

MULTI-TRIANGULATIONS AS COMPLEXES OF STAR POLYGONS

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ABSTRACT

The maximum possible number of diagonals that can be drawn in a convex polygon with $k + 1$ of them mutually crossing is $k(2n - 2k - 1)$ ([CP92]). Maximal such subsets of edges (called here k -triangulations) are known to generalize nicely some properties of triangulations of a convex n -gon ([NAK00],[DKM03],[JON05]).

In this poster, we present proofs of basic properties of k -triangulations (number of edges, flip), using the new tool of stars that generalize triangles for multi-triangulations. We also discuss open problems that may hopefully be easier to analyze using this new tool.

DEFINITIONS

A k -triangulation of a convex n -gon is a maximal subset of (diagonal) edges without any $(k + 1)$ -crossing (that is, subset of $k+1$ mutually intersecting edges).

A k -star is a polygon formed by connecting $2k+1$ vertices s_0, \dots, s_{2k} (cyclically ordered) with the edges $[s_0, s_k], [s_1, s_{1+k}], \dots, [s_k, s_{2k}], [s_{k+1}, s_0], \dots, [s_{2k}, s_{k-1}]$.

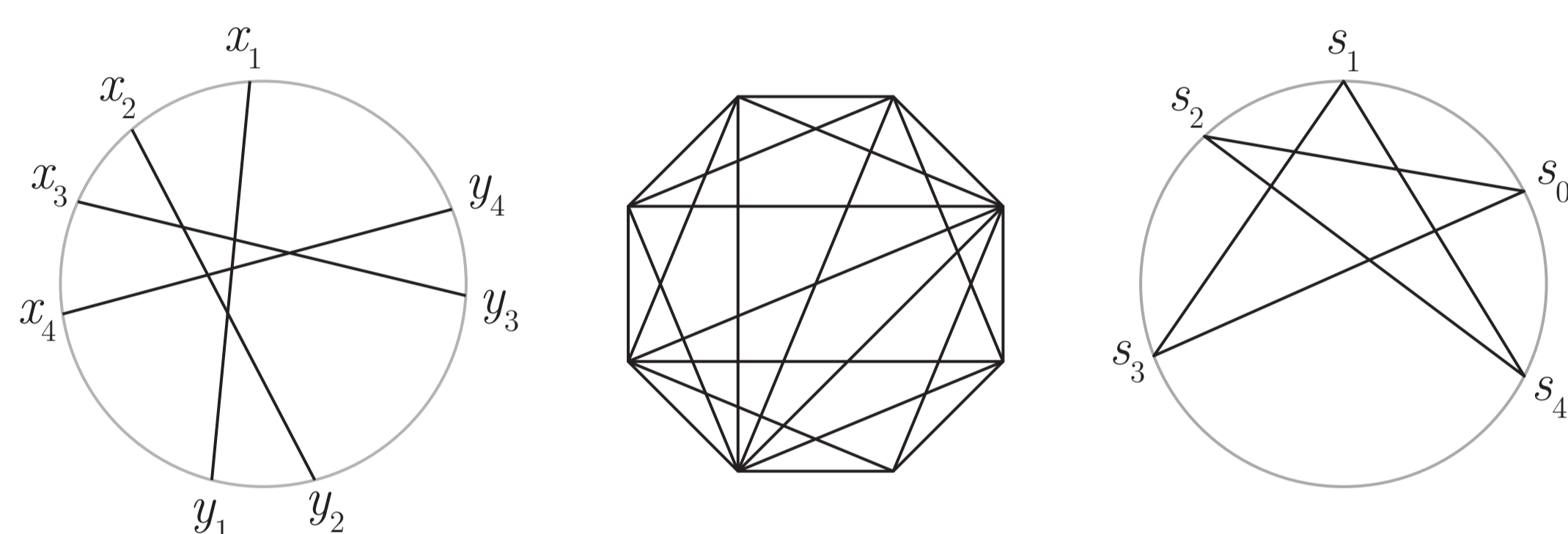


FIG. 1: A 4-crossing – a 2-triangulation of the octagon – a 2-star.

COMPLEXES OF k -STARS

THEOREM 1. An edge of a k -triangulation T is contained in zero, one or two k -stars of T , depending on whether its length is smaller, equal or greater than k .

