Shard polytopes

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ANR CHARMS kickoff meeting
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TWO CLASSICAL LATTICES AND POLYTOPES
lattice = partially ordered set \( L \) where any \( X \subseteq L \) admits a meet \( \bigwedge X \) and a join \( \bigvee X \)
lattice congruence = equivalence relation on \( L \) compatible with meets and joins
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lattice congruence = equivalence relation on \( L \) compatible with meets and joins

\[
\begin{align*}
4321 \\
3421 & \quad 4231 \quad 4312 \\
3241 & \quad 2431 \quad 3412 \quad 4213 \quad 4132 \\
3214 & \quad 2341 \quad 3142 \quad 2413 \quad 4123 \quad 4132 \\
2314 & \quad 1324 \quad 2143 \quad 1342 \quad 1423\end{align*}
\]

weak order = permutations of \( \mathcal{S}_n \) ordered by inclusion of inversion sets

Tamari lattice = binary trees on \([n]\) ordered by paths of right rotations
**LATTICES: WEAK ORDER AND TAMARI LATTICE**

A lattice is a partially ordered set $L$ where any $X \subseteq L$ admits a meet $\bigwedge X$ and a join $\bigvee X$.

A lattice congruence is an equivalence relation on $L$ compatible with meets and joins.

**Weak order** is permutations of $\mathfrak{S}_n$ ordered by inclusion of inversion sets.

**Tamari lattice** is binary trees on $[n]$ ordered by paths of right rotations.
lattice = partially ordered set $L$ where any $X \subseteq L$ admits a meet $\bigwedge X$ and a join $\bigvee X$

lattice congruence = equivalence relation on $L$ compatible with meets and joins

weak order = permutations of $S_n$

ordered by inclusion of inversion sets

Tamari lattice = binary trees on $[n]$

ordered by paths of right rotations

sylvester congruence = equivalence classes are fibers of BST insertion

= rewriting rule $UacVbW \equiv_{\text{sylv}} UcaVbW$ with $a < b < c$
lattice = partially ordered set \( L \) where any \( X \subseteq L \) admits a meet \( \bigwedge X \) and a join \( \bigvee X \).

lattice congruence = equivalence relation on \( L \) compatible with meets and joins.

weak order = permutations of \( S_n \)
ordered by inclusion of inversion sets

Tamari lattice = binary trees on \([n]\)
ordered by paths of right rotations

sylvester congruence = equivalence classes are fibers of BST insertion
= rewriting rule \( UacVbW \equiv_{\text{sylv}} UcaVbW \) with \( a < b < c \)
fan = collection of polyhedral cones closed by faces and intersecting along faces
polytope = convex hull of a finite set = intersection of finitely many affine half-space
polyhedral cone = positive span of a finite set of \( \mathbb{R}^n \)
= intersection of finitely many linear half-spaces

fan = collection of polyhedral cones closed by faces
and where any two cones intersect along a face
polytope = convex hull of a finite set of $\mathbb{R}^n$
= bounded intersection of finitely many affine half-spaces

face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations
face \( F \) of polytope \( P \)

normal cone of \( F \) = positive span of the outer normal vectors of the facets containing \( F \)

normal fan of \( P \) = \{ normal cone of \( F \) | \( F \) face of \( P \) \}
fan $=$ collection of polyhedral cones closed by faces and intersecting along faces
polytope $=$ convex hull of a finite set $=$ intersection of finitely many affine half-space
**POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON**

\[ \text{fan} = \text{collection of polyhedral cones closed by faces and intersecting along faces} \]
\[ \text{polytope} = \text{convex hull of a finite set} = \text{intersection of finitely many affine half-space} \]

\[
\begin{align*}
\text{braid fan} &= \mathcal{C}(\sigma) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \right\} \\
\text{svester fan} &= \mathcal{C}(T) = \left\{ \mathbf{x} \in \mathbb{R}^n \mid x_i \leq x_j \text{ if } i \to j \text{ in } T \right\}
\end{align*}
\]
fan = collection of polyhedral cones closed by faces and intersecting along faces
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braid fan = \( \mathcal{C}(\sigma) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \} \)

sylvester fan = \( \mathcal{C}(T) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_i \leq x_j \text{ if } i \rightarrow j \text{ in } T \} \)
**POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON**

