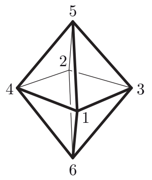
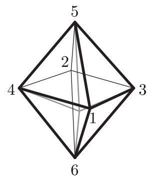
Combinatoire des polytopes
TD E – Gale duality

1 Examples of Gale diagrams

Exercice 1 (Gale diagrams of two octahedra)**.** From G. Ziegler, *Lectures on polytopes, Chapter 6.* Consider the octahedra whose vertices are the column vectors of the matrices



$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1/6 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$



(1) Describe the circuits and cocircuits of these two point configurations.
(2) Compute and represent Gale diagrams and affine Gale diagrams of these point configurations.
(3) Can you read the circuits and cocircuits of (1) on the affine Gale diagrams of the configurations?

Exercice 2 (Dual oriented matroid and dual graph of a planar graph)**.** Consider a planar directed graph $G = (V, E)$ and its dual planar directed graph $G^* = (V^*, E^*)$, where each edge $(f, g) \in E^*$ dual to an edge $e \in E$ is oriented from the face f on the left of e to the face g on the right of e. Show the oriented matroids corresponding to the incidence configurations of G and G^* are duals.

2 Polytopes with few vertices

In this section, we use Gale diagrams to understand combinatorial properties of high dimensional polytopes with few vertices.

Exercice 3 (Polytopes with $d + 2$ vertices)**.**

(1) What is the Gale diagram of a d-simplex $\triangle_d$?

(2) Let $P \subset \mathbb{R}^d$ be a full-dimensional d-polytope with vertices $(\overline{p}_1, \dots, \overline{p}_n) \in \mathbb{R}^{d \times n}$ and Gale diagram $G = (\overline{g}_1, \dots, \overline{g}_n) \in \mathbb{R}^{(n-d-1)\times n}$. Give a Gale diagram for the pyramid $\mathrm{pyr}(P)$ in terms of G.

(3) Let $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$ be full-dimensional polytopes containing the origin in their interior. Let $(\overline{p}_1, \dots, \overline{p}_n) \in \mathbb{R}^{d\times n}$ and $(\overline{q}_1, \dots, \overline{q}_m) \in \mathbb{R}^{e\times m}$ denote their vertices, and $G = (\overline{g}_1, \dots, \overline{g}_n) \in \mathbb{R}^{(n-d-1)\times n}$ and $H = (\overline{h}_1, \dots, \overline{h}_m) \in \mathbb{R}^{(m-e-1)\times m}$ denote their Gale diagrams. Give a Gale diagram of the direct sum $P \oplus Q$ in terms of G and H.

(4) For $1 \le s, t$ with $s + t = d$, what is the Gale diagram of $\triangle_s \oplus \triangle_t$?

(5) Prove that every simplicial d-polytope with $d + 2$ vertices is combinatorially equivalent to a direct sum of simplices $\triangle_s \oplus \triangle_t$ for some $1 \le s, t$ with $s + t = d$.

(6) Prove that the number of combinatorial types of simplicial d-polytopes with $d + 2$ vertices is $\left\lfloor \frac{d}{2} \right\rfloor$.

(7) Prove that every d-polytope with $d + 2$ vertices is combinatorially equivalent to $\mathrm{pyr}_k(\triangle_s \oplus \triangle_t)$ for some $0 \le k$, $1 \le s, t$ with $s + t + k = d$, where pyr_k denotes the iteration of the operation of taking a pyramid k times.

(8) Prove that the number of d-polytopes with $d + 2$ vertices is $\left\lfloor \frac{d^2}{4} \right\rfloor$.

3 Unprescribable facets

In this section, we show that there exists combinatorial types of polytopes for which the geometry of certain facets cannot be prescribed. This is somewhat the first manifestation of universality in polytopes.

