Signed tree associahedra

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Spines

T a tree on a signed ground set V = $V^- \sqcup V^+$.

Spine on T = directed and labeled tree S such that

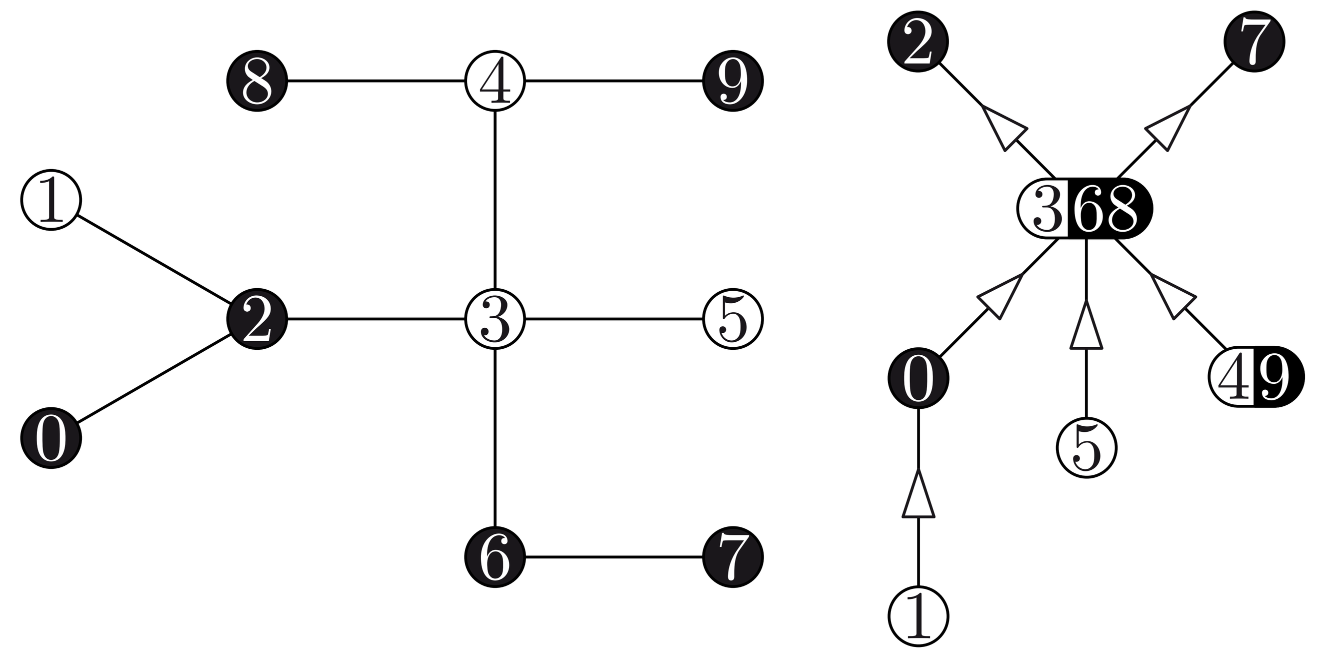
- the labels of the nodes of S form a partition of the signed ground set V,
- at a node labeled by $U = U^- \sqcup U^+$, the source label sets of the incoming arcs are subsets of distinct connected components of T r U^-, and the sink label sets of the outgoing arcs are subsets of distinct connected components of T r U^+.

Spine poset $\mathcal{S}$(T) = poset of arc contractions on signed spines of T.

Prop. *The spine poset $\mathcal{S}$(T) is a pure graded poset of rank |V|.*

Signed nested complex = simplicial complex $\mathcal{N}$(T) = {N(S) | S ∈ $\mathcal{S}$(T)}, where N(S) = collection of source sets of S.

Exm. $V^- = \{1,3,4,5\}$ $V^+ = \{0,2,6,7,8,9\}$



graph G spine S on G

Spine fan

Ambiant space H = $\left\{\mathbf{x} \in \mathbb{R}^V \;\middle|\; \sum_{v\in V} x_v = \binom{|V|+1}{2}\right\}$.

Cone C(S) of a spine S = $\{\mathbf{x} \in H \mid x_u \le x_v$ for all $u \to v$ in S$\}$.

Theo. *The collection of cones $\mathcal{F}$(T) = {C(S) | S ∈ $\mathcal{S}$(T)} defines a complete simplicial fan on H, called the* spine fan *of* T.

The spine fan $\mathcal{F}$(T) coarsens the braid fan on H. It defines a map κ from linear orders on V to maximal spines on T.