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\[
\begin{align*}
C(\sigma) &= \{ \mathbf{x} \in \mathbb{R}^n \mid x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \} \\
C(T) &= \{ \mathbf{x} \in \mathbb{R}^n \mid x_i \leq x_j \text{ if } i \rightarrow j \text{ in } T \}
\end{align*}
\]

\[ \text{braid fan} = C(\sigma) \]

\[ \text{sylvester fan} = C(T) \]

\[ \text{quotient fan} = C(T) \text{ obtained by glueing } C(\sigma) \text{ for all } \sigma \text{ in the same BST insertion fiber} \]
fan = collection of polyhedral cones closed by faces and intersecting along faces
polytope = convex hull of a finite set = intersection of finitely many affine half-space
**POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON**

fan = collection of polyhedral cones closed by faces and intersecting along faces

polytope = convex hull of a finite set = intersection of finitely many affine half-space

\[
x_1 < x_2 \quad x_3 < x_4
\]

\[
x_2 < x_3
\]

\[
4231
\]

\[
4321
\]

\[
x_1 < x_2 \quad x_3 < x_4
\]

\[
3421
\]

\[
4132
\]

\[
4213
\]

\[
2431
\]

\[
3241
\]

\[
x_1 > x_3 \quad x_2 > x_4
\]

\[
3412
\]

\[
24314213
\]

\[
12434132
\]

\[
4321
\]

\[
\text{braid fan} = \mathbb{C}(\sigma) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \}\]

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**POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON**

\[ \text{fan} = \text{collection of polyhedral cones closed by faces and intersecting along faces} \]

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**permahedron**  \( \text{Perm}(n) \)

\[ = \text{conv} \left\{ \left[ \sigma^{-1}(i) \right]_{i \in [n]} \right\} \sigma \in \mathfrak{S}_n \]

\[ = \mathbb{H} \cap \bigcap_{\emptyset \neq J \subseteq [n]} H_J \]

where  \( H_J = \left\{ x \in \mathbb{R}^n \mid \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\} \)

**associahedron**  \( \text{Asso}(n) \)

\[ = \text{conv} \left\{ \left[ \ell(T, i) \cdot r(T, i) \right]_{i \in [n]} \right\} T \text{ binary tree} \]

\[ = \mathbb{H} \cap \bigcap_{1 \leq i < j \leq n} H_{i,j} \]

*Stasheff ('63)*  
*Shnider–Sternberg ('93)*  
*Loday ('04)*
**POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON**

fan = collection of polyhedral cones closed by faces and intersecting along faces
polytope = convex hull of a finite set = intersection of finitely many affine half-space

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permutahedron $\text{Perm}(n)$

$$= \text{conv} \left\{ [\sigma^{-1}(i)]_{i \in [n]} \mid \sigma \in S_n \right\}$$

$$= H \cap \bigcap_{\varnothing \neq J \subseteq [n]} H_J$$

where $H_J = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{j \in J} x_j \geq \left( \frac{|J|+1}{2} \right) \right\}$

associahedron $\text{Asso}(n)$

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Stasheff ('63)
Shnider–Sternberg ('93)
Loday ('04)
**Polytopes: Permutahedron and Associahedron**

*fan* = collection of polyhedral cones closed by faces and intersecting along faces

*polytope* = convex hull of a finite set = intersection of finitely many affine half-space

**Permutahedron** $\mathbf{Perm}(n)$

$$= \text{conv} \left\{ [\sigma^{-1}(i)]_{i \in [n]} \mid \sigma \in \mathcal{S}_n \right\}$$

$$= \mathbb{H} \cap \bigcap_{\emptyset \neq J \subseteq [n]} \mathbb{H}_J$$

where $\mathbb{H}_J = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{j \in J} x_j \geq \frac{|J|+1}{2} \right\}$

