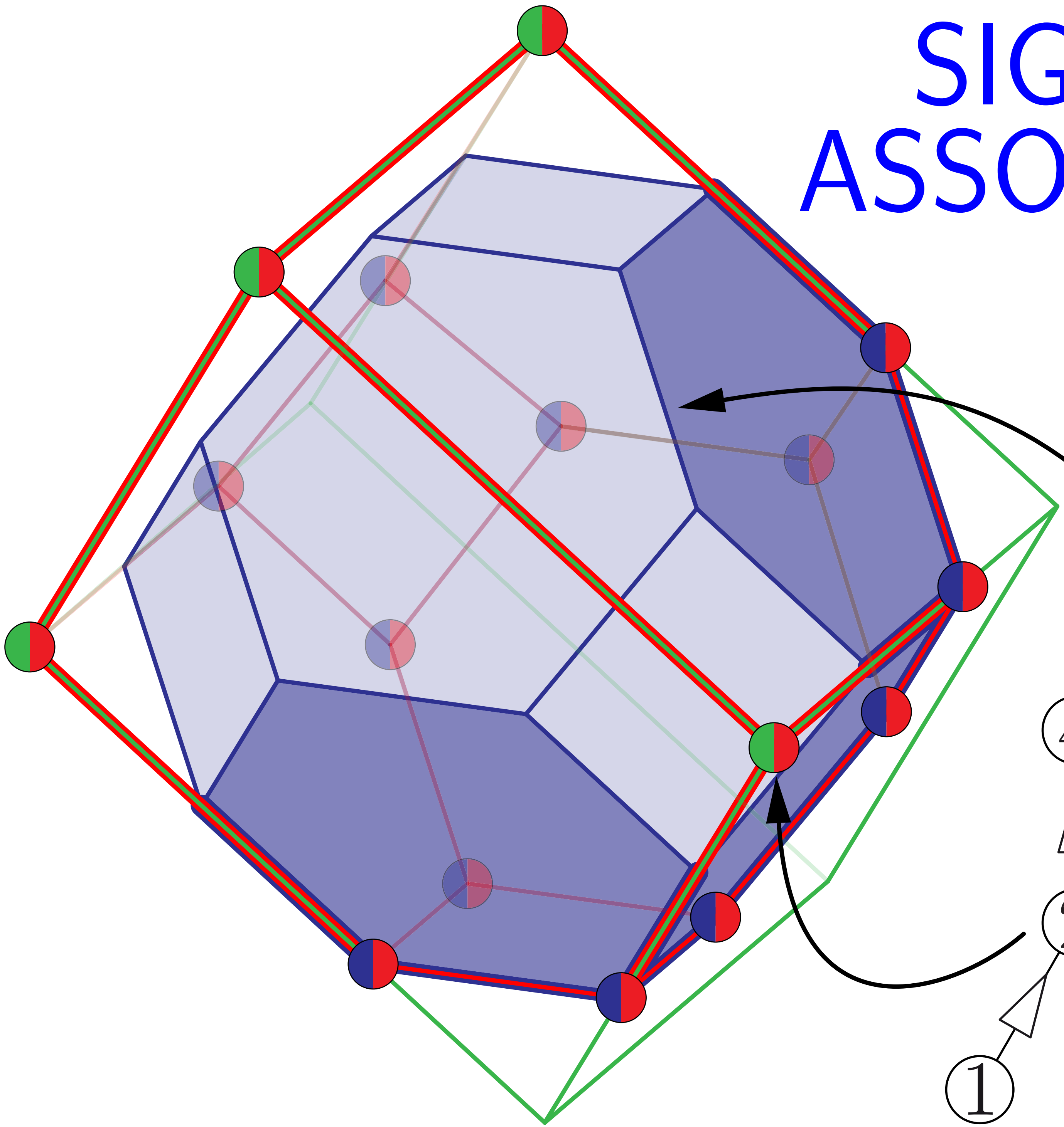
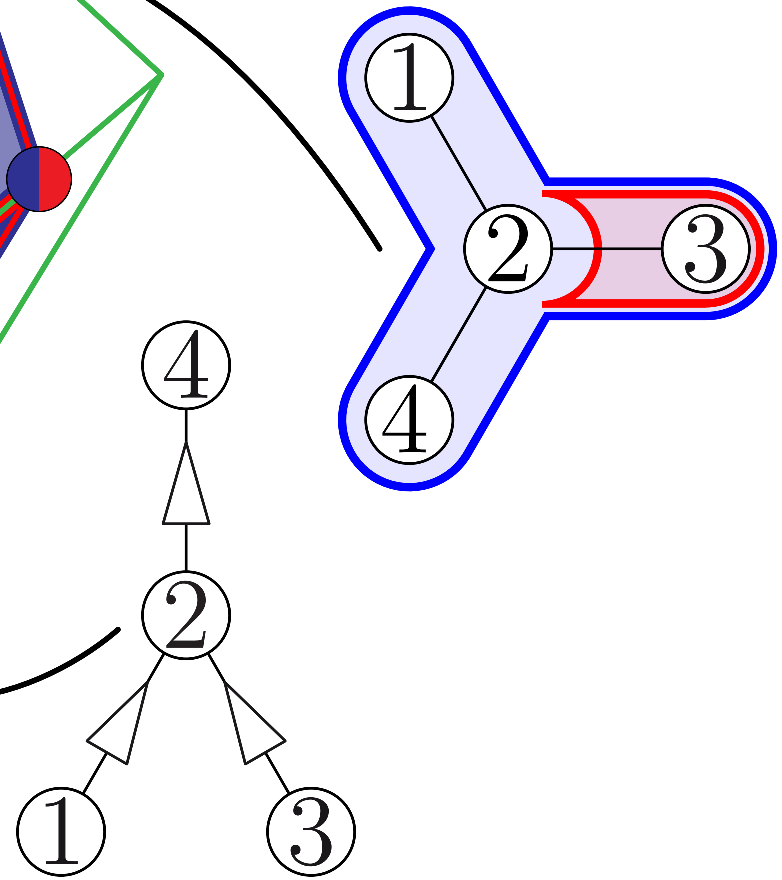
SIGNED TREE ASSOCIAHEDRA

Vincent PILAUD
(CNRS & LIX)





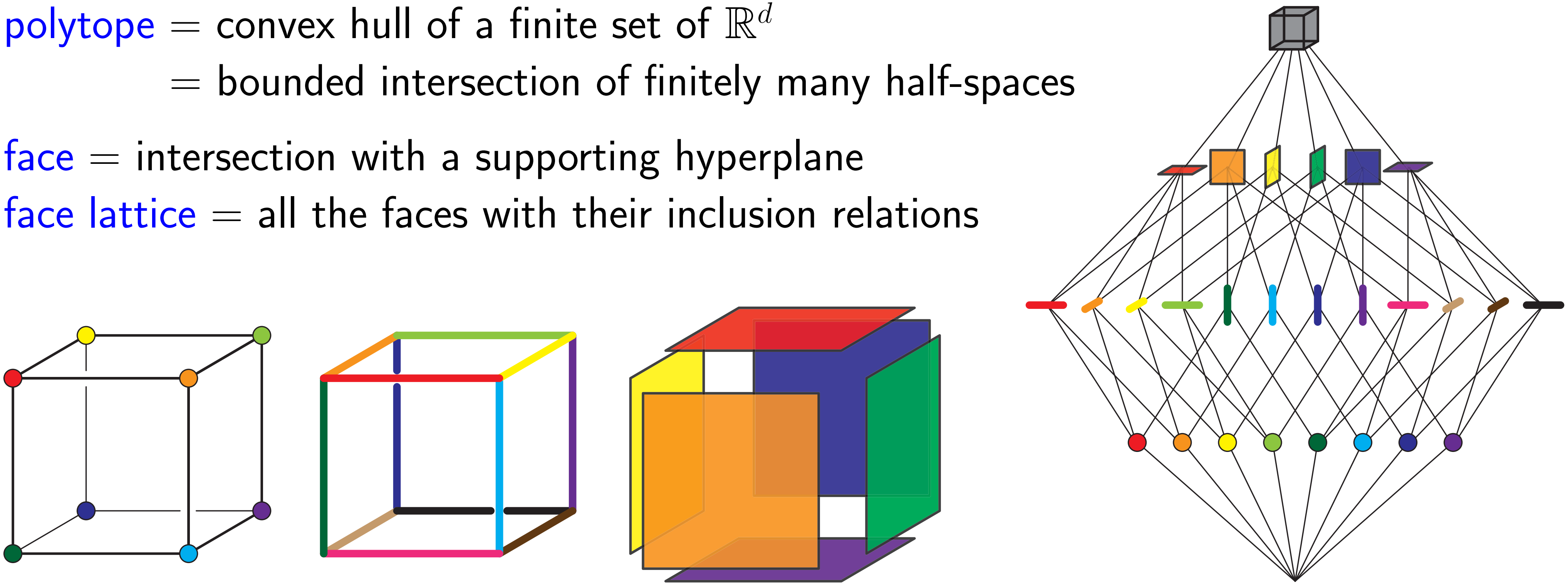
POLYTOPES FROM COMBINATORICS

POLYTOPES & COMBINATORICS

polytope = convex hull of a finite set of $\mathbb{R}^d$
= bounded intersection of finitely many half-spaces

face = intersection with a supporting hyperplane

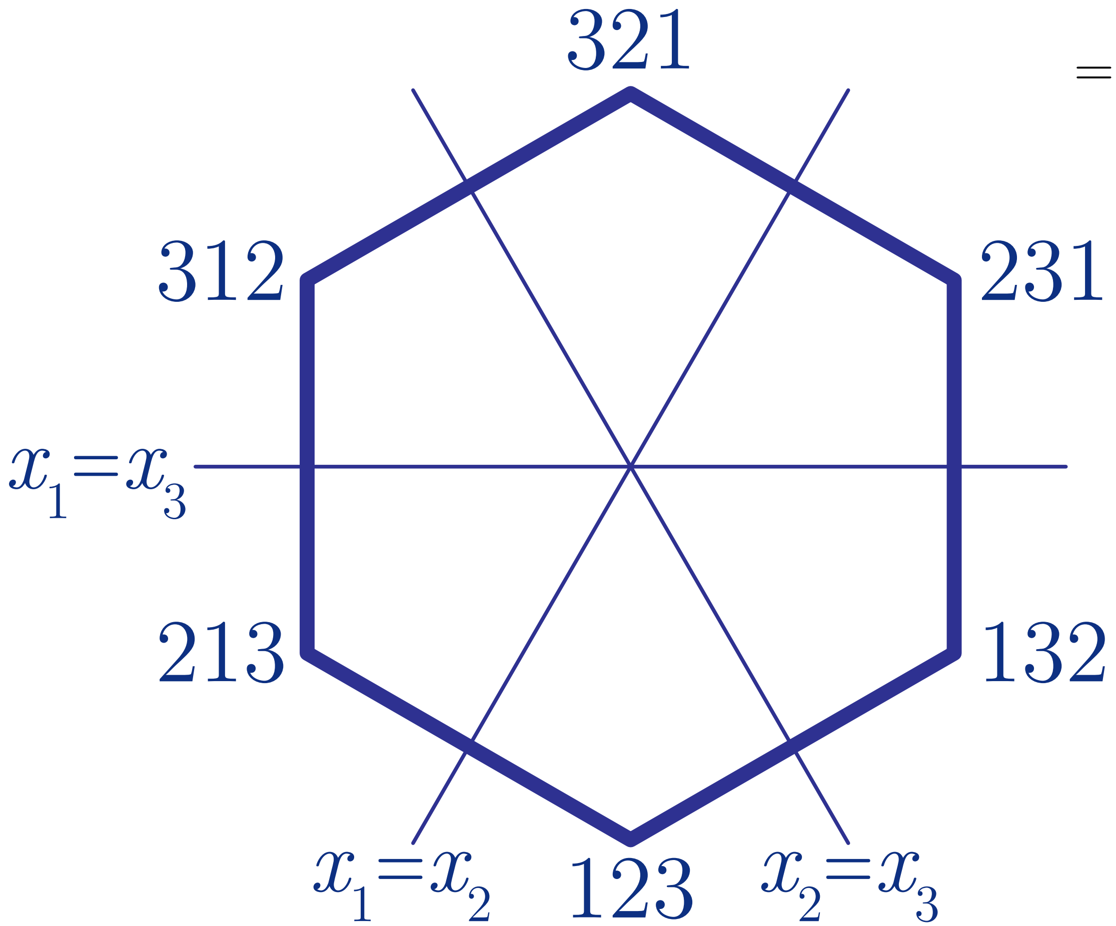
face lattice = all the faces with their inclusion relations



Given a set of points, determine the face lattice of its convex hull.

Given a lattice, is there a polytope which realizes it?

PERMUTAHEDRON

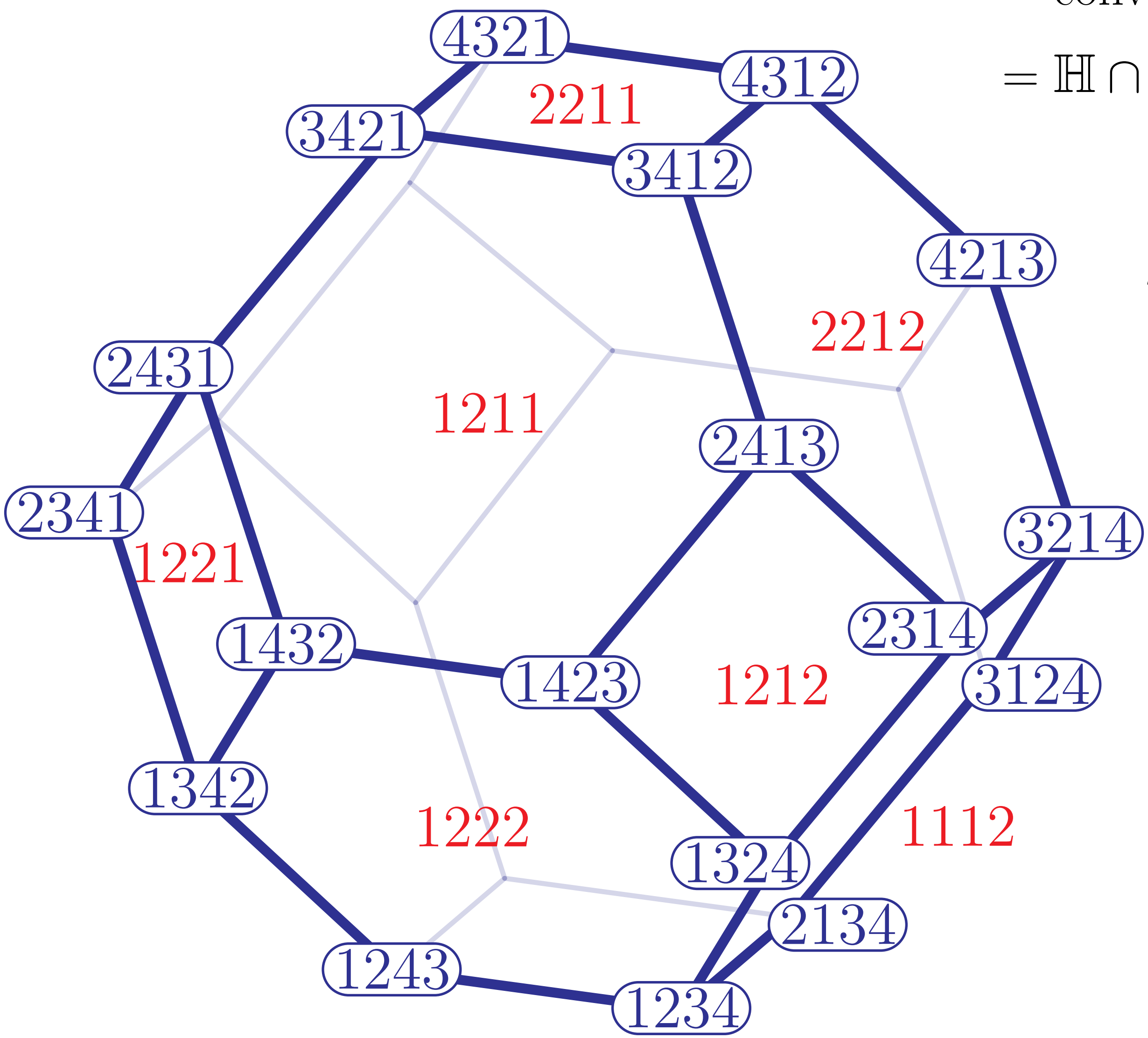


Permutohedron $\mathsf{Perm}(n)$

$$= \operatorname{conv}\left\{(\sigma(1), \dots, \sigma(n+1)) \mid \sigma \in \Sigma_{n+1}\right\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n+1]} \mathbf{H}^{\geq}(J)$$

