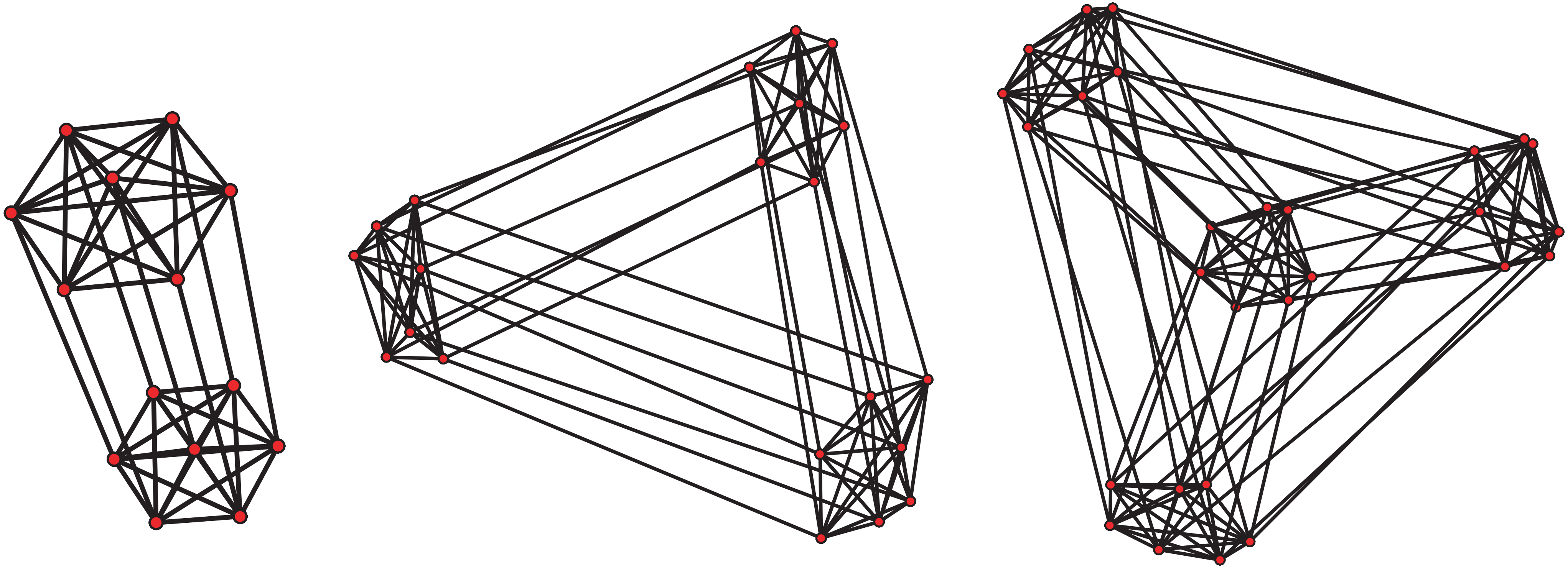


POLYTOPALITY AND CARTESIAN PRODUCTS

Vincent Pilaud (Université Paris 7)

