POLYTOPALITY AND CARTESIAN PRODUCTS

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**COMBINATORICS OF POLYTOPES**
**POLYTOPES FROM COMBINATORICS**

polytope = convex hull of a finite set of $\mathbb{R}^d$
= bounded intersection of finitely many half-spaces

face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations

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Given a set of points, determine the face lattice of its convex hull.

Given (part of) a face lattice, is there a polytope which realizes it?
In which dimension(s)?
A graph is \textit{d-polytopal} if it is the graph of a \textit{d-dimensional polytope}.

One of these graphs is polytopal. Can you guess which?
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The polytopality range of a graph is the set of dimensions in which it is polytopal.

Which dimension can have a polytope with this graph?
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### GENERAL POLYTOPES

**THEOREM.** $3$-polytopal $\iff$ simple, planar and $3$-connected.  
E. Steinitz 1922

**THEOREM.** A $d$-polytopal graph satisfies the following properties:

- **Balinski’s Theorem.** $G$ is $d$-connected.  
  M. Balinski 1961

- **Principal Subdivision Property.** Every vertex of $G$ is the principal vertex of a principal subdivision of $K_{d+1}$ contained in $G$.  
  D. Barnette 1967

### SIMPLE POLYTOPES

**THEOREM.** Two simple polytopes are combinatorially equivalent if and only if they have the same graph.  

**LEMMA.** All induced $3$-, $4$- and $5$-cycles in the graph of a simple polytope are $2$-faces.
**LEMMA.** All induced 3-, 4- and 5-cycles in the graph of a simple polytope are 2-faces.

**EXAMPLE.** None of the graphs of the following family is polytopal:
Polytopality of Cartesian products of graphs

Julian Pfeifle
Francisco Santos
**Cartesian product of polytopes:** \( P \times Q := \{(p, q) \mid p \in P, q \in Q\} \).

**Cartesian product of graphs:**

\[
\begin{align*}
V(G \times H) &:= V(G) \times V(H), \\
E(G \times H) &:= (V(G) \times E(H)) \cup (E(G) \times V(H)).
\end{align*}
\]

**Remark.** graph of \( P \times Q = (\text{graph of } P) \times (\text{graph of } Q) \).

**Problem.** Does the polytopality of \( G \times H \) imply that of \( G \) and \( H \)?
PROBLEM. Does the polytopality of $G \times H$ imply that of $G$ and $H$?

THEOREM. $G \times H$ simply polytopal $\iff$ $G$ and $H$ simply polytopal.

THEOREM. The product of a $d$-polytopal graph by the graph of a regular subdivision of an $e$-polytope is $(d+e)$-polytopal.

J. Pfeifle, V. P. & F. Santos, On polytopality of Cartesian products of graphs, 2010
POLYTOPALITY AND CARTESIAN PRODUCTS

THEOREM. The product of a $d$-polytopal graph by the graph of a regular subdivision of an $e$-polytope is $(d + e)$-polytopal.

EXAMPLE. The product of two domino graphs is polytopal.

J. Pfeifle, V. P. & F. Santos, On polytopality of Cartesian products of graphs, 2010
THEOREM. The product of a $d$-polytopal graph by the graph of a regular subdivision of an $e$-polytope is $(d + e)$-polytopal.

EXAMPLE. Polytopal product of regular non-polytopal graphs.

J. Pfeifle, V. P. & F. Santos, On polytopality of Cartesian products of graphs, 2010
THEOREM. The graph $K_{n,n} \times K_2$ is not polytopal for $n \geq 3$.

THEOREM. There is a unique combinatorial 3-dimensional manifold whose graph is $K_{3,3} \times K_3$. It is homeomorphic to $\mathbb{RP}^2 \times S^1$. 

A. Guedes de Oliveira, E. Kim, M. Noy, A. Padrol, J. Pfeifle & V. P.
SOME CHALLENGING EXAMPLES

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PROBLEM. Is the product of two Petersen graphs the graph of a polytope?