**Associahedron** $\mathbf{Asso}(n)$

$$= \text{conv} \left\{ [\ell(T,i) \cdot r(T,i)]_{i \in [n]} \mid T \text{ binary tree} \right\}$$

$$= \mathbb{H} \cap \bigcap_{1 \leq i < j \leq n} \mathbb{H}_{[i,j]}$$

Stasheff ('63)
Shnider–Sternberg ('93)
Loday ('04)
fan = collection of polyhedral cones closed by faces and intersecting along faces
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\[\begin{align*}
&\text{permutahedron } \text{Perm}(n) \\
&\implies \text{weak order on permutations} \\
&\text{Hasse diagram of weak order } = \text{graph of Tamari lattice}
\end{align*}\]

\[\begin{align*}
&\text{associahedron } \text{Asso}(n) \\
&\implies \text{Tamari lattice on binary trees} \\
&\text{permutahedron oriented associahedron } \text{oriented left } \rightarrow \text{ right comb}
\end{align*}\]
QUOTIENT FANS AND QUOTIENTOPES
QUOTIENT FAN

lattice congruence $\equiv$ equivalence relation on $L$ compatible with meets and joins:

$x \equiv x'$ and $y \equiv y'$ implies $x \land y \equiv x' \land y'$ and $x \lor y \equiv x' \lor y'$

quotient fan $\mathcal{F}_\equiv = \text{chambers are obtained by glueing the chambers } C(\sigma) \text{ of the permutations } \sigma \text{ in the same congruence class of } \equiv$

Reading ('05)
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\[ x \equiv x' \text{ and } y \equiv y' \text{ implies } x \land y \equiv x' \land y' \text{ and } x \lor y \equiv x' \lor y' \]

quotient fan $\mathcal{F}_{\equiv}$ = chambers are obtained by glueing the chambers $C(\sigma)$ of the permutations $\sigma$ in the same congruence class of $\equiv$

$W_{\equiv}$ = walls of the quotient fan $\mathcal{F}_{\equiv}$

Describe the possible sets of walls $W_{\equiv}$
lattice congruence = equivalence relation on $L$ compatible with meets and joins:

$x \equiv x'$ and $y \equiv y'$ implies $x \land y \equiv x' \land y'$ and $x \lor y \equiv x' \lor y'$

quotient fan $F_\equiv =$ chambers are obtained by glueing the chambers $C(\sigma)$ of the permutations $\sigma$ in the same congruence class of $\equiv$

$W_\equiv =$ walls of the quotient fan $F_\equiv$
Describe the possible sets of walls $W_\equiv$
\textbf{ARCS AND SHARDS}

\texttt{arc} \ (a, b, A, B) \ with \ 1 \leq a < b \leq n \ and \ A \sqcup B = ]a, b[

\texttt{shard} \ \Sigma(a, b, A, B) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_{a'} \leq x_a = x_b \leq x_{b'} \ \text{for all} \ a' \in A \ \text{and} \ b' \in B \}
\textbf{ARCS AND SHARDS}

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\textbf{ARCS AND SHARDS}

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The set of walls \(\mathcal{W}_\pi\) of the quotient fan \(\mathcal{F}_\pi\) is a union of shards \(\Sigma_\pi\)\hfill\text{Reading (‘05)}
\[ \Sigma(a, b, A, B) \text{ forces } \Sigma(c, d, C, D) = \]
\[ c \leq a < b \leq d \text{ and } A \subseteq C \text{ and } B \subseteq D \]
\[ \Sigma(a, b, A, B) \text{ forces } \Sigma(c, d, C, D) = \]
c \leq a < b \leq d \text{ and } A \subseteq C \text{ and } B \subseteq D

Reading ('15)

TFAE for a set of shards \( \Sigma \):
- there is a congruence \( \equiv \) with \( \Sigma = \Sigma_{\equiv} \)
- \( \Sigma \) is an upper ideal in forcing order
shard ideal = upper ideal in forcing order

essential congruences:
1, 1, 4, 47, 3322, ...
OEIS A330039

all congruences
1, 2, 7, 60, 3444, ...
OEIS A091687
shard ideal = upper ideal in forcing order

essential congruences:
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quotientope = polytope whose normal fan is $\mathcal{F}_\equiv$
quotientope = polytope whose normal fan is $\mathcal{F}$