Prop. *The fibers of κ are the classes of* T-congruence *defined by $XuvY \equiv_T XvuY$ iff there is $w \in$ V in between u and v in* T *and such that $w \in X \cap V^+$ or $w \in Y \cap V^-$.*

Signed tree associahedron

Theo. *The spine fan $\mathcal{F}$(T) is the normal fan of the* signed tree associahedron Asso(T) *with*

- *a vertex* $\mathbf{a}$(S) ∈ $\mathbb{R}^V$ *for each maximal* S ∈ $\mathcal{S}$(T), *with coordinates*

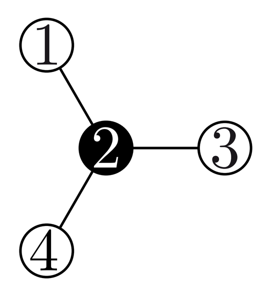
$$\mathbf{a}(S)_v = \begin{cases} \left|\{\pi \in \ (S) \mid v \in \pi \text{ and } r_v \notin \pi\}\right| & \text{if } v \in V^- \\ |V| + 1 - \left|\{\pi \in \ (S) \mid v \in \pi \text{ and } r_v \notin \pi\}\right| & \text{if } v \in V^+ \end{cases}$$

where r_v = unique incoming (outgoing) arc at $v \in V^-$ ($v \in V^+$),
(S) = {*(undirected) paths in* S},

- *a facet for each $B \in \bigcup_{S\in\mathcal{S}(T)}$* N(S) *defined by the half-space*

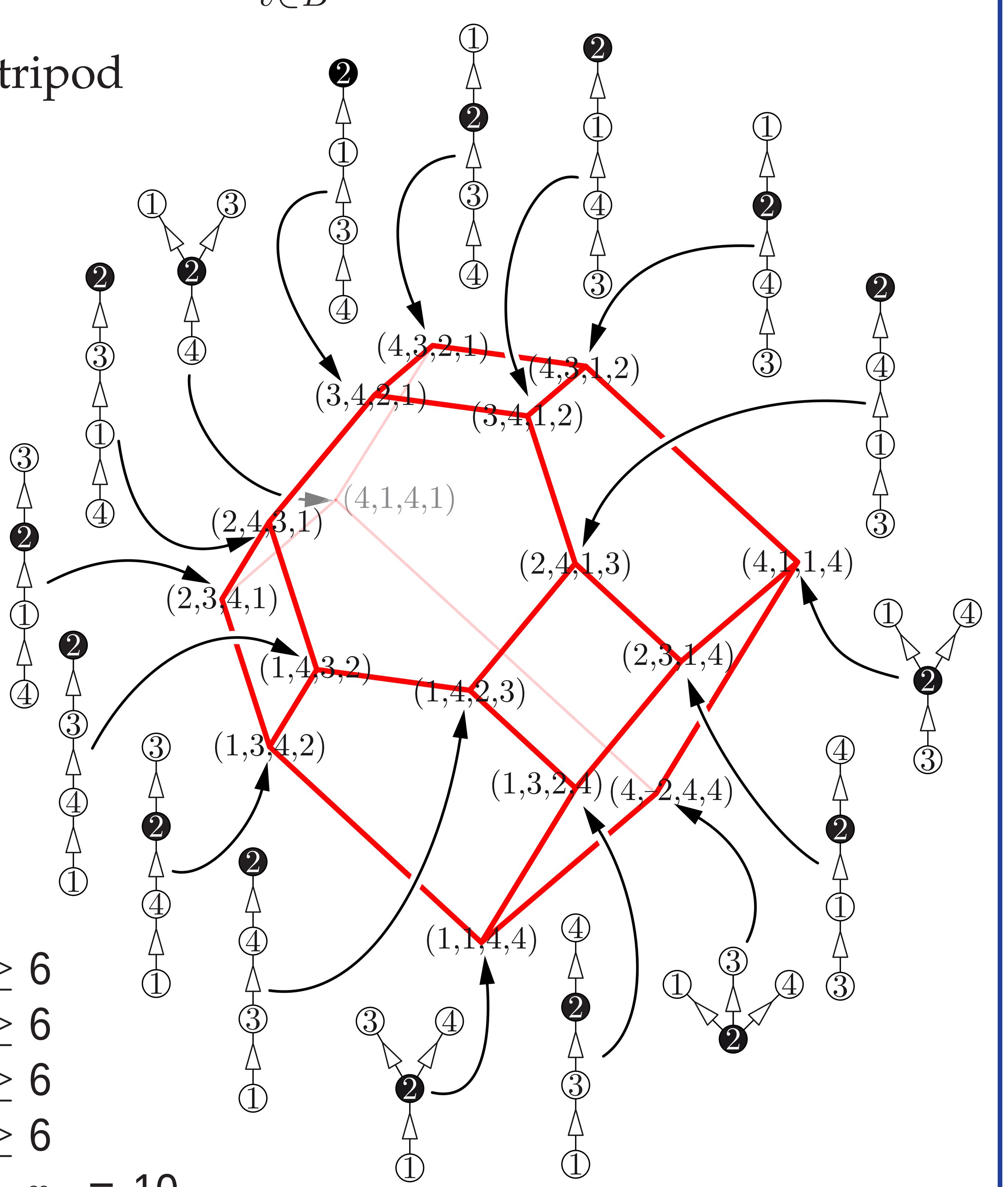
$$\left\{\mathbf{x} \in \mathbb{R}^V \;\middle|\; \sum_{v\in B} x_v \ge \binom{|B|+1}{2}\right\}.$$

