Signaletic operads

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slides: http://www.lri.fr/~hivert/FPSAC20.pdf
preprint: https://arxiv.org/pdf/1906.02228.pdf

Recall the classical shuffle:

$$12 \sqcup\!\sqcup 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$$

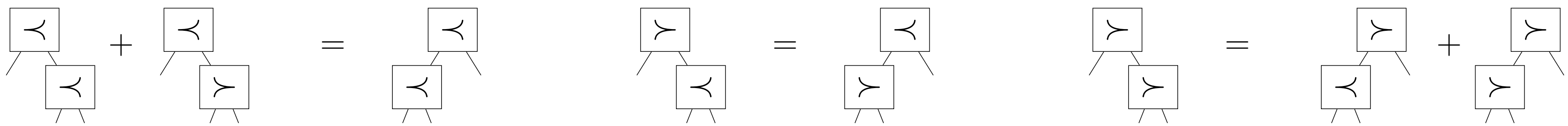
<u>shuffle</u> <u>dendriform operators</u>

$$\begin{aligned}\sigma_1 \dots \sigma_m \sqcup\!\sqcup \tau_1 \dots \tau_n &= \sigma_1 \dots \sigma_m \prec \tau_1 \dots \tau_n \quad \cup \quad \sigma_1 \dots \sigma_m \succ \tau_1 \dots \tau_n \\ &= \sigma_1 \big(\sigma_2 \dots \sigma_m \sqcup\!\sqcup \tau_1 \dots \tau_n\big) \quad \cup \quad \tau_1 \big(\sigma_1 \dots \sigma_m \sqcup\!\sqcup \tau_2 \dots \tau_n\big)\end{aligned}$$

Dendriform relations:

$$a \prec (b \sqcup\!\sqcup c) = (a \prec b) \prec c \qquad a \succ (b \succ c) = (a \sqcup\!\sqcup b) \succ c$$

$$a \prec (b \prec c) + a \prec (b \succ c) = (a \prec b) \prec c \qquad a \succ (b \prec c) = (a \succ b) \prec c \qquad a \succ (b \succ c) = (a \prec b) \succ c + (a \succ b) \succ c$$