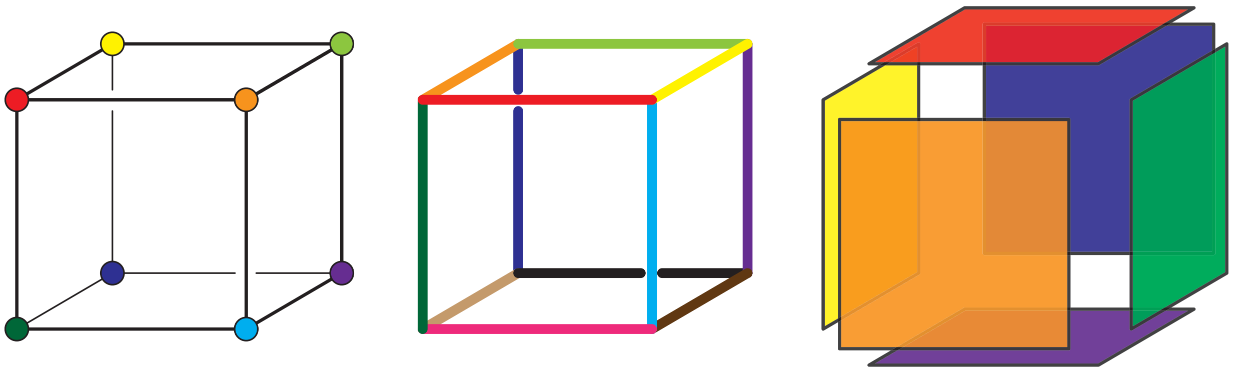
COMBINATORICS OF POLYTOPES
POLYTOPES FROM COMBINATORICS

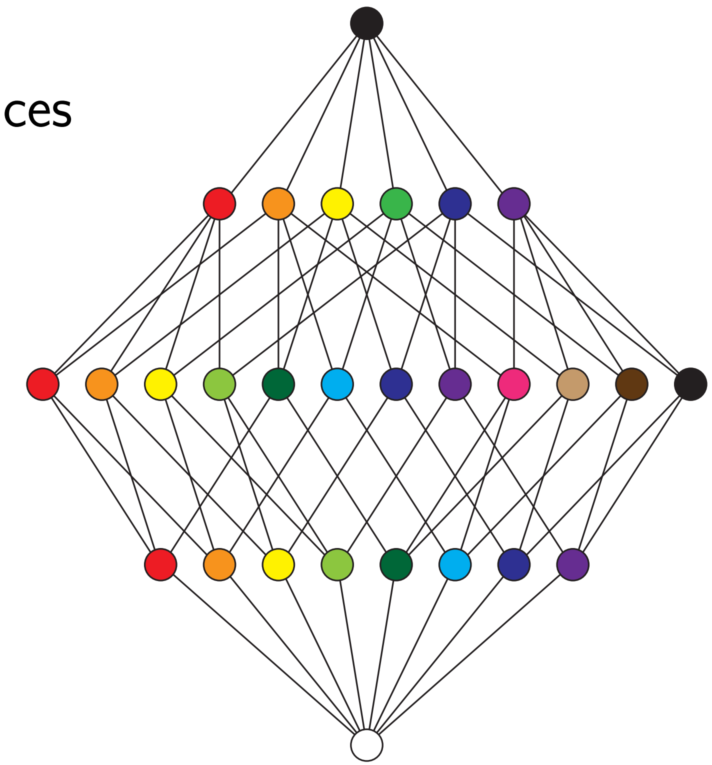
polytope = convex hull of a finite set of $\mathbb{R}^d$