This polytope could have dimension 4 or 5.
Prodsimplicial Neighborly Polytopes

Benjamin Matschke
Julian Pfeifle
PRODSIMPLICIAL NEIGHBORLY POLYTOPES

\( k \geq 0 \) and \( \underline{n} := (n_1, \ldots, n_r) \).

A polytope is \((k, \underline{n})\)-prodsimplicial-neighborly if its \(k\)-skeleton is combinatorially equivalent to that of the product of simplices \( \triangle_n := \triangle_{n_1} \times \cdots \times \triangle_{n_r} \).

**EXAMPLE.**

(i) **neighborly** polytopes arise when \( r = 1 \).

For example, the cyclic polytope \( C_{2k+2}(n+1) \) is \((k, n)\)-PSN.

(ii) **neighborly cubical** polytopes arise when \( \underline{n} = (1, 1, \ldots, 1) \).

M. Joswig & G. Ziegler, Neighborly cubical polytopes, 2000

**PROBLEM.** What is the minimal dimension of a \((k, n)\)-PSN polytope?
$k \geq 0$ and $\underline{n} := (n_1, \ldots, n_r)$.

A polytope is $(k, \underline{n})$-prodsimplicial-neighborly if its $k$-skeleton is combinatorially equivalent to that of the product of simplices $\triangle_{\underline{n}} := \triangle_{n_1} \times \cdots \times \triangle_{n_r}$.

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M. Joswig & G. Ziegler, Neighborly cubical polytopes, 2000

**PROBLEM.** What is the minimal dimension of a $(k, \underline{n})$-PSN polytope?

A $(k, \underline{n})$-PSN polytope is $(k, \underline{n})$-projected-prodsimplicial-neighborly if it is a projection of a polytope combinatorially equivalent to $\triangle_{\underline{n}}$.

**PROBLEM.** What is the minimal dimension of a $(k, \underline{n})$-PPSN polytope?
**PRODUCT OF CYCLIC POLYTOPES**

\[ C_d(n) := \text{conv} \{\mu_d(t_i) \mid i \in [n]\} \text{ the } d\text{-dimensional cyclic polytope with } n \text{ vertices,} \]

where \( \mu_d(t) = (t, t^2, \ldots, t^d)^T \) and \( t_1, t_2, \ldots, t_n \in \mathbb{R} \) distinct.

**PROPOSITION.** Any subset of at most \( \left\lfloor \frac{d}{2} \right\rfloor \) vertices of \( C_d(n) \) forms a face of \( C_d(n) \).

\( F \subset [n] \) defines a facet of \( C_d(n) \) \( \iff \) \( |F| = d \) and all inner blocs are even.

The normal vector of this facet is given by the coefficients of the polytope

\[
\prod_{i \in F} (t - t_i) = \sum_{i=1}^{d} \gamma_i(F) t^i = \begin{pmatrix} \gamma_1(F) \\ : \\ \gamma_d(F) \end{pmatrix} \cdot \begin{pmatrix} t^1 \\ : \\ t^d \end{pmatrix} + \gamma_0(F).
\]

**PROPOSITION.** Let \( k \geq 0 \) and \( \underline{n} := (n_1, \ldots, n_r) \). Let \( I := \{i \in [n] \mid n_i \geq 2k + 3\} \). The product

\[
\prod_{i \in I} C_{2k+2}(n_i + 1) \times \prod_{i \notin I} \Delta_{n_i}
\]

is a \((k, \underline{n})\)-PPSN polytope of dimension \((2k + 2)|I| + \sum_{i \notin I} n_i \leq (2k + 2)r\).
PROPOSITION. Let $k \geq 0$ and $n := (n_1, \ldots, n_r)$. Define

$$v_{a_1, \ldots, a_r} := \left( \begin{array}{c}
\sum_{i \in [r]} a_i \\
\sum_{i \in [r]} a_i^2 \\
\vdots \\
\sum_{i \in [r]} a_i^{2k+2r}
\end{array} \right) \in \mathbb{R}^{2k+2r}.$$ 