Exercice 4 (Gale diagrams and unprescribable facets)**.** Consider the following affine Gale diagram

(1) Show that this is the affine Gale diagram of a polytope P. What is its number of vertices and its dimension?
(2) Is it simplicial? Is it simple?
(3) For each $i \in [7]$ describe the combinatorial type of the vertex figure P/i. (Notice that this is a 3-dimensional polytope, and hence it can be drawn as a planar graph.)
(4) Show that the vertex figure $P/4$ cannot be arbitrarily prescribed. That is, that there exists a polytope $Q \cong P/4$ that cannot be extended to a realization of P.

This example is due to Bernd Sturmfels, *Some applications of affine Gale diagrams to polytopes with few vertices*, 1988.

Exercice 5 (Schlegel diagrams and unprescribable facets)**.**

(1) Consider a d-dimensional polytope $P = \{\overline{x} \in \mathbb{R}^d \mid \langle \overline{a}_i \mid \overline{x} \rangle \leq b_i \text{ for all } i \in I\}$, and let F be the facet of P defined by the inequality $\langle \overline{a}_0 \mid \overline{x} \rangle \leq b_0$.

   Show that there is a point $\overline{y}_F$ such that $\langle \overline{a}_0 \mid \overline{y}_F \rangle > b_0$ and $\langle \overline{a}_i \mid \overline{y}_F \rangle \leq b_i$ for all $i \in I \smallsetminus \{0\}$.

(2) For this facet F and this point $\overline{y}_F$, we define the map $p : \mathbb{R}^d \to \mathbb{R}^d$ by

$$p(\overline{x}) = \overline{y}_F + \frac{b_0 - \langle \overline{a}_0 \mid \overline{y}_F \rangle}{\langle \overline{a}_0 \mid \overline{x} - \overline{y}_F \rangle}(\overline{x} - \overline{y}_F).$$

   Describe this map geometrically for $x \in P$.

(3) Show that for each proper face G of P other than F, the projection $p(G)$ is a polytope combinatorially equivalent to G.

(4) A *Schlegel diagram* $\mathcal{D}(P, F)$ of P based at the facet F, is the image under the projection map p of all proper faces of P other than F:

$$\mathcal{D}(P, F) = \{p(G) \mid G \in \mathcal{F}(P) \smallsetminus \{P, F\}\}.$$

   This is a polytopal subdivision of F, that is, a finite collection $\mathcal{S}$ of polytopes such that:
   - for every $Q \in \mathcal{S}$ and every face G of Q, $G \in \mathcal{S}$,
   - for every $Q_1, Q_2 \in \mathcal{S}$, their intersection $Q_1 \cap Q_2$ is a face of Q_1 and Q_2 and
   - the union of $\mathcal{S}$ covers F: $\bigcup_{Q \in \mathcal{S}} Q = F$.

   Draw all combinatorially different Schlegel diagrams of the cube $\square_3$, the octahedron $\diamond_3$ and the triangular prism $\triangle_1 \times \triangle_2$.

(5) Let $\text{prism}\big(\text{pyr}(\square_2)\big) = \triangle_1 \times \big(\triangle_0 * (\triangle_1 \times \triangle_1)\big)$ denote the prism over the square pyramid. Draw a Schlegel diagram of $\text{prism}\big(\text{pyr}(\square_2)\big)$ with respect to its cubical facet ($\cong \square_3$).

(6) Let P be a polytope combinatorially equivalent to the triangular prism $\triangle_1 \times \triangle_2$. Let E_0, E_1, E_2 be the three lines spanned by the edges e_0, e_1, e_2 corresponding to the edges of $\triangle_1 \times \triangle_2$ of the form $\triangle_1 \times v$ for $v \in F_0(\triangle_2)$. Show that E_0, E_1, E_2 either intersect at a point or are parallel (intersect at a point at infinity).

(7) Prove that in any Schlegel diagram of a polytope combinatorially equivalent to $\text{prism}\big(\text{pyr}(\square_2)\big)$, the cubical facet has four parallel edges.

(8) Conclude that one cannot prescribe the cubical facet of $\text{prism}\big(\text{pyr}(\square_2)\big)$. In other words, there are polytopes $Q \cong \square_3$ such that there is no polytope combinatorially equivalent to $\text{prism}\big(\text{pyr}(\square_2)\big)$ that has Q as a facet.