= bounded intersection of finitely many half-spaces

face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations



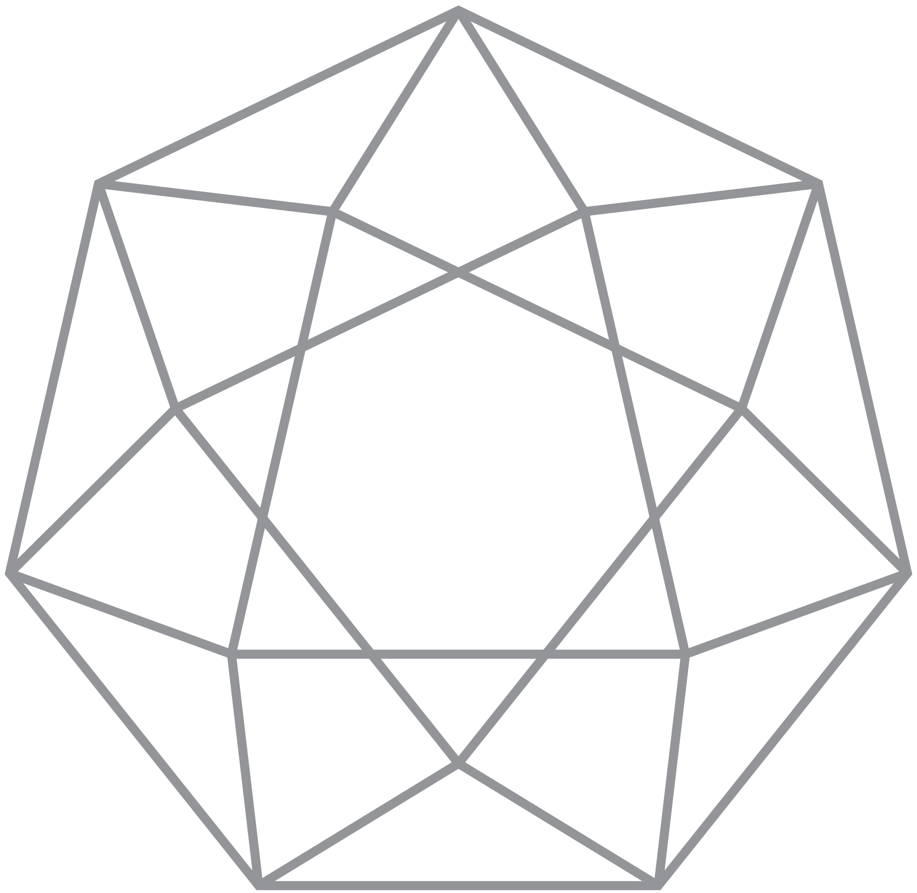


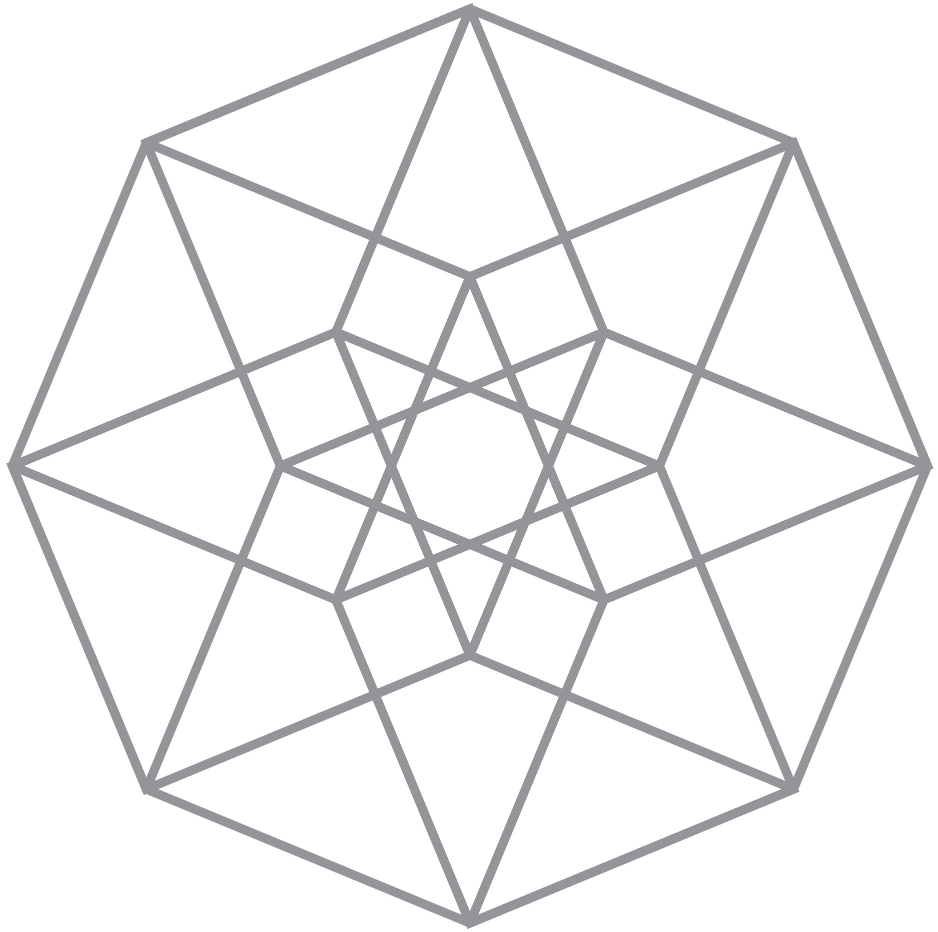
Given a set of points, determine the face lattice of its convex hull.

Given (part of) a face lattice, is there a polytope which realizes it?
In which dimension(s)?

POLYTOPALITY

A graph is d-polytopal if it is the graph of a d-dimensional polytope.