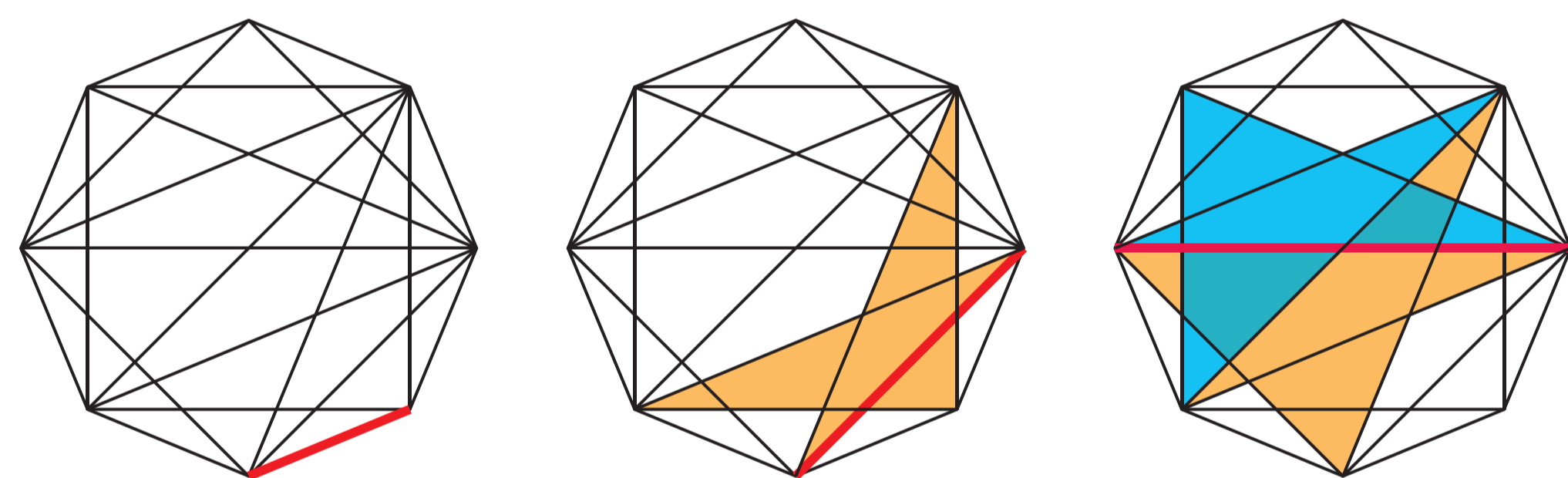


FIG. 2: 2-stars containing a given (red) edge.

THEOREM 2. Let T be a k -triangulation.

- (i) Every pair of k -stars of T have a unique common bisector.
- (ii) Any edge which is not in T is the common bisector of a unique pair of k -stars of T .

