_0(\mathbb{P}') &= f_0(\mathbb{P}) + 1, \\ f_i(\mathbb{P}') &= f_i(\mathbb{P}) + f_{i-1}(\mathbb{F}), \qquad \text{for } 0 \le i \le d-2, \\ f_{d-1}(\mathbb{P}') &= f_{d-1}(\mathbb{P}) + f_{d-2}(\mathbb{F}) - 1. \end{aligned}$$

STACKED POLYTOPES

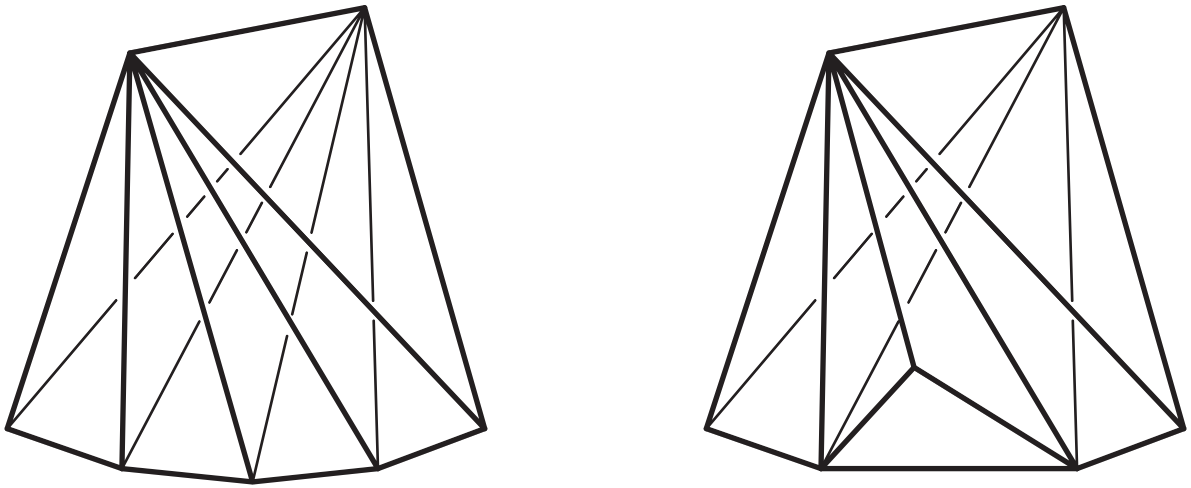
DEF. stacked polytope $=$ polytope arising from a d-simplex by stacking $(n-1)$ times.



QU. f-vector of stacked polytopes?

F-VECTORS OF STACKED POLYTOPES

DEF. stacked polytope $=$ polytope arising from a d-simplex by stacking n times.



LEM. The f-vector of a stacked polytope on $d+n$ vertices is

$$f_0 = d+1+n,$$

$$f_i = \binom{d+1}{i+1} + n\binom{d}{i} \quad \text{for } 0 \le i \le d-2,$$

$$f_{d-1} = d+1+n(d-1).$$

LOWER BOUND THEOREM

THM. (Lower Bound Theorem, Barnette) For any simplicial d-polytope $\mathbb{P}$ with n vertices:

$$f_i(\mathbb{P}) \geq f_i(\mathbb{Q})$$

where $\mathbb{Q}$ is a stacked polytope on n vertices.
Moreover, equality holds $\iff$ $d = 3$ or $d \geq 4$ and $\mathbb{P}$ is stacked.

GRAPHS OF POLYTOPES

POLYTOPE SKELETA

DEF. $\mathbb{P}$ d-polytope, $k \leq d$.
graph of $\mathbb{P}$ $=$ graph with same vertices and edges as $\mathbb{P}$.
k-skeleton of $\mathbb{P}$ $=$ all $\leq k$-dimensional faces of $\mathbb{P}$.