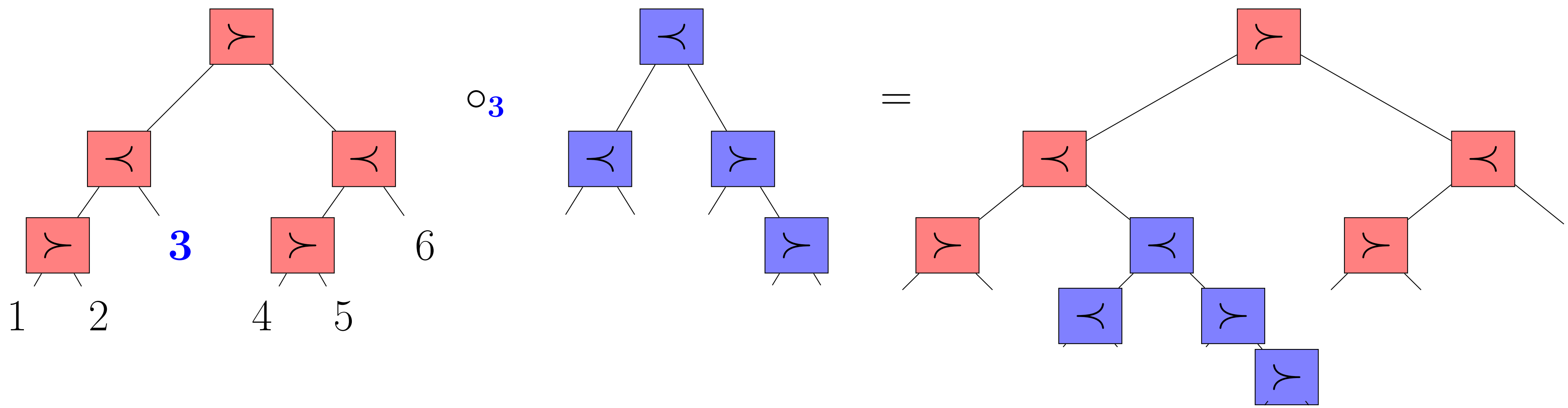


operad $=$ algebraic structure abstracting a type of algebras
$=$ graded vector space of operations $\mathcal{O} = \oplus_{p \ge 1} \mathcal{O}(p)$ with a unit $\mathbb{1} \in \mathcal{O}(1)$
and partial compositions $\circ_i : \mathcal{O}(p) \otimes \mathcal{O}(q) \to \mathcal{O}(p+q-1)$ for $p, q \ge 1$ and $i \in [p]$
such that for all $\mathfrak{p} \in \mathcal{O}(p)$, $\mathfrak{q} \in \mathcal{O}(q)$, $\mathfrak{r} \in \mathcal{O}(r)$:

(unitality)	$\mathbb{1} \circ_1 \mathfrak{p} = \mathfrak{p} = \mathfrak{p} \circ_i \mathbb{1}$	for all $i \in [p]$,
(series composition)	$(\mathfrak{p} \circ_i \mathfrak{q}) \circ_{i+j-1} \mathfrak{r} = \mathfrak{p} \circ_i (\mathfrak{q} \circ_j \mathfrak{r})$	for all $i \in [p]$, $j \in [q]$,
(parallel composition)	$(\mathfrak{p} \circ_i \mathfrak{q}) \circ_{j+q-1} \mathfrak{r} = (\mathfrak{p} \circ_j \mathfrak{r}) \circ_i \mathfrak{q}$	for all $i < j \in [p]$.

Hilbert series $= \sum_{p \ge 1} \dim \mathcal{O}(p)\, t^p$

free operad $=$ syntax trees on $\mathcal{O}(1)$ with grafting



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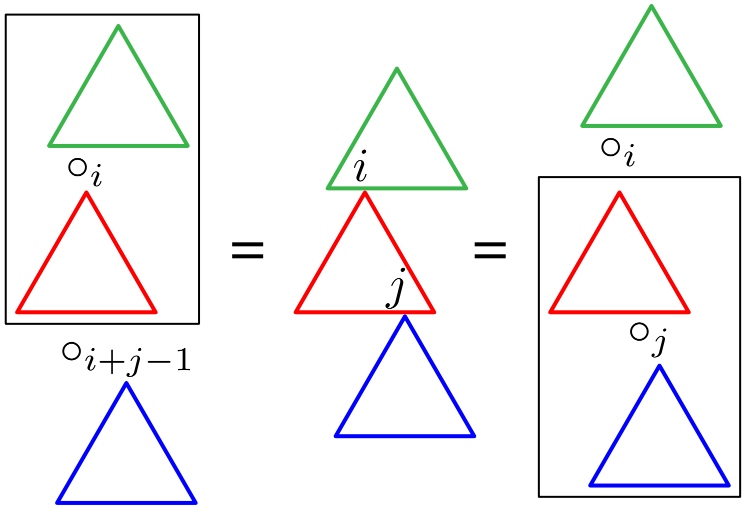
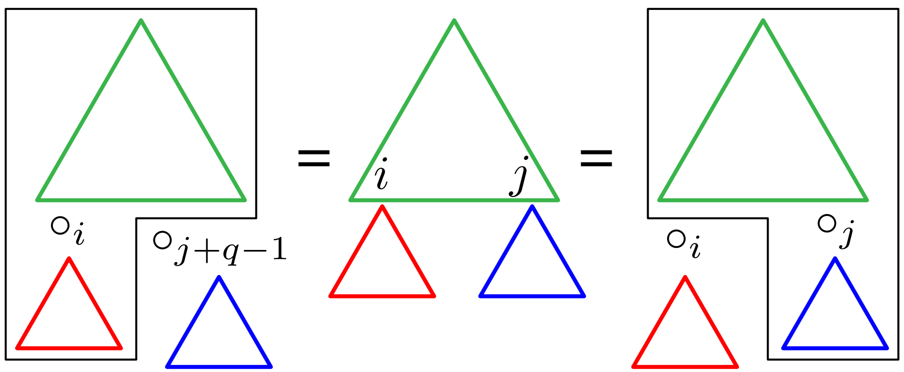


