Combinatoire des polytopes
Examen du 21/02/2019

Exercice 1 (Trivalent vertices and faces)**.**

(1) Show that for any 3-dimensional polytope with v_3 vertices of degree 3 and f_3 facets of degree 3 (*i.e.* triangles), we have the inequality $v_3 + f_3 \geq 8$.

(2) Give examples of 3-dimensional polytopes with $(v_3, f_3) = (8,0), (6,2), (4,4), (2,6)$ and $(0,8)$.

(3) Can the other pairs (v_3, f_3) with $v_3 + f_3 = 8$ be reached?

Solution.

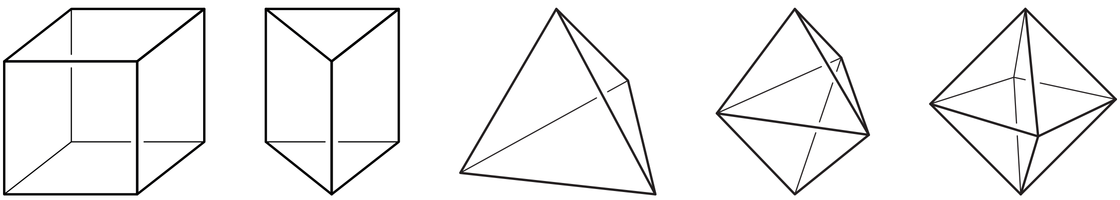
(1) Let P be a 3-dimensional polytope. For $i \geq 3$, we denote by v_i (resp. f_i) the number of vertices (resp. facets) of P of degree i. We also denote by $v := \sum_{i\geq 3} v_i$ the number of vertices, by e the number of edges, and by $f := \sum_{i\geq 3} f_i$ the number of facets of P. By Euler's relation, we have $v - e + f = 2$, and by double counting the number of vertex-edge and edge-facet incidences, we obtain $\sum_{i\geq 3} i v_i = 2e = \sum_{i\geq 3} i f_i$. We therefore obtain that

$$8 = 4v - 4e + 4f = 4\sum_{i\geq 3} v_i - \sum_{i\geq 3} i v_i - \sum_{i\geq 3} i f_i + 4\sum_{i\geq 3} f_i = \sum_{i\geq 3}(4-i)(v_i + f_i)$$

and thus

$$v_3 + f_3 = 8 + \sum_{i\geq 5}(i-4)(v_i + f_i) \geq 8.$$

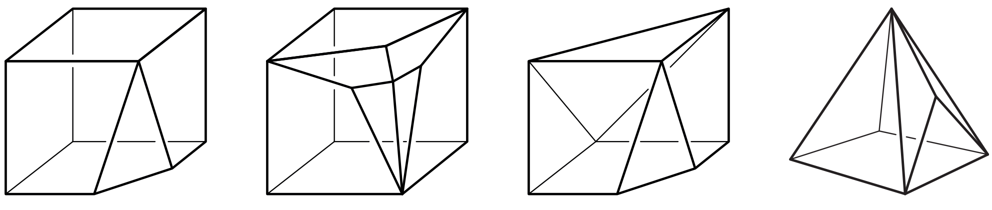
(2) Cube, triangular prism, tetrahedron, triangular bipyramid, octahedron. See Figure **??**.



(8, 0) (6, 2) (4, 4) (2, 6) (0, 8)

Figure 1: Examples of 3-dimensional polytopes with $v_3 + f_3 = 8$.

(3) Assume now that v_3 is odd. Since $2e = \sum_{i\geq 3} i v_i$, it implies that there is another odd integer i such that v_i is odd, and thus non-zero. We obtain that $v_3 + f_3 \geq 8 + (i-4)v_i \geq 9$. The proof is similar when f_i is odd. Note that all pairs (v_3, f_3) with $v_3 + f_3 = 9$ are possible. We have represented some of them in Figure **??**, the others are obtained by polarity.



(8, 1) (7, 2) (4, 5) (3, 6)

Figure 2: Examples of 3-dimensional polytopes with $v_3 + f_3 = 9$.