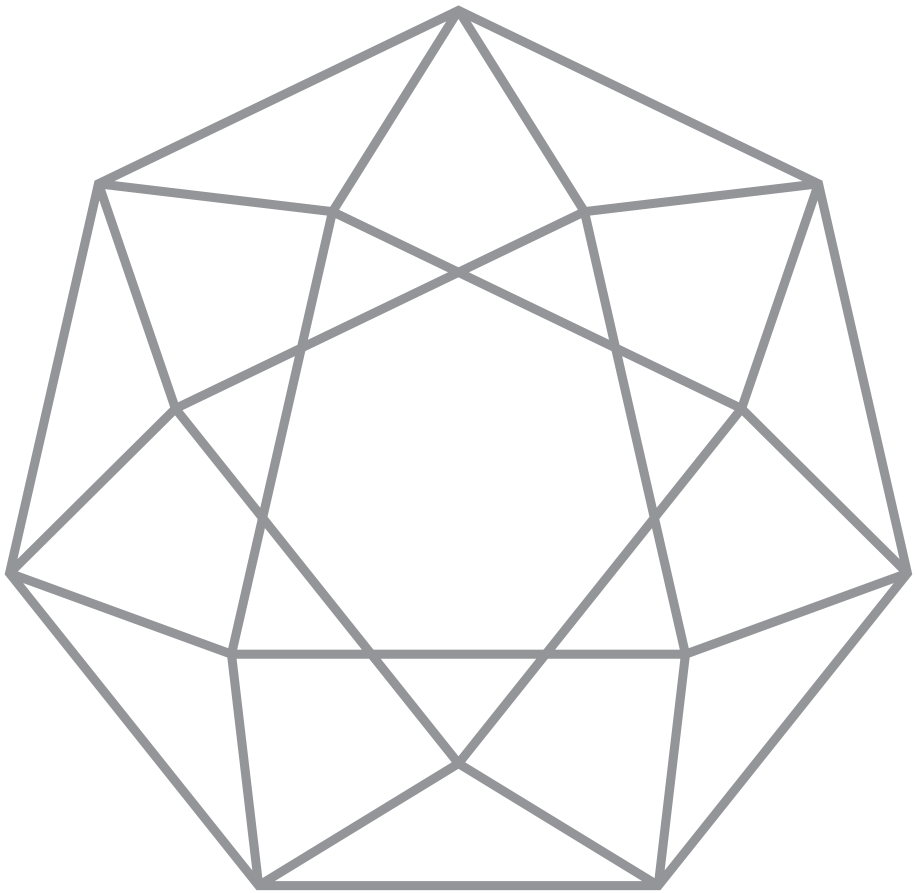


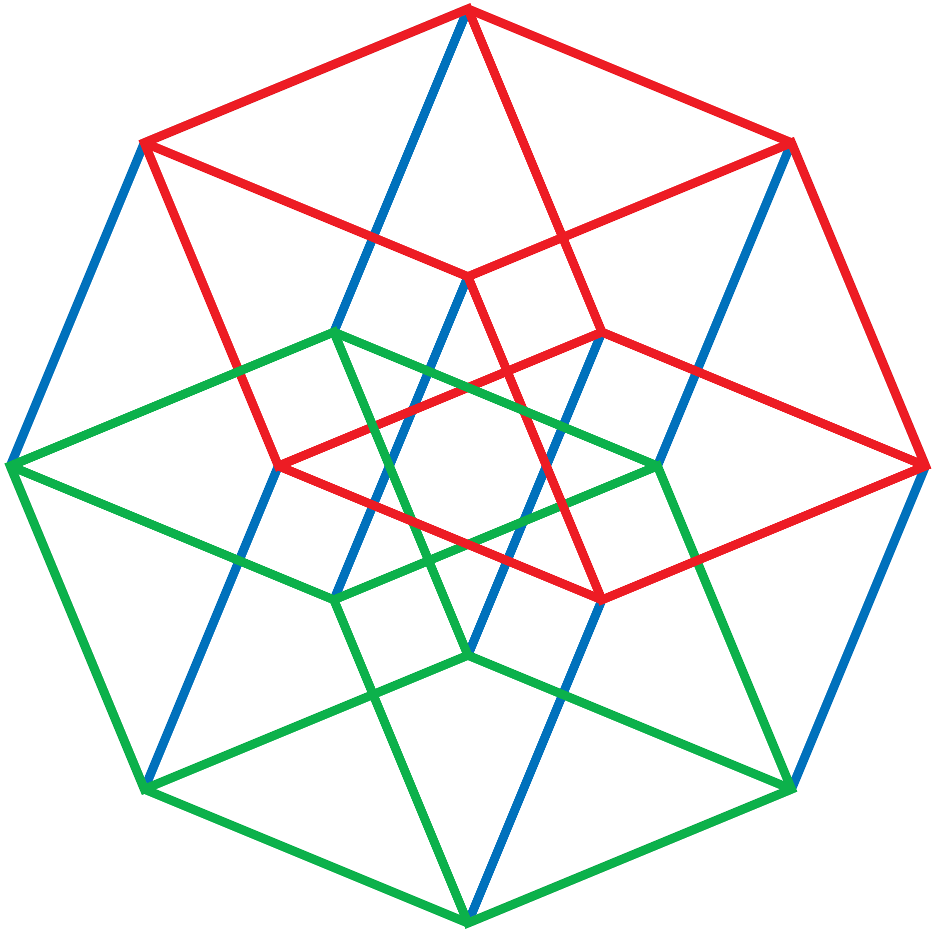


One of these graphs is polytopal. Can you guess which?

POLYTOPALITY

A graph is *d-polytopal* if it is the graph of a d-dimensional polytope.