Exm. For the tripod



$V^- = \{1,3,4\}$
$V^+ = \{2\}$

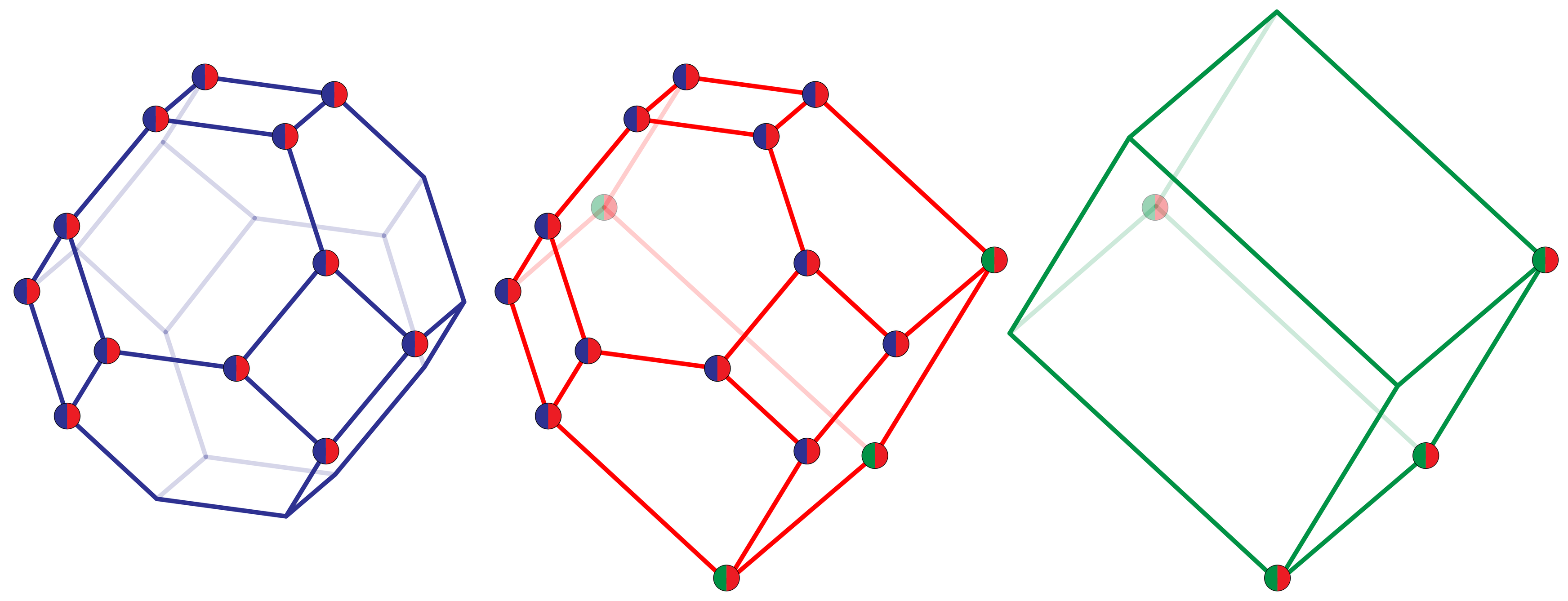
half-spaces:
$x_1 \ge 1$
$x_2 \ge 1$
$x_4 \ge 1$
$x_1 + x_4 \ge 3$
$x_1 + x_3 \ge 3$
$x_3 + x_4 \ge 3$
$x_1 + x_2 + x_3 \ge 6$
$x_1 + x_2 + x_4 \ge 6$
$x_1 + x_3 + x_4 \ge 6$
$x_2 + x_3 + x_4 \ge 6$
$x_1 + x_2 + x_3 + x_4 = 10$