P.–Santos ('19)
quotientope = polytope whose normal fan is $\mathcal{F}_\equiv$
MINKOWSKI SUMS OF ASSOCIAHEDRA
If the congruence $\equiv$ is the intersection of the congruences $\equiv_1, \ldots, \equiv_k$, then the quotient fan $\mathcal{F}_\equiv$ is the common refinement of the quotient fans $\mathcal{F}_{\equiv_1}, \ldots, \mathcal{F}_{\equiv_k}$. 
If the congruence $\equiv$ is the intersection of the congruences $\equiv_1, \ldots, \equiv_k$, then the quotient fan $\mathcal{F}_\equiv$ is the common refinement of the quotient fans $\mathcal{F}_\equiv_1, \ldots, \mathcal{F}_\equiv_k$.

Minkowski sum $P + Q = \{ p + q \mid p \in P, \ q \in Q \}$
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**Minkowski sum** $\mathbb{P} + \mathbb{Q} = \{ p + q | p \in \mathbb{P}, q \in \mathbb{Q} \}$
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Minkowski sum $\mathbb{P} + \mathbb{Q} = \{p + q \mid p \in \mathbb{P}, q \in \mathbb{Q}\}$

Normal fan of $\mathbb{P} + \mathbb{Q}$ = common refinement of normal fans of $\mathbb{P}$ and $\mathbb{Q}$
If the congruence $\equiv$ is the intersection of the congruences $\equiv_1, \ldots, \equiv_k$, then the quotient fan $\mathcal{F}_\equiv$ is the common refinement of the quotient fans $\mathcal{F}_\equiv_1, \ldots, \mathcal{F}_\equiv_k$, and a Minkowski sum of quotientopes for $\mathcal{F}_\equiv_1, \ldots, \mathcal{F}_\equiv_k$ is a quotientope for $\mathcal{F}_\equiv$. 

\[ \bigcup \quad = \quad = \]
MINKOWSKI SUMS OF ASSOCIAHEDRA

If the congruence $\equiv$ is the intersection of the congruences $\equiv_1, \ldots, \equiv_k$, then the quotient fan $\mathcal{F}_\equiv$ is the common refinement of the quotient fans $\mathcal{F}_\equiv_1, \ldots, \mathcal{F}_\equiv_k$, and a Minkowski sum of quotientopes for $\mathcal{F}_\equiv_1, \ldots, \mathcal{F}_\equiv_k$ is a quotientope for $\mathcal{F}_\equiv$.

Principal arc ideals are Cambrian congruences.

Any quotient fan is realized by a Minkowski sum of (low dim.) associahedra.

Padrol-P.-Ritter (‘20+).
quotientope = polytope whose normal fan is $\mathcal{F}_{≡}$
TROU NORMAND — QUESTIONS?
SHARD POLYTOPES
for a shard $\Sigma = \Sigma(a, b, A, B)$, define

- $\Sigma$-matching = sequence $a \leq a_1 < b_1 < \cdots < a_k < b_k \leq b$ where $\{a_i \in \{a\} \cup A$
  
  \[b_i \in B \cup \{b\}\] for all $i$

- Characteristic vector $\chi(M) = \sum_{i \in [k]} e_{a_i} - e_{b_i}$

**Shard polytope** $\text{SP}(\Sigma) = \text{conv} \\{ \chi(M) \mid M \Sigma$-matching $\}$

\[
\begin{align*}
\text{SP}(\Sigma) = \text{conv} \{ & x \in \mathbb{R}^n \mid \\
& x_j = 0 \quad \text{for all } j \in [n] \setminus [a, b] \\
& 0 \leq x_{a'} \leq 1 \quad \text{for all } a' \in \{a\} \cup A \\
& -1 \leq x_{b'} \leq 0 \quad \text{for all } b' \in B \cup \{b\} \\
& 0 \leq \sum_{i \leq j} x_i \leq 1 \quad \text{for all } j \in [n]
\end{align*}
\]