Exercice 2 (Gram formula for angles)**.** Consider a d-dimensional polytope P, a face F of P, and a sufficiently small ball B_F centered at a point in the relative interior of F. We call the *solid angle* of P at F the fraction α_F of B_F that is contained in P. We denote by α_i the sum of the solid angles of P at its i-dimensional faces. We want to prove the following analogue of Euler's formula for solid angles:

$$\sum_{i=0}^{d}(-1)^i\alpha_i = 0.$$

(1) Show that this formula is equivalent to

$$\sum_{i=0}^{d-2}(-1)^i\alpha_i = (-1)^d\big(f_{d-1}/2-1\big)$$

where f_{d-1} is the number of $(d-1)$-dimensional faces of P.

(2) Show the result for a 2-dimensional polytope P.

(3) Consider now a 3-dimensional polytope P. Choose a random direction $\overline{u}$ on the 2-dimensional sphere and project P orthogonally to this direction $\overline{u}$ to a polygon $P_{\overline{u}}$.
- What is the probability that a vertex v of P does not project to a vertex in the projected polygon $P_{\overline{u}}$ in terms of the solid angle of P at v?
- Deduce the expected number of vertices of the projected polygon $P_{\overline{u}}$.
- What is the expected number of edges of the projected polygon $P_{\overline{u}}$?
- Using these expectations, show that $\alpha_0-\alpha_1=-f_2/2+1$.

(4) Extend this method to any dimension d.

Solution.

(1) Since the whole ball centered at a point in the relative interior of P is included in P, we have $\alpha_d=1$. Similarly, precisely half of a small ball centered at a point in the relative interior of a facet is contained in P, so that $\alpha_{d-1}=f_{d-1}/2$. The formulas are thus clearly equivalent.

(2) In dimension 2, consider an n-gon P, and let w be a point in its relative interior. Cut the polygon P into triangles formed by an edge of P and w. Since the sum of the angles in a triangle is π and the sum of the angles around w is 2π, we obtain that $2\pi\cdot\alpha_1=n\pi-2\pi$ and therefore $\alpha_1=n/2-1$. This is precisely the second formula for $d=2$.

(3) A vertex v of P does not project to a vertex of the projected polygon $P_{\overline{u}}$ if and only if the direction $\overline{u}$ or its opposite belongs to the solid angle of P at v. Therefore, the probability that v does not project to a vertex of $P_{\overline{u}}$ is $2\alpha_v$. Thus, the expected number of vertices is $\sum_{v\in V}(1-2\alpha_v)=f_0-2\alpha_0$. Similarly, the expected number of edges is $\sum_{e\in E}(1-2\alpha_e)=f_1-2\alpha_1$. Since the projected polygon $P_{\overline{u}}$ has the same number of vertices and edges, these two expectations coincide, so that $f_0-2\alpha_0=f_1-2\alpha_1$. Equivalently, $\alpha_0-\alpha_1=(f_0-f_1)/2=f_2/2-1$ by Euler's formula. This is precisely the second formula for $d=3$.

(4) We apply the same strategy. We choose a random direction $\overline{u}$ and consider the $(d-1)$-dimensional polytope $P_{\overline{u}}$ obtained by projection of P orthogonally to this direction $\overline{u}$. For any $i\le d-2$, an i-dimensional face F of P is not projected to an i-dimensional face in the projected polytope $P_{\overline{u}}$ if and only if the direction $\overline{u}$ or its opposite belongs to the solid angle of P at F. Therefore, the expected number of i-dimensional faces of the projected polytope $P_{\overline{u}}$ is $\sum_{F\ i\text{-face}}(1-2\alpha_F)=f_i-2\alpha_i$. By Euler's formula on the projected polytope $P_{\overline{u}}$ and linearity of the expectation, we obtain that

$$\sum_{i=0}^{d-2}(-1)^i(f_i-2\alpha_i)=(-1)^d,$$

which gives by Euler's formula on the polytope P

$$\sum_{i=0}^{d-2}(-1)^i\alpha_i = \frac{1}{2}\Big((-1)^{d+1} + \sum_{i=0}^{d-2}(-1)^i f_i\Big) = \frac{1}{2}\Big((-1)^{d-1} - (-1)^d - (-1)^{d-1}f_{d-1}\Big)$$
$$= (-1)^d\big(f_{d-1}/2 - 1\big).$$

