EL-labelings and canonical spanning trees for subword complexes

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SUBWORD COMPLEXES

$(W;S)$ nite Coxeter system, $Q = q_1q_2 \cdots q_m$ 2 S , and 2 W.

Subword complex $SC(Q; \ ) =$ simplicial complex with
- vertices $= [m] =$ positions in Q,
- facets $= F(Q; \ ) =$ complements of reduced expressions of in Q.

Exm. $Q^{ex} =$ $2\ 3\ 1\ 3\ 2\ 1\ 2\ 3\ 1$ in $(S_4; f(i\ \ i+1)g)$
$^{ex} = [4;1;3;2] =$ $2\ 3\ 2\ 1 =$ $3\ 2\ 3\ 1 =$ $3\ 2\ 1\ 3$
$F(Q^{ex}; \ ^{ex}) =$ f1; 2; 3; 5; 6g; f1; 2; 3; 6; 7g; f1; 2; 3; 7; 9g;
f1; 3; 4; 5; 6g; f1; 3; 4; 6; 7g; f1; 3; 4; 7; 9g; : : :

Inductive structure: if $Q_a = q_1 \cdots q_{m-1}$, then
$$F(Q; \ ) = F(Q_a; \ q_m) \sqcup F(Q_a; \ ) ? m :$$

Theo. [KM04] The subword complex $SC(Q; \ )$ is either a simplicial sphere or a simplicial ball.

Type A spherical subword complexes provide combinatorial models for families of geometric objects:

A. Knutson and E. Miller. Subword complexes in Coxeter groups. 2004.

EL-LABELINGS OF GRAPHS AND POSETS

$G = (V;E)$ nite, acyclic, directed graph.

EL-labeling of G = edge labeling $:E\ !\ \ N$ of G such that
- there is a unique -rising path p between any $u \dashrightarrow v$ in G,
- (p) lexicographically rst among the (p^0) for $p^0 : u \dashrightarrow v$.

De nes two canonical spanning trees on any interval $[u;v]$ of G:
- -source tree of $[u;v]$ = union of all -rising paths from u,
- -sink tree of $[u;v]$ = union of all -rising paths towards v.

If G is the Hasse diagram of a poset P, EL-labelings carry information on its Möbius function and the topology of its order complex.

Prop. [BW96] For an EL-labeling of P, and u $_P$ v in P,
$$(u;v) = \text{even}\ (u;v) \quad \text{odd}\ (u;v);$$
where even $(u;v)$ and odd $(u;v)$ = numbers of even and odd length -falling paths from u to v in the Hasse diagram of P.

A. Björner and M. Wachs. Shellable nonpure complexes and posets I. 199

RESULTS

1. EL-labelings of the increasing ip graph

$Q^{ex} =$ $2\ 3\ 1\ 3\ 2\ 1\ 2\ 3\ 1$, $^{ex} = [4;1;3;2]$

Increasing ip graph $G(Q; \ )$ = directed labeled graph with
- nodes = facets of $SC(Q; \ )$,
- arcs = $I \to J$ if $9 i\ 2\ I; j\ \ 2\ J$ such that $I\ r\ i = J\ r\ j$ and $i < j$.
  - $i = p(I \to J) =$ positive edge label
  - $j = n(I \to J) =$ negative edge label

Theo. The positive and negative edge labelings p, n are EL-labelings of the increasing ip graph $G(Q; \ )$.

2. Greedy facets and spanning trees of $SC(Q; \ )$

Prop. The lexicographically smallest (resp. largest) facet of $SC(Q; \ )$ is the unique source (resp. sink) of $G(Q; \ )$.

Positive/negative source/sink trees of $SC(Q; \ )$ = canonical spanning trees oriented from/towards the source/sink of $G(Q; \ )$.

Simple inductive descriptions of the rst and last trees, and characterizations of the father of a given node in these four trees.
It yields a greedy ip algorithm to generate $F(Q; \ )$ in polynomial running time and working space.

3. Double root free subword complexes

Increasing ip poset $(Q; \ )$ = transitive closure of the increasing ip graph $G(Q; \ )$.

Prop. $SC(Q; \ )$ is double root free $()$ G $(Q; \ )$ coincides with the Hasse diagram of $(Q; \ )$.

Theo. If $SC(Q; \ )$ is double root free and $I;J$ are facets of $SC(Q; \ )$, then
- There is at most one p-falling (resp. n-falling) path between I and J.
- The Möbius function on $(Q; \ )$ is given by $(I;J) = (\ \ 1)^{jJ\ r\ Ij}$ if there is a p-falling (resp. n-falling) path from I to J, and 0 otherwise.

Relevant Examples:
- $SC(w\ \ (c); w\ \ ) =$ Cluster complex
- $(w\ \ (c); w\ \ ) =$ Cambrian lattice
- see also M. Kallipoliti and H. Mühle's poster
- Duplicated words (boolean lattices)

V. Pilaud and C. Stump. EL-labelings and greedy ip trees for subword complexes. Discrete Geometry and Optimization. Bezdek, Deza & Ye (eds.). 2013.