PERMUTAHEDRON



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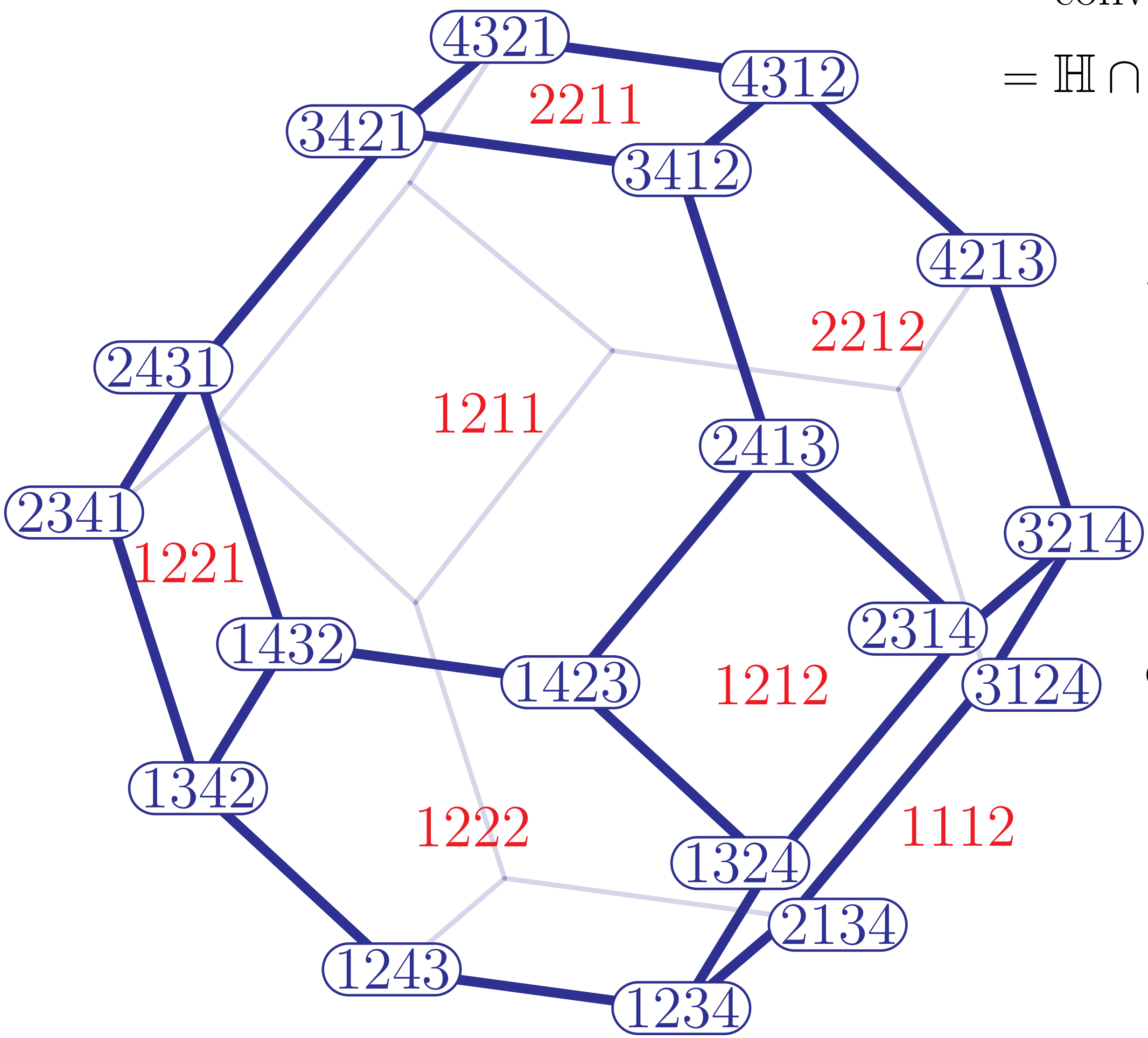
k-faces of Perm(n)

$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

PERMUTAHEDRON



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k-faces of Perm(n)

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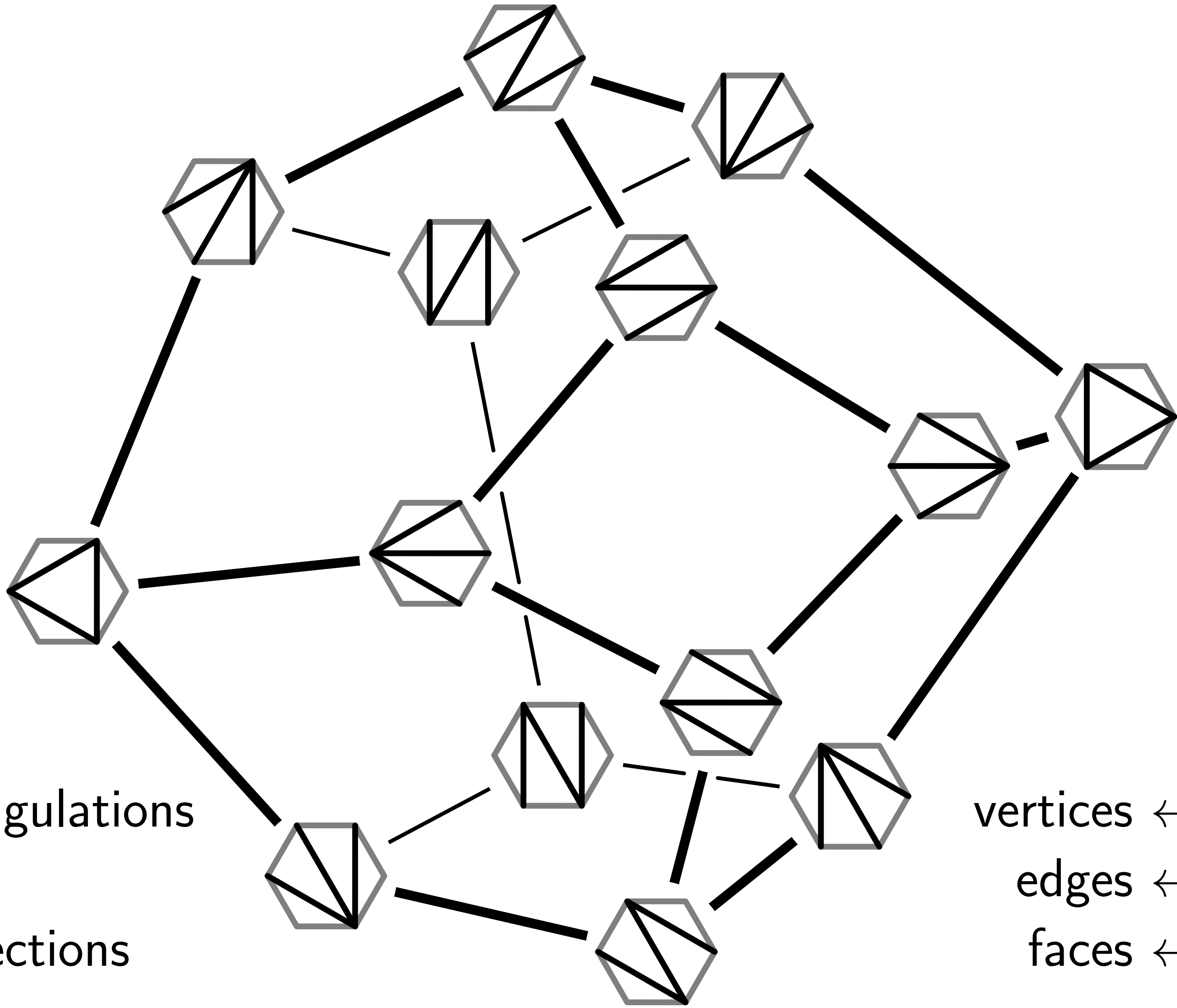
connections to

- inversion sets
- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group

ASSOCIAHEDRA

ASSOCIAHEDRON

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion

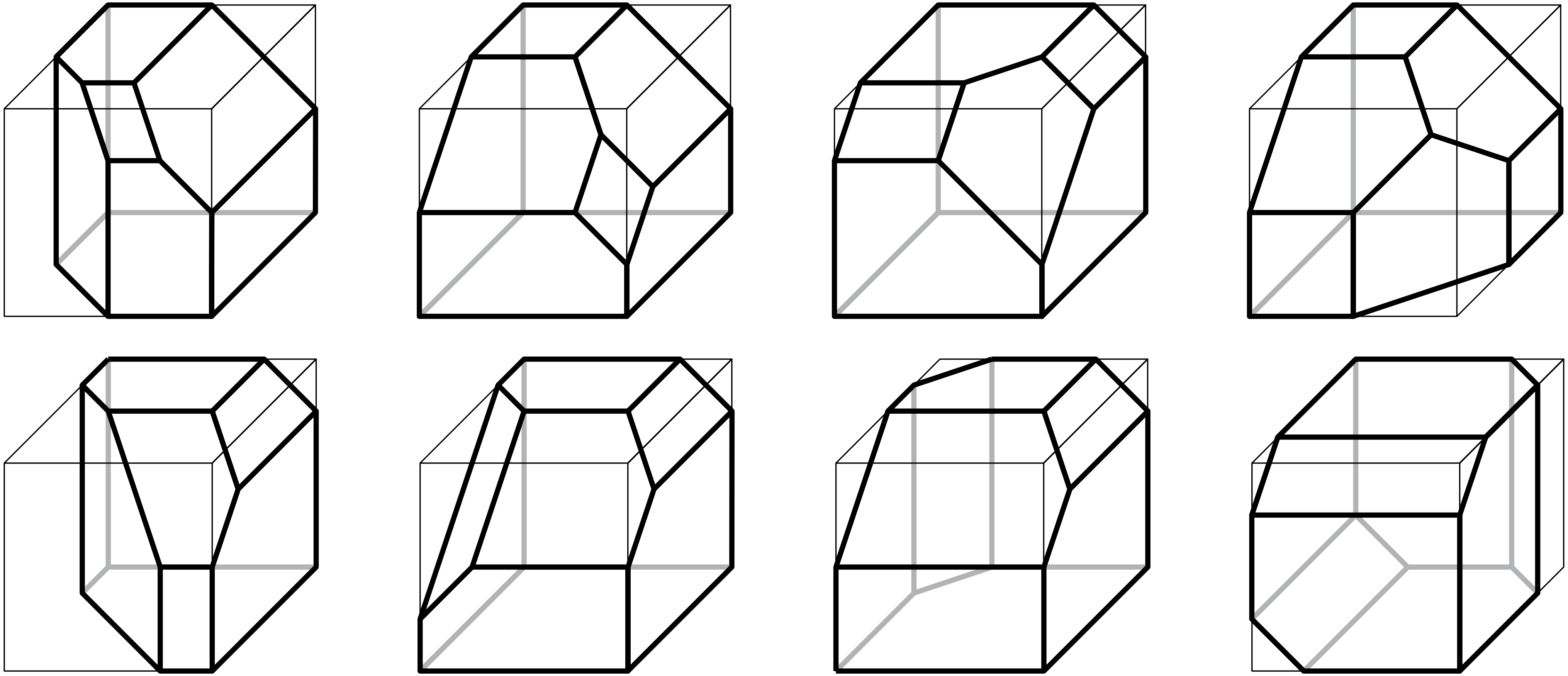


vertices $\leftrightarrow$ triangulations
edges $\leftrightarrow$ flips
faces $\leftrightarrow$ dissections

vertices $\leftrightarrow$ binary trees
edges $\leftrightarrow$ rotations
faces $\leftrightarrow$ Schröder trees

VARIOUS ASSOCIAHEDRA

Associahedron $=$ polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n + 3)$-gon, ordered by reverse inclusion



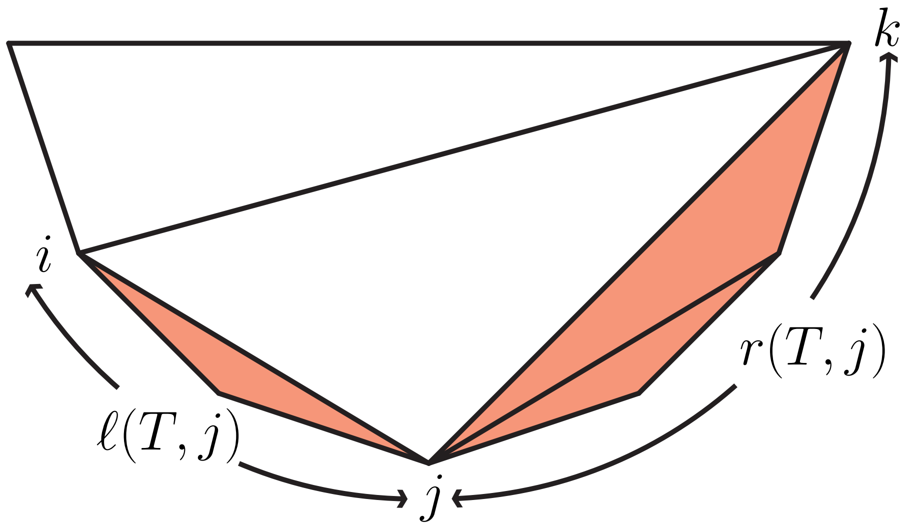
(Pictures by Ceballos-Santos-Ziegler)

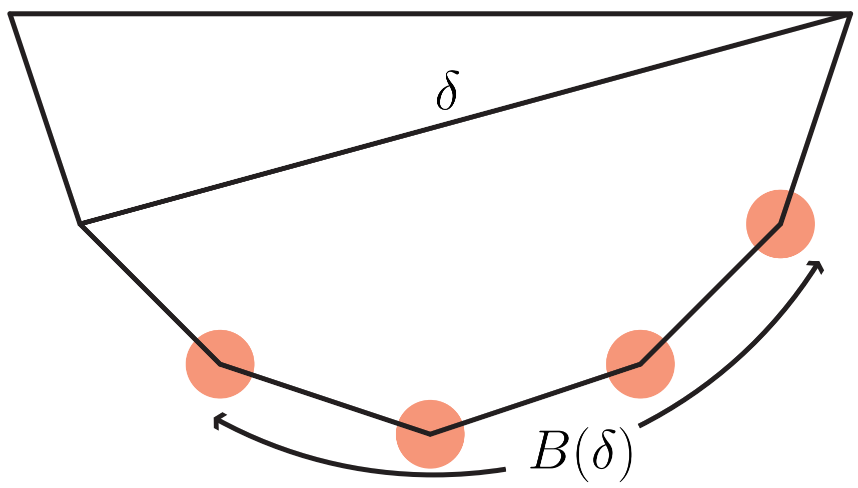
Lee ('89), Gel'fand-Kapranov-Zelevinski ('94), Billera-Filliman-Sturmfels ('90), . . . , Ceballos-Santos-Ziegler ('11)
Loday ('04), Hohlweg-Lange ('07), Hohlweg-Lange-Thomas ('12), P.-Santos ('12), P.-Stump ('12$^+$), Lange-P. ('13$^+$)

LODAY'S ASSOCIAHEDRON

Loday's associahedron $= \operatorname{conv}\{L(T) \mid T$ triangulation of the $(n+3)$-gon$\}$

$$= \mathbb{H} \cap \bigcap_{\substack{\delta \text{ diagonal} \\ \text{of the } (n+3)\text{-gon}}} \mathbf{H}^{\geq}(\delta)$$