Exercice 3 (A 4-dimensional polytope with a non-prescribable 2-face)**.**

(1) Consider a polytope P with *vertex-facet incidence graph* $\mathcal{I}$. In other words, $\mathcal{I}$ is the bipartite graph whose nodes are the vertices of P and the facets of P, and with an arc from a vertex v to a facet F if and only if v belongs to F.

- Show that the faces of P are in bijection with the maximal complete bipartite subgraphs of $\mathcal{I}$, *i.e.* with inclusion maximal pairs $(\mathcal{V}, \mathcal{F})$ where $\mathcal{V}$ is a subset of vertices of P and $\mathcal{F}$ is a subset of facets of P such that $v \in F$ for any $v \in \mathcal{V}$ and $F \in \mathcal{F}$.
- Deduce that the face lattice of P is completely determined by the vertex-facet incidences of P.

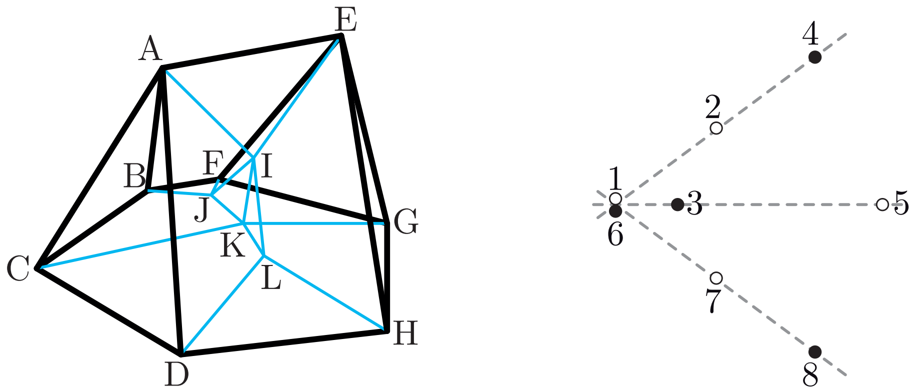


Figure 3: A Schlegel diagram (left) and a Gale diagram (right).

(2) Consider the Schlegel diagram of a 4-dimensional polytope Q on the left of Figure 1.

- What is the number of vertices and facets of Q?
- List all facets of Q (for each facet F, just list the vertices of F in alphabetical order). Label these facets from 1 to 8 in lexicographic order.

(3) Consider the planar affine Gale diagram G of a polytope R on the right of Figure 1.

- What is the dimension and the number of vertices of R?
- List all circuits C of G for which $C_4 \neq 0$ and $C_6 \neq 0$.
- List all cocircuits X of G for which $X_1 = 0$.
- What are the facets of R?

(4) Show that the face lattices of the polytopes Q and R are opposite.

(5) Show that the polytope Q has an hexagonal 2-dimensional face whose geometry cannot be prescribed, meaning that there are hexagons which cannot appear as a 2-face of any polytope combinatorially equivalent to Q. For this, prove that

- any convex hexagon with alternating black and white vertices is the affine Gale diagram of a polytope,
- for any polytope combinatorially equivalent to R, the three lines passing through the vertices 2 and 4, through the vertices 3 and 5, and through the vertices 7 and 8, of the Gale diagram G must be concurrent,
- the iterated vertex figure $(R/v_1)/v_2$ cannot be prescribed for R,

- and conclude by polarity.

(6) We recall from TD F that a polytope is neighborly if and only if its Gale diagram is balanced, meaning that there are at least $\lfloor \frac{n-r+1}{2} \rfloor$ vectors on each side of any hyperplane spanned by $r-1$ vectors (where r is the dimension of G). Show that any convex polygon with alternating black and white vertices is the affine Gale diagram of a neighborly polytope. What are the dimension, the number of vertices and the number of facets of this polytope?

