SIGNED TREE ASSOCIAHEDRA

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POLYTOPES FROM COMBINATORICS
polytope = convex hull of a finite set of $\mathbb{R}^d$

= bounded intersection of finitely many half-spaces

face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations

Given a set of points, determine the face lattice of its convex hull.

Given a lattice, is there a polytope which realizes it?
Permutohedron $\text{Perm}(n)$

$$= \text{conv} \{(\sigma(1), \ldots, \sigma(n + 1)) \mid \sigma \in \Sigma_{n+1}\}$$

$$= \mathbb{H} \cap \bigcap_{\emptyset \neq J \subseteq [n+1]} \mathbb{H}^\geq(J)$$
Permutohedron $\text{Perm}(n)$

$$= \text{conv}\ \{\sigma(1), \ldots, \sigma(n+1)\mid \sigma \in \Sigma_{n+1}\}$$

$$= H \cap \bigcap_{\emptyset \neq J \subset [n+1]} H^\geq(J)$$

$k$-faces of $\text{Perm}(n)$

$\equiv$ ordered partitions of $[n + 1]$ into $n + 1 - k$ parts

$\equiv$ surjections from $[n + 1]$ to $[n + 1 - k]$
Permutohedron $\text{Perm}(n)$

$$\text{Perm}(n) = \text{conv} \left\{ (\sigma(1), \ldots, \sigma(n+1)) \mid \sigma \in \Sigma_{n+1} \right\} = \mathbb{H} \cap \bigcap_{\emptyset \neq J \subseteq [n+1]} \mathbb{H}^{\geq}(J)$$

$k$-faces of $\text{Perm}(n)$

$$\equiv$$

ordered partitions of $[n+1]$ into $n+1-k$ parts

surjections from $[n+1]$ to $[n+1-k]$

connections to

• inversion sets
• weak order
• reduced expressions
• braid moves
• cosets of the symmetric group
ASSOCIAHEDRA
Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex \((n + 3)\)-gon, ordered by reverse inclusion.
**VARIOUS ASSOCIAHEDRA**

**Associahedron** = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex \((n + 3)\)-gon, ordered by reverse inclusion

(Pictures by Ceballos-Santos-Ziegler)

Lee (’89), Gel’fand-Kapranov-Zelevinski (’94), Billera-Filliman-Sturmfels (’90), . . . , Ceballos-Santos-Ziegler (’11)
Loday (’04), Hohlweg-Lange (’07), Hohlweg-Lange-Thomas (’12), P.-Santos (’12), P.-Stump (’12+), Lange-P. (’13+)
Loday’s associahedron

\[ \text{Loday’s associahedron} = \text{conv} \{ L(T) \mid T \text{ triangulation of the } (n+3)\text{-gon} \} \]

\[ = \mathbb{H} \bigcap \bigcap_{\delta \text{ diagonal of the } (n+3)\text{-gon}} \mathbb{H}^\geq(\delta) \]

\[ L(T) = (\ell(T, \hat{j}) \cdot r(T, \hat{j}))_{j \in [n+1]} \]

\[ \mathbb{H}^\geq(\delta) = \left\{ x \in \mathbb{R}^{n+1} \right\mid \sum_{j \in B(\delta)} x_j \geq \left( |B(\delta)| + 1 \right) \right\} \]

Loday, Realization of the Stasheff polytope ('04)
Loday’s associahedron

\[
\text{Loday’s associahedron} = \text{conv} \left\{ L(T) \mid T \text{ binary tree on } n + 1 \text{ nodes} \right\}
\]

\[
= \mathbb{H} \cap \bigcap_{I \text{ interval of } [n+1]} H^\geq(I)
\]

\[
L(T') - L(T) \in \mathbb{R}_{>0}(e_i - e_j)
\]

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Loday’s associahedron = \( \text{conv} \{ L(T) \mid T \text{ binary tree on } n+1 \text{ nodes} \} \)

\[ = \bigcap \bigcap \text{ H}^\geq(I) \]

\[ \bigcap \bigcap \text{ I interval of } [n+1] \]

\[ \ell(T, j) \quad r(T', j) \]

\[ L(T') - L(T) \in \mathbb{R}_{>0}(e_i - e_j) \]

Loday, *Realization of the Stasheff polytope* (’04)
The associahedron is obtained from the permutahedron by removing facets.
Relevant connections to combinatorial properties:

• the normal fan of $\text{Perm}(n)$ refines that of $\text{Asso}(P)$

• it defines a surjection $\kappa : \mathcal{S}_{n+1} \rightarrow \{\text{triangulations}\}$ (connection to linear extensions and insertion in binary search trees)

• $\kappa$ defines a lattice homomorphism from the weak order to the Tamari lattice
LODAY’S ASSOCIAHEDRON AND PERMUTAHEDRON
Can also replace Loday’s \( (n + 3) \)-gon by others...