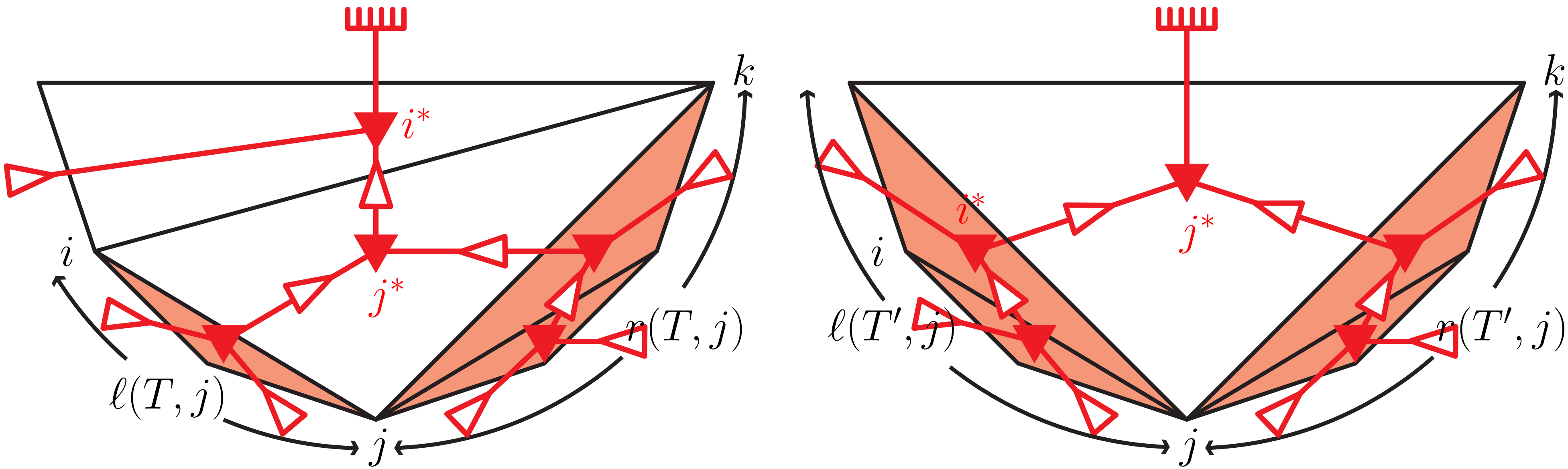
$$L(T) = \big(\ell(T,j)\cdot r(T,j)\big)_{j\in[n+1]}$$

$$\mathbf{H}^{\geq}(\delta) = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j\in B(\delta)} x_j \geq \binom{|B(\delta)|+1}{2} \right\}$$

Loday, *Realization of the Stasheff polytope* ('04)

LODAY'S ASSOCIAHEDRON

$$\text{Loday's associahedron} = \operatorname{conv}\{L(T) \mid T \text{ binary tree on } n+1 \text{ nodes}\}$$
$$= \mathbb{H} \cap \bigcap_{\substack{I \text{ interval} \\ \text{of } [n+1]}} \mathbf{H}^{\geq}(I)$$



$$L(T') - L(T) \in \mathbb{R}_{>0}(e_i - e_j)$$

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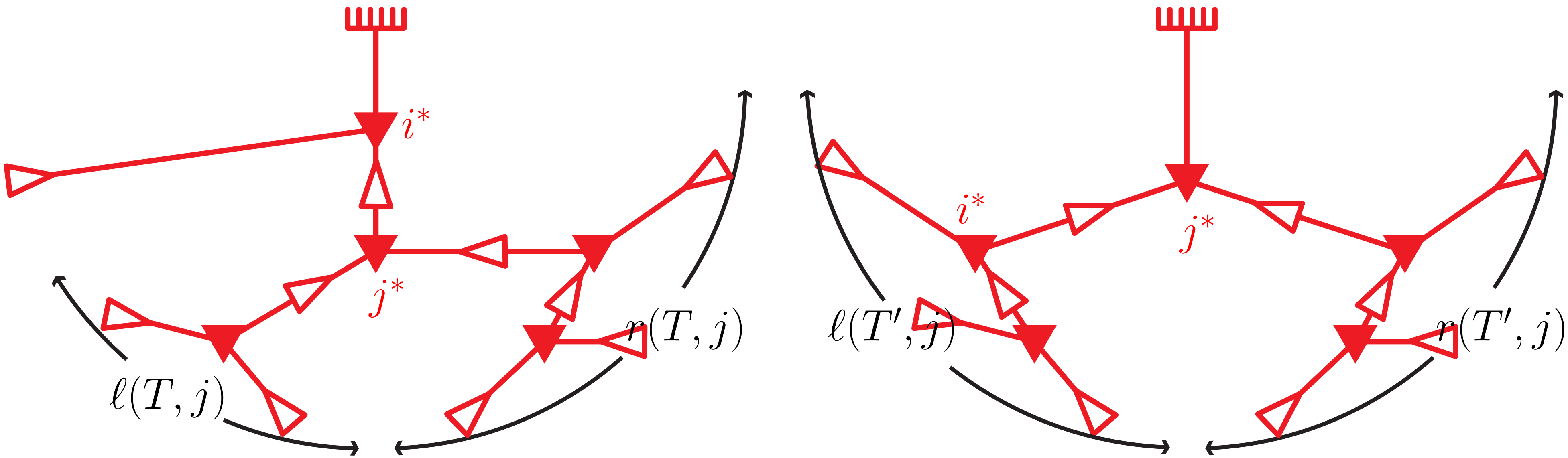

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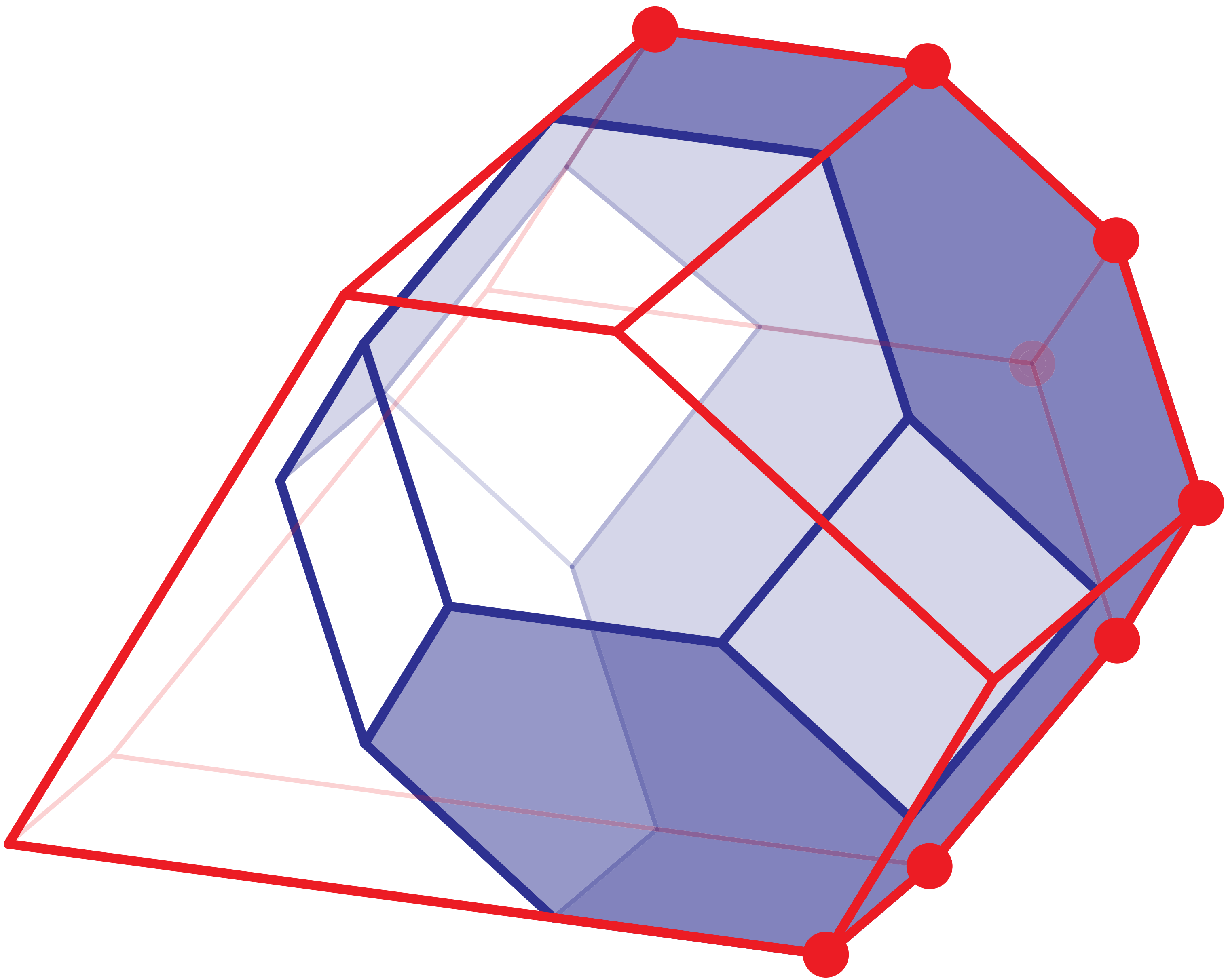
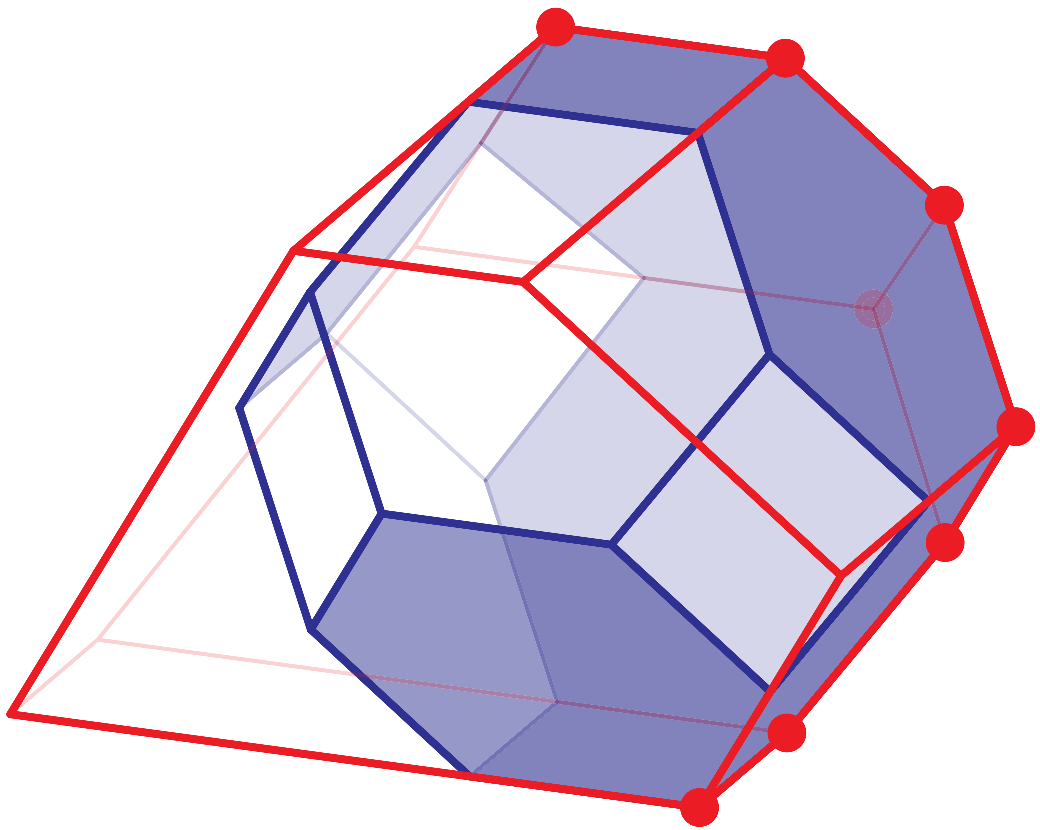
ASSOCIAHEDRON AND PERMUTAHEDRON



The associahedron is obtained from the permutahedron by removing facets

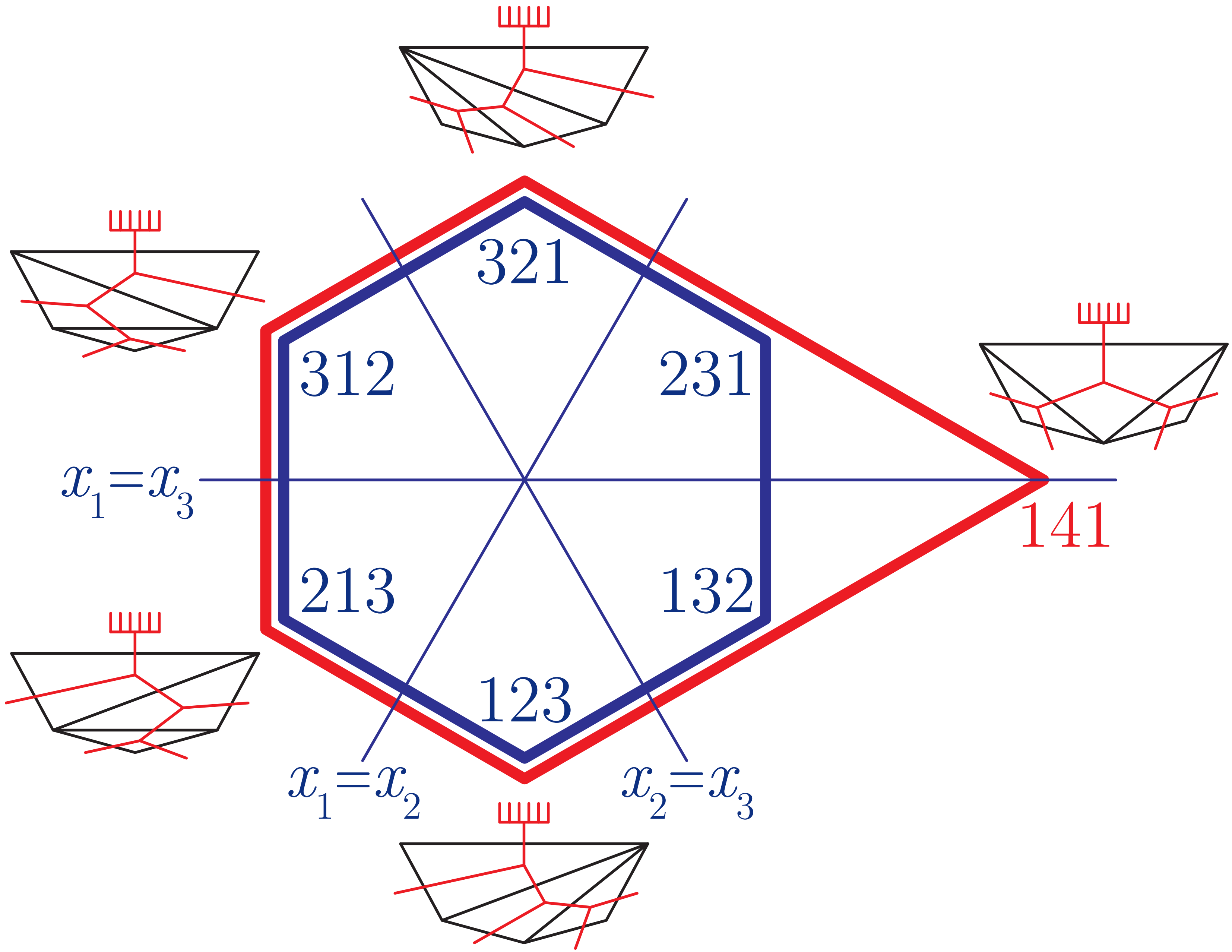
ASSOCIAHEDRON AND PERMUTAHEDRON



Relevant connections to combinatorial properties:

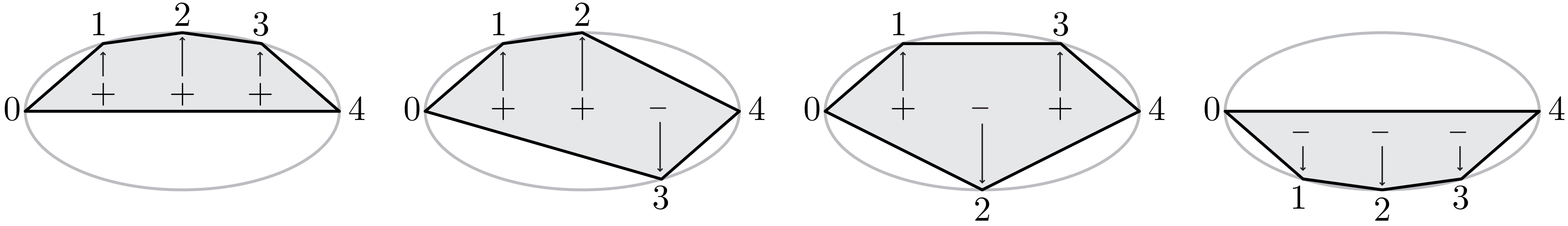
- the normal fan of Perm(n) refines that of Asso(P)
- it defines a surjection $\kappa : \mathfrak{S}_{n+1} \to$ {triangulations} (connection to linear extensions and insertion in binary search trees)
- κ defines a lattice homomorphism from the weak order to the Tamari lattice