POLYTOPAL GRAPHS

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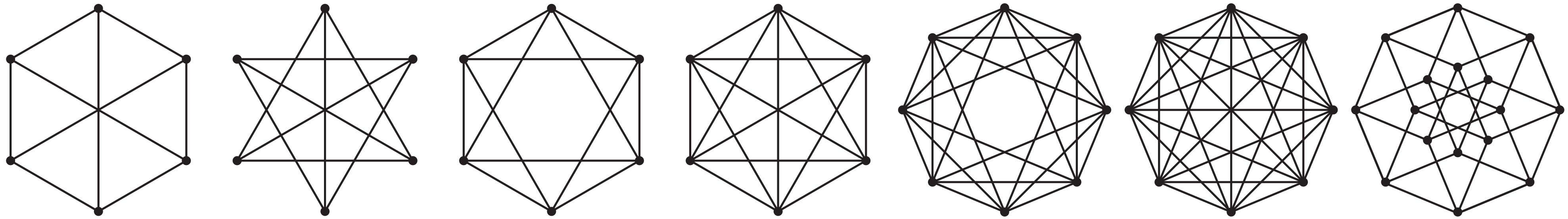
QU. Which of the following graphs are graphs of polytopes? In which dimension?



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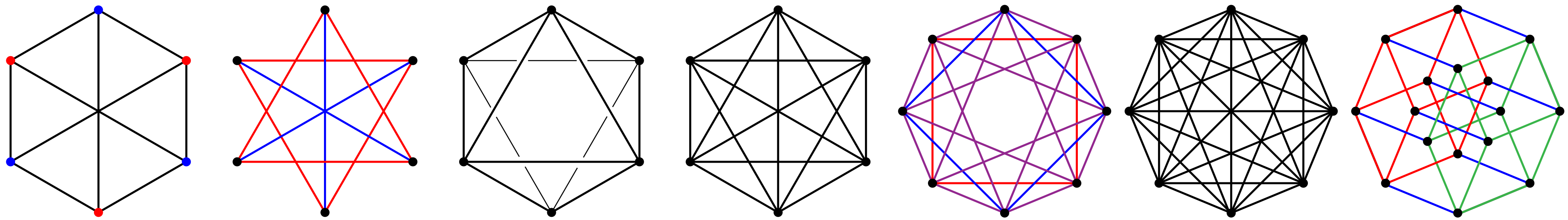
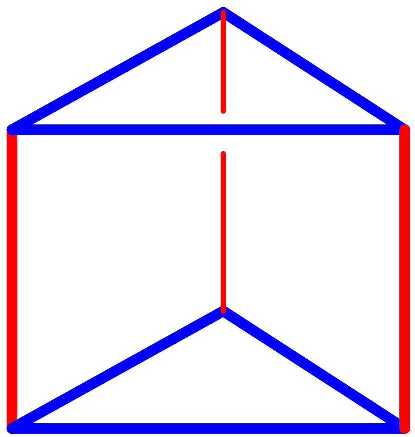
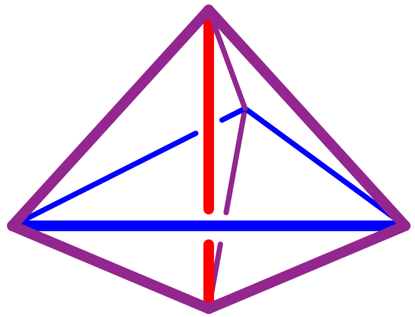


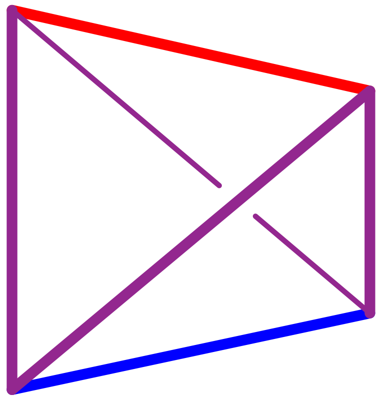
$\varnothing$	$\triangle_2 \times \triangle_1$	$\Diamond_3$	$\mathbb{C}\mathrm{yc}_4(6)$, $\triangle_5$	$\Diamond_4$, $\Diamond_2 * \Diamond_2$	$\mathbb{C}\mathrm{yc}_d(8)$	$\square_4$
$\varnothing$	3	3	$\{4, 5\}$	$\{4, 5\}$	$\{4, 5, 6, 7\}$	4