...to obtain different realizations of the associahedron

Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* (’07)
Asso\((P) = \text{conv}\{HL(T) \mid T \text{ triangulation of } P\} = H \cap \bigcap_{\delta \text{ diagonal of } P} H^{\geq}(\delta)\)

\[HL(T)_j = \begin{cases} \ell(T, j) \cdot r(T, j) & \text{if } j \text{ down} \\ n + 2 - \ell(T, j) \cdot r(T, j) & \text{if } j \text{ up} \end{cases}\]

\[H^{\geq}(\delta) = \left\{ x \middle| \sum_{j \in B(\delta)} x_j \geq \left(\frac{|B(\delta)| + 1}{2}\right) \right\}\]

Hohlweg-Lange, Realizations of the associahedron and cyclohedron ('07)
Spines = labeled and oriented dual binary trees
REM. 1. Spines can be defined without their triangulations…
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2. Alternative vertex description of Hohlweg-Lange’s associahedra:

$$a(S)_j = \begin{cases} 
|\{\pi \text{ maximal path in } S \text{ with 2 incoming arcs at } j\}| & \text{if } j \text{ down} \\
 n + 2 - |\{\pi \text{ maximal path in } S \text{ with 2 outgoing arcs at } j\}| & \text{if } j \text{ up}
\end{cases}$$
REM. 1. Spines can be defined without their triangulations...

2. Alternative vertex description of Hohlweg-Lange’s associahedra:

\[ a(S)_j = \begin{cases} 
|\{ \pi \text{ path in } S \text{ not using the outgoing arc at } j \}| & \text{if } j \text{ down} \\
n + 2 - |\{ \pi \text{ path in } S \text{ not using the incomming arc at } j \}| & \text{if } j \text{ up} 
\end{cases} \]
GRAPH ASSOCIAHEDRA
NESTED COMPLEX AND GRAPH ASSOCIAHEDRON

$G$ graph on ground set $V$

**Tube** on $V = $ connected induced subgraph of $G$

Compatible tubes $=$ nested, or disjoint and non-adjacent

Nested complex $\mathcal{N}(G) =$ simplicial complex of sets of pairwise compatible tubes

$= $ clique complex of the compatibility relation on tubes

$G$-associahedron $=$ polytopal realization of the nested complex on $G$

Carr-Devadoss, Coxeter complexes and graph associahedra ('06)
EXM: GRAPH ASSOCIAHEDRON
**TWO QUESTIONS**

**Qu 1.** Which graph associahedra can be realized by removahedra?

Lange-P., Which nestohedra are removahedra? ('14+)

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**Qu 2.** Can we obtain distinct realizations of graph associahedra?

Yes for trees... 

P., Signed tree associahedra ('13+)
SIGNATURE TREE ASSOCIATION
T tree on the signed ground set $V = V^- \sqcup V^+$ (negative in white, positive in black)

**Signed spine** on $T = \text{directed and labeled tree } S$

(i) the labels of the nodes of $S$ form a partition of the signed ground set $V$

(ii) at a node of $S$ labeled by $U = U^- \sqcup U^+$, the source label sets of the different incoming arcs are subsets of distinct connected components of $T \setminus U^-$, while the sink label sets of the different outgoing arcs are subsets of distinct connected components of $T \setminus U^+$
LEM. Contracting an arc in a signed spine on $T$ leads to a new signed spine on $T$.

LEM. Let $S$ be a signed spine on $T$ with a node labeled by a set $U$ containing at least two elements. For any $u \in U$, there exists a signed spine on $T$ whose nodes are labeled exactly as that of $S$, except that the label $U$ is partitioned into $\{u\}$ and $U \setminus \{u\}$. 