LODAY'S ASSOCIAHEDRON AND PERMUTAHEDRON

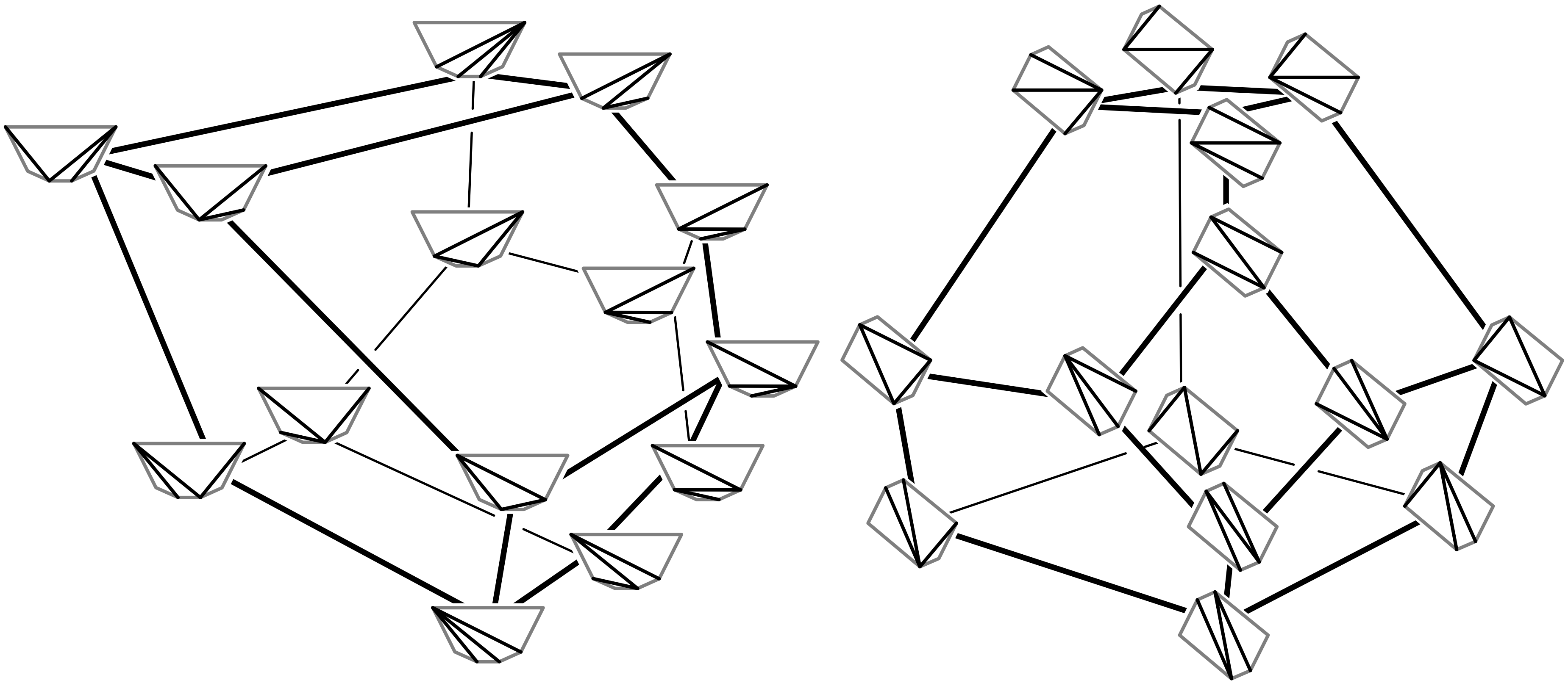


HOHLWEG & LANGE'S ASSOCIAHEDRA

Can also replace Loday's $(n+3)$-gon by others. . .



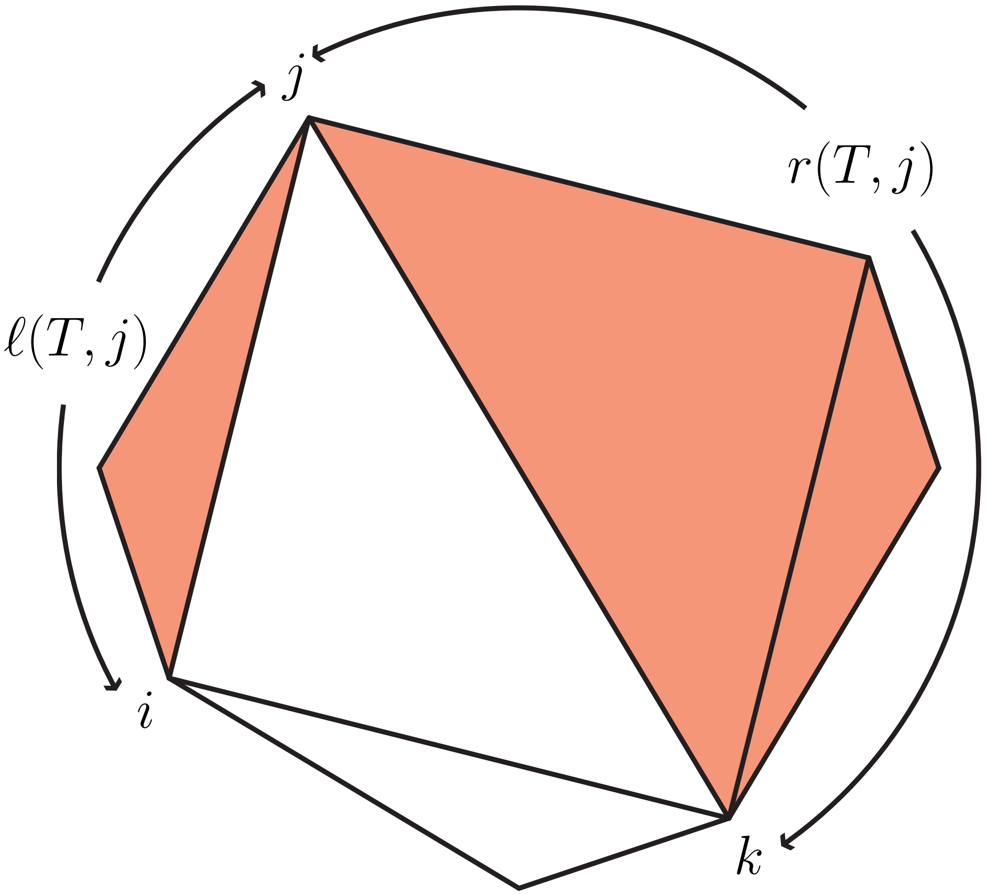
. . . to obtain different realizations of the associahedron

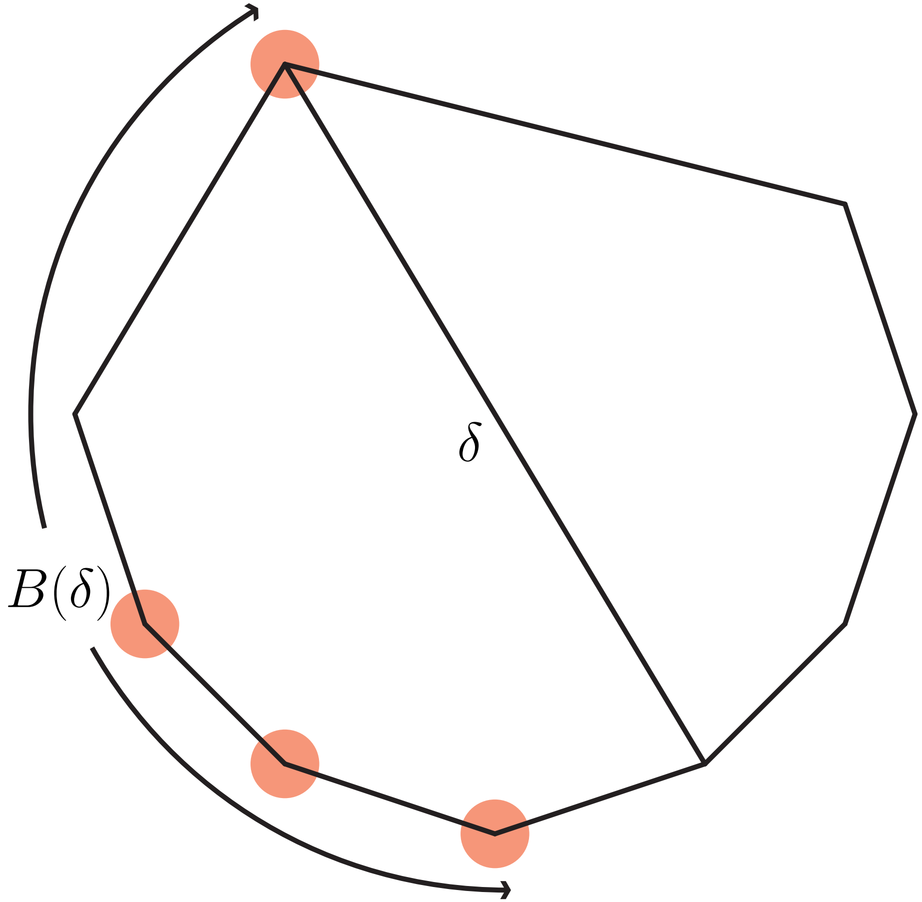


Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)

HOHLWEG & LANGE'S ASSOCIAHEDRA

$$\text{Asso}(P) = \text{conv}\{HL(T) \mid T \text{ triangulation of } P\} = \mathbb{H} \cap \bigcap_{\delta \text{ diagonal of } P} \mathbf{H}^{\geq}(\delta)$$





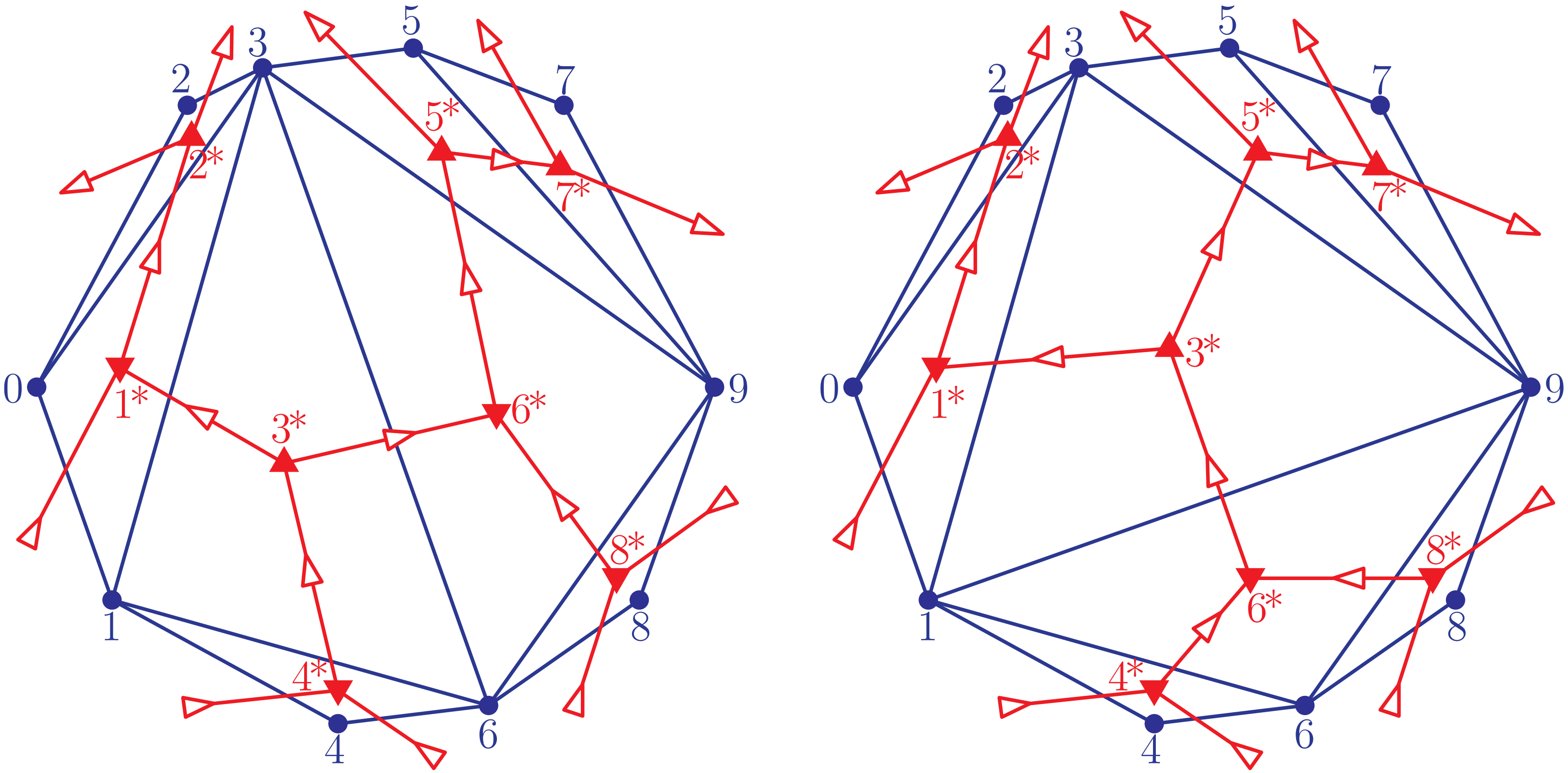
$$HL(T)_j = \begin{cases} \ell(T,j) \cdot r(T,j) & \text{if } j \text{ down} \\ n+2-\ell(T,j) \cdot r(T,j) & \text{if } j \text{ up} \end{cases}$$

$$\mathbf{H}^{\geq}(\delta) = \left\{ \mathbf{x} \;\middle|\; \sum_{j \in B(\delta)} x_j \geq \binom{|B(\delta)|+1}{2} \right\}$$

Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)

SPINES

Lange-P., *Using spines to revisit a construction of the associahedron* ('13$^+$)



Spines = labeled and oriented dual binary trees

SPINES

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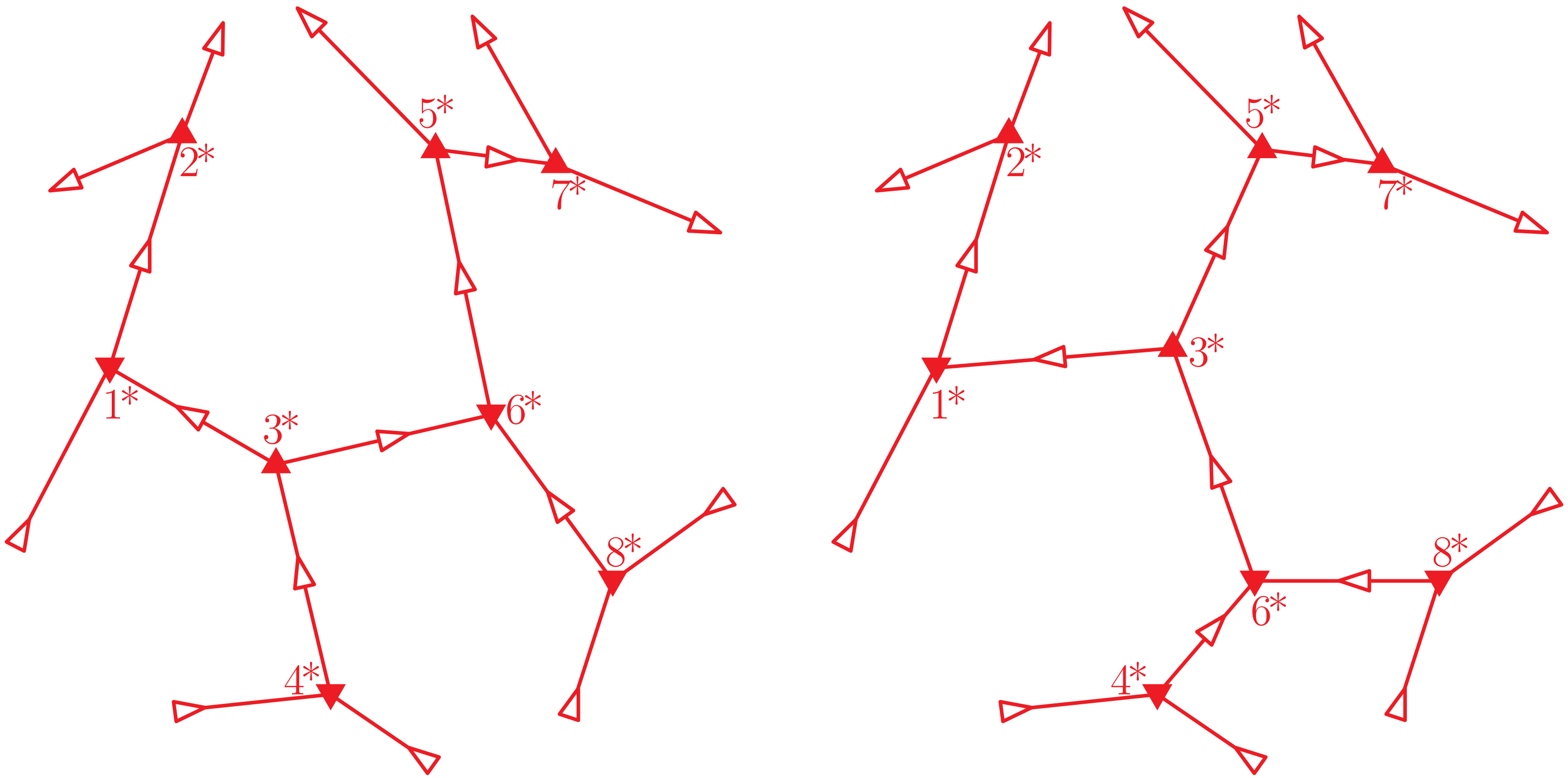
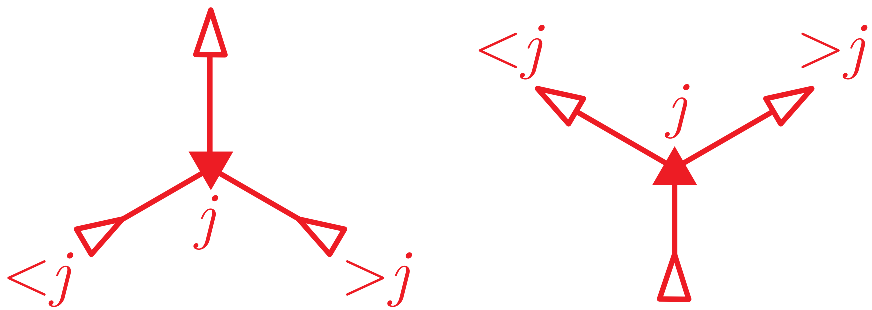


REM. 1. Spines can be defined without their triangulations. . .



