# Combinatoire des polytopes
# TD F – Universality

*The exercices in this sheet are interdependent. They should be done in the given order and use previous answers for the proofs. Most of the statements in these exercices are also valid for non-realizable oriented matroids, but we have not seen the necessary tools to prove them, so we restrict to the realizable case.*

**Exercice 1** (Duals of neighborly polytopes)**.** Let $P$ be a $d$-polytope with $n$ vertices $\{p_1, \dots, p_n\}$, and let $G = \{g_1, \dots, g_n\}$ be its Gale dual, of dimension $r = n - d - 1$.

(1) Prove that $P$ is $k$-neighborly if and only if there are at least $k+1$ vectors of $G$ in every open linear halfspace of $\mathbb{R}^r$: $|G \cap H^+| > k$ for every linear hyperplane $H \subset \mathbb{R}^r$.

(2) State this property in terms of the cocircuits of $G$, and give a direct primal proof in terms of the circuits of $P$.

An $r$-dimensional vector configuration on $n$ vectors is *balanced* if for every hyperplane spanned by $r-1$ vectors there are at least $\left\lfloor \frac{n-(r-1)}{2} \right\rfloor$ vectors on either open side of the hyperplane.

(3) What are the balanced configurations of rank $r = 2$?

(4) Prove that the Gale dual of a neighborly polytope is balanced.

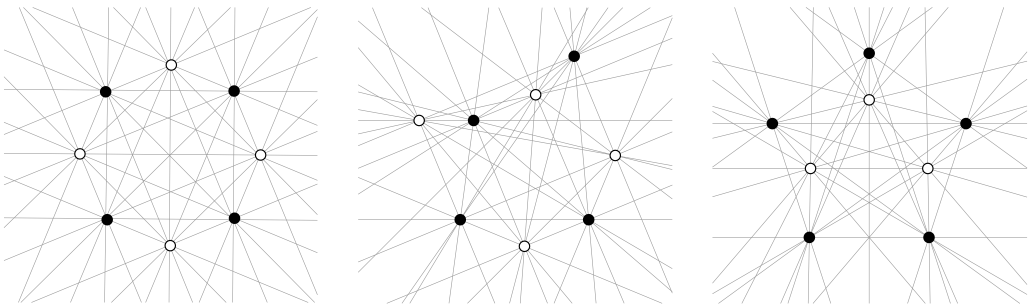
(5) Deduce that, if $d$ is even, the vertices of every neighborly $d$-polytope are in general position, and hence the polytope is simplicial. Show that this is not true for odd $d$.

**Exercice 2** (Inseparability graph)**.** Consider a realizable oriented matroid $\mathcal{M} := \mathcal{M}(V)$, and a sign $\alpha \in \{+, -\}$. We say that two elements $i$ and $j$ of $\mathcal{M}$ are *$\alpha$-inseparable* if $C_i = \alpha C_j$ for each circuit $C \in \mathcal{C}(\mathcal{M})$ such that $C_i, C_j \neq 0$. The *$\alpha$-inseparability graph* of $\mathcal{M}$ is the graph whose vertices are the elements of $\mathcal{M}$ and whose edges are pairs of $\alpha$-inseparable elements. For a sign vector $C \in \{+, -, 0\}^n$, we set $C^s = \{i \mid C_i = s\}$ for $s \in \{+, -, 0\}$, and $\underline{C} = C^+ \cup C^-$.

(1) Show that, for any cocircuit $X \in \mathcal{C}^*(\mathcal{M})$ and $i, j \in \underline{X}$ there is a circuit $C \in \mathcal{C}(\mathcal{M})$ such that $\underline{C} \cap \underline{X} = \{i, j\}$. State the dual property.

(2) Show that $i$ and $j$ are $\alpha$-inseparable in $\mathcal{M}$ if and only if they are $(-\alpha)$-inseparable in $\mathcal{M}^*$.

(3) What are the $+$-inseparability graphs of the following three affine Gale diagrams?