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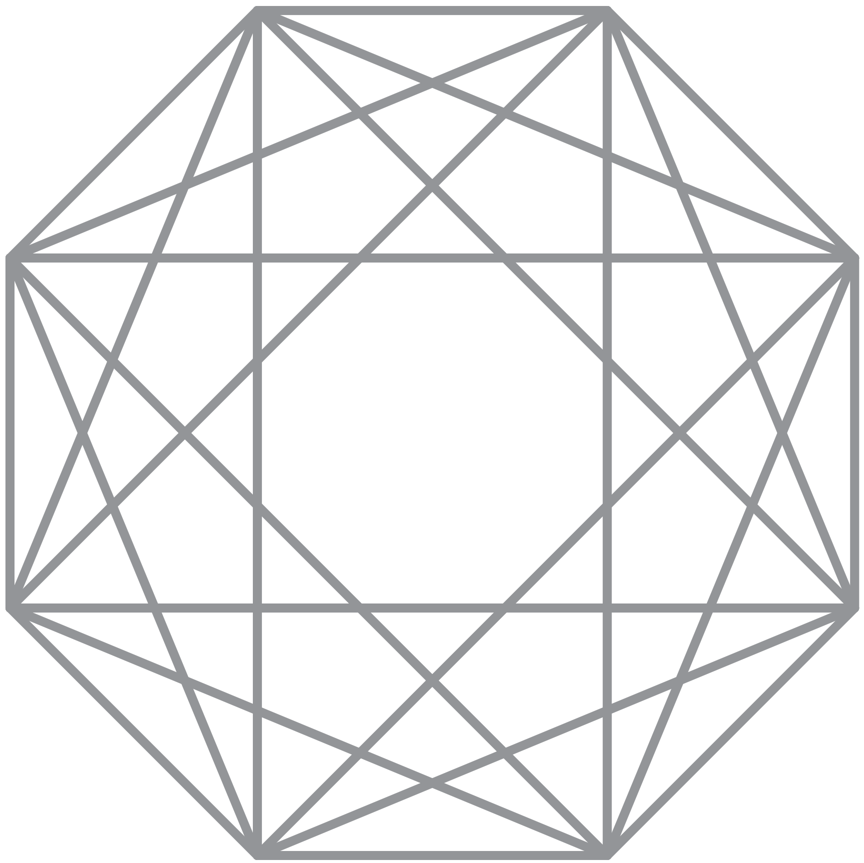
POLYTOPALITY RANGE

The polytopality range of a graph is the set of dimensions in which it is polytopal.



Which dimension can have a polytope with this graph?

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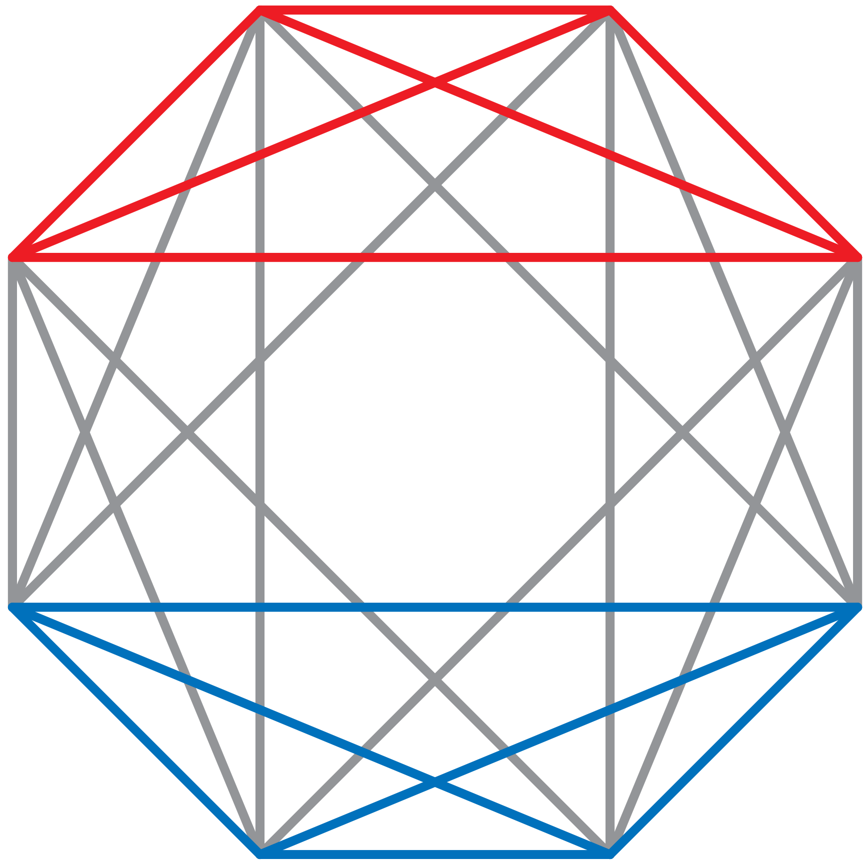