Solution.

(1) A face X can be seen as the convex hull of its set $\mathcal{V}$ of vertices or as the intersection of the set $\mathcal{F}$ of facets containing X. For any $v \in \mathcal{V}$ and $F \in \mathcal{F}$, we have $v \in X \subseteq F$. Therefore, the face X defines a complete bipartite subgraph of $\mathcal{I}$. It is maximal since we have taken all vertices of X in $\mathcal{V}$ and all facets containing X in $\mathcal{F}$. Conversely, consider a maximal complete bipartite subgraph of $\mathcal{I}$ with nodes $\mathcal{V} \cup \mathcal{F}$. Since it contains all vertices of $\mathcal{V}$, the intersection of the facets of $\mathcal{F}$ is non-empty, and thus defines a face X of P. This face X contains precisely the vertices of $\mathcal{V}$ and is contained precisely in the facets of $\mathcal{F}$.

(2) The polytope Q has 12 vertices and 8 facets (don't forget the exterior facet!). Its facets are: 1:ABCDEFGH, 2:ABCIJK, 3:ABEFIJ, 4:ACDIKL, 5:ADEHIL, 6:BCDFGHJKL, 7:EFGIJK, 8:EGHIKL.

(3) G is an affine Gale diagram of rank $r = 2$ with $n = 8$ vertices. Therefore, the polytope R has dimension $d = n - r - 2 = 4$ and $n = 8$ vertices. The circuits of G correspond to support minimal affine dependences, and the cocircuits of G correspond to support minimal affine evaluations (being careful in both cases with the signs of the vertices). Therefore we obtain:

- the circuits C with $C_4 \neq 0 \neq C_6$ are 0+0+0+00, 00−+0+−0, 00−+0+0+, and their opposites,
- the cocircuits X with $X_1 = 0$ are 00+0−0−+, 0+0−00−+, 0−++−000, and their opposites.

The facets of R are given by the complements of its positive cocircuits, thus by the complements of the positive circuits of its Gale diagram. The positive circuits of G are:
A:678, B:4578, C:3578, D:2378, E:246, F:2458, G:2345, H:2347, I:16, J:1458, K:135, L:1237.
Therefore, the facets are:
A:12345, B:1236, C:1246, D:1456, E:13578, F:1367, G:1678, H:1568, I:234578, J:2367, K:24678, L:4568.

(4) From Questions (2) and (3), we obtain that the polytopes Q and R have opposite vertex-facet incident graphs. The result thus follows from Question (1).

(5) Consider any convex polygon X with alternating black and white vertices. Any line passing through two points of X has

- either two black points and two white points on one side,
- or one black point on one side and two black points and one white point on the other side,
- or one white point on one side an two white points and one black point on the other side,
- or one black point and one white point on each side.

This shows the criterion for a colored point set to be an affine Gale diagram.

Consider now the affine Gale diagram of any polytope combinatorially equivalent to R. Since 12345 forms a facet of this polytope, its complement 678 defines a positive cocircuit, and thus a positive circuit of its Gale diagram. Therefore, the points 6, 7 and 8 are aligned in the affine Gale diagram of this polytope. We obtained similarly (considering the facets 13578, 24678 and 234578 of the polytope) that the points 2, 4 and 6 are aligned, that the points 1, 3 and 5 are aligned, and that the points 1 and 6 coincide. Therefore, the lines passing through the vertices 2 and 4, through the vertices 3 and 5, and through the vertices 7 and 8, must be concurrent.

Consider now a convex hexagon X with alternating black and white vertices labeled 2, 3, 7, 8, 5 and 4, in which the three lines 24, 35 and 78 are not concurrent. We obtain from the first observation that this hexagon X is the affine Gale diagram of an hexagon Y. Since X cannot be completed to a Gale diagram of R, Y cannot be the hexagonal iterated vertex figure of a polytope combinatorially equivalent to R (recall that contraction and deletion are dual, and hence that removing elements in the dual corresponds to taking vertex figures).

Finally, face figures of R correspond to the polars of faces of Q, by polarity. Hence the polar of Y cannot be an hexagonal 2-dimensional face of Q.