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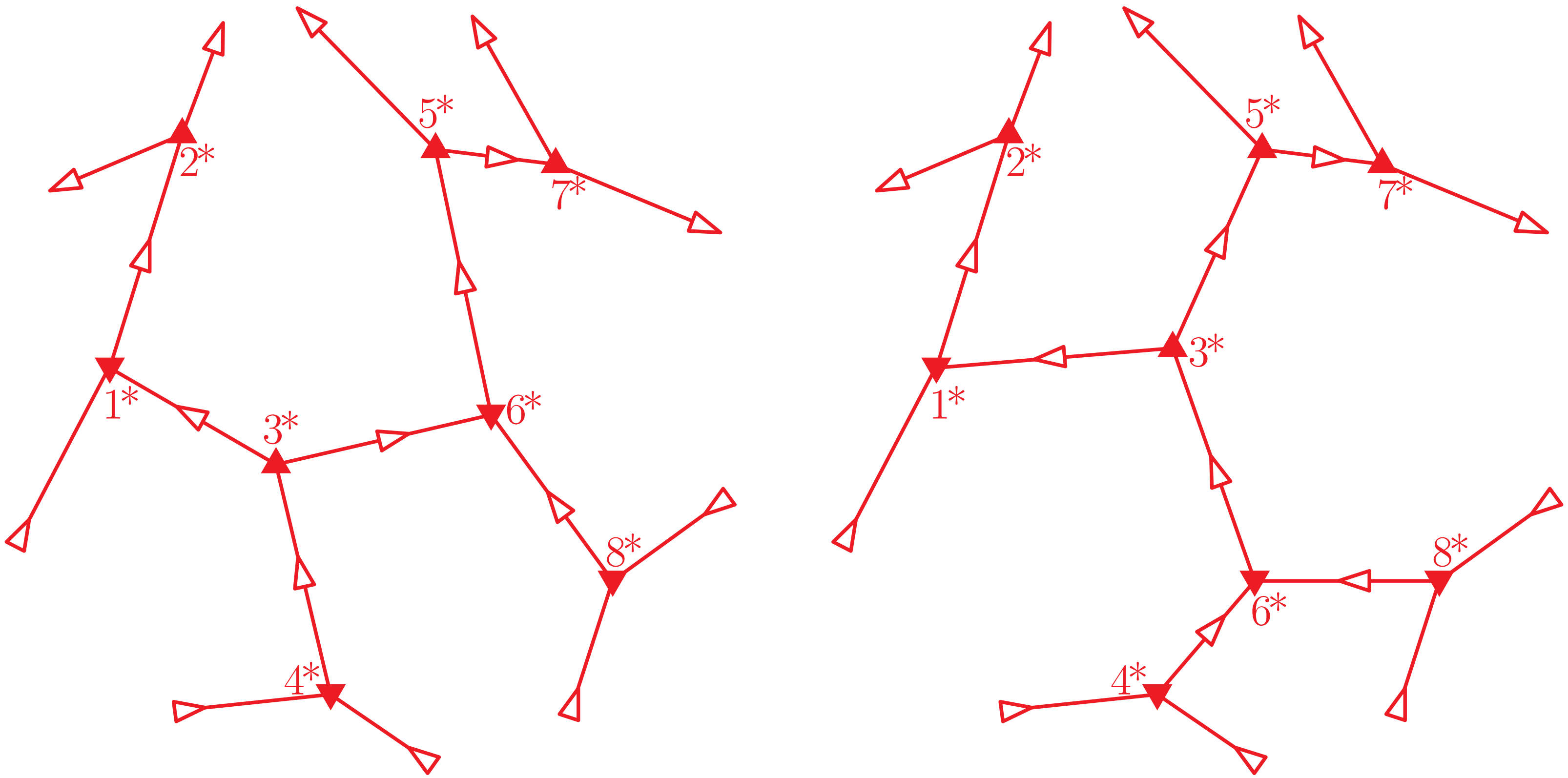

REM. 1. Spines can be defined without their triangulations...

2. Alternative vertex description of Hohlweg-Lange's associahedra:

$$\mathbf{a}(\mathrm{S})_j = \begin{cases} |\{\pi \text{ maximal path in } \mathbb{S} \text{ with 2 incoming arcs at } j\}| & \text{if } j \text{ down} \\ n+2-|\{\pi \text{ maximal path in } \mathbb{S} \text{ with 2 outgoing arcs at } j\}| & \text{if } j \text{ up} \end{cases}$$

SPINES

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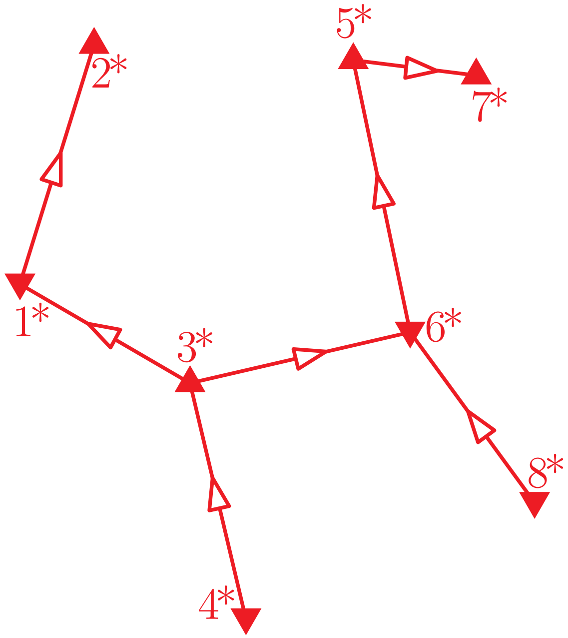
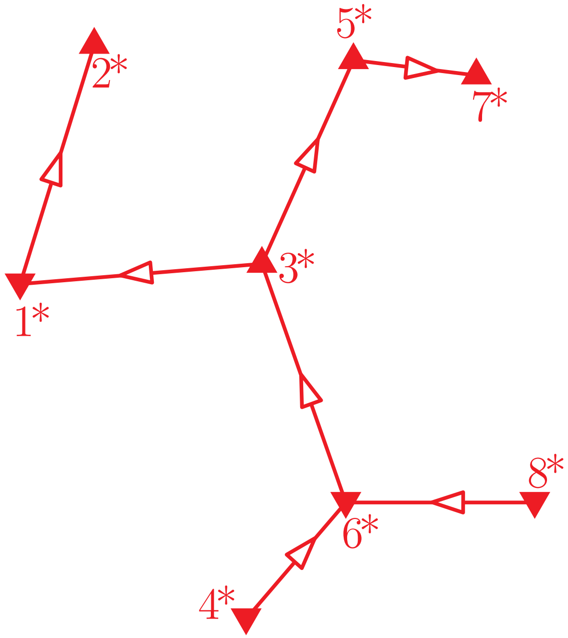




REM. 1. Spines can be defined without their triangulations...

2. Alternative vertex description of Hohlweg-Lange's associahedra:

$$\mathbf{a}(\mathrm{S})_j = \begin{cases} |\{\pi \text{ path in } \mathrm{S} \text{ not using the outgoing arc at } j\}| & \text{if } j \text{ down} \\ n+2-|\{\pi \text{ path in } \mathrm{S} \text{ not using the incomming arc at } j\}| & \text{if } j \text{ up} \end{cases}$$

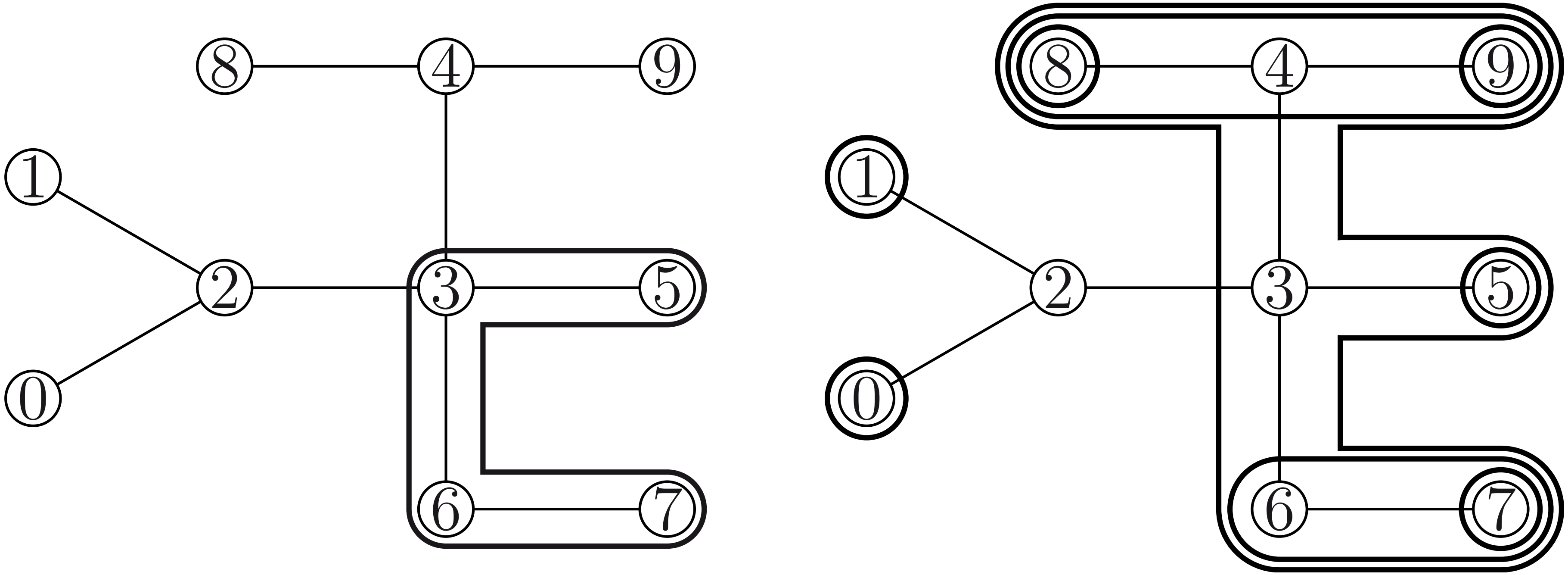
GRAPH ASSOCIAHEDRA

NESTED COMPLEX AND GRAPH ASSOCIAHEDRON

G graph on ground set V

Tube on V = connected induced subgraph of G

Compatible tubes = nested, or disjoint and non-adjacent

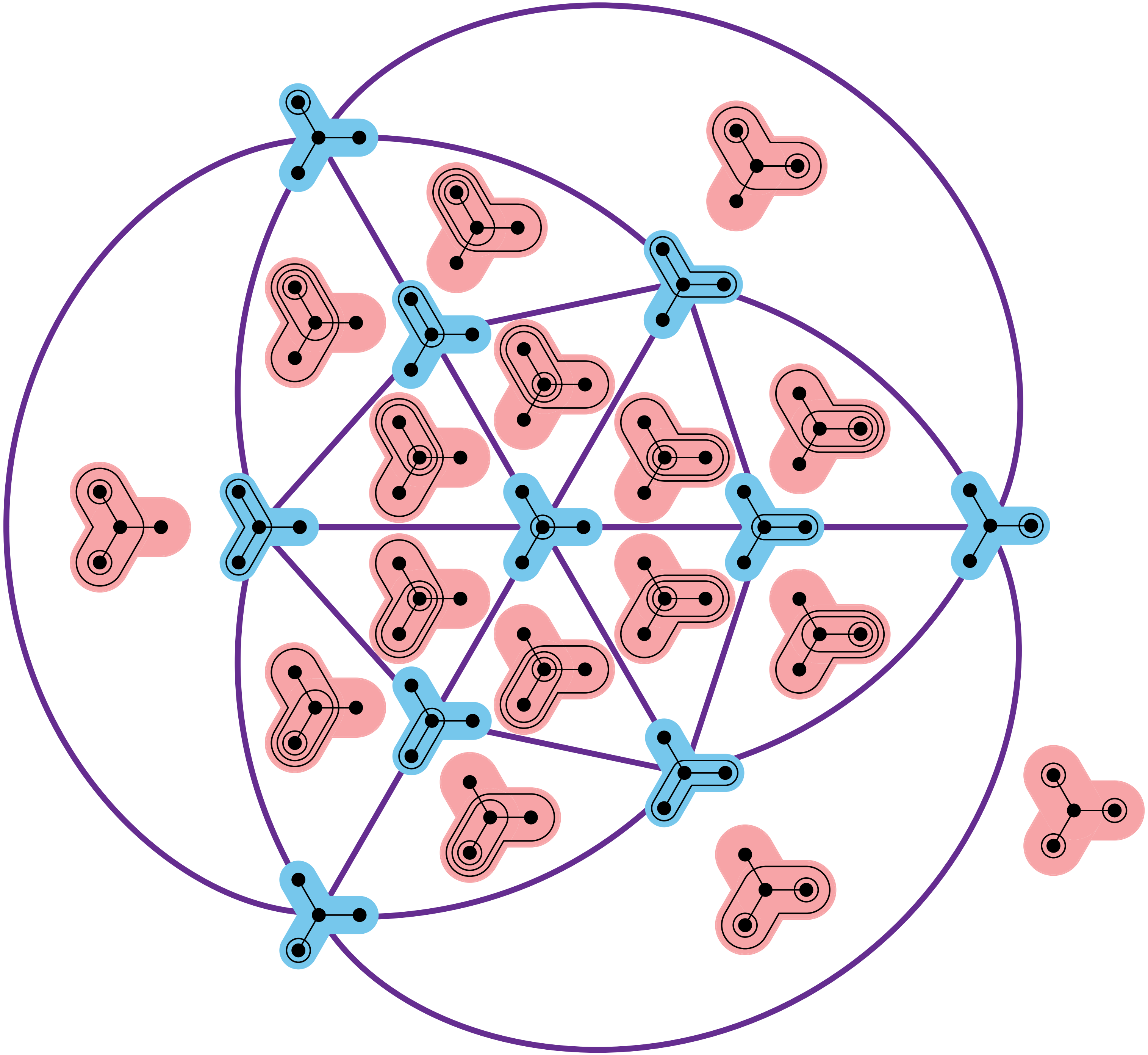


Nested complex $\mathcal{N}(\mathrm{G})$ = simplicial complex of sets of pairwise compatible tubes
= clique complex of the compatibility relation on tubes