$\mathbb{P} \times \mathbb{P}'$



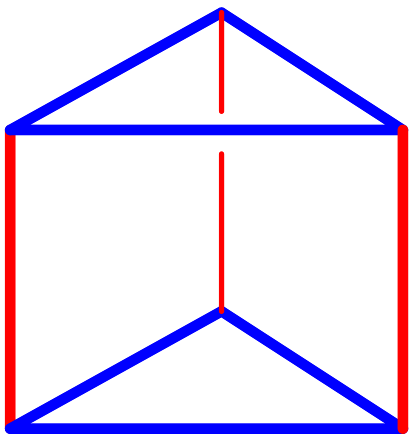
$\mathbb{P} \oplus \mathbb{P}'$



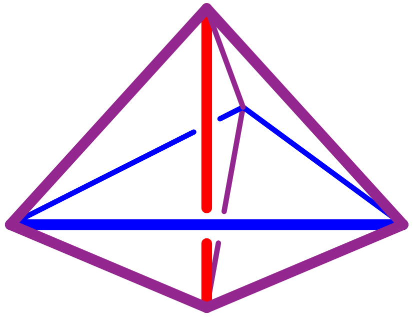
$\mathbb{P} * \mathbb{P}'$

QU. Describe the graphs of the Cartesian product $\mathbb{P} \times \mathbb{P}'$, the direct sum $\mathbb{P} \oplus \mathbb{P}'$ and the join $\mathbb{P} * \mathbb{P}'$ in terms of that of $\mathbb{P}$ and $\mathbb{P}'$.

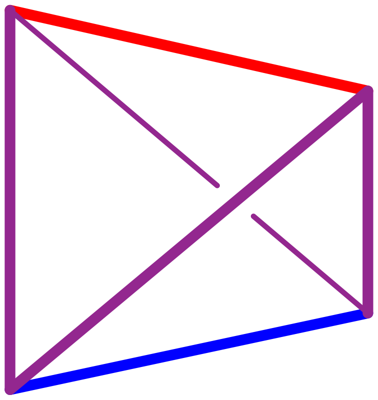
GRAPHS & POLYTOPE OPERATIONS



$\mathbb{P} \times \mathbb{P}'$



$\mathbb{P} \oplus \mathbb{P}'$



$\mathbb{P} * \mathbb{P}'$

PROP. Define $E^\star(\mathbb{P}) = E(\mathbb{P}) \smallsetminus \{\mathbb{P}\}$ (if $\dim \mathbb{P} = 1$, then $E^\star(\mathbb{P}) = \varnothing$).

$$V(\mathbb{P} \times \mathbb{P}') = V(\mathbb{P}) \times V(\mathbb{P}') \qquad E(\mathbb{P} \times \mathbb{P}') = \big(V(\mathbb{P}) \times E(\mathbb{P}')\big) \cup \big(E(\mathbb{P}) \times V(\mathbb{P}')\big)$$

$$V(\mathbb{P} \oplus \mathbb{P}') = V(\mathbb{P}) \cup V(\mathbb{P}') \qquad E(\mathbb{P} \oplus \mathbb{P}') = E^\star(\mathbb{P}) \cup E^\star(\mathbb{P}') \cup \big(V(\mathbb{P}) \times V(\mathbb{P}')\big)$$

$$V(\mathbb{P} * \mathbb{P}') = V(\mathbb{P}) \cup V(\mathbb{P}') \qquad E(\mathbb{P} * \mathbb{P}') = E(\mathbb{P}) \cup E(\mathbb{P}') \cup \big(V(\mathbb{P}) \times V(\mathbb{P}')\big)$$

GRAPHS OF 3-POLYTOPES

THM. (Steinitz) 3-polytopal $\iff$ planar and 3-connected.