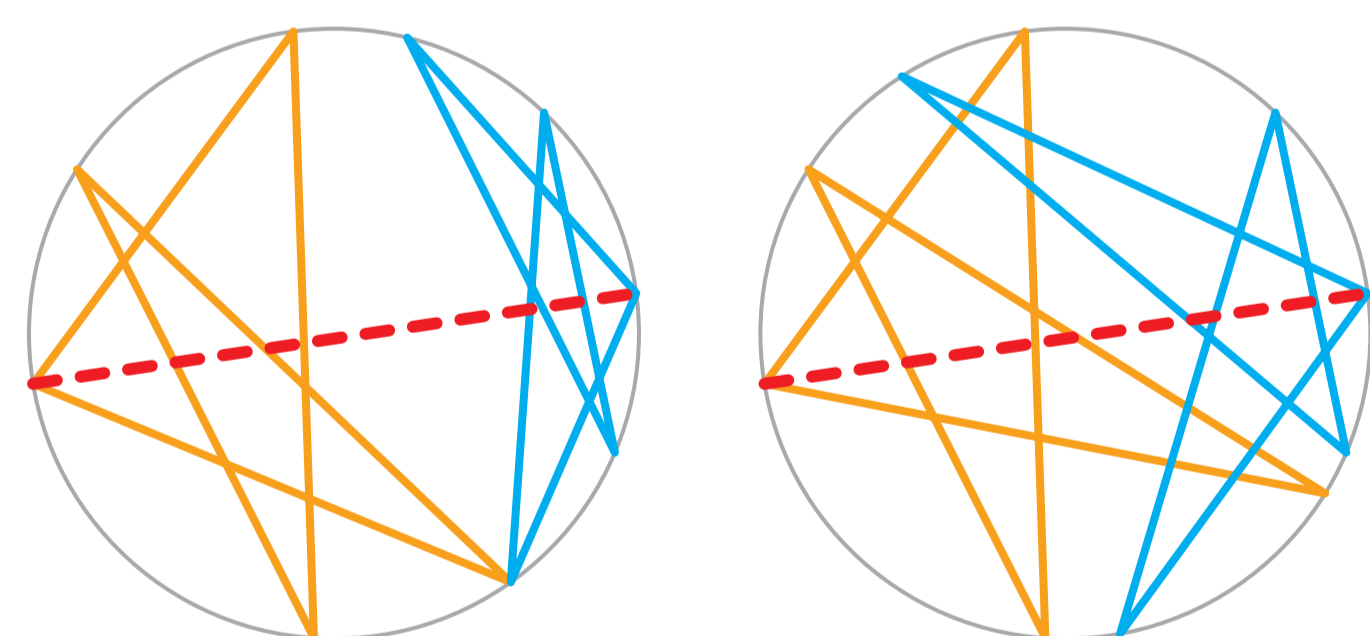


FIG. 3: Common bisector of two 2-stars.

COROLLARY. Any k -triangulation of the convex n -gon contains exactly $n - 2k$ k -stars and thus $k(2n - 2k - 1)$ edges.

FLIPS

THEOREM 3. If e is an edge contained in two stars of T with common bisector f , then:
 (i) $T \Delta \{e, f\}$ is also a k -triangulation and (ii) no other k -triangulation contains $T \setminus \{e\}$.
 The k -triangulation $T \Delta \{e, f\}$ is obtained by flipping the edge e in the k -triangulation T .