exm: for an up shard $(a, b,]a, b[, \emptyset)$, we get the standard simplex $\Delta_{[a,b]} - e_b$
The normal fan of the shard polytope $\mathcal{SP}(\Sigma)$
- contains the shard $\Sigma$,
- is contained in the union of the shards forcing $\Sigma$
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- contains the shard $\Sigma$,
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For any lattice congruence $\equiv$, the quotient fan $\mathcal{F}_\equiv$ is the normal fan of the Minkowski sum of the shard polytopes $\mathcal{SP}(\Sigma)$ for $\Sigma \in \Sigma_\equiv$

Padrol-P.-Ritter (20+)}
The normal fan of the shard polytope $\mathcal{SP}(\Sigma)$

- contains the shard $\Sigma$,
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Padrol-P.-Ritter (20+)}
SHARD POLYTOPES AND TYPE CONES
CHOOSING RIGHT-HAND-SIDES

\( \mathcal{F} = \) complete simplicial fan in \( \mathbb{R}^n \) with \( N \) rays

\( G = (N \times n) \)-matrix whose rows are representatives of the rays of \( \mathcal{F} \)

for a height vector \( h \in \mathbb{R}_>^N \), consider the polytope

\[ \mathbb{P}_h = \{ x \in \mathbb{R}^n \mid Gx \leq h \} \]
CHOOSING RIGHT-HAND-SIDES

\( \mathcal{F} = \) complete simplicial fan in \( \mathbb{R}^n \) with \( N \) rays

\( \mathbf{G} = (N \times n) \)-matrix whose rows are representatives of the rays of \( \mathcal{F} \)

for a height vector \( \mathbf{h} \in \mathbb{R}_0^N \), consider the polytope \( P_h = \{ x \in \mathbb{R}^n \mid \mathbf{G}x \leq h \} \)
$\mathcal{F}$ = complete simplicial fan in $\mathbb{R}^n$ with $N$ rays
$G = (N \times n)$-matrix whose rows are representatives of the rays of $\mathcal{F}$
for a height vector $h \in \mathbb{R}_+^N$, consider the polytope $P_h = \{x \in \mathbb{R}^n \mid Gx \leq h\}$

When is $\mathcal{F}$ the normal fan of $P_h$?
\( \mathcal{F} = \) complete simplicial fan in \( \mathbb{R}^n \) with \( N \) rays

\( \mathbf{G} = (N \times n) \)-matrix whose rows are representatives of the rays of \( \mathcal{F} \)

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$\mathcal{F} = \text{complete simplicial fan in } \mathbb{R}^n \text{ with } N \text{ rays}$

$G = (N \times n)$-matrix whose rows are representatives of the rays of $\mathcal{F}$

for a height vector $h \in \mathbb{R}^N_{>0}$, consider the polytope $P_h = \{ x \in \mathbb{R}^n \mid Gx \leq h \}$

wall-crossing inequality for a wall $R = \sum_{s \in R \cup \{r, r'\}} \alpha_{R,s} h_s > 0$ where

- $r, r' = \text{rays such that } R \cup \{r\} \text{ and } R \cup \{r'\} \text{ are chambers of } \mathcal{F}$
- $\alpha_{R,s} = \text{coeff. of unique linear dependence } \sum_{s \in R \cup \{r, r'\}} \alpha_{R,s} s = 0 \text{ with } \alpha_{R,r} + \alpha_{R,r'} = 2$
$\mathcal{F}$ = complete simplicial fan in $\mathbb{R}^n$ with $N$ rays

$G = (N \times n)$-matrix whose rows are representatives of the rays of $\mathcal{F}$

for a height vector $h \in \mathbb{R}_{>0}^N$, consider the polytope $\mathbb{P}_h = \{x \in \mathbb{R}^n \mid Gx \leq h\}$

wall-crossing inequality for a wall $R = \sum_{s \in R \cup \{r, r'\}} \alpha_{R,s} h_s > 0$ where