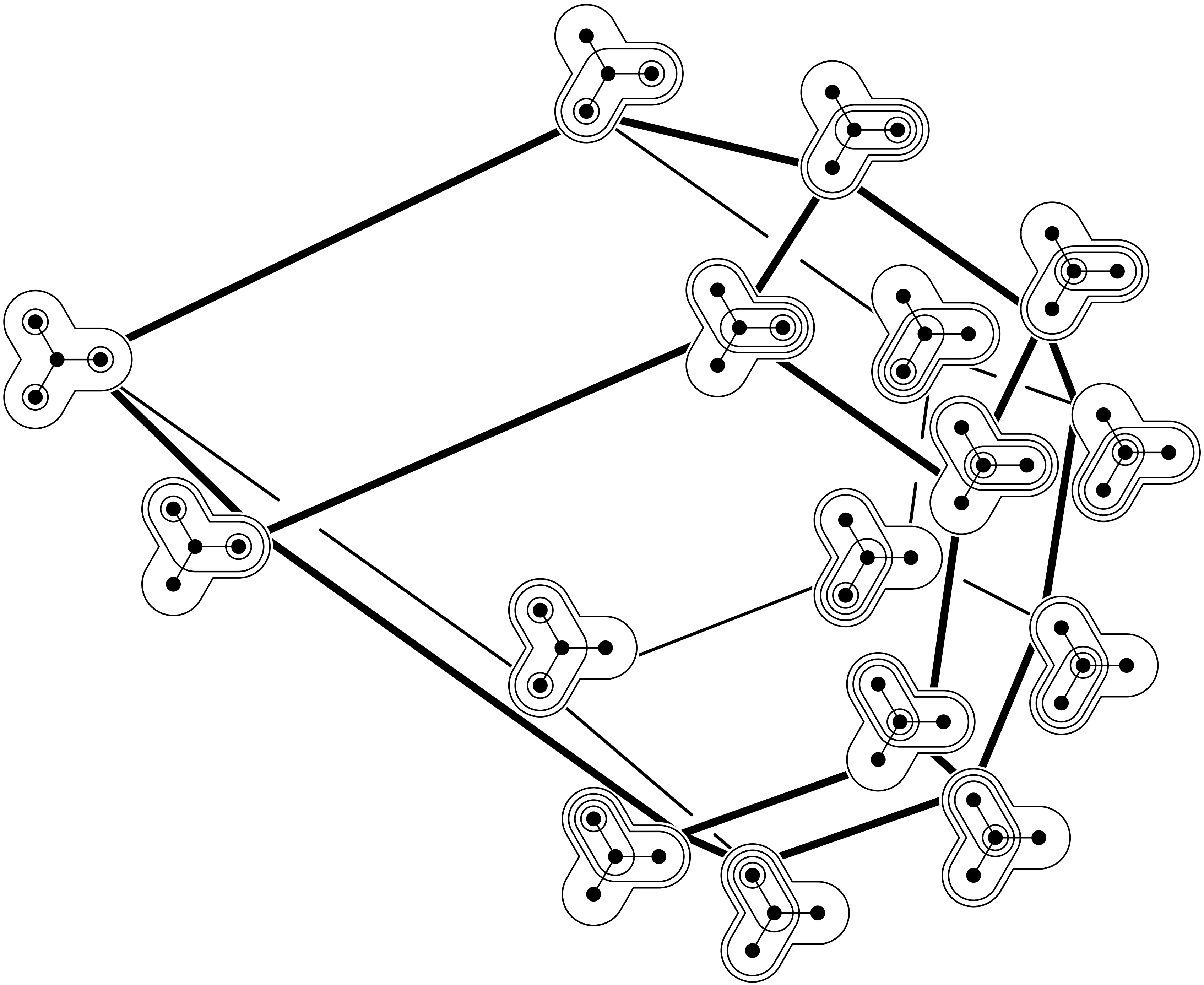
G-associahedron = polytopal realization of the nested complex on G

Carr-Devadoss, Coxeter complexes and graph associahedra ('06)

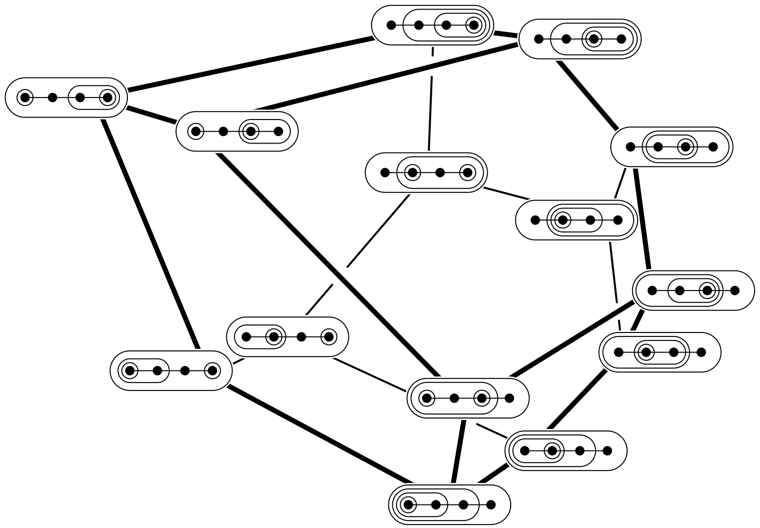
EXM: NESTED COMPLEX

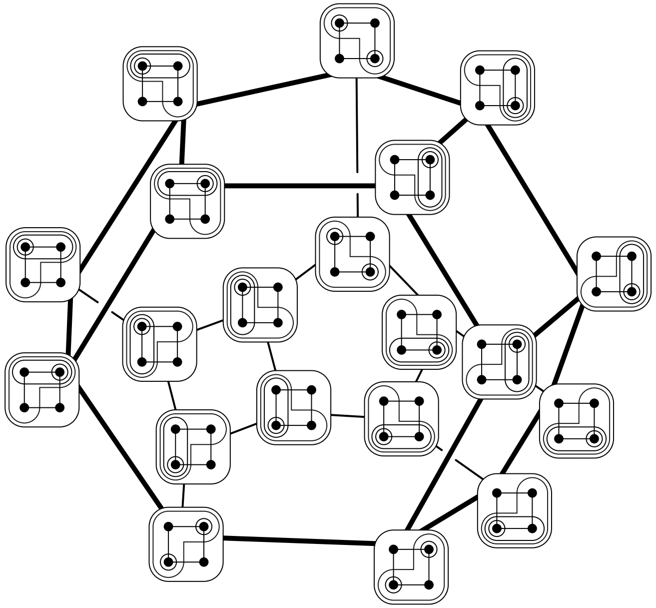


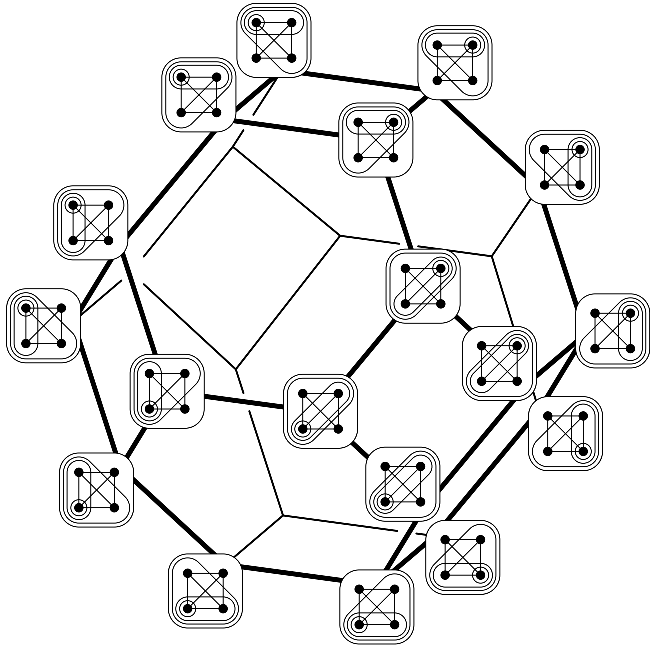
EXM: GRAPH ASSOCIAHEDRON



SPECIAL GRAPH ASSOCIAHEDRA



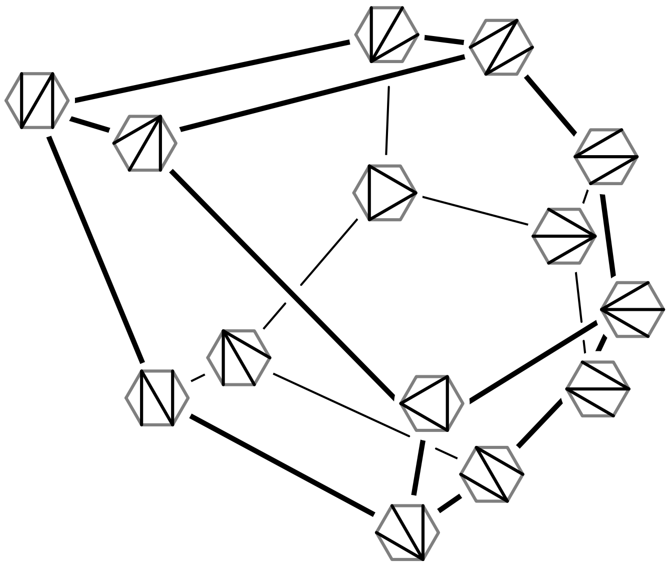


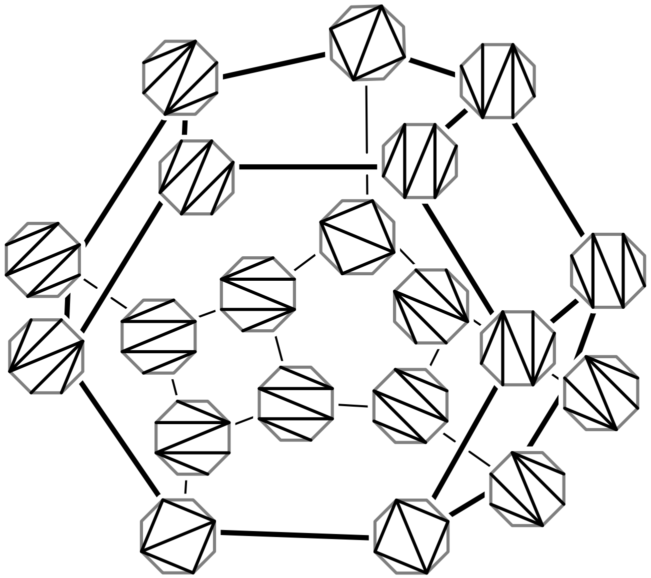


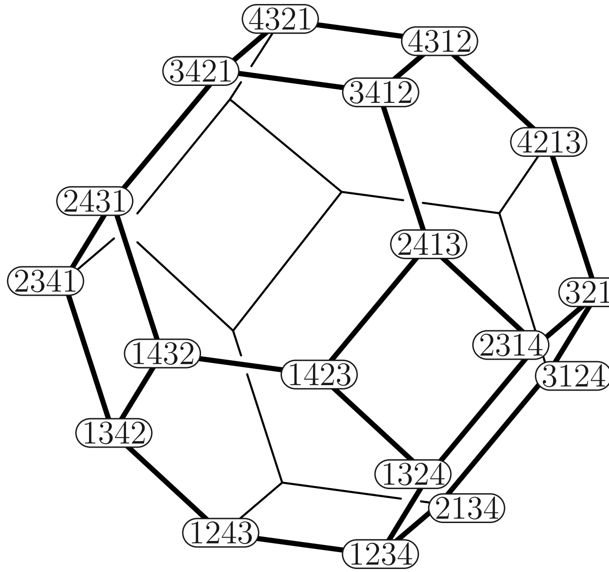
Path associahedron
= associahedron

Cycle associahedron
= cyclohedron

Complete graph associahedron
= permutahedron

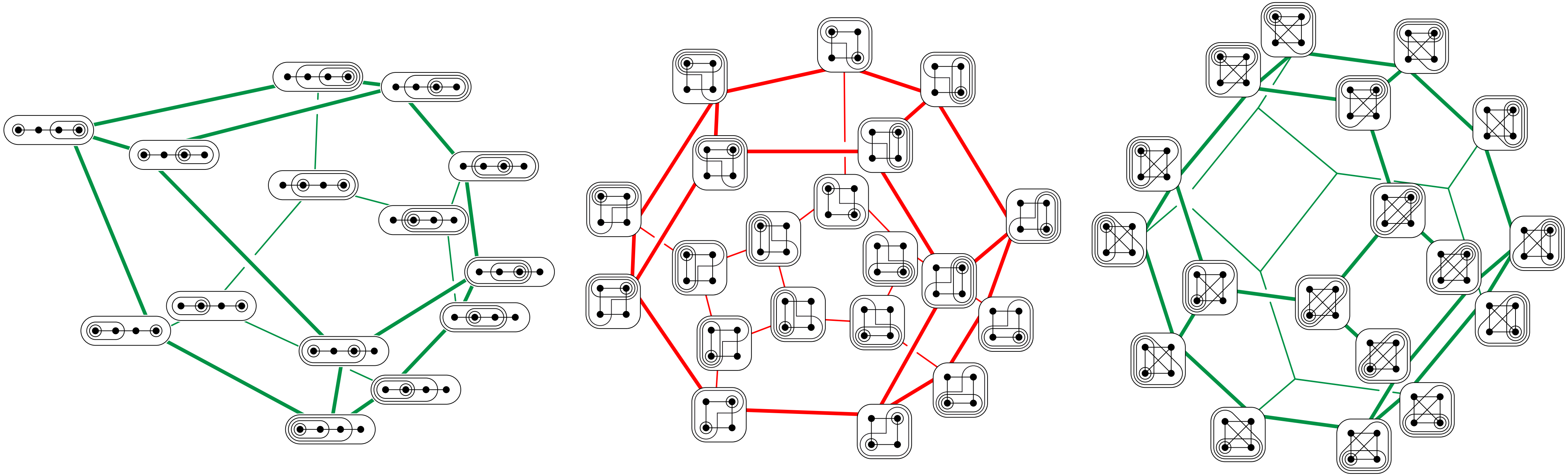






TWO QUESTIONS

Qu 1. Which graph associahedra can be realized by removahedra?



Lange-P., Which nestohedra are removahedra? ('14$^+$)

Qu 2. Can we obtain distinct realizations of graph associahedra?

Yes for trees. . .

P., Signed tree associahedra ('13$^+$)

SIGNED TREE ASSOCIAHEDRON

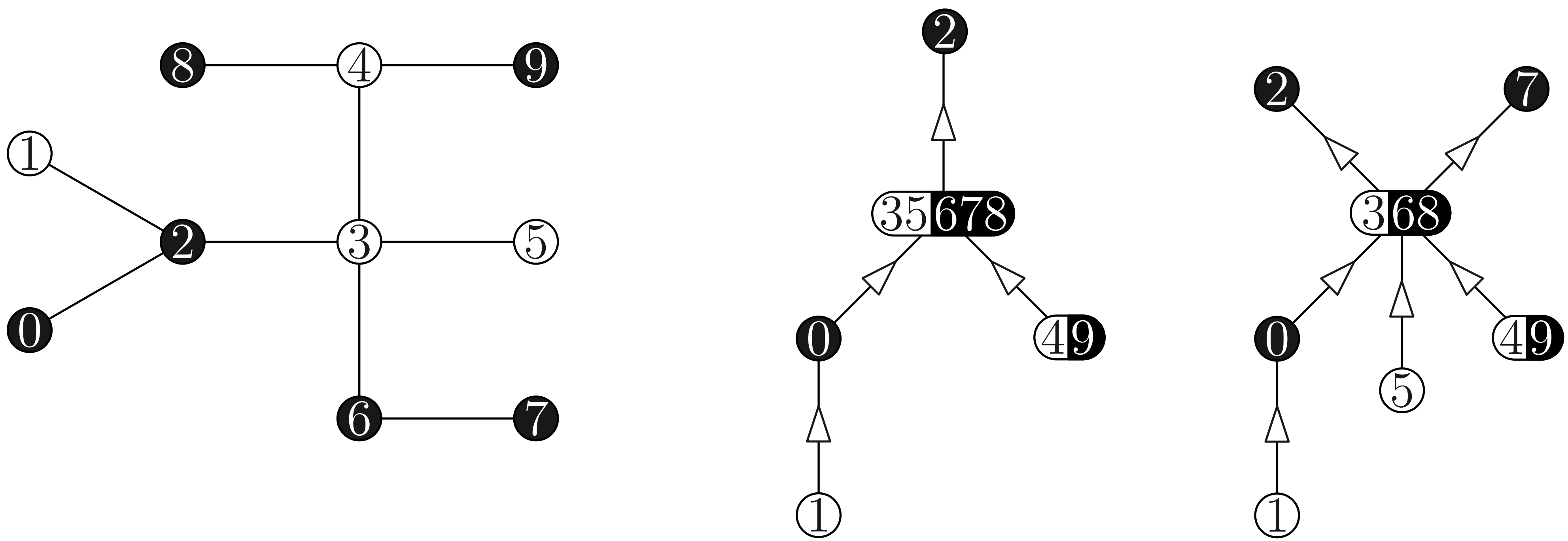
SIGNED SPINES

T tree on the signed ground set $\mathrm{V} = \mathrm{V}^- \sqcup \mathrm{V}^+$ (negative in white, positive in black)