(6) We just need to check that a convex $2m$-gon with alternating black and white vertices is balanced, *i.e.* that for any line passing through two of its points, the number of black points on one side plus the number of white points on the other side is at least $m-1$. There are three types of such lines:

- lines with two black endpoints: it has k black points and $k+1$ white points on one side, and $m-k-2$ black points and $m-k-1$ white points on the other side. Therefore, the number of black points on one side plus black points on the other side is $m-1$.
- lines with two white endpoints: the argument is symmetric.
- lines with endpoints of distinct colors: it has k black points and k white points on one side, and $m-k-1$ black points and $m-k-1$ white points on the other side. Therefore, the number of black points on one side plus white points on the other side is $m-1$.

Let P be a polytope whose affine Gale diagram is a $2m$-gon with alternating black and white vertices. The number of vertices of P is $2m$ and the dimension of P is $d = 2m-2-2 = 2m-4$. Finally, the facets of P correspond to positive circuits in the bicolored polygon, that is pairs of crossing diagonals $(\delta_\bullet, \delta_\circ)$, where $\delta_\bullet$ has two black endpoints and $\delta_\circ$ has two white endpoints. To count these pairs, we choose one vertex of the diagonal $\delta_\bullet$, the length ℓ of $\delta_\bullet$ and it determines the number of possible diagonals $\delta_\circ$. We obtain that the number f of facets of P is

$$f = \frac{m}{2}\sum_{\ell=1}^{m-1} i(m-i) = \frac{m}{2}\left(\frac{m^2(m-1)}{2} - \frac{(m-1)m(2m-1)}{6}\right) = \frac{(m-1)m^2(m+1)}{12}.$$

Note that this is also the number of facets of the cyclic polytope of dimension $2m-4$ with $2m$ vertices, given according to Exercise 7(4)(c) of TD C by

$$\binom{2m-2-\frac{2m-6}{2}}{\frac{2m-6}{2}} + \binom{2m-\frac{2m-4}{2}}{\frac{2m-4}{2}} = \binom{m+1}{m-3} + \binom{m+2}{m-2} = \frac{(m-1)m^2(m+1)}{12}.$$

Exercice 4 (Realization space of a polytope)**.** Let $\overline{v}_1, \dots, \overline{v}_d$ be d affinely independent points in $\mathbb{R}^d$, and H be the hyperplane they span.

(1) Given a point $\overline{p} \in \mathbb{R}^d$, how can you check (algebraically) whether $\overline{p} \in H$, and if $\overline{p} \notin H$, in which of the open halfspaces defined by H does $\overline{p}$ lie?

(2) Given $\overline{p}, \overline{q} \in \mathbb{R}^d$, how can you check (algebraically) whether $\overline{p}$ and $\overline{q}$ lie in the same open halfspace defined by H?

(3) Prove that the realization space of a polytope is a primary basic semialgebraic set.

(4) (If you did not already do it in the previous point.) Prove that the realization space of a (simplicial) d-dimensional polytope is a primary basic semialgebraic set defined by polynomials of degree at most d.

(Do it only for simplicial polytopes if you find it easier.)

Solution.

(1) We need to check the sign of the determinant of the $(d+1)\times(d+1)$-matrix $\begin{pmatrix} \overline{v}_1 & \cdots & \overline{v}_d & \overline{p} \\ 1 & \cdots & 1 & 1 \end{pmatrix}$. If it is 0, then $\overline{p} \in H$, and otherwise the sign determines the side.

(2) We need to check that the determinants $\begin{vmatrix} \overline{v}_1 & \cdots & \overline{v}_d & \overline{p} \\ 1 & \cdots & 1 & 1 \end{vmatrix}$ and $\begin{vmatrix} \overline{v}_1 & \cdots & \overline{v}_d & \overline{q} \\ 1 & \cdots & 1 & 1 \end{vmatrix}$ are non-zero and have the same sign. That is, that their product is positive.