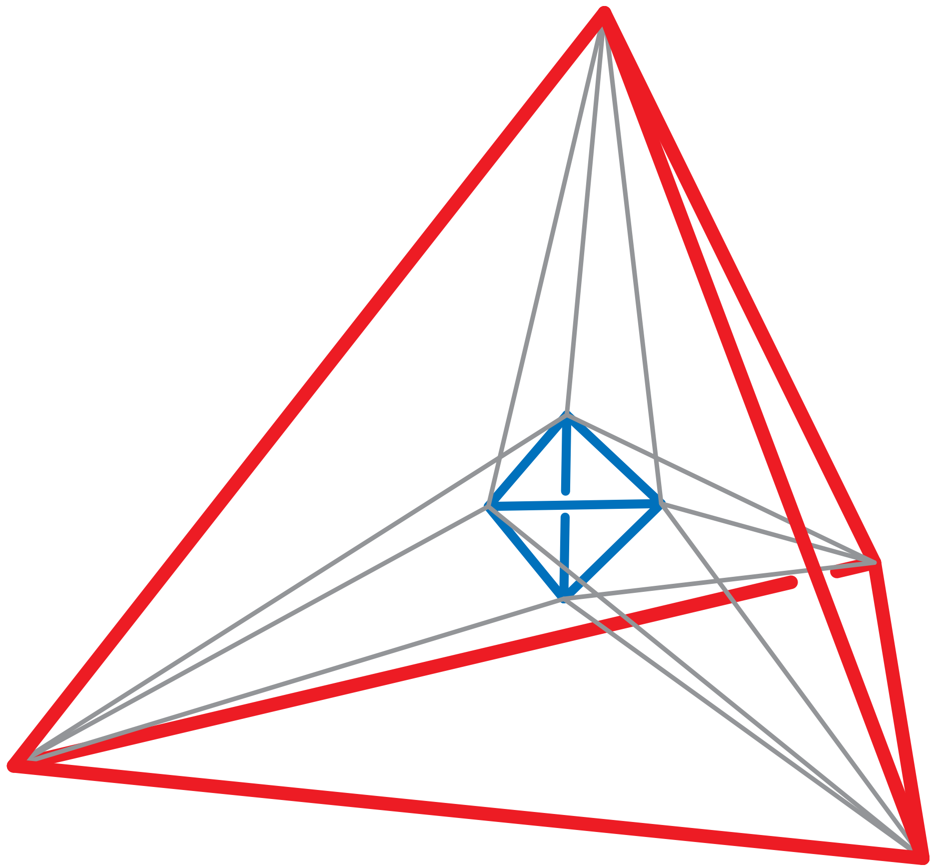

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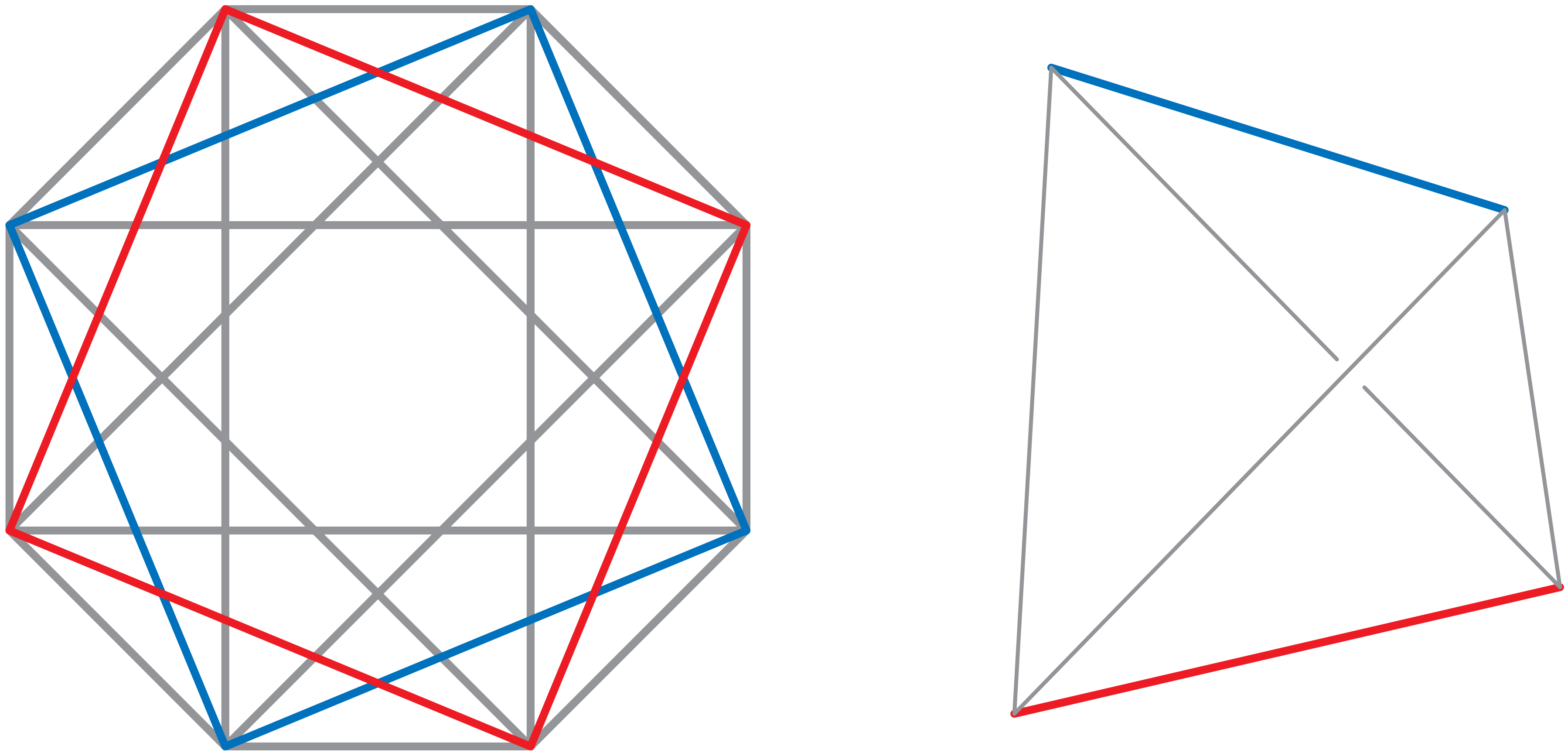