Different proofs are possible:

- See Ziegler, Lect. 4 for the proof based on ΔY operations.
- Lift Tutte's barycentric embedding.

THM. (Mnëv, Richter-Gebert) Polytopality of graphs is NP-hard.

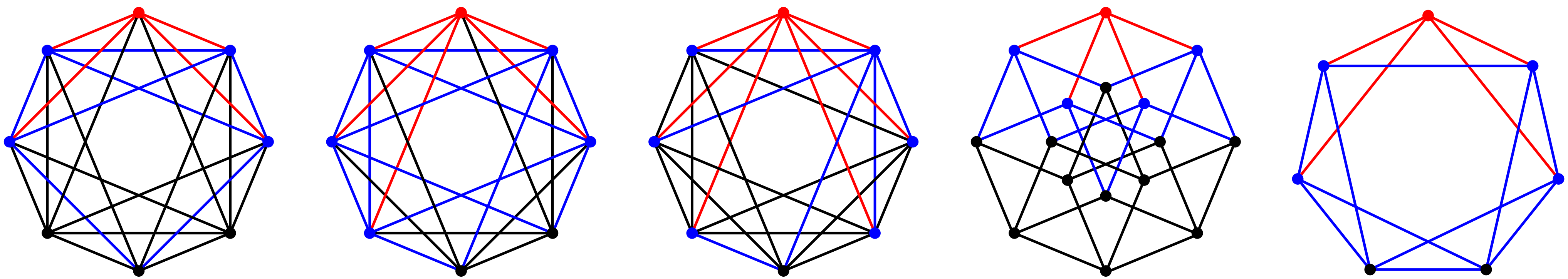
SOME NECESSARY CONDITIONS

THM. If G is the graph of a d-polytope, then

(1) Balinski's Theorem: G is d-connected.

(2) Principal Subdivision Property: Every vertex of G is the principal vertex of a principal subdivision of K_{d+1}.

(3) Separation Property: The maximal number of components into which G may be separated by removing $n > d$ vertices equals $f_{d-1}\big(\mathbb{C}\mathrm{yc}_d(n)\big)$.



DEDUCING THE FACES FROM THE GRAPH

THM. (Whitney) In a 3-polytope, graphs of faces $=$ non-separating induced cycles.

REM. In general, the graph does not determine the face lattice of the polytope (even for a fixed dimension).

THM. (Blind & Mani-Levitska, Kalai)
Two simple polytopes with isomorphic graphs have isomorphic face lattices.

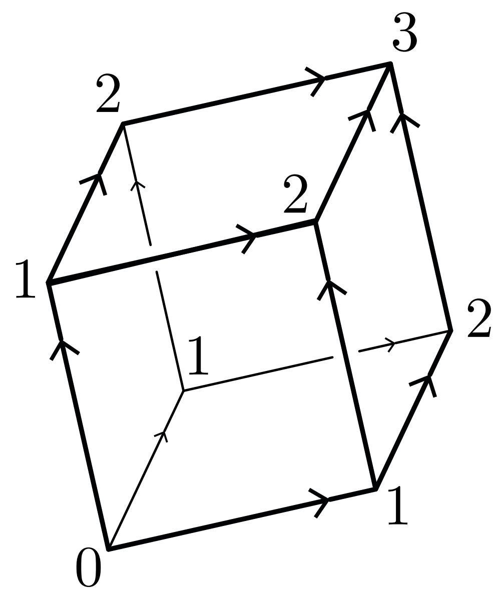
DEDUCING THE FACES FROM THE GRAPH

THM. (Blind & Mani-Levitska, Kalai)
Two simple polytopes with isomorphic graphs have isomorphic face lattices.

proof: G graph of a simple d-polytope $\mathbb{P}$. An orientation $\mathcal{O}$ of G is:

- acyclic $=$ no oriented cycle,
- good $=$ each face of $\mathbb{P}$ has a unique sink.