For any pairwise disjoint index sets $I_1, \ldots, I_r \subset \mathbb{R}$, with $|I_i| = n_i$ for all $i \in [r]$, the polytope $\text{conv} \{ v_{a_1, \ldots, a_r} \mid (a_1, \ldots, a_r) \in I_1 \times \cdots \times I_r \} \subset \mathbb{R}^{2k+2r}$ is a $(k, n)$-PPSN $(2k + 2r)$-dimensional polytope.
PROPOSITION. Let \( k \geq 0 \) and \( \underline{n} := (n_1, \ldots, n_r) \). Define

\[
\begin{pmatrix}
a_1 \\
\vdots \\
a_r \\
\sum_{i \in [r]} a_i^2 \\
\vdots \\
\sum_{i \in [r]} a_i^{2k+2}
\end{pmatrix} \in \mathbb{R}^{2k+r+1}.
\]

There exists pairwise disjoint index sets \( I_1, \ldots, I_r \subset \mathbb{R} \), with \( |I_i| = n_i \) for all \( i \in [r] \), such that the polytope \( \text{conv} \left\{ w_{a_1,\ldots,a_r} \mid (a_1, \ldots, a_r) \in I_1 \times \cdots \times I_r \right\} \subset \mathbb{R}^{2k+r+1} \) is a \((k, \underline{n})\)-PPSN \((2k + r + 1)\)-dimensional polytope.

B. Matschke, J. Pfeifle & V. P., Prodsimplicial neighborly polytopes, 2010
$n > d$.

$\pi : \mathbb{R}^n \to \mathbb{R}^d$ the orthogonal projection on the first $d$ coordinates.

$\tau : \mathbb{R}^n \to \mathbb{R}^{n-d}$ the dual projection on the last $n-d$ coordinates.

A proper face $F$ of a polytope $P$ is strictly preserved under $\pi$ if:

(i) $\pi(F)$ is a face of $\pi(P)$,

(ii) $F$ and $\pi(F)$ are combinatorially isomorphic, and

(iii) $\pi^{-1}(\pi(F))$ equals $F$. 

\begin{align*}
Q & \quad S \\
\quad & \quad R \\
\quad & \quad p \\
q & \quad s \\
\quad & \quad r
\end{align*}
$n > d$.

$\pi : \mathbb{R}^n \rightarrow \mathbb{R}^d$ the orthogonal projection on the first $d$ coordinates.

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Let $F_1, \ldots, F_m$ be the facets of $P$. Let $f_i$ be the normal vector of $F_i$ and $g_i = \tau(f_i)$.

For any face $F$ of $P$, let $\phi(F) = \{i \in [m] | F \subset F_i\}$. In other words, $F = \cap_{i \in \phi(F)} F_i$.

**LEMMA.** $F$ face of $P$ is strictly preserved $\iff$ $\{g_i | i \in \phi(F)\}$ is positively spanning.

N. Amenta & G. Ziegler, Deformed products and maximal shadows of polytopes, 1999
G. Ziegler, Projected products of polytopes, 2004
**DEFORMED PRODUCTS**

$P_1, \ldots, P_r$ simple polytopes, with facet description:

$$P_i := \{ x \in \mathbb{R}^{n_i} \mid A_i x \leq b_i \}, \text{ where } A_i \in \mathbb{R}^{m_i \times n_i} \text{ and } b_i \in \mathbb{R}^{m_i}.$$  

The product $P := P_1 \times \cdots \times P_r$ has dimension $\sum_{i \in [r]} n_i$ and is defined by the $\sum_{i \in [r]} m_i$ inequalities:

$$\begin{pmatrix}
A_1 \\
\cdots \\
A_r
\end{pmatrix} x \leq 
\begin{pmatrix}
b_1 \\
\vdots \\
b_r
\end{pmatrix}.$$

**THEOREM. (DEFORMED PRODUCT CONSTRUCTION)**

For any matrix $A^\sim := \begin{pmatrix}
A_1 & \ast & \ast \\
\cdots & \ast \\
A_r
\end{pmatrix}$ obtained by arbitrarily changing the 0’s above the diagonal blocks, there exists $b^\sim$ such that the polytope defined by $A^\sim x \leq b^\sim$ is combinatorially equivalent to $P_1 \times \cdots \times P_r$.

