Signed tree associahedra

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Spines

T a tree on a signed ground set \( V = V^- \cup V^+ \).

Spine on \( T \) = directed and labeled tree \( S \) such that
- the labels of the nodes of \( S \) form a partition of the signed ground set \( V \),
- at a node labeled by \( U = U^- \cup U^+ \), the source label sets of the incoming arcs are subsets of distinct connected components of \( T \setminus U^- \), and the sink label sets of the outgoing arcs are subsets of distinct connected components of \( T \setminus U^+ \).

**Spine poset** \( S(T) = \) poset of arc contractions on signed spines of \( T \).

**Prop.** The spine poset \( S(T) \) is a pure graded poset of rank \( |V| \).

**Signed nested complex** = simplicial complex \( \mathcal{N}(S) = \{ NS | S \in S(T) \} \), where \( N(S) = \) collection of source sets of \( S \).

Spine fan

**Ambient space** \( H = \{ x \in \mathbb{R}^V | \sum_{v \in V} x_v = (|V|+1) \} \).

**Cone** \( C(S) \) of a spine \( S = \{ x \in H | x_u \leq x_v \} \) for all \( u \rightarrow v \) in \( S \).

**Theo.** The collection of cones \( \mathcal{F}(T) = \{ C(S) | S \in S(T) \} \) defines a complete simplicial fan on \( H \), called the spine fan of \( T \).

The spine fan \( \mathcal{F}(T) \) coarsens the braid fan on \( \mathbb{R}^2 \). It defines a map \( \kappa \) from linear orders on \( V \) to maximal spines on \( T \).

**Prop.** The fibers of \( \kappa \) are the classes of \( T \)-congruence defined by \( X_uY \equiv_T X_vY \) iff there is \( w \in V \) in between \( u \) and \( v \) in \( T \) and such that \( w \in X \cap V^+ \) or \( w \in Y \cap V^- \).

Signed tree associahedron

**Theo.** The spine fan \( \mathcal{F}(T) \) is the normal fan of the signed tree associahedron \( Asso(T) \) with
- a vertex \( a(S) \in \mathbb{R}^V \) for each maximal \( S \in S(T) \), with coordinates \( a(S)_v = \left\{ \begin{array}{ll}
\{ \pi \in \Pi(S) | v \in \pi \; \text{and} \; r_v \notin \pi \} & \text{if } v \in V^- \\
\{ \pi \in \Pi(S) | v \in \pi \; \text{and} \; r_v \notin \pi \} & \text{if } v \in V^+ 
\end{array} \right. 
\)
where \( r_v \) = unique incoming (outgoing) arc at \( v \in V^- \) \( (v \in V^+) \), \( \Pi(S) = \{ \text{(undirected) paths in } S \} \),
- a facet for each \( B \in \bigcup_{S \in \mathcal{S}(T)} N(S) \) defined by the half-space \( \left\{ x \in \mathbb{R}^V | \sum_{v \in B} x_v \geq \left( |B| + 1 \right) / 2 \right\} \).

**Exm.** For the tripod

- \( V^- = \{ 1, 3, 4 \} \)
- \( V^+ = \{ 2 \} \)

- half-spaces:
  - \( x_1 \geq 1 \)
  - \( x_2 \geq 1 \)
  - \( x_3 \geq 1 \)
  - \( x_1 + x_3 \geq 3 \)
  - \( x_1 + x_3 \geq 3 \)
  - \( x_3 + x_3 \geq 3 \)
  - \( x_1 + x_3 + x_3 \geq 6 \)
  - \( x_1 + x_3 + x_3 \geq 6 \)
  - \( x_2 + x_3 + x_3 \geq 6 \)
  - \( x_1 + x_2 + x_3 + x_4 = 10 \)

Some properties

**Prop.** The signed tree associahedron \( Asso(T) \) is sandwiched between the permutahedron \( \text{Perm}(V) \) and the parallel piped \( \text{Para}(T) \):

\[
\sum_{u,v \in T} \gamma_{uv} = \text{Perm}(T) \subset Asso(T) \subset \text{Para}(T) = \sum_{u,v \in T} \pi_{uv} \gamma_{uv}
\]

- Common vertices of \( Asso(T) \) and \( Para(T) \) \( \equiv \) orientations of \( T \) which are spines on \( T \), \( Asso(T) \) and \( \text{Perm}(V) \) \( \equiv \) linear orders on \( V \) which are spines on \( T \), 
  \( \Rightarrow \) no common vertex of the three polytopes except if \( T \) = signed path.

**Prop.** \( Asso(T) \) and \( Asso(T') \) isometric \( \iff \) \( T \) and \( T' \) isomorphic or anti-isomorphic up to the signs of their leaves, i.e. there is a bijection \( \theta : V \to V' \) st. \( \forall u,v \in V \)
- \( u \to v \) edge in \( T \) \( \iff \theta(u) \to \theta(v) \) edge in \( T' \),
- if \( u \) is not a leaf of \( T \), the signs of \( u \) and \( \theta(u) \) coincide (resp. differ).

Examples

For a signed path \( P \), \( Asso(P) \) is the classical associahedron faces \( \iff \) dissections \( \iff \) Schröder trees, vertices \( \iff \) triangulations \( \iff \) binary trees.

Hohlweg & Lange, Realizations of the associahedron and cyclohedron, 2007

Loday, Realization of the Stasheff polytope, 2004

For an unsigned tree \( T \), \( Asso(T) \) is the \( T \)-associahedron faces \( \iff \) tubes = connected induced subgraphs of \( T \), faces \( \iff \) tubings = collections of tubes which are pairwise nested, or disjoint and non-adjacent.

Carr & Devadoss, Coxeter complexes and graph associahedra, 2006