(3) From the face lattice we can determine an affine basis for every facet F. Assume that $B = (\overline{v}_1, \dots, \overline{v}_d)$ is an affine basis for F, and let $s_B(\overline{p}) = \begin{vmatrix} \overline{v}_1 & \cdots & \overline{v}_d & \overline{p} \\ 1 & \cdots & 1 & 1 \end{vmatrix}$. Then for every vertex $\overline{p}$ of F, we require $s_B(\overline{p}) = 0$, and for every pair of vertices $\overline{p}, \overline{q} \notin F$, we require $s_B(\overline{p}) \cdot s_B(\overline{q}) > 0$. (If there is a single vertex outside F, then it must belong to the affine basis of P and the sign of the determinant is already determined by the definition of the realization space.) These polynomial equations and inequalities describe the realization space of P as a (primary basic) semi-algebraic set, as they certify the vertex-facet incidences.

(4) Note that the description given in the previous answer involves a product of two determinants, and hence polynomials of degree $2d$. However, if one already knows the sign of $s_B(\overline{p})$ for a vertex $\overline{p} \notin F$, then one can simplify the inequalities. Indeed, if for instance we know that $s_B(\overline{p}) > 0$, then the inequalities corresponding to the facet F become $s_B(\overline{q}) > 0$ for all $\overline{q} \notin F$, which is a polynomial inequality of degree d. We thus need a way to determine for each facet F of P an affine basis B for F and the sign $s_B(\overline{p})$ for some point $\overline{p} \notin F$.

Assume first that P is simplicial and that $\overline{v}_1, \dots, \overline{v}_d, \overline{v}_{d+1}$ are an affine basis of P arising from a flag of faces. In particular, since P is simplicial, $B = (\overline{v}_1, \dots, \overline{v}_d)$ is the vertex set and an affine basis of a facet F of P. The condition of fixing the coordinates of the affine basis of P implies $\begin{vmatrix} \overline{v}_1 & \cdots & \overline{v}_d & \overline{v}_{d+1} \\ 1 & \cdots & 1 & 1 \end{vmatrix} > 0$, and hence we impose $s_B(\overline{p}) > 0$ for every vertex $\overline{p} \notin F$. Now, let F' be a facet sharing a $(d-1)$-face with F. This means that there is $i \in [d]$ such that $\overline{v}_i \notin F'$ and a vertex $\overline{w} \in F' \smallsetminus F$. Hence $B' = (\overline{v}_1, \dots, \overline{v}_{i-1}, \overline{w}, \overline{v}_{i+1}, \dots, \overline{v}_d)$ is the vertex set and an affine basis of P'. We know that

$$\begin{aligned} 0 < s_B(\overline{w}) &= \begin{vmatrix} \overline{v}_1 & \cdots & \overline{v}_{i-1} & \overline{v}_i & \overline{v}_{i+1} & \dots & \overline{v}_d & \overline{w} \\ 1 & \cdots & 1 & 1 & 1 & \dots & 1 & 1 \end{vmatrix} \\ &= - \begin{vmatrix} \overline{v}_1 & \cdots & \overline{v}_{i-1} & \overline{w} & \overline{v}_{i+1} & \dots & \overline{v}_d & \overline{v}_i \\ 1 & \cdots & 1 & 1 & 1 & \dots & 1 & 1 \end{vmatrix} = -s_{B'}(\overline{v}_i), \end{aligned}$$

and we impose hence that $s_{B'}(\overline{p}) < 0$ for every vertex $\overline{p} \notin F'$. We can keep propagating the sign to every facet that shares a $(d-1)$-face with one of the facets whose orientation is already fixed. Note that we can reach every facet of P this way, since we are just walking on the graph of $P^{\triangle}$, which is connected by Balinski's theorem. Hence, we know the sign of the determinants associated to all the inequalities describing the realization space, and hence we can express them by degree d polynomials of the form $0 < \pm s_B(\overline{w})$.