![Diagram](image-url)
CONTRACTIONS AND SPINE COMPLEX

**LEM.** Contracting an arc in a signed spine on $T$ leads to a new signed spine on $T$

**LEM.** Let $S$ be a signed spine on $T$ with a node labeled by a set $U$ containing at least two elements. For any $u \in U$, there exists a signed spine on $T$ whose nodes are labeled exactly as that of $S$, except that the label $U$ is partitioned into $\{u\}$ and $U \setminus \{u\}$

**Signed spine complex** $S(T) =$ simplicial complex whose inclusion poset is isomorphic to the poset of edge contractions on the signed spines of $T$

**CORO.** The signed spine complex $S(T)$ is a pure simplicial complex of rank $|V|$
Braid arrangement on $\mathbb{R}^V = \{x \in \mathbb{R}^V | x_u = x_v\}$ for $u \neq v \in V$.

Braid fan $BF = \text{complete simplicial fan defined by the braid arrangement on }$ $H := \left\{ x \in \mathbb{R}^V \middle| \sum_{v \in V} x_v = \binom{|V|+1}{2} \right\}$.
For $S$ spine on $T$, define $C(S) := \{ x \in \mathbb{H} \mid x_u \leq x_v, \text{ for all arcs } u \to v \text{ in } S \}$

**THEO.** The collection of cones $\mathcal{F}(T) := \{ C(S) \mid S \in \mathcal{S}(T) \}$ defines a complete simplicial fan on $\mathbb{H}$, which we call the spine fan

**CORO.** For any signed tree $T$, the signed nested complex $\mathcal{N}(T)$ is a simplicial sphere
Signed tree associahedron $\text{Asso}(T) = \text{convex polytope with}$

(i) a vertex $a(S) \in \mathbb{R}^V$ for each maximal signed spine $S \in S(T)$, with coordinates

$$a(S)_v = \begin{cases} 
| \{ \pi \in \Pi(S) \mid v \in \pi \text{ and } r_v \notin \pi \} | & \text{if } v \in V^- \\
|V| + 1 - | \{ \pi \in \Pi(S) \mid v \in \pi \text{ and } r_v \notin \pi \} | & \text{if } v \in V^+ 
\end{cases}$$

where $r_v = \text{unique incoming (resp. outgoing) arc when } v \in V^- \ (\text{resp. when } v \in V^+)$

$\Pi(S) = \text{set of all (undirected) paths in } S, \text{ including the trivial paths}$

(ii) a facet defined by the half-space

$$H^\geq(B) := \left\{ x \in \mathbb{R}^V \mid \sum_{v \in B} x_v \geq \binom{|B|+1}{2} \right\}$$

for each signed building block $B \in B(T)$
EXM: FACET DESCRIPTION
**MAIN RESULT**

**THM.** The spine fan $\mathcal{F}(T)$ is the normal fan of the signed tree associahedron $\text{Asso}(T)$, defined equivalently as

(i) the convex hull of the points

$$a(S)_v = \begin{cases} | \{ \pi \in \Pi(S) \mid v \in \pi \text{ and } r_v \notin \pi \} | & \text{if } v \in V^- \\ |V| + 1 - | \{ \pi \in \Pi(S) \mid v \in \pi \text{ and } r_v \notin \pi \} | & \text{if } v \in V^+ \end{cases}$$

for all maximal signed spines $S \in \mathcal{S}(T)$

(ii) the intersection of the hyperplane $\mathbb{H}$ with the half-spaces

$$H^\geq(B) := \left\{ x \in \mathbb{R}^V \mid \sum_{v \in B} x_v \geq \binom{|B|+1}{2} \right\}$$

for all signed building blocks $B \in \mathcal{B}(T)$

**CORO.** The signed tree associahedron $\text{Asso}(T)$ realizes the signed nested complex $\mathcal{N}(T)$
STEP 1. We have
\[ \sum_{v \in V} a(S)_v = \left( \frac{|V| + 1}{2} \right) \quad \text{and} \quad \sum_{v \in sc(r)} a(S)_v = \left( \frac{|sc(r)| + 1}{2} \right) \]
for any arc \( r \) of \( S \). In other words, “each vertex \( a(S) \) belongs to the hyperplanes \( H^=(B) \) it is supposed to”. Proof by double counting.
SKETCH OF THE PROOF

STEP 1. We have
\[
\sum_{v \in V} a(S)_v = \left( |V| + 1 \right) \quad \text{and} \quad \sum_{v \in \text{sc}(r)} a(S)_v = \left( |\text{sc}(r)| + 1 \right)
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for any arc \( r \) of \( S \). In other words, “each vertex \( a(S) \) belongs to the hyperplanes \( H^\mp(B) \) it is supposed to”. Proof by double counting.