Which dimension can have a polytope with this graph?

POLYTOPALITY OF GRAPHS

GENERAL POLYTOPES

THEOREM. 3-polytopal $\iff$ simple, planar and 3-connected. E. Steinitz 1922

THEOREM. A d-polytopal graph satisfies the following properties:

Balinski's Theorem. G is d-connected. M. Balinski 1961

Principal Subdivision Property. Every vertex of G is the principal vertex of a principal subdivision of K_{d+1} contained in G. D. Barnette 1967

SIMPLE POLYTOPES

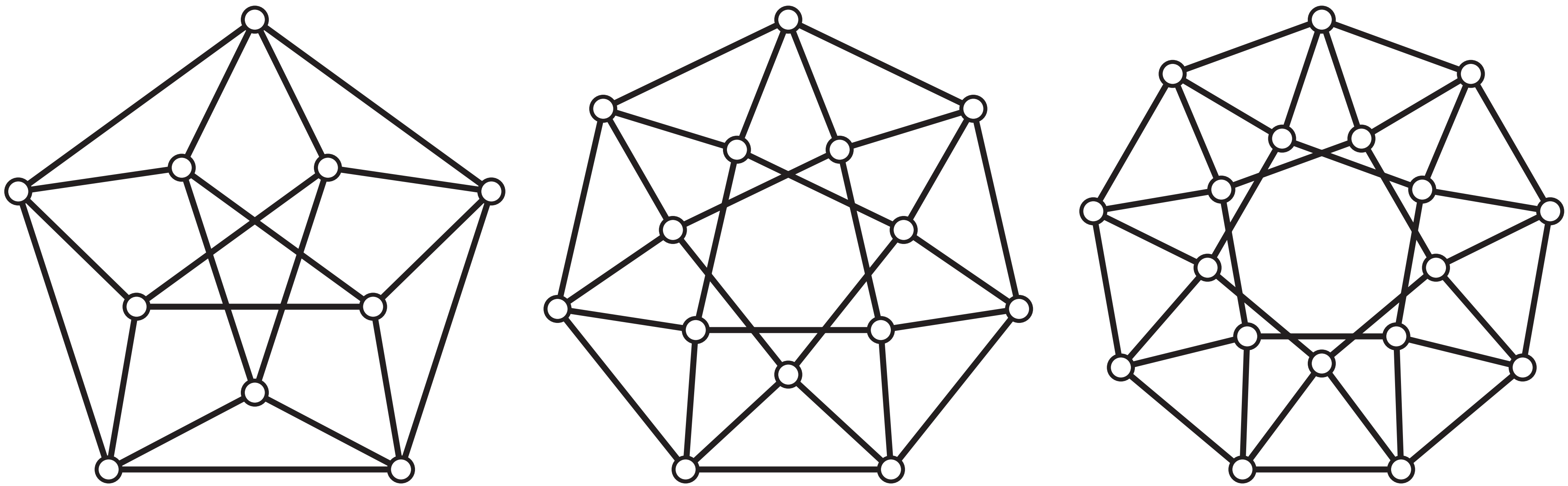
THEOREM. Two simple polytopes are combinatorially equivalent if and only if they have the same graph. R. Blind and P. Mani 1987, G. Kalai 1988

