Signed tree associahedra

**Spines**

A tree on a signed ground set \( V = V^- \sqcup V^+ \).

Spine on \( T \) = directed and labeled tree \( S \) such that:
- the labels of the nodes of \( S \) form a partition of the signed ground set \( V \),
- at a node labeled by \( U = U^- \sqcup U^+ \), the source label sets of the incoming arcs are subsets of distinct connected components of \( T \setminus U^- \), and the sink label sets of the outgoing arcs are subsets of distinct connected components of \( T \setminus U^+ \).

**Spine poset** \( S(T) \) = poset of arc contractions on signed spines of \( T \).

Prop. The spine poset \( S(T) \) is a pure graded poset of rank \( |V| \).

Signed nested complex = simplicial complex \( \mathcal{N}(T) = \{ N(S) \mid S \in S(T) \} \), where \( N(S) \) = collection of source sets of \( S \).

**Spine fan**

Ambiant space \( H = \{ x \in \mathbb{R}^V \mid \sum_{v \in V} x_v = (|V|+1) \} \).

Cones \( C(S) \) of a spine \( S = \{ x_u \in \mathbb{R} \mid x_u \leq x_v \} \) for all \( u \to v \) in \( S \).

Theo. The collection of cones \( \mathcal{F}(T) = \{ C(S) \mid S \in S(T) \} \) defines a complete simplicial fan on \( \mathbb{R} \), called the spine fan of \( T \).

The spine fan \( \mathcal{F}(T) \) coarsens the braid fan on \( \mathbb{R} \). It defines a map \( \kappa \) from linear orders on \( V \) to maximal spines on \( T \).

Prop. The fibers of \( \kappa \) are the classes of \( T \)-congruence defined by \( X \sim Y \iff X \psi Y \) if \( \psi \) is w \in V in between \( u \) and \( v \) in \( T \) and such that \( w \in X \cap V^+ \) or \( w \in Y \cap V^- \).

**Signed tree associahedron**

Theo. The spine fan \( \mathcal{F}(T) \) is the normal fan of the signed tree associahedron \( \text{Asso}(T) \) with:
- a vertex \( a(S) \in \mathbb{R}^V \) for each maximal \( S \in S(T) \), with coordinates
  \[
  a(S)_v = \begin{cases} 
  \{ \pi \in \Pi(S) \mid v \in \pi \text{ and } r_v \notin \pi \} & \text{if } v \in V^- \\
  |V| + 1 - \{ \pi \in \Pi(S) \mid v \in \pi \text{ and } r_v \notin \pi \} & \text{if } v \in V^+ 
  \end{cases} 
  \]
- \( \mathcal{F}(T) \) an unoriented fan with \( \mathcal{S} \mid S \in S(T) \)
- a facet for each \( B \in \bigcup_{S \in S(T)} N(S) \) defined by the half-space
  \[
  \{ x \in \mathbb{R}^V \mid \sum_{v \in B} x_v \geq \binom{|B|+1}{2} \}.
  \]

Exm. For the tripod

\[
\begin{align*}
V^- = \{1, 3, 4\} \\
V^+ = \{2\}
\end{align*}
\]

half-spaces:

\[
\begin{align*}
x_1 & \geq 1 \\
x_2 & \geq 1 \\
x_3 & \geq 1 \\
x_1 + x_2 + x_3 & \geq 3 \\
x_1 + x_3 & \geq 3 \\
x_3 + x_4 & \geq 3 \\
x_1 + x_2 + x_3 + x_4 & \geq 6 \\
x_1 + x_2 + x_4 & \geq 6 \\
x_1 + x_3 + x_4 & \geq 6 \\
x_2 + x_3 + x_4 & \geq 6 \\
x_1 + x_2 + x_3 + x_4 & = 10
\end{align*}
\]

Some properties

Prop. The signed tree associahedron \( \text{Asso}(T) \) is sandwiched between the permutahedron \( \text{Perm}(V) \) and the parallelepiped \( \text{Para}(T) \):

\[
\sum_{uv \in V} [e_u, e_v] = \text{Perm}(T) \subset \text{Asso}(T) \subset \text{Para}(T) = \sum_{uv \in T} \pi_{uv}[e_u, e_v]
\]

Common vertices of:
- \( \text{Asso}(T) \) and \( \text{Para}(T) \) \( \equiv \) orientations of \( T \) which are spines on \( T \),
- \( \text{Asso}(T) \) and \( \text{Perm}(V) \) \( \equiv \) linear orders on \( V \) which are spines on \( T \),
- no common vertex of the three polytopes except if \( T = \text{signed path} \).

Prop. \( \text{Asso}(T) \) and \( \text{Asso}(T') \) isomorphic \( \iff \ T \text{ and } T' \text{ isomorphic up to the signs of their leaves, i.e. there is a bijection } \theta : V \to V' \text{ st. } \forall u, v \in V \)

\[
\begin{align*}
& u \prec v \in T \iff \theta(u) \prec \theta(v) \in T' \\
& \text{if } u/v \text{ is not a leaf of } T, \text{the signs of } u \text{ and } \theta(u) \text{ coincide (resp. differ).}
\end{align*}
\]

Examples

For a signed path \( P \), \( \text{Asso}(P) \) is the classical associahedron

faces \( \leftrightarrow \) dissections \( \leftrightarrow \) Schröder trees,

vertices \( \leftrightarrow \) triangulations \( \leftrightarrow \) binary trees.

Loday, Realization of the Stasheff polytope, 2004

Hohlweg & Lange, Realizations of the associahedron and cyclohedron, 2007

For an unsigned tree \( T \), \( \text{Asso}(T) \) is the \( T \)-associahedron facets \( \leftrightarrow \) tubes = connected induced subgraphs of \( T \),

faces \( \leftrightarrow \) tubings = collections of tubes which are pairwise nested, or disjoint and non-adjacent.

Carr & Devadoss, Coxeter complexes and graph associahedra, 2006