If the polytope is not simplicial, it suffices to triangulate the boundary into $(d-1)$-simplices. To this end, perturb the vertices of any realization of P. This gives a simplicial polytope $\widetilde{P}$ with triangulated boundary. We walk through the graph of $\widetilde{P}^{\triangle}$. This provides an affine basis and an orientation for each facet of P. For those facets that are not a simplex, we get several basis, but this does not pose any problem as the corresponding orientations are compatible. For an oriented base B of a facet of P, we set $s_B(\overline{p}) = 0$ for $\overline{p} \in F$ and $s_B(\overline{p}) > 0$ or $s_B(\overline{p}) < 0$ for $\overline{p} \notin F$, according to its orientation.

Exercice 5 (Beneath-beyond and realization spaces of stacked polytopes)**.**

(1) Let F be a face of a d-dimensional polytope $P \subseteq \mathbb{R}^d$. Consider the set $\mathcal{N}_F$ of the vectors $(\overline{a}, b) \in \mathbb{R}^{d+1}$ such that $\langle \overline{a} \mid \overline{x} \rangle = b$ for all $\overline{x} \in F$ and $\langle \overline{a} \mid \overline{x} \rangle \le b$ for all $\overline{x} \in P$. Show that $\mathcal{N}_F$ is a polyhedral cone. What are its generating rays?

(2) Let P be a d-dimensional polytope, let $\overline{q} \in \mathbb{R}^d \smallsetminus P$ and let $Q := \operatorname{conv}(P \cup \{\overline{q}\})$. Prove that every face G of Q is either a face of P or the convex hull of the union of a face of P with $\{\overline{q}\}$.

(3) Let P be a d-dimensional polytope, let $\overline{q} \in \mathbb{R}^d \smallsetminus P$ and let $Q := \operatorname{conv}(P \cup \{\overline{q}\})$. Let H be a supporting hyperplane such that $P \subset \overline{H}^-$. We say that $\overline{q}$ is beneath / on / beyond H if $\overline{q}$ is in H^- / H / H^+, respectively. If F is a facet of P, we say that $\overline{q}$ is beneath / on / beyond F if it is beneath / on / beyond its supporting hyperplane H (oriented so that $P \subset \overline{H}^-$).

Prove that a facet F of P is also a facet of Q if and only if $\overline{q}$ is beneath F.

(4) Let P be a d-dimensional polytope, let $\overline{q} \in \mathbb{R}^d \setminus P$, let $Q := \operatorname{conv}(P \cup \{\overline{q}\})$, and let G be a face of P. Prove that

- G is a face of Q if and only if there is a facet F of P, with $G \subseteq F$, such that $\overline{q}$ is beneath F.
- $\operatorname{conv}(G \cup \{\overline{q}\})$ is a face of Q if and only if
  (i) either $\overline{q} \in \operatorname{aff}(G)$ (equivalently, $\overline{q}$ is on every facet of P containing G),
  (ii) or $\overline{q}$ is beneath at least one of the facets of P containing F and beyond at least one of the facets of P containing F.

(5) Let G be a face of P. Show that there is a point $\overline{q}$ beyond all the facets of P containing G and beneath all the facets of P not containing G. We then say that the polytope $\operatorname{conv}(P \cup \{\overline{q}\})$ is obtained from P by *stacking* a vertex over G.

(6) Let P be a d-dimensional polytope with vertex set $V = \{\overline{v}_1, \dots, \overline{v}_n\}$. Is the combinatorial type of $\operatorname{conv}(V \setminus \{\overline{v}_n\})$ always determined by the combinatorial type of P?

(7) A *stacked* polytope is a polytope obtained from a simplex by iterative stacking operations over arbitrary facets. Show that the realization spaces of stacked polytopes are trivial (stably equivalent to a point).

Solution.

(1) It is the polyhedral cone defined by the linear inequalities $\langle \overline{a} \mid \overline{v} \rangle \leq b$ for each vertex $\overline{v}$ of P, and the equations $\langle \overline{a} \mid \overline{v} \rangle = b$ for each vertex $\overline{v}$ of F. Its rays are the supporting hyperplanes of the facets of P containing F.