Intuitively, good acyclic orientations of G $\longleftrightarrow$ linear orientations of $\mathbb{P}$



DEDUCING THE FACES FROM THE GRAPH

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Two simple polytopes with isomorphic graphs have isomorphic face lattices.

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- acyclic $=$ no oriented cycle,
- good $=$ each face of $\mathbb{P}$ has a unique sink.

1. Good acyclic orientations can be recognized from G:
   $h_j(\mathcal{O})$ $=$ number indegree j vertices for $\mathcal{O}$.
   $F(\mathcal{O}) := h_0(\mathcal{O}) + 2\, h_1(\mathcal{O}) + \cdots + 2^d\, h_d(\mathcal{O})$.
   Since $\mathbb{P}$ is simple, each indegree j vertex is a sink in 2^j faces.
   Thus $F(\mathcal{O}) \geq$ number of faces of $\mathbb{P}$ with equality iff $\mathcal{O}$ is good.

DEDUCING THE FACES FROM THE GRAPH

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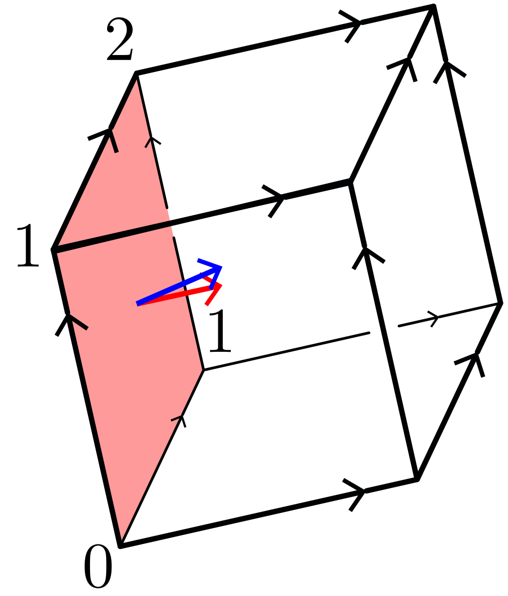

1. Good acyclic orientations can be recognized from G
2. Faces of $\mathbb{P}$ can be determined from good acyclic orientations:
   H regular induced subgraph of G, with vertices W.
   H is the graph of a face of $\mathbb{P}$
   $\iff$ W is initial wrt some good acyclic orientation.

$\Longrightarrow$ perturb a linear functional defining the face

$\Longleftarrow$ assume H k-regular subgraph of G induced by W initial for $\mathcal{O}$.

Let $\boldsymbol{v}$ be a sink of H, and $\mathbb{F}$ be the k-face containing the k edges of H incident to $\boldsymbol{v}$.

Since $\mathcal{O}$ is good, $\boldsymbol{v}$ is the unique sink of the graph of $\mathbb{F}$.

Since W is initial, all vertices of $\mathbb{F}$ are in W.

Since H and the graph of $\mathbb{F}$ are k-regular, they coincide.

DIAMETERS OF POLYTOPES & THE SIMPLEX METHOD

DEF. diameter of G = minimum δ such that any two vertices are connected by a path with at most δ edges.

$\Delta(d, n)$ = maximal diameter of a d-polytope with at most n facets.

remark: diameters of polytopes are important in linear programming and its resolution via the classical simplex algorithm.

CONJ. (Hirsh, disproved by Santos) $\quad \Delta(d, n) \le n - d.$

PROB. Is $\Delta(d, n)$ bounded polynomially in both n and d.

THM. (Kalai and Kleitman) $\quad \Delta(d, n) \le n^{\log_2(d)+1}.$

THM. (Barnette, Larman) $\quad \Delta(d, n) \le \frac{2^{d-2}}{3} n.$

SOME REFERENCES

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