Combinatoire des polytopes
TD B – Faces and operations

1 Faces and f-vectors

Consider a d-dimensional polytope P. Its *f-vector* is the vector $(f_0(P), f_1(P), \dots, f_d(P))$ where $f_i(P)$ is the number of i-dimensional faces of P. Its *f-polynomial* is the polynomial $f(P,x) := \sum_{i=0}^{d} f_i(P)x^i$.

Exercice 1 (f-vectors of the simplex, the cube, and the cross-polytope)**.** What are the f-vectors and f-polynomials of the d-dimensional simplex, cube, and cross-polytope?

Exercice 2 (f-vectors and polarity)**.** Let P be a d-dimensional polytope and $P^\triangle$ denote the polar polytope of P. Show that $f(P^\triangle, x) = x^d - 1/x + x^{d-1} f(P, 1/x)$.

Exercice 3 (f-vectors of 3-polytopes)**.** Prove that the f-vectors of 3-dimensional polytopes are precisely the integer vectors $(f_0, f_1, f_2, 1)$ such that

$$f_0 - f_1 + f_2 = 2 \qquad f_0 \leq 2f_2 - 4 \qquad \text{and} \qquad f_2 \leq 2f_0 - 4.$$

[*Hint: For the difficult direction, compute the f-vector of a pyramid over a p-gon, and study the effect on the f-vector of the two polar operations of simple vertex truncation and simplicial facet stacking, see Exercice 5.*]

Which polytopes satisfy the first (resp. second) inequality?

Exercice 4 (Non-unimodality of f-vector)**.** A sequence $x_0, \dots, x_d$ is *unimodal* if there exists $0 \leq i \leq d$ such that $x_0 \leq x_1 \leq \dots \leq x_{i-1} \leq x_i \geq x_{i+1} \geq \dots \geq x_{d-1} \geq x_d$ or the opposite. We say that i is the *unimodality peak*.

(1) Show that the f-vectors of d-polytopes are unimodal for $d \leq 5$.
(2) Using Exercice 1, show that the f-vectors of simplices, cubes and cross-polytopes are unimodal and study their unimodality peak.
(3) Show that there exist simplicial polytopes whose f-vector is not unimodal.
[*Hint: use iterative stackings on facets of the cross-polytope, see Exercice 5.*]
(4)* Are the f-vectors of d-polytopes all unimodal for $d = 6$, $d = 7$, $d = 8$, …?

2 Operations on polytopes

Exercice 5 (Truncating and stacking)**.** Let P be a d-dimensional polytope, v be a simple vertex of P (contained in precisely d facets) and f be a simplicial facet of P (containing precisely d vertices). We consider the polytopes obtained by

- truncating the vertex v of P by a hyperplane separating v from all other vertices of P,
- stacking a vertex w on the facet f of P, where the vertex w is separated from P by f, but close enough to f so that it sees the vertices of f and no other vertex of P.

These operations are illustrated on Figure 4.

Observe that truncating and stacking are dual operations: the polytope obtained by stacking a vertex on the facet $v^\diamond$ of $P^\triangle$ is the polar of the polytope obtained by truncating vertex v of P. Describe the f-vector of the resulting polytopes in terms of the f-vector of P. What can you say in the case v is not simple or f is not simplicial?

Exercice 6 (Cartesian product of polygons)**.** (1) Describe the i-dimensional faces of the Cartesian product $P \times Q$ in terms of the faces of the two polytopes P and Q.

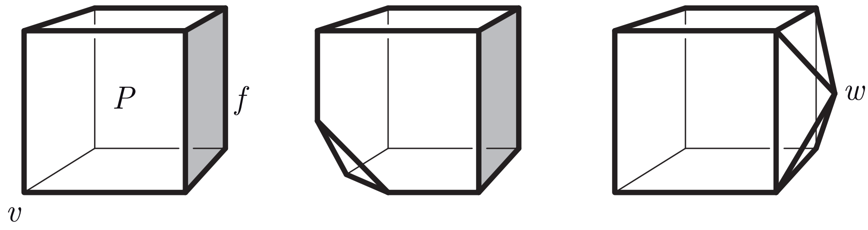

Figure 1: Truncating a vertex v (middle) and stacking over a facet f (right).

(2) Describe the f-vector of the product $P \times Q$ in terms of f-vector of the two polytopes P and Q.
(3) What is the f-vector of a product of q copies of a p-gon? Do you recognize something when $p = 4$?

Exercice 7 (Normal fans and polytope operations)**.** Describe the normal fans of the Cartesian product $P \times Q$ and the Minkowski sum $P + Q$ in terms of the normal fans of the polytopes P and Q.

Exercice 8 (Hanner polytopes)**.** A *Hanner polytope* is either the interval $I := [-1, 1]$ or a product or a direct sum of two Hanner polytopes.
(1) What are the Hanner polytopes of dimension $1, 2, 3, 4$?
(2) Show that each Hanner polytope has $3^d + 1$ faces.
(3) Is it true that each Hanner polytope is a prism or a bipyramid?