series composition

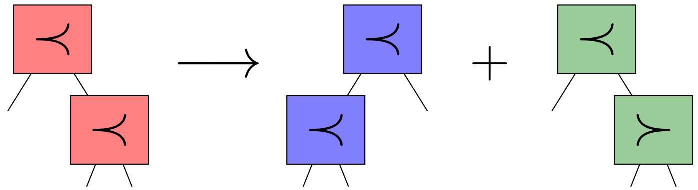


parallel composition

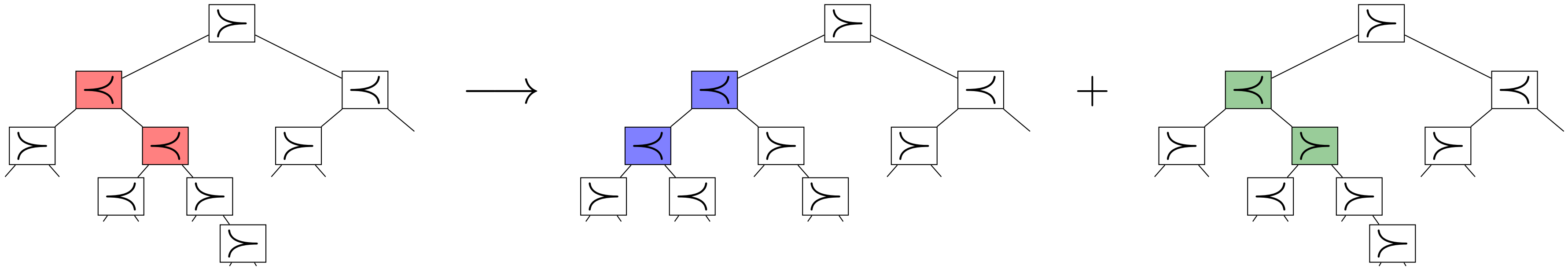
OPERADS AND SYNTAX TREES

Any operad is a quotient of the free operad compatible with grafting

quadratic rewriting rule = rewrites a syntax tree on two nodes into a linear combination of syntax trees on two nodes:



... used internally in a syntax tree:



normal form = unrewritable syntax tree

convergent rewriting system = any syntax trees rewrites as a unique linear combination of normal forms

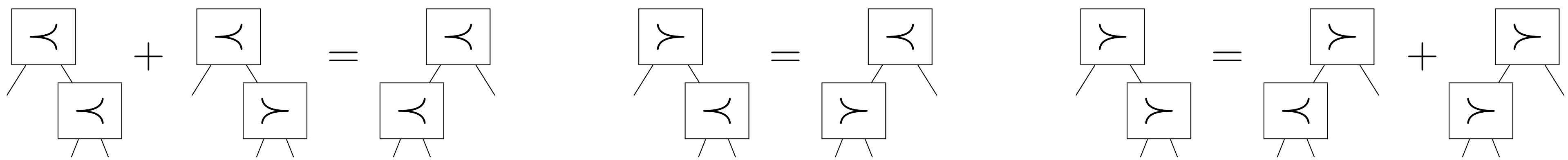
KOSZUL DUALITY

Koszul operad = admits a quadratic presentation whose relations can be oriented to obtain a convergent rewriting system