- $r, r' =$ rays such that $R \cup \{r\}$ and $R \cup \{r'\}$ are chambers of $\mathcal{F}$
- $\alpha_{R,s} =$ coeff. of unique linear dependence $\sum_{s \in R \cup \{r, r'\}} \alpha_{R,s} s = 0$ with $\alpha_{R,r} + \alpha_{R,r'} = 2$

$\mathcal{F}$ is the normal fan of $\mathbb{P}_h \iff h$ satisfies all wall-crossing inequalities of $\mathcal{F}$
$\mathcal{F}$ = complete simplicial fan in $\mathbb{R}^n$ with $N$ rays
$G = (N \times n)$-matrix whose rows are representatives of the rays of $\mathcal{F}$
for a height vector $h \in \mathbb{R}_>^N$, consider the polytope $P_h = \{x \in \mathbb{R}^n \mid Gx \leq h\}$

wall-crossing inequalities:

wall 1: $h_2 + h_5 > 0$
wall 2: $h_1 + h_3 > h_2$
wall 3: $h_2 + h_4 > h_3$
wall 4: $h_3 + h_5 > h_4$
wall 5: $h_1 + h_4 > 0$
**TYPE CONE**

\( \mathcal{F} = \) complete simplicial fan in \( \mathbb{R}^n \) with \( N \) rays  
\( G = (N \times n) \)-matrix whose rows are representatives of the rays of \( \mathcal{F} \)  
for a height vector \( \mathbf{h} \in \mathbb{R}_0^N \), consider the polytope  
\( P_{\mathbf{h}} = \{ \mathbf{x} \in \mathbb{R}^n \mid G\mathbf{x} \leq \mathbf{h} \} \)

- **type cone** \( \text{TC}(\mathcal{F}) \) = realization space of \( \mathcal{F} \)  
  = \( \{ \mathbf{h} \in \mathbb{R}^N \mid \mathcal{F} \text{ is the normal fan of } P_{\mathbf{h}} \} \)  
  = \( \{ \mathbf{h} \in \mathbb{R}^N \mid \mathbf{h} \text{ satisfies all wall-crossing inequalities of } \mathcal{F} \} \)  

McMullen ('73)
\( \mathcal{F} = \) complete simplicial fan in \( \mathbb{R}^n \) with \( N \) rays

\( \mathbf{G} = (N \times n) \)-matrix whose rows are representatives of the rays of \( \mathcal{F} \)

for a height vector \( \mathbf{h} \in \mathbb{R}^N_{>0} \), consider the polytope \( \mathbb{P}_h = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{G}\mathbf{x} \leq \mathbf{h} \} \)

\[ \text{type cone } \mathbb{T}_C(\mathcal{F}) = \text{realization space of } \mathcal{F} \]
\[ = \{ \mathbf{h} \in \mathbb{R}^N \mid \mathcal{F} \text{ is the normal fan of } \mathbb{P}_h \} \]
\[ = \{ \mathbf{h} \in \mathbb{R}^N \mid \mathbf{h} \text{ satisfies all wall-crossing inequalities of } \mathcal{F} \} \]

some properties of \( \mathbb{T}_C(\mathcal{F}) \):

- \( \mathbb{T}_C(\mathcal{F}) \) is an open cone (dilations preserve normal fans)
- \( \mathbb{T}_C(\mathcal{F}) \) has lineality space \( \mathbf{G}\mathbb{R}^n \) (translations preserve normal fans)
- dimension of \( \mathbb{T}_C(\mathcal{F})/\mathbf{G}\mathbb{R}^n = \mathbb{R}^N = N - n \)
**TYPE CONE**

\( \mathcal{F} = \) complete simplicial fan in \( \mathbb{R}^n \) with \( N \) rays