(4) Let $\mathcal{M}$ be a uniform matroid ($V$ in general position), and $i, j$ be $\alpha$-inseparable in $\mathcal{M}$. Show that for any circuit $C \in \mathcal{C}(\mathcal{M})$ with $C_i = 0$ and $C_j \neq 0$, there exists a circuit $C' \in \mathcal{C}(\mathcal{M})$ with $C'_i = -\alpha C_j$, $C'_j = 0$ and $C'_k = C_k$ for all $k \neq i, j$. State the dual property.

(5) Show that if $\mathcal{M}$ is balanced and $n-r-1>0$ is even, then $\mathcal{M}$ does not have $-$-inseparable pairs.

(6) Show that when $n$ is odd, there is a unique balanced oriented matroid of rank 2 up to isomorphism.

(7) Show that if $\mathcal{M}$ is balanced and $n-r-1>0$ is even, then $\{i,j\}$ form an inseparable pair if and only if $\mathcal{M} \setminus \{i,j\}$ is balanced.

(8) Show that the three affine Gale diagrams of Question (3) correspond to three non-isomorphic neighborly polytopes.

**Exercice 3** (Missing faces and rigidity)**.** Let $P$ be a simplicial $d$-polytope with vertex set $\{p_1,\dots,p_n\}$ labelled by $[n]$. For any $I \subseteq [n]$, define $P_I := \operatorname{conv}\{p_i \mid i \in I\}$. A *missing face* of $P$ is an inclusion minimal subset $I \subseteq [n]$ such that $P_I$ is not a face of $P$.

(1) Show that $P_I$ is not a proper face of $P$ if and only if $P_I \cap \operatorname{relint}(P) \neq \varnothing$.

(2) Show that $P_I$ is not a proper face of $P$ if and only if $\operatorname{aff}(P_I) \cap P_{\overline{I}} \neq \varnothing$.

(3) Show that the latter condition cannot be replaced by $P_I \cap P_{\overline{I}} \neq \varnothing$.

(4) Prove that, if $I$ is a missing face, then $\operatorname{relint}(P_I) \cap P_{\overline{I}} \neq \varnothing$. Conclude that there is a circuit $X$ such that $X^+ = I$

(5) Show that the set of missing faces determines the face lattice of $P$, and vice-versa.

(6) Prove that a simplicial $2k$-polytope is neighborly if and only if all its missing faces are of cardinality $k+1$.

(7) Show that if $P$ is neighborly then $P_I$ is neighborly for each $I \subseteq [n]$.

(8) If $P$ is a neighborly $2k$-polytope, show that $I = \{i_1,\dots,i_{k+1}\} \subseteq [n-1]$ is a missing face of $P_{[n-1]}$ if and only if there is an $j \in [k+1]$ such that both $I$ and $I \setminus \{i_j\} \cup \{n\}$ are missing faces of $P$.

(9) Deduce that neighborly $2k$-polytopes are rigid: the face lattice determines the oriented matroid.

**Exercice 4** (Single element extensions of oriented matroids)**.** Let $\mathcal{M} = \mathcal{M}(V)$ be a realizable oriented matroid with ground set $[n]$. The oriented matroid $\mathcal{M}'$ with ground set $[n+1]$ is a *single element extension* of $\mathcal{M}$ if $\mathcal{M} = \mathcal{M}' \setminus (n+1)$.

(1) Prove that there is a function $\sigma : \mathcal{C}^*(\mathcal{M}) \to \{0,+,-\}$ such that $(C, \sigma(C))$ is a cocircuit of $\mathcal{C}^*(\mathcal{M}')$ for all $C \in \mathcal{C}^*(\mathcal{M})$. This function is called the signature of the extension.

(2) Prove that $\mathcal{M}$ and $\sigma$ completely determine $\mathcal{M}'$.