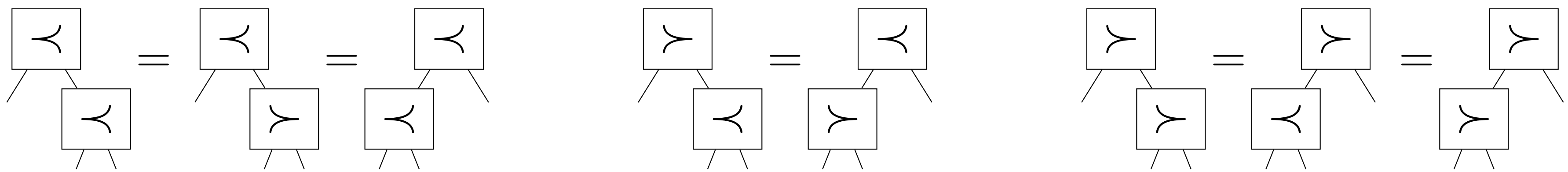
Koszul dual $\mathcal{O}^!$ = operad presented by relations given the orthogonal complement of the relations of $\mathcal{O}$ for the scalar product defined by

$$\langle\ \mathfrak{a} \circ_i \mathfrak{b} \mid \mathfrak{c} \circ_j \mathfrak{d}\ \rangle = \begin{cases} 1 & \text{if } i = j = 1 \\ -1 & \text{if } i = j = 2 \\ 0 & \text{otherwise} \end{cases}$$

DEF. dendriform operad = quadratic operad over $\{\prec, \succ\}$ defined by:



DEF. diassociative operad = quadratic operad over $\{\prec, \succ\}$ defined by:



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PROP. The Hilbert series of two Koszul dual Koszul operads $\mathcal{O}$ and $\mathcal{O}^!$ are related by Lagrange inversion:

$$\mathcal{H}_{\mathcal{O}}(-\mathcal{H}_{\mathcal{O}^!}(-t)) = t$$

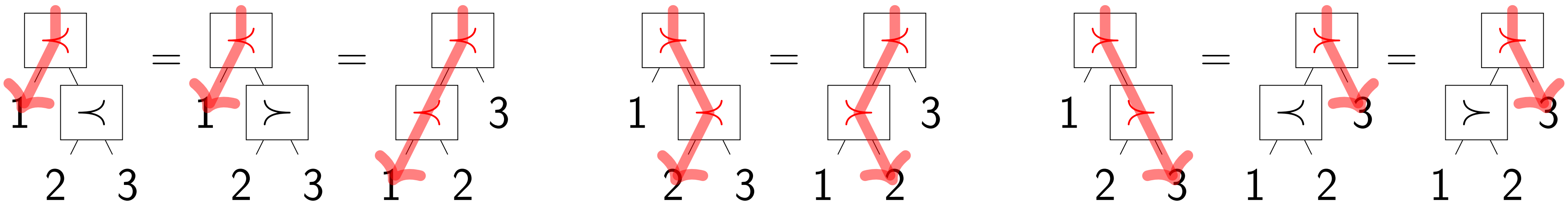
exm: dendriform and diassociative operads

$$\mathcal{H}_{\mathsf{Dend}}(t) = \sum_{p \geq 1} C_p\, t^p = \frac{1 - \sqrt{1-4t}}{2t} - 1 \qquad \text{and} \qquad \mathcal{H}_{\mathsf{Diass}}(t) = \sum_{p \geq 1} p\, t^p = \frac{t}{(1-t)^2},$$