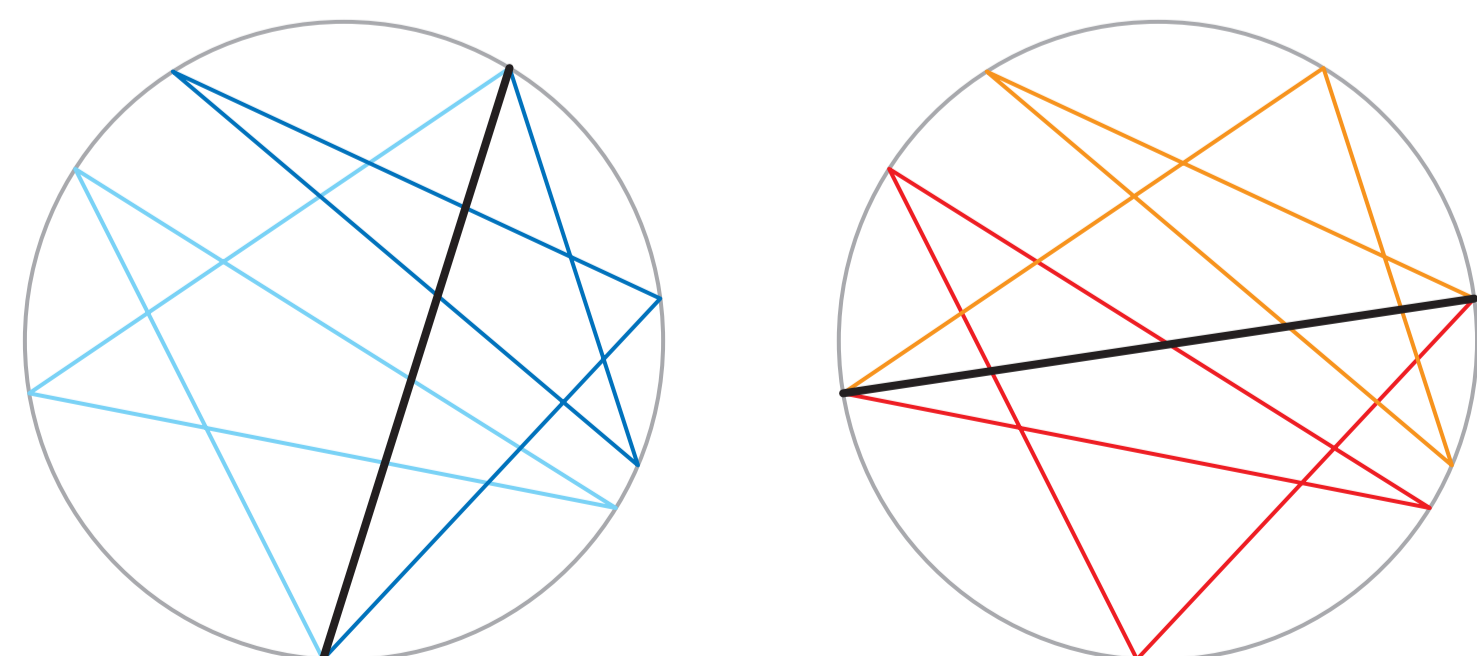


FIG. 4: The flip of an edge.

THEOREM 4. The graph of flips on the set of k -triangulations of the convex n -gon is connected, regular of degree $k(n - 2k - 1)$, and its diameter is at most $2k(n - 2k - 1)$.

RELATED TOPICS AND OPEN PROBLEMS

1. Multi-Dyck-paths

THEOREM 5. [JON05] The number of k -triangulations of the convex n -gon is $\det(C_{n-i-j})_{1 \leq i, j \leq k}$ (where $C_m = \frac{1}{m+1} \binom{2m}{m}$ denotes the m -th Catalan number).

This determinant is also known to count k -tuples of non-crossing Dyck paths of semi-length $n - 2k$ (k -Dyck-path).

PROBLEM 1. Find an explicit bijection between k -triangulations and k -Dyck-paths (done when $k = 2$ in [ELI07]).

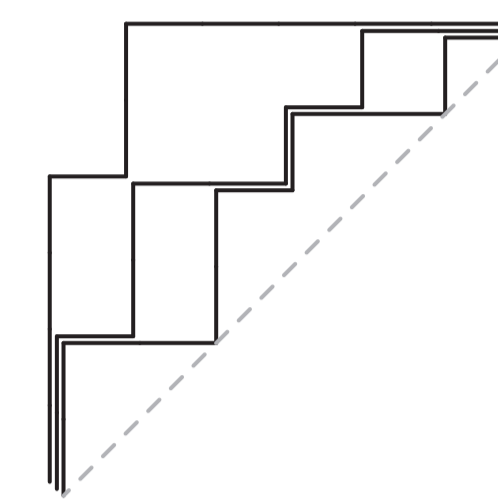


FIG. 5: 3-Dyck-path.

2. Sparsity & rigidity

A graph $G = (V, E)$ is (p, q) -sparse if for any subset F of E , $|F| \leq p|V(F)| - q$ (where $V(F)$ denotes the set of vertices of F). A (p, q) -sparse graph $G = (V, E)$ is (p, q) -tight when furthermore $|E| = p|V| - q$.

Depending on the parameters (p, q) , sparsity is related to different subjects ([LS07]):

- (i) a (generically minimally) rigid graph in dimension d is $(d, \binom{d+1}{2})$ -tight;
- (ii) a graph is an ℓ -arborescence if and only if it is (ℓ, ℓ) -tight.