STEP 2. If \( S \) and \( S' \) are two adjacent maximal spines on \( T \), such that \( S' \) is obtained from \( S \) by flipping an arc joining node \( u \) to node \( v \), then
\[
a(S') - a(S) \in \mathbb{R}_{>0} \cdot (e_u - e_v)
\]

\[
a(S') - a(S) = (|U| + 1) \cdot (|V| + 1) \cdot (e_u - e_v)
\]
SKETCH OF THE PROOF

STEP 1. We have

$$\sum_{v \in V} a(S)_v = \left( \frac{|V| + 1}{2} \right)$$
and

$$\sum_{v \in \text{sc}(r)} a(S)_v = \left( \frac{|\text{sc}(r)| + 1}{2} \right)$$

for any arc $r$ of $S$. In other words, “each vertex $a(S)$ belongs to the hyperplanes $H^\mp(B)$ it is supposed to”. Proof by double counting.

STEP 2. If $S$ and $S'$ are two adjacent maximal spines on $T$, such that $S'$ is obtained from $S$ by flipping an arc joining node $u$ to node $v$, then

$$a(S') - a(S) \in \mathbb{R}_{>0} \cdot (e_u - e_v)$$

STEP 3. A general theorem concerning realizations of simplicial fan by polytopes
In other words, a characterization of when is a simplicial fan regular

Hohlweg-Lange-Thomas, Permutahedra and generalized associahedra (’11)
De Loera-Rambau-Santos, Triangulations: Structures for Algorithms and Applications (’10)
**PROP.** The signed tree associahedron Asso(T) is sandwiched between the permutahedron Perm(V) and the parallelepiped Para(T)

\[
\sum_{u \neq v \in V} [e_u, e_v] = \text{Perm}(T) \subset \text{Asso}(T) \subset \text{Para}(T) = \sum_{u \sim v \in T} \pi(u - v) \cdot [e_u, e_v]
\]
FURTHER GEOMETRIC PROPERTIES

PROP. The signed tree associahedron $\text{Asso}(T)$ is sandwiched between the permutahedron $\text{Perm}(V)$ and the parallelepiped $\text{Para}(T)$

$$\sum_{u \neq v \in V} [e_u, e_v] = \text{Perm}(T) \subset \text{Asso}(T) \subset \text{Para}(T) = \sum_{u \rightarrow v \in T} \pi(u - v) \cdot [e_u, e_v]$$

Common vertices of $\text{Asso}(T)$ and $\text{Para}(T)$ $\equiv$ orientations of $T$ which are spines on $T$

Common vertices of $\text{Asso}(T)$ and $\text{Perm}(T)$ $\equiv$ linear orders on $V$ which are spines on $T$

$\Rightarrow$ no common vertex of the three polytopes except if $T$ is a signed path
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$\Rightarrow$ no common vertex of the three polytopes except if $T$ is a signed path

**PROP.** $\text{Asso}(T)$ and $\text{Asso}(T')$ isometric $\iff$ $T$ and $T'$ isomorphic or anti-isomorphic, up to the sign of their leaves, ie. $\exists$ bijection $\theta : V \to V'$ st. $\forall u, v \in V$

- $u-v$ edge in $T$ $\iff$ $\theta(u)-\theta(v)$ edge in $T'$
- if $u$ is not a leaf of $T$, the signs of $u$ and $\theta(u)$ coincide (resp. are opposite)
FURTHER GEOMETRIC PROPERTIES

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- if $u$ is not a leaf of $T$, the signs of $u$ and $\theta(u)$ coincide (resp. are opposite)

REM. The vertex barycenter of $\text{Asso}(T)$ does not necessarily coincide with that of the permutahedron (but it lies on the linear span of the characteristic vectors of the orbits of $V$ under the automorphism group of $T$)
THANK YOU