where $C_p = \frac{1}{p+1}\binom{2p}{p}$ = p-th Catalan number

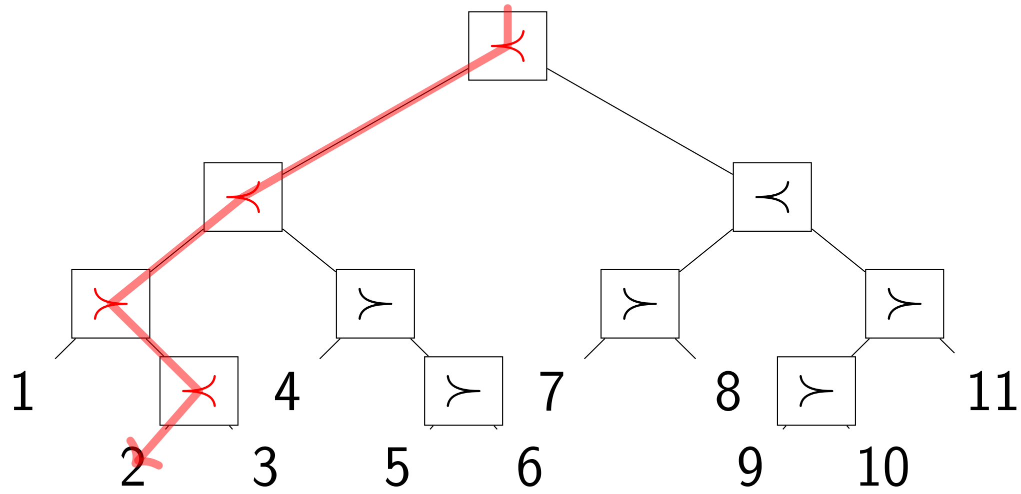
SIGNALETIC INTERPRETATION OF DIASSOCIATIVE OPERAD

DEF. diassociative operad = quadratic operad over $\{\prec, \succ\}$ defined by:



Signaletic interpretation

- a binary road
- $\prec$ and $\succ$ signals at each branching node
- a cyclist follows the signals



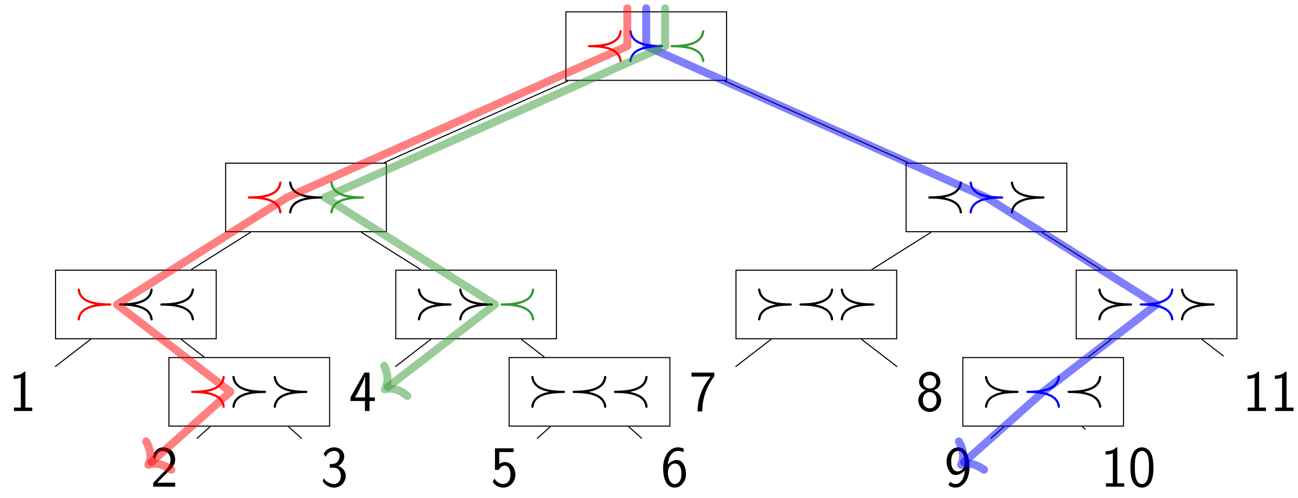
two signaletic trees are equivalent $\iff$
the cyclist reaches the same destination

Hilbert series $\mathcal{H}_{\text{Diass}}(t) = \sum_{p \geq 1} p\, t^p = \dfrac{t}{(1-t)^2}$ (in arity p, there are p possible destinations)