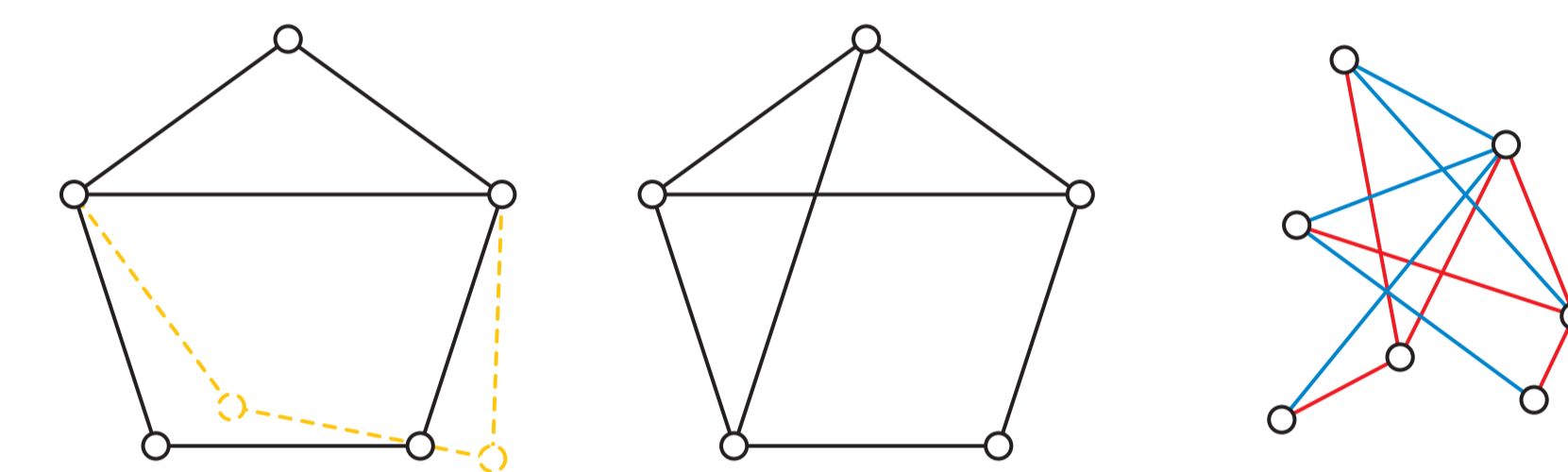


FIG. 6: A non-rigid graph – a rigid graph – a 2-arborescence.

LEMMA. (i) A k -triangulation of a convex polygon is $(2k, \binom{2k+1}{2})$ -tight.

(ii) The dual graph of a k -triangulation is (k, k) -tight.

PROBLEM 2. Is a k -triangulation always (generically minimally) rigid in dimension $2k$?

3. Multi-associahedron

Let $\Delta_{n,k}$ be the complex of all subsets of edges that do not contain any $(k + 1)$ -crossing.

PROBLEM 3. Is there a polytope of dimension $k(n - 2k - 1)$ with boundary $\Delta_{n,k}$?

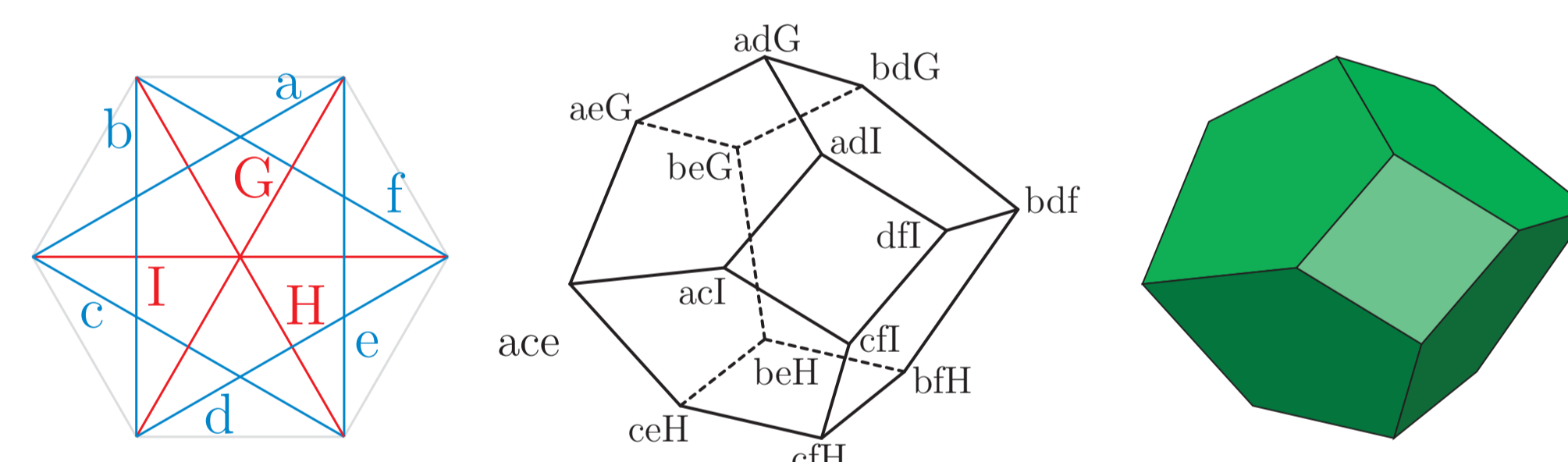


FIG. 7: A realization of the polar of $\Delta_{6,1}$ (the associahedron).