Signed spine on T = directed and labeled tree S st

(i) the labels of the nodes of S form a partition of the signed ground set V

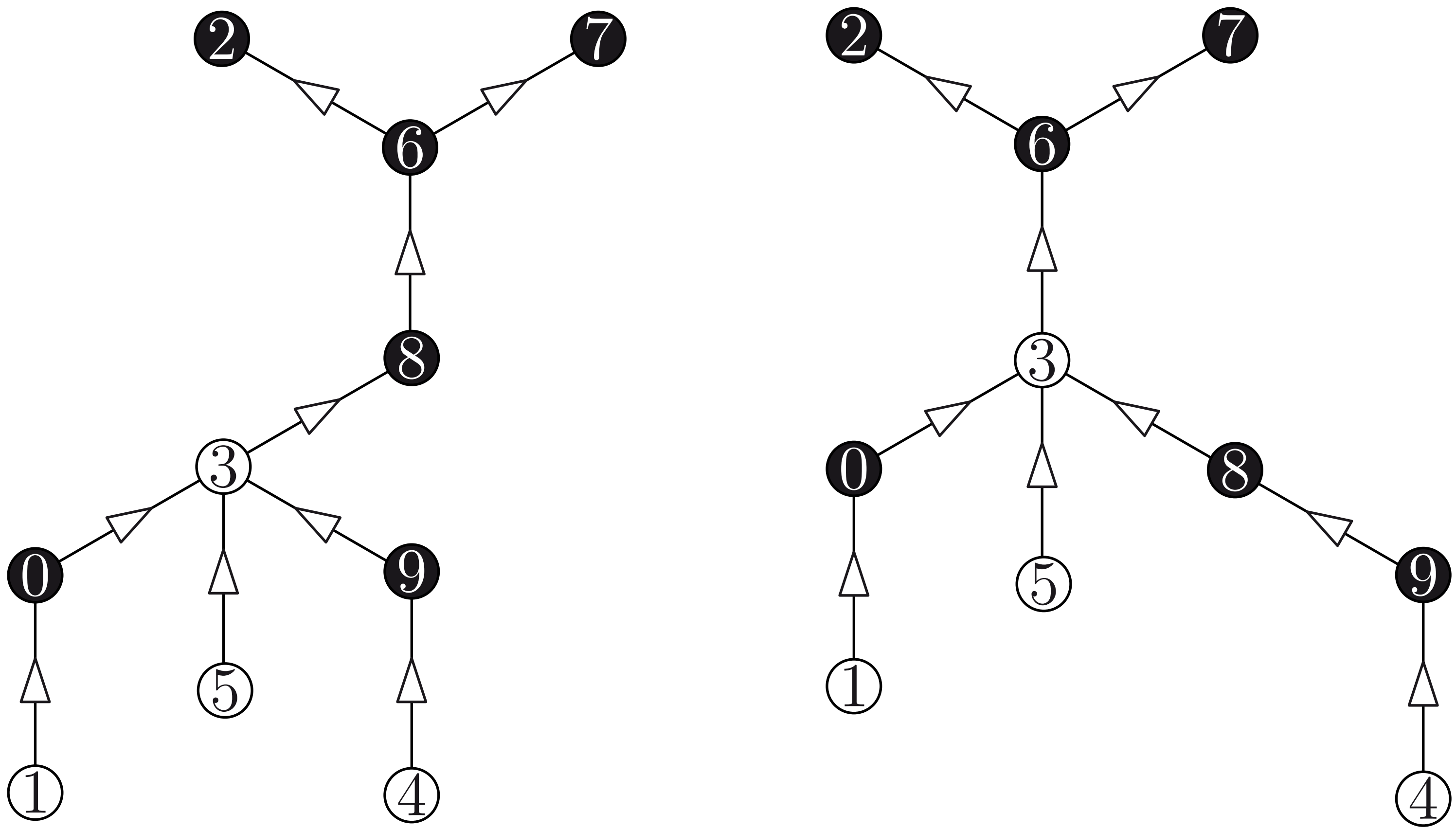
(ii) at a node of S labeled by $U = U^- \sqcup U^+$, the source label sets of the different incoming arcs are subsets of distinct connected components of $\mathrm{T} \smallsetminus U^-$, while the sink label sets of the different outgoing arcs are subsets of distinct connected components of $\mathrm{T} \smallsetminus U^+$



CONTRACTIONS AND SPINE COMPLEX

LEM. Contracting an arc in a signed spine on T leads to a new signed spine on T

LEM. Let S be a signed spine on T with a node labeled by a set U containing at least two elements. For any $u \in U$, there exists a signed spine on T whose nodes are labeled exactly as that of S, except that the label U is partitioned into $\{u\}$ and $U \smallsetminus \{u\}$



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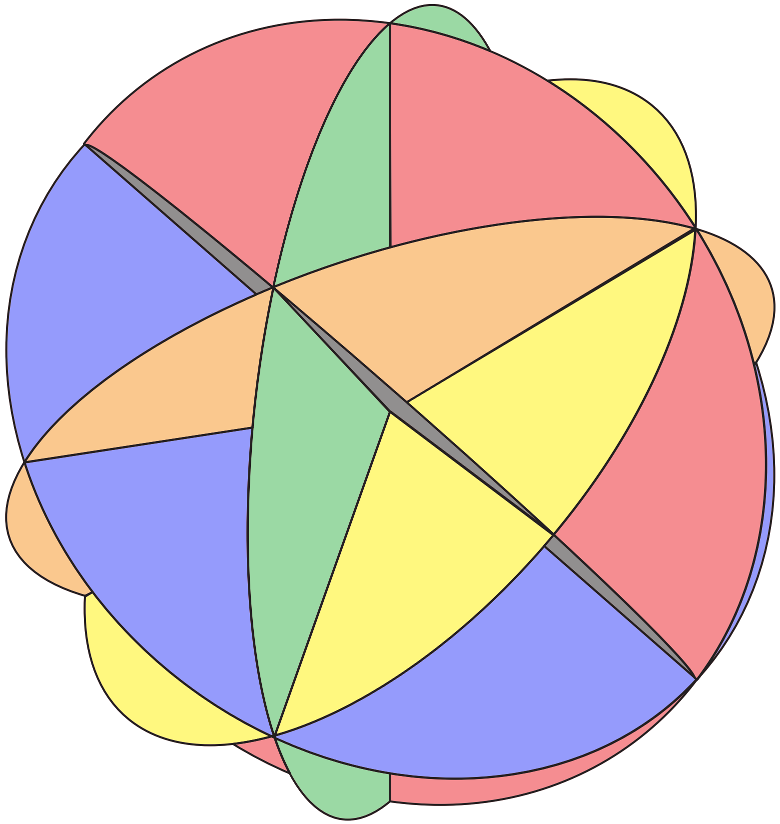
Signed spine complex $\mathcal{S}(\mathrm{T})$ = simplicial complex whose inclusion poset is isomorphic to the poset of edge contractions on the signed spines of T

CORO. The signed spine complex $\mathcal{S}(\mathrm{T})$ is a pure simplicial complex of rank $|\mathrm{V}|$

BRAID FAN



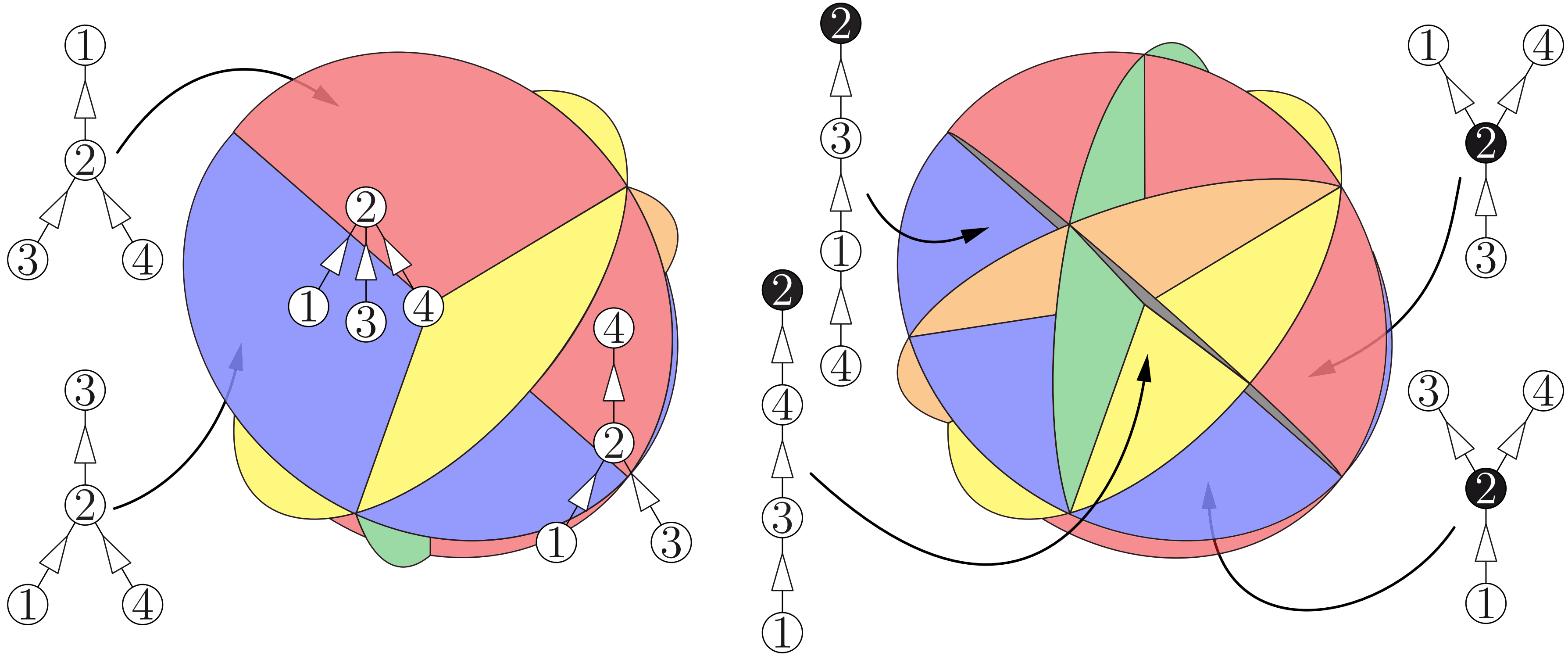
Braid arrangement on $\mathbb{R}^V$ = collection of hyperplanes $\{\mathbf{x} \in \mathbb{R}^V \mid x_u = x_v\}$ for $u \neq v \in V$

Braid fan $\mathcal{BF}$ = complete simplicial fan defined by the braid arrangement on

$$\mathbb{H} := \left\{ \mathbf{x} \in \mathbb{R}^V \;\middle|\; \sum_{v \in V} x_v = \binom{|V|+1}{2} \right\}$$

SPINE FAN

For S spine on T, define $\mathrm{C}(\mathrm{S}) := \{\mathbf{x} \in \mathbb{H} \mid x_u \le x_v, \text{ for all arcs } u \to v \text{ in } \mathrm{S}\}$



THEO. The collection of cones $\mathcal{F}(\mathrm{T}) := \{\mathrm{C}(\mathrm{S}) \mid \mathrm{S} \in \mathcal{S}(\mathrm{T})\}$ defines a complete simplicial fan on $\mathbb{H}$, which we call the spine fan

CORO. For any signed tree T, the signed nested complex $\mathcal{N}(\mathrm{T})$ is a simplicial sphere

SIGNED TREE ASSOCIAHEDRON

Signed tree associahedron Asso(T) = convex polytope with

(i) a vertex $\mathbf{a}(\mathrm{S}) \in \mathbb{R}^{\mathrm{V}}$ for each maximal signed spine $\mathrm{S} \in \mathcal{S}(\mathrm{T})$, with coordinates

$$\mathbf{a}(\mathrm{S})_v = \begin{cases} \big| \{\pi \in \Pi(\mathrm{S}) \mid v \in \pi \text{ and } r_v \notin \pi\} \big| & \text{if } v \in \mathrm{V}^- \\ |\mathrm{V}| + 1 - \big| \{\pi \in \Pi(\mathrm{S}) \mid v \in \pi \text{ and } r_v \notin \pi\} \big| & \text{if } v \in \mathrm{V}^+ \end{cases}$$

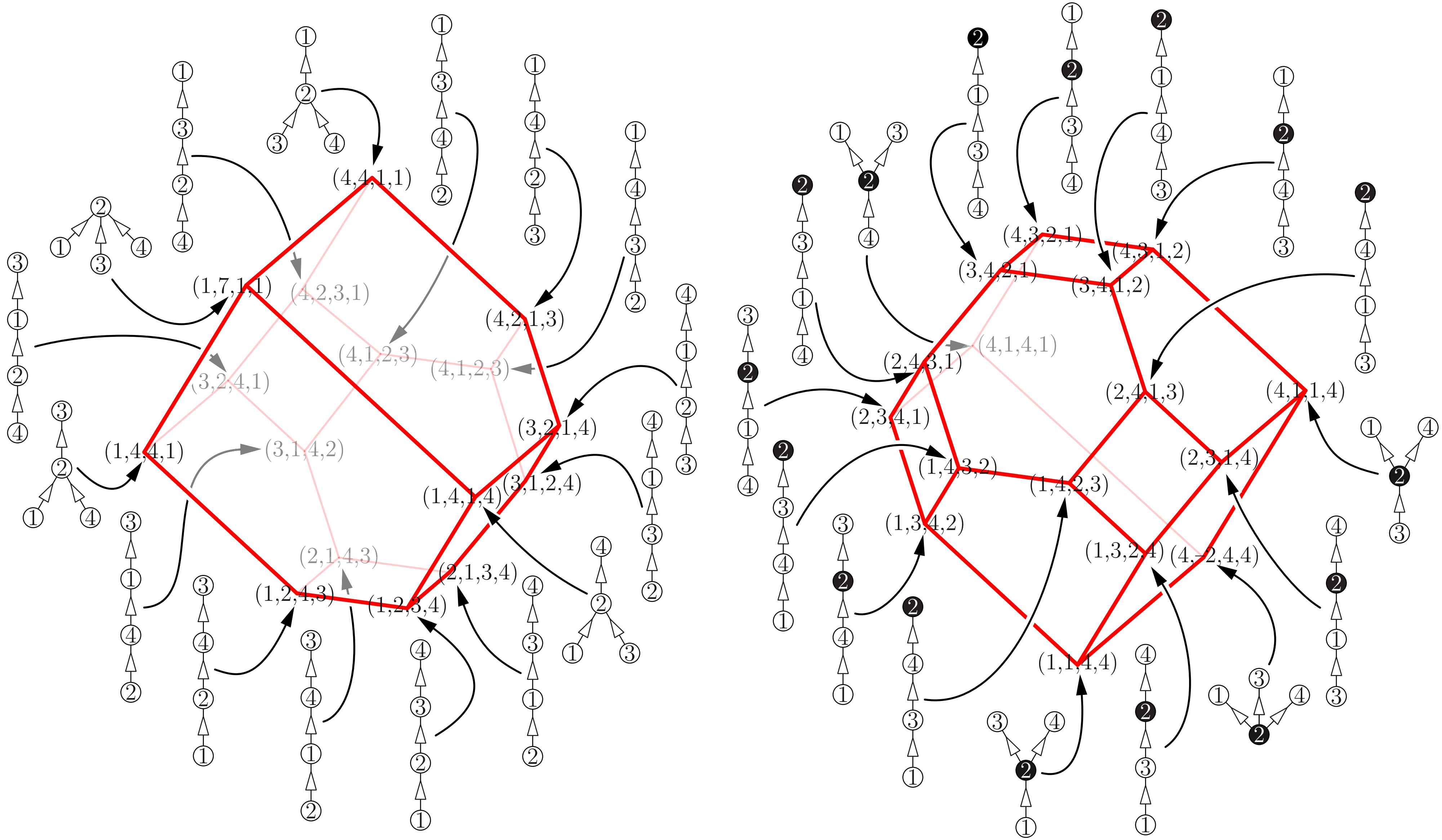
where r_v = unique incoming (resp. outgoing) arc when $v \in \mathrm{V}^-$ (resp. when $v \in \mathrm{V}^+$)
$\Pi(\mathrm{S})$ = set of all (undirected) paths in S, including the trivial paths