\( \mathbf{G} = (N \times n) \)-matrix whose rows are representatives of the rays of \( \mathcal{F} \)

for a height vector \( \mathbf{h} \in \mathbb{R}^N_\geq 0 \), consider the polytope \( \mathbb{P}_h = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{Gx} \leq \mathbf{h} \} \)

\[
\text{type cone } \mathcal{T} \mathcal{C}(\mathcal{F}) = \text{realization space of } \mathcal{F} \\
= \{ \mathbf{h} \in \mathbb{R}^N \mid \mathcal{F} \text{ is the normal fan of } \mathbb{P}_h \} \\
= \{ \mathbf{h} \in \mathbb{R}^N \mid \mathbf{h} \text{ satisfies all wall-crossing inequalities of } \mathcal{F} \}
\]

some properties of \( \mathcal{T} \mathcal{C}(\mathcal{F}) \):

- closure of \( \mathcal{T} \mathcal{C}(\mathcal{F}) = \) polytopes whose normal fan coarsens \( \mathcal{F} = \) deformation cone
- Minkowski sums \( \leftarrow \rightarrow \) positive linear combinations

McMullen (’73)
Assume that the type cone $\mathcal{T}C(\mathcal{F})$ is simplicial

$K = (N-n) \times N$-matrix whose rows are inner normal vectors of the facets of $\mathcal{T}C(\mathcal{F}(\delta))$

All polytopal realizations of $\mathcal{F}$ are affinely equivalent to

$$\mathbb{R}_\ell = \left\{ z \in \mathbb{R}^N \mid Kz = \ell \text{ and } z \geq 0 \right\}$$

for any positive vector $\ell \in \mathbb{R}^{N-n}_{>0}$

Padrol–Palu–P.–Plamondon ('19+)

Fundamental exms: $g$-vector fans of cluster-like complexes

sylvester fans
genette fans wrt any seed (acyclic or not)

finite gentle fans for brick and 2-acyclic quivers

Arkani-Hamed–Bai–He–Yan ('18)  
BMDMTY ('18+)  
Palu–P.–Plamondon ('18)
closed type cone of braid fan = \{\text{deformed permutahedra}\} = \{\text{submodular functions}\}

deformed permutahedron = \text{polytope whose normal fan coarsens the braid fan}

\[ \text{Defo}(z) = \{ \mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{1} \mid \mathbf{x} \rangle = z_{[n]} \text{ and } \langle \mathbf{1}_R \mid \mathbf{x} \rangle \geq z_R \text{ for all } R \subseteq [n] \} \]

for some vector \( z \in \mathbb{R}^{2^n} \) such that \( z_R + z_S \leq z_{R \cup S} + z_{R \cap S} \) and \( z_{\emptyset} = 0 \)

Postnikov ('09) Postnikov–Reiner–Williams ('08)
closed type cone of braid fan = \{\text{deformed permutahedra}\} = \{\text{submodular functions}\}

deformed permutahedron = \text{polytope whose normal fan coarsens the braid fan}

\[
\text{Defo}(z) = \{ \mathbf{x} \in \mathbb{R}^n_{\geq 0} \mid \langle \mathbf{1} | \mathbf{x} \rangle = z_{[n]} \text{ and } \langle \mathbf{1}_R | \mathbf{x} \rangle \geq z_R \text{ for all } R \in \mathcal{J} \}
\]

for some vector \( z \in \mathbb{R}^{2^n} \) such that \( z_R + z_S \leq z_{R \cup S} + z_{R \cap S} \) and \( z_{\emptyset} = z_{\{i\}} = 0 \),

where \( \mathcal{J} = \{ J \subseteq [n] \mid |J| \geq 2 \} \)
SUBMODULAR FUNCTIONS

\[ \text{dim } \text{TC}(\mathcal{F}) = N - n = 6 - 2 = 4 \]
SUBMODULAR FUNCTIONS

\[ SP(\cdot) = \]

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all quotientopes of PS ('18) are Minkowski sums of scaled shard polytopes

Padrol-P.-Ritter (20+)
all quotientopes of PS ('18) are Minkowski sums of scaled shard polytopes

shard polytopes are Minkowski indecomposable (thus rays of the type cone)
⇒ Newton polytopes $F$-polyn.
⇒ brick polytope summands
all quotientopes of PS ('18) are Minkowski sums of scaled shard polytopes

shard polytopes are Minkowski indecomposable (thus rays of the type cone)