SIGNALETIC OPERADS

<ins>k-Signaletic operad</ins>

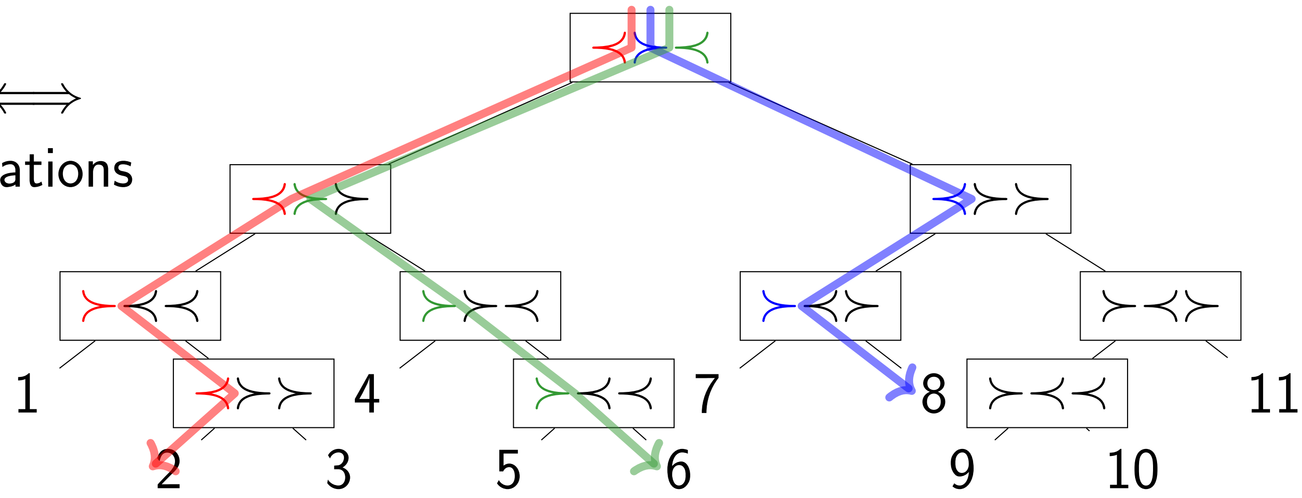
- a binary road
- signals in $\{\prec, \succ\}^k$ on nodes
- k cyclists follow the signals...



... in parallel

(i-th cyclist follows ith signal)

k-signaletic trees are equivalent $\iff$ the cyclists reach the same destinations (same destination vector)