N. Amenta & G. Ziegler, Deformed products and maximal shadows of polytopes, 1999
PROJECTED DEFORMED PRODUCTS

IDEA. Use your freedom on the upper part of the matrix $A\sim$ to obtain a polytope $P\sim := \{ x \in \mathbb{R}^{\sum n_i} \mid A\sim x \leq b\sim \}$ such that:

(i) $P\sim$ is a deformed product combinatorially equivalent to $P := P_1 \times \cdots \times P_r$; and

(ii) the projection of $P\sim$ on the first $d$ coordinates preserves its $k$-skeleton.

EXAMPLE. Let $P_1, \ldots, P_r$ be $r$ simple polytopes of respective dimension $n_i$ and with $m_i$ many facets. If $d = \sum_{i \in [t]} n_i$, then there exists a $d$-dimensional polytope whose $k$-skeleton is combinatorially equivalent to that of $P_1 \times \cdots \times P_r$ provided

$$k \leq \sum_{i \in [r]} n_i - \sum_{i \in [r]} m_i + \left\lfloor \frac{\sum_{i \in [t]} m_i - 1}{2} \right\rfloor.$$  

For improvements, see  

B. Matschke, J. Pfeifle & V. P., Prodsimplicial neighborly polytopes, 2010
SANYAL’S TOPOLOGICAL OBSTRUCTION METHOD

\( n > d. \)
\( \pi : \mathbb{R}^n \rightarrow \mathbb{R}^d \) the orthogonal projection on the first \( d \) coordinates.
\( \tau : \mathbb{R}^n \rightarrow \mathbb{R}^{n-d} \) the dual projection on the last \( n-d \) coordinates.

Let \( P \) be a simple full-dimensional polytope whose vertices are strictly preserved by \( \pi \). Let \( F_1, \ldots, F_m \) be the facets of \( P \). Let \( f_i \) be the normal vector of \( F_i \) and \( g_i = \tau(f_i) \).

For any face \( F \) of \( P \), let \( \phi(F) = \{ i \in [m] \mid F \subset F_i \} \). In other words, \( F = \cap_{i \in \phi(F)} F_i \).

**LEMMA.** The vector configuration \( \{ g_i \mid i \in [m] \} \) is the Gale transform of the vertex set \( \{ a_i \mid i \in [m] \} \) of a \( (m-n+d-1) \)-dimensional (simplicial) polytope \( Q \).

A face \( F \) of \( P \) is strictly preserved by \( \pi \)

\[ \iff \{ g_i \mid i \in \phi(F) \} \text{ is positively spanning} \]
\[ \iff \{ a_i \mid i \in [m] \setminus \phi(F) \} \text{ is a face of } Q. \]

R. Sanyal, Topological obstructions for vertex numbers of Minkowski sums, 2009
Projection preserving the $k$-skeleton of $\Delta_n$
\[\longrightarrow\] simplicial complex embeddable in a certain dimension (Gale duality)
\[\longrightarrow\] topological obstruction (Sarkaria’s criterion).

**THEOREM.** (Topological obstruction for low-dimensional skeleta)
Let $\underline{n} := (n_1, \ldots, n_r)$ and $R := \{i \in [r] \mid n_i \geq 2\}$. If $0 \leq k \leq \sum_{i \in R} \left\lfloor \frac{n_i - 2}{2} \right\rfloor$, then the dimension of any $(k, \underline{n})$-PPSN polytope is at least $2k + |R| + 1$.

**THEOREM.** (Topological obstruction for high-dimensional skeleta)
Let $\underline{n} := (n_1, \ldots, n_r)$. If $k \geq \left\lceil \frac{1}{2} \sum_{i \in [r]} n_i \right\rceil$, then any $(k, \underline{n})$-PPSN polytope is combinatorially equivalent to $\Delta_n$.

B. Matschke, J. Pfeifle & V. P., Prodsimplicial neighborly polytopes, 2010
THANK YOU