⇒ Newton polytopes $F$-polyn.
⇒ brick polytope summands

Any deformed permutahedron is a Minkowski sum and difference of shard polytopes

$$\text{Defo}(z) = \sum_{J \in J} y_J \triangle J = \sum_{I \in J} s_I \text{SP} (\Sigma_I)$$

with explicit (combinatorial) exchange matrices between the parameters $s$, $y$ and $z$
OPEN QUESTIONS
**QUOTIENTOPES FOR HYPERPLANE ARRANGEMENTS**

\(\mathcal{H}\) hyperplane arrangement in \(\mathbb{R}^n\)

base region \(B = \) distinguished region of \(\mathbb{R}^n \setminus \mathcal{H}\)

inversion set of a region \(C = \) set of hyperplanes of \(\mathcal{H}\) that separate \(B\) and \(C\)

poset of regions \(\text{PR}(\mathcal{H}, B) = \) regions of \(\mathbb{R}^n \setminus \mathcal{H}\) ordered by inclusion of inversion sets

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The poset of regions \(\text{PR}(\mathcal{H}, B)\)

- is never a lattice when \(B\) is not a simplicial region
- is always a lattice when \(\mathcal{H}\) is a simplicial arrangement

---

If \(\text{PR}(\mathcal{H}, B)\) is a lattice, and \(\equiv\) is a congruence of \(\text{PR}(\mathcal{H}, B)\), the cones obtained by glueing the regions of \(\mathbb{R}^n \setminus \mathcal{H}\) in the same congruence class form a complete fan \(\mathcal{F}_\equiv\)

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Is the quotient fan \(\mathcal{F}_\equiv\) always polytopal?
**SHARDS FOR HYPERPLANE ARRANGEMENTS**

shard = piece of hyperplane obtained after cutting all rank 2 subgroups

shard poset = (pre)poset of forcing relations among shards
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SHARDS FOR HYPERPLANE ARRANGEMENTS

Reading (‘03)
SHARDS FOR HYPERPLANE ARRANGEMENTS

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Reading (’03)
SHARD POLYTOPES FOR HYPERPLANE ARRANGEMENTS?

**shard** = piece of hyperplane obtained after cutting all rank 2 subgroups

**shard poset** = (pre)poset of forcing relations among shards

**shard polytope** for a shard \( \Sigma \) = polytope whose normal fan

- contains the shard \( \Sigma \),
- is contained in the union of the shards forcing \( \Sigma \)

Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions
SHARD POLYTOPES FOR HYPERPLANE ARRANGEMENTS?

shard = piece of hyperplane obtained after cutting all rank 2 subgroups
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shard polytope for a shard $\Sigma = \text{polytope whose normal fan}$
  • contains the shard $\Sigma$,
  • is contained in the union of the shards forcing $\Sigma$

Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions

If any shard $\Sigma$ admits a shard polytope $\mathbb{SP}(\Sigma)$, then
  • for any lattice congruence $\equiv$ of $\text{PR}(\mathcal{H}, B)$, the quotient fan $\mathcal{F}_\equiv$ is the normal of the Minkowski sum of the shard polytopes $\mathbb{SP}(\Sigma)$ for $\Sigma$ in the shard ideal $\Sigma_\equiv$
  • if the arrangement $\mathcal{H}$ is simplicial, then the shard polytopes $\mathbb{SP}(\Sigma)$ form a basis for the type cone of the fan defined by $\mathcal{H}$ (up to translation)

Padrol-P.-Ritter (20+)
SHARD POLYTOPES FOR HYPERPLANE ARRANGEMENTS?

shard = piece of hyperplane obtained after cutting all rank 2 subgroups
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shard polytope for a shard $\Sigma$ = polytope whose normal fan
- contains the shard $\Sigma$,
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Find shard polytopes for arbitrary hyperplane arrangement with a lattice of regions

For crystallographic arrangements, Newton polytopes of $F$-polynomials all seem to be shard polytopes, but some shards are missing...
THANKS