(2) Let V be the vertex set of P. Then $Q := \operatorname{conv}(V \cup \{\overline{q}\})$. Let G be a face of Q with supporting hyperplane H. Note that H is also supporting for P. Hence, $V \cap H$ must be the vertex set of a face F of P. If $\overline{q}$ is in H, then $G = \operatorname{conv}(F \cup \{\overline{q}\})$, otherwise $G = F$.

(3) Follows because all the vertices of a polytope that do not belong to a facet lie beneath its supporting hyperplane.

(4)
- If $\overline{q}$ is beneath a facet F, then F is a facet of Q. Since G is a face of F, it is also a face of Q. For the converse, if G is still a face of Q, then $G \subseteq F$ for some facet of Q that does not contain $\overline{q}$. Therefore $\overline{q}$ is beneath F, who was also a facet of P.
- We first prove one direction
  (i) Let H a supporting hyperplane for G in P. Then $\overline{q} \in H$ and H is a supporting hyperplane for $\operatorname{conv}(G \cup \{\overline{q}\})$ in Q.
  (ii) Let H be a supporting hyperplane for G, H_1 be a supporting hyperplane for a facet containing G that has $\overline{q}$ beneath, and H_2 a supporting hyperplane for a facet containing G that has $\overline{q}$ beyond. Then for $\varepsilon > 0$ sufficiently small, $\widetilde{H}_1 = H_1 + \varepsilon H$ and $\widetilde{H}_2 = H_2 + \varepsilon H$ are supporting hyperplanes for G with $\overline{q}$ beneath and beyond, respectively. In the convex segment between $\widetilde{H}_1$ and $\widetilde{H}_2$ there must be a supporting hyperplane for G containing $\overline{q}$, showing that $\operatorname{conv}(G \cup \{\overline{q}\})$ is a face of Q.

  For the converse, assume that $\overline{q} \notin \operatorname{aff}(G)$ but that $\operatorname{conv}(G \cup \{\overline{q}\})$ is a face of Q. Then G must be a face of $\operatorname{conv}(G \cup \{\overline{q}\})$. Let H be a supporting hyperplane for $\operatorname{conv}(G \cup \{\overline{q}\})$ in Q and H' a supporting hyperplane for G in $\operatorname{conv}(G \cup \{\overline{q}\})$. By pivoting $H \pm \varepsilon H'$, we obtain supporting hyperplanes for G in P with $\overline{q}$ beneath and beyond. This means that there are vectors $(\overline{a}, b) \in \mathcal{N}_G$ with $\langle \overline{a} \mid \overline{q} \rangle < b$ and also vectors $(\overline{a}, b) \in \mathcal{N}_G$ with $\langle \overline{a} \mid \overline{q} \rangle > b$. Hence there must be rays of $\mathcal{N}_G$ in each open halfspace defined by $\langle \overline{a} \mid \overline{q} \rangle = b$. These are facets containing G with $\overline{q}$ beneath and beyond them.

(5) Let $\overline{p}_G \in \operatorname{relint} G$ and $\overline{p}_P \in \operatorname{relint} P$. Then for $\varepsilon > 0$ small enough, $\overline{p}_G + \varepsilon(\overline{p}_G - \overline{p}_P)$ is the point we are looking for.

(6) Not always. For example, if P is an octahedron, the convex hull of 7 of its vertices can be a triangular bipyramid or a square pyramid.

(7) We will prove this by induction on the number of vertices. First, the realization space of a simplex is trivial. Let now P be a stacked polytope, and let P' be the stacked polytope obtained by stacking a vertex $\overline{q}$ beyond a facet F of P. Observe that, in contrast with the previous question, removing $\overline{q}$ in a realization of P' always gives rise to a realization of P, because the d neighbors of $\overline{q}$ always span a hyperplane. We have therefore a natural projection $\pi : \mathcal{R}(P', B) \to \mathcal{R}(P, B)$, where B is any affine basis of P. This projection is surjective by Question (5), and the fiber is a relatively open polyhedron depending polynomially on the coordinates of the vertices of P by Question (4) (it is the intersection of the open halfspace beyond F and the open halfspaces beneath the remaining facets). We deduce that π is a stable projection and that $\mathcal{R}(P', B) \approx \mathcal{R}(P, B)$.