Some properties

Prop. *The signed tree associahedron* Asso(T) *is sandwiched between the permutahedron* Perm(V) *and the parallelepiped* Para(T)*:*

$$\sum_{u\neq v\in V} [e_u, e_v] = \text{Perm}(T) \subset \text{Asso}(T) \subset \text{Para}(T) = \sum_{uv\in T} \pi_{uv}[e_u, e_v]$$



Common vertices of

- Asso(T) *and* Para(T) ≡ *orientations of* T *which are spines on* T,
- Asso(T) *and* Perm(T) ≡ *linear orders on* V *which are spines on* T,

⇒ *no common vertex of the three polytopes except if* T = *signed path.*

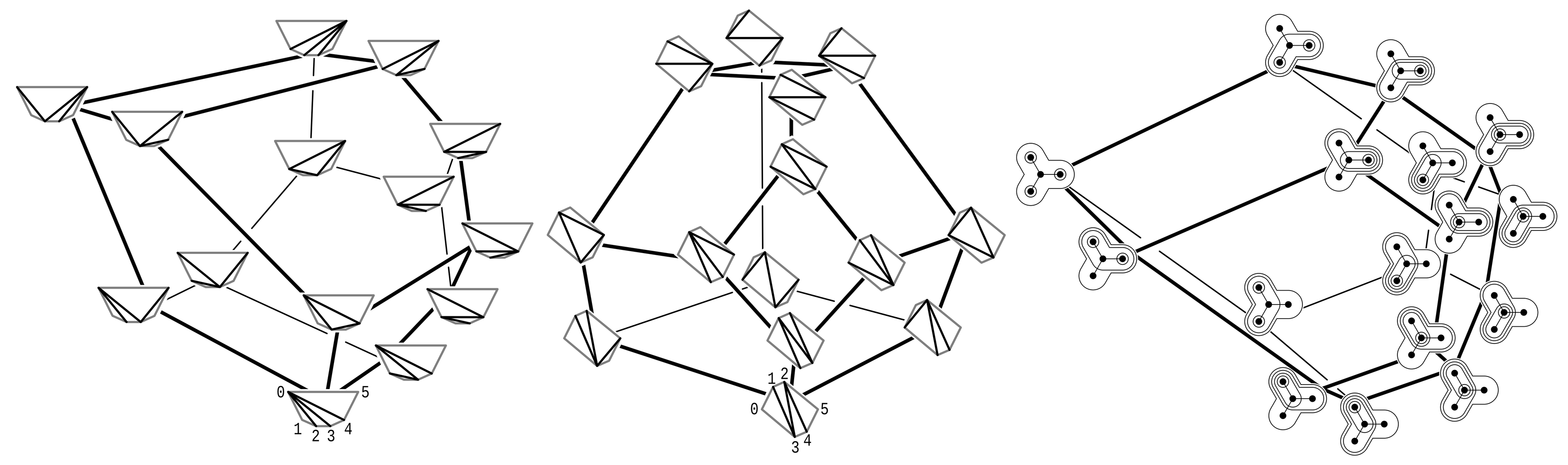
Prop. Asso(T) *and* Asso(T′) *isometric* ⟺ T *and* T′ *isomorphic or anti-isomorphic up to the signs of their leaves, i.e. there is a bijection θ :* V → V′ *st.* ∀$u, v \in$ V

- *$u - v$ edge in* T ⟺ *$\theta(u) - \theta(v)$ edge in* T′,
- *if u is not a leaf of* T, *the signs of u and $\theta(u)$ coincide (resp. differ).*

Examples

For a signed path P, Asso(P) is the classical associahedron
faces ⟷ dissections ⟷ Schröder trees,
vertices ⟷ triangulations ⟷ binary trees.

Loday, Realization of the Stasheff polytope, 2004
Hohlweg & Lange, Realizations of the associahedron and cyclohedron, 2007



For an unsigned tree T, Asso(T) is the T-associahedron
facets ⟷ *tubes* = connected induced subgraphs of T,
faces ⟷ *tubings* = collections of tubes which are pairwise nested, or disjoint and non-adjacent.

Carr & Devadoss, Coxeter complexes and graph associahedra, 2006

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