(3) Let $\mathcal{R}_{om}(\mathcal{M})$ denote the realization space of an oriented matroid, and let $\pi : \mathcal{R}_{om}(\mathcal{M}') \to \mathcal{R}_{om}(\mathcal{M})$ the natural projection that removes the $(n+1)$-th vector. Prove that all non-empty fibers of $\pi$ are polynomially defined relatively open polyhedra. Deduce that $\mathcal{R}_{om}(\mathcal{M}) \approx \mathcal{R}_{om}(\mathcal{M}')$ whenever each realization of $\mathcal{M}$ can be extended to a realization of $\mathcal{M}'$.

(4) Let $(a_1,\dots,a_k)$ be an ordered subset of $[n]$ and let $(\varepsilon_1,\dots,\varepsilon_k) \in \{+,-\}^k$. The lexicographic extension $\mathcal{M}[a_1^{\varepsilon_1},\dots,a_k^{\varepsilon_k}]$ of $\mathcal{M}$ by $p = [a_1^{\varepsilon_1},\dots,a_k^{\varepsilon_k}]$ is the single element extension of $\mathcal{M}$ with signature $\sigma : \mathcal{C}^*(\mathcal{M}) \to \{+,-,0\}$ given by

$$\sigma(C) \mapsto \begin{cases} \varepsilon_i C_{a_i} & \text{if } i \text{ is minimal with } C_{a_i} \neq 0 \\ 0 & \text{if } C_{a_i} = 0 \text{ for } 1 \leq i \leq k. \end{cases}$$

Show that every single element extension of a (realizable) rank 2 oriented matroid is lexicographic.

(5) Show that if $\mathcal{M}$ is uniform of rank $r$ (all bases of cardinal $r$, all circuits of cardinal $r+1$, all cocircuits of cardinal $n-r-1$) and $k \geq r$, then $\mathcal{M}[a_1^{\varepsilon_1},\dots,a_k^{\varepsilon_k}]$ is uniform.

(6) Let $\mathcal{M}'$ be a lexicographic extension of $\mathcal{M}$ by $p = [a_1^{\varepsilon_1}, \dots, a_k^{\varepsilon_k}]$. Show that $p$ and $a_1$ are $(-\varepsilon_1)$-inseparable in $\mathcal{M}'$.

(7) Show that if $\mathcal{M} = \mathcal{M}(V)$ is realizable then every lexicographic extension of $\mathcal{M}$ is realizable. Deduce that $\mathcal{R}_{om}(\mathcal{M}) \approx \mathcal{R}_{om}(\mathcal{M}')$ for every lexicographic extension $\mathcal{M}'$ of $\mathcal{M}$.

(8) Let $\mathcal{M} = \mathcal{M}(V)$ be a realizable uniform oriented matroid of rank 3 with $n$ elements $\{v_1, \dots, v_n\}$. Define $\kappa(\mathcal{M}) = \mathcal{M}^{(n)}$ as follows.

Let $\mathcal{M}^{(1)}$ be the lexicographic extension of $\mathcal{M}$ by $w_i = [v_1^+, v_2^+, v_3^+]$; and for $2 \leq i \leq n$, let $\mathcal{M}^{(i)}$ the lexicographic extension of $\mathcal{M}^{(i-1)}$ by $w_i = [v_i^-, v_1^-, w_1^-]$.

Draw an affine diagram of the construction when $n = 3$.

(9) Prove that $\kappa(\mathcal{M})$ is balanced.

(10) Prove that every simplicial $d$-polytope with $d + 4$ vertices appears as an iterated vertex figure of a neighborly $d$-polytope with $d + 4$ vertices.

**Exercice 5** (Universality)**.** Assuming the Universality Theorem for Uniform Oriented Matroids, that states that every open primary basic semi-algebraic set is stably equivalent to the realization space of a uniform oriented matroid of rank 3, prove the Universality Theorem for Simplicial Polytopes: every open primary basic semi-algebraic set is stably equivalent to the realization space of a simplicial polytope.