(ii) a facet defined by the half-space

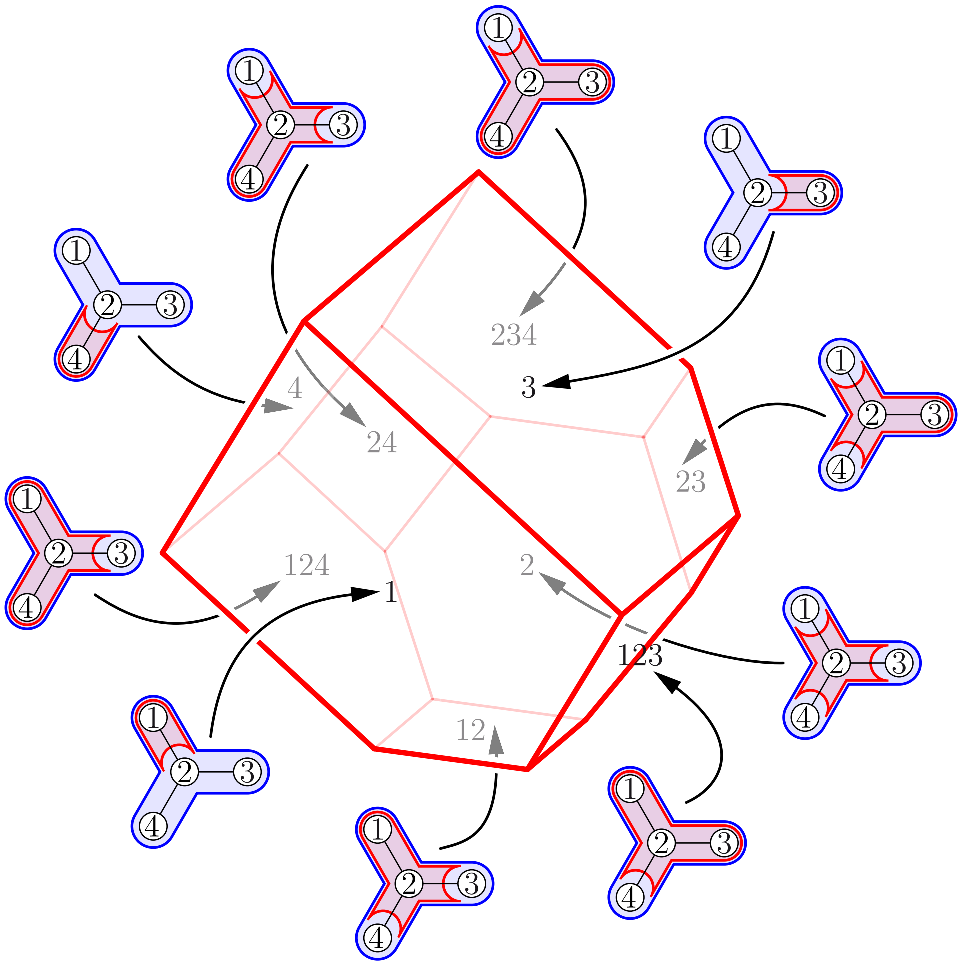
$$\mathbf{H}^{\geq}(B) := \left\{ \mathbf{x} \in \mathbb{R}^{\mathrm{V}} \;\middle|\; \sum_{v \in B} x_v \geq \binom{|B|+1}{2} \right\}$$

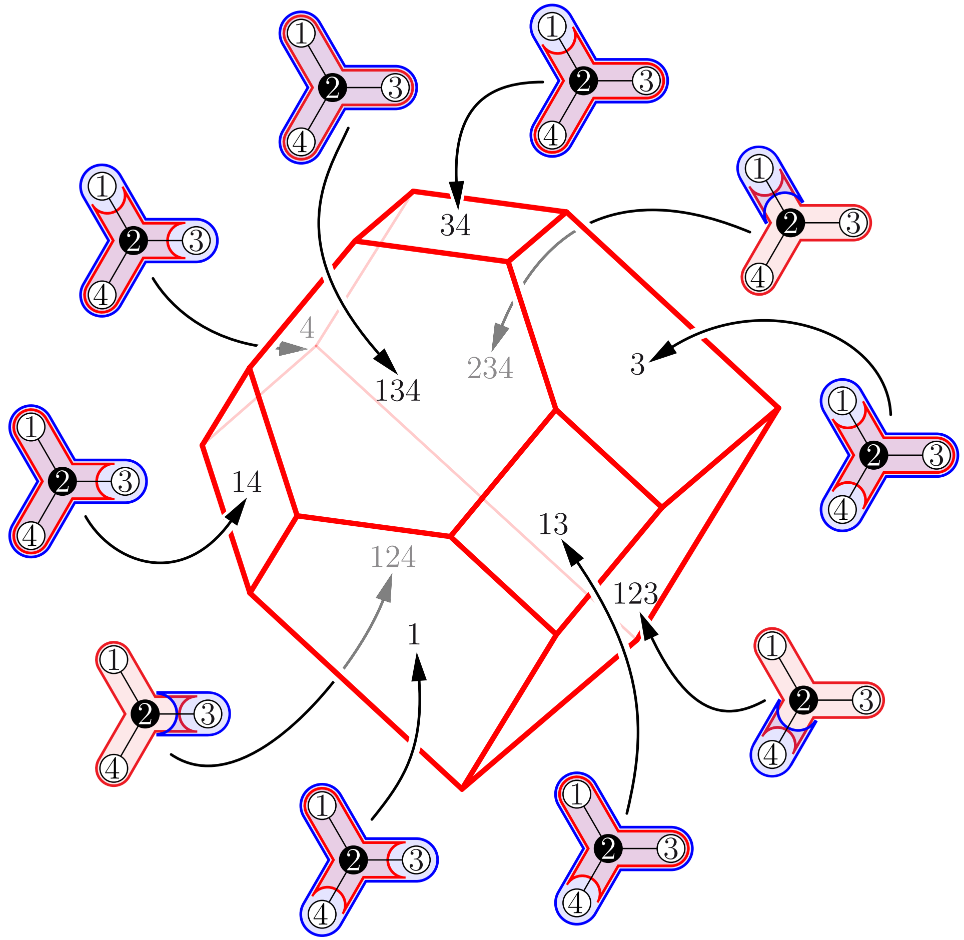
for each signed building block $B \in \mathcal{B}(\mathrm{T})$

EXM: VERTEX DESCRIPTION



EXM: FACET DESCRIPTION





MAIN RESULT

THM. The spine fan $\mathcal{F}(\mathrm{T})$ is the normal fan of the signed tree associahedron $\mathrm{Asso}(\mathrm{T})$, defined equivalently as

(i) the convex hull of the points

$$\mathbf{a}(\mathrm{S})_v = \begin{cases} \big|\, \{\pi \in \Pi(\mathrm{S}) \mid v \in \pi \text{ and } r_v \notin \pi\} \,\big| & \text{if } v \in \mathrm{V}^- \\ |\mathrm{V}| + 1 - \big|\, \{\pi \in \Pi(\mathrm{S}) \mid v \in \pi \text{ and } r_v \notin \pi\} \,\big| & \text{if } v \in \mathrm{V}^+ \end{cases}$$

for all maximal signed spines $\mathrm{S} \in \mathcal{S}(\mathrm{T})$

(ii) the intersection of the hyperplane $\mathbb{H}$ with the half-spaces

$$\mathbf{H}^{\geq}(B) := \left\{ \mathbf{x} \in \mathbb{R}^{\mathrm{V}} \;\middle|\; \sum_{v \in B} x_v \geq \binom{|B|+1}{2} \right\}$$

for all signed building blocks $B \in \mathcal{B}(\mathrm{T})$

CORO. The signed tree associahedron $\mathrm{Asso}(\mathrm{T})$ realizes the signed nested complex $\mathcal{N}(\mathrm{T})$

SKETCH OF THE PROOF

STEP 1. We have

$$\sum_{v \in \mathrm{V}} \mathbf{a}(\mathrm{S})_v = \binom{|\mathrm{V}|+1}{2} \qquad \text{and} \qquad \sum_{v \in \mathrm{sc}(r)} \mathbf{a}(\mathrm{S})_v = \binom{|\mathrm{sc}(r)|+1}{2}$$

for any arc r of S. In other words, "each vertex $\mathbf{a}(\mathrm{S})$ belongs to the hyperplanes $\mathbf{H}^{=}(B)$ it is supposed to". Proof by double counting

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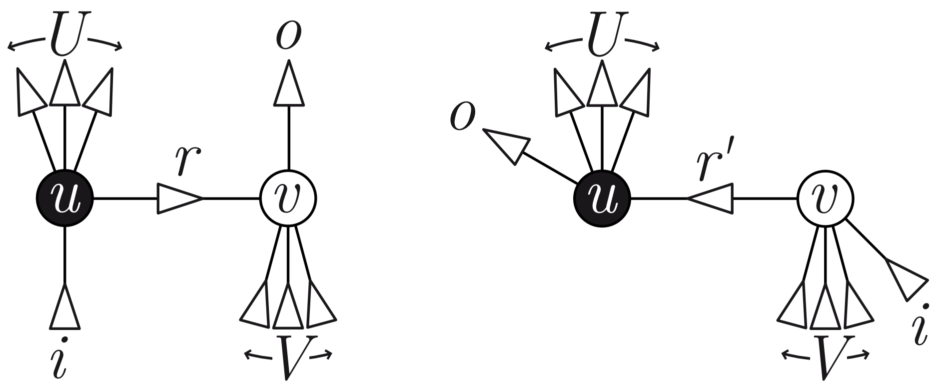
STEP 2. If S and S' are two adjacent maximal spines on T, such that S' is obtained from S by flipping an arc joining node u to node v, then

$$\mathbf{a}(\mathrm{S}') - \mathbf{a}(\mathrm{S}) \in \mathbb{R}_{>0} \cdot (e_u - e_v)$$



$$\mathbf{a}(\mathrm{S}') - \mathbf{a}(\mathrm{S}) = (|U|+1) \cdot (|V|+1) \cdot (e_u - e_v)$$

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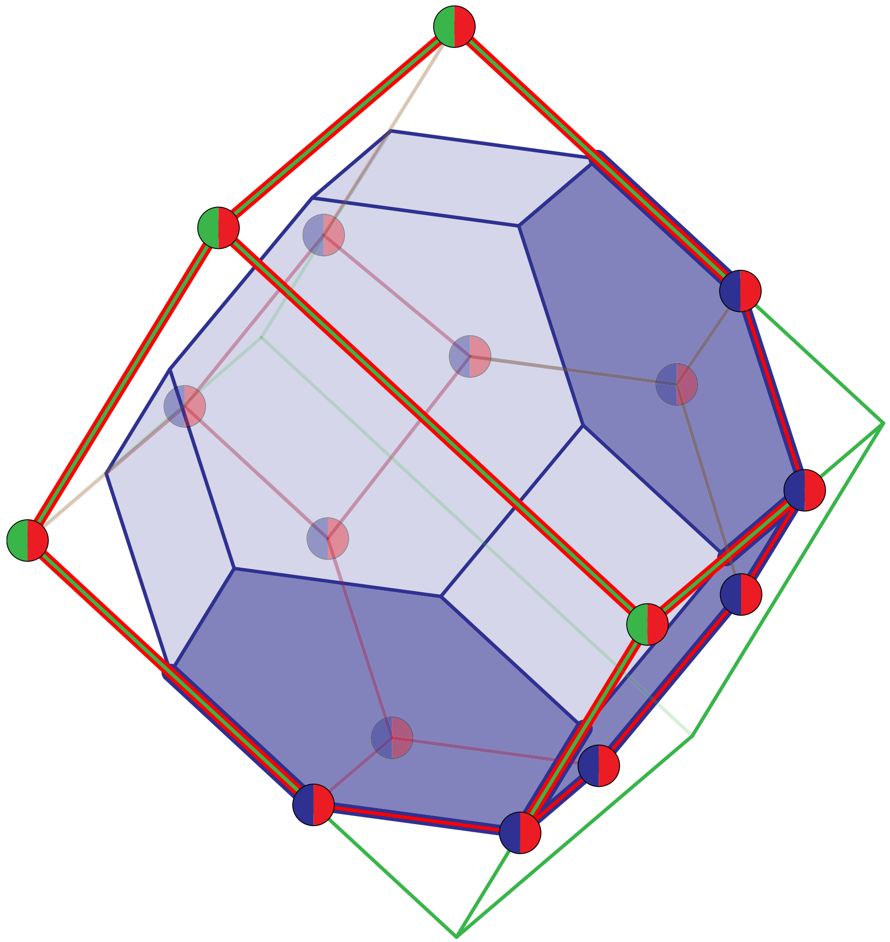
STEP 3. A general theorem concerning realizations of simplicial fan by polytopes
In other words, a characterization of when is a simplicial fan regular

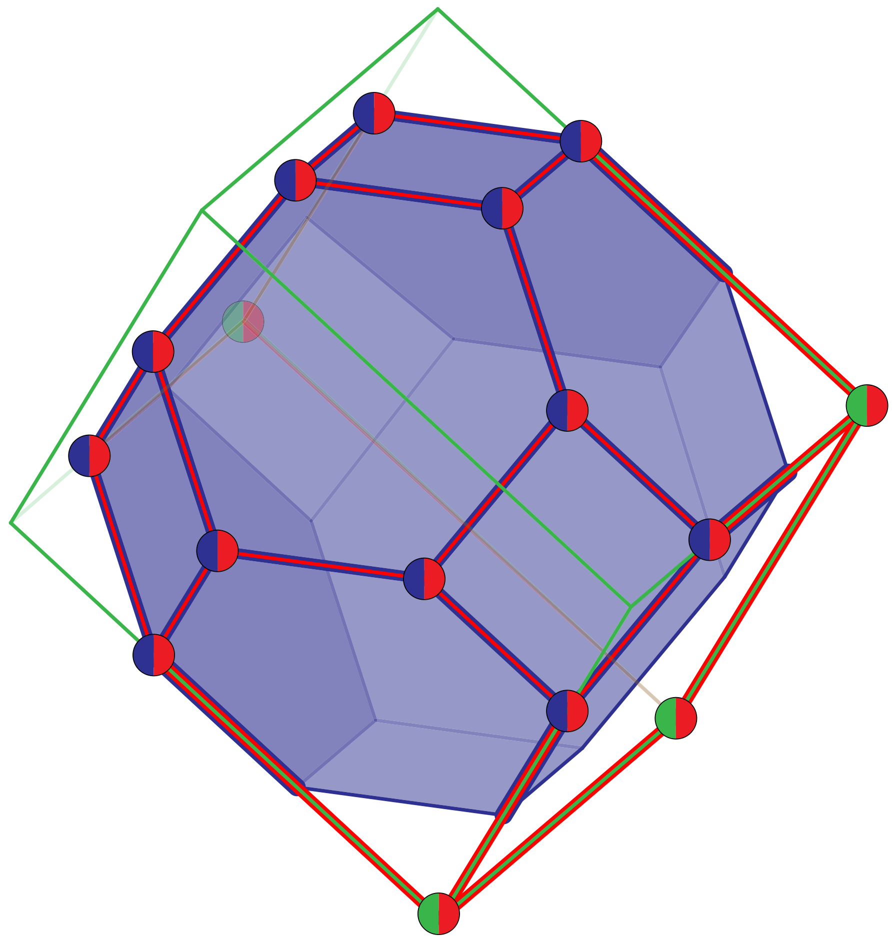
Hohlweg-Lange-Thomas, Permutahedra and generalized associahedra ('11)
De Loera-Rambau-Santos, Triangulations: Structures for Algorithms and Applications ('10)

FURTHER GEOMETRIC PROPERTIES

PROP. The signed tree associahedron $\mathsf{Asso}(\mathrm{T})$ is sandwiched between the permutahedron $\mathsf{Perm}(\mathrm{V})$ and the parallelepiped $\mathsf{Para}(\mathrm{T})$

$$\sum_{u \neq v \in \mathrm{V}} [e_u, e_v] = \mathsf{Perm}(\mathrm{T}) \quad \subset \quad \mathsf{Asso}(\mathrm{T}) \quad \subset \quad \mathsf{Para}(\mathrm{T}) = \sum_{u - v \in \mathrm{T}} \pi(u - v) \cdot [e_u, e_v]$$





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PROP. The signed tree associahedron Asso(T) is sandwiched between the permutahedron Perm(V) and the parallelepiped Para(T)

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Common vertices of Asso(T) and Para(T) $\equiv$ orientations of T which are spines on T
Common vertices of Asso(T) and Perm(T) $\equiv$ linear orders on V which are spines on T
$\Rightarrow$ no common vertex of the three polytopes except if T is a signed path

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$\Rightarrow$ no common vertex of the three polytopes except if T is a signed path

PROP. $\mathsf{Asso}(\mathrm{T})$ and $\mathsf{Asso}(\mathrm{T}')$ isometric $\iff$ T and T' isomorphic or anti-isomorphic, up to the sign of their leaves, ie. $\exists$ bijection $\theta : \mathrm{V} \to \mathrm{V}'$ st. $\forall u, v \in \mathrm{V}$

- $u - v$ edge in T $\iff$ $\theta(u) - \theta(v)$ edge in T'
- if u is not a leaf of T, the signs of u and $\theta(u)$ coincide (resp. are opposite)

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REM. The vertex barycenter of $\mathsf{Asso}(\mathrm{T})$ does not necessarily coincide with that of the permutahedron (but it lies on the linear span of the characteristic vectors of the orbits of V under the automorphism group of T)

THANK YOU