4. Surfaces

The polygonal complex $\mathcal{C}(T)$ associated to a k -triangulation T is a polygonal decomposition of an orientable surface with boundary $\mathcal{S}_{n,k}$.

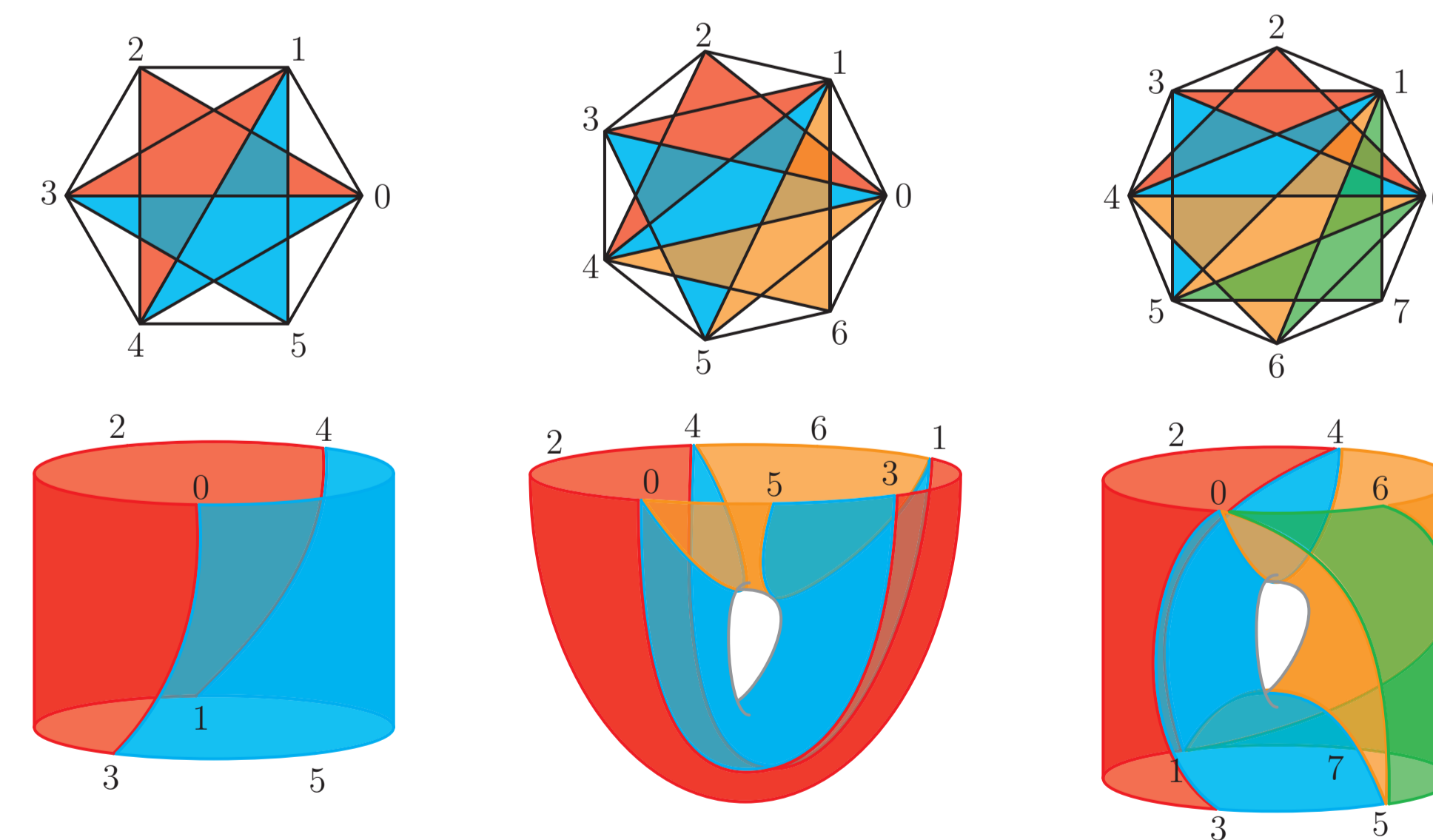


FIG. 8: Decomposition of $\mathcal{S}_{n,2}$ ($n=6,7,8$) associated to the greedy 2-triangulation.

PROBLEM 4. Characterize the decompositions of $\mathcal{S}_{n,k}$ that correspond to k -triangulations.

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