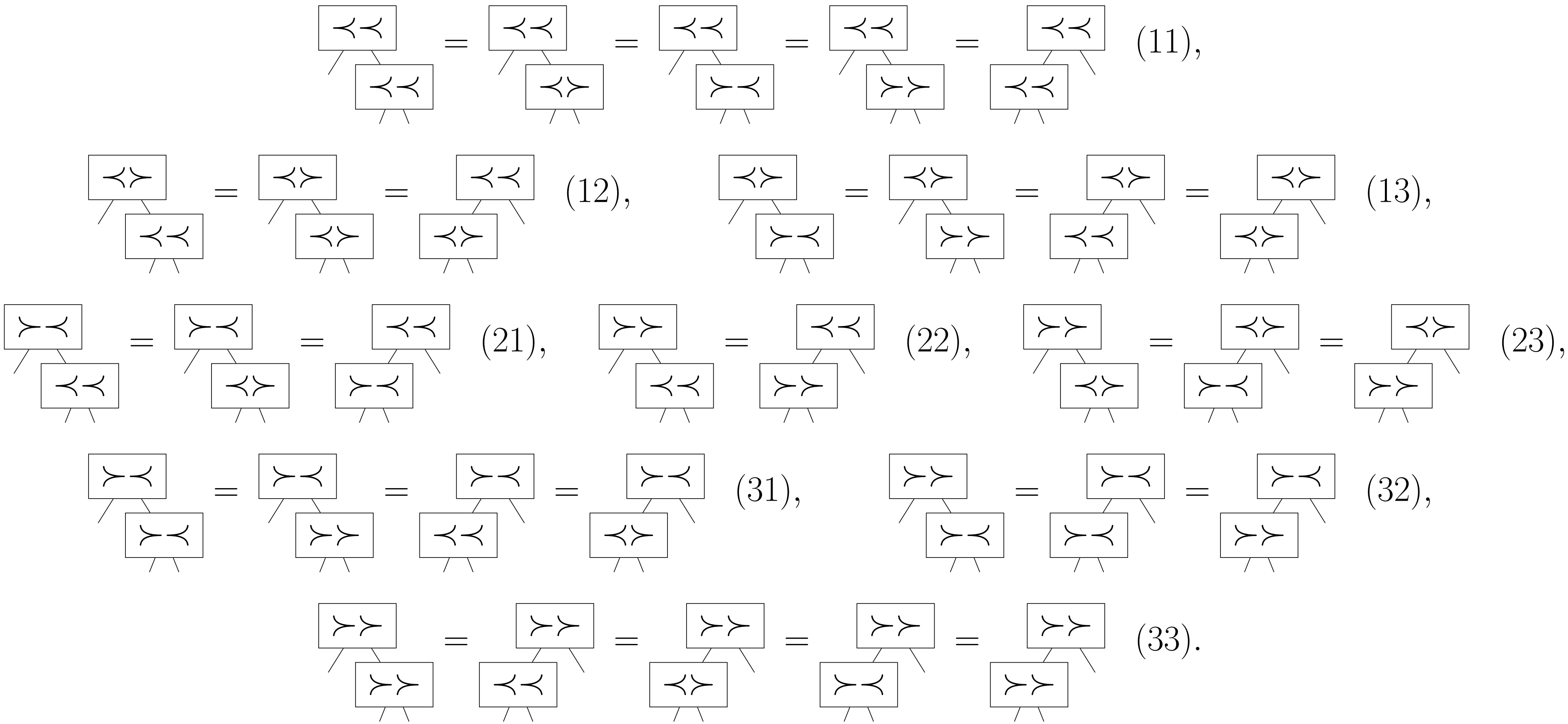


... in series

(ith cyclist follows leftmost remaining signal and erases it)

THM. Both parallel and series k-signaletic operads are quadratic and Koszul. Therefore, they admit a presentation by the quadratic parallel and series k-signaletic relations (same k-destination vector)

exm: series 2-signaletic relations



SIGNALETIC OPERADS

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PROP. The Hilbert series of both parallel and series k-signaletic operads are

$$\mathcal{H}(t) = \sum_{p \geq 1} p^k t^p = \frac{1}{(1-t)^{k+1}} \sum_{p \geq 0} \left\langle {k \atop p} \right\rangle t^p$$

where $\left\langle {k \atop p} \right\rangle$ is the number of permutations of $\mathfrak{S}_k$ with p descents (Eulerian numbers)

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DEF. k-citelangis operad = Koszul dual of k-signaletic operad

$d(n, p)$ = dimension of degree p component of k-citelangis operad =

$k \backslash p$	1	2	3	4	5	6	7	8	OEIS ref
1	1	2	5	14	42	132	429	1430	A000108
2	1	4	23	156	1162	9192	75819	644908	A007297
3	1	8	101	1544	26190	474144	8975229	175492664	A291536
4	1	16	431	14256	525682	20731488	855780699	36512549680	—
5	1	32	1805	125984	9825222	820259712	71710602189	6481491238880	—

CITELANGIS OPERADS : A COMBINATORIAL MODEL

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DEF. A k-permutation is fully k-cuttable if its restriction to any interval (or equivalently any subset) of $[n]$ of size at least 2 has a k-cut

exm: a 3-permutation of degree 5: 3 5 5 1 1 2 1 2 2 4 3 3 4 4 5

it has a 3-cut: ~~3~~ ~~5~~ ~~5~~ 1 1 2 1 2 2 | 4 3 3 4 4 5

its restriction to $[1, 2, 3]$ also has a 3-cut: ~~3~~ ~~1~~ ~~1~~ 2 1 2 2 | 3 3

its restriction to $[3, 4, 5]$ also has a 3-cut: ~~3~~ ~~5~~ ~~5~~ 4 3 3 4 4 | 5

The restriction to $[1, 2]$ also has a 3-cut: ~~1~~ ~~1~~ ~~2~~ 1 | 2 2

CITELANGIS OPERADS : A COMBINATORIAL MODEL

$d(n,p)$ = dimension of degree p component of k-citelangis operad =

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THM. $d(n,p)$ = number of fully k-cutable k-permutations of degree p

Idea: action on k-permutations + leading term