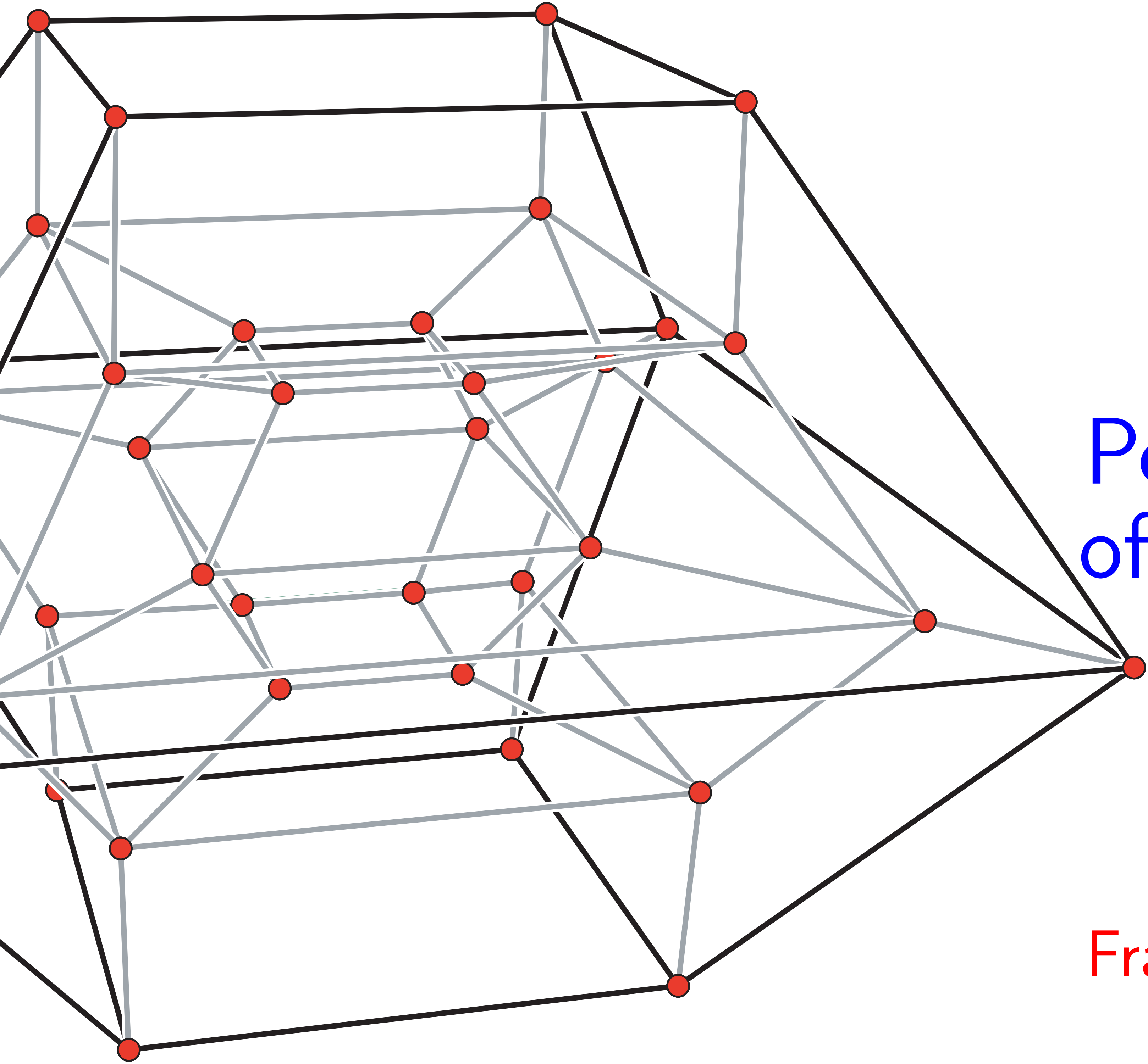
LEMMA. All induced 3-, 4- and 5-cycles in the graph of a simple polytope are 2-faces.

POLYTOPALITY OF GRAPHS

> LEMMA. All induced 3-, 4- and 5-cycles in the graph of a simple polytope are 2-faces.

EXAMPLE. None of the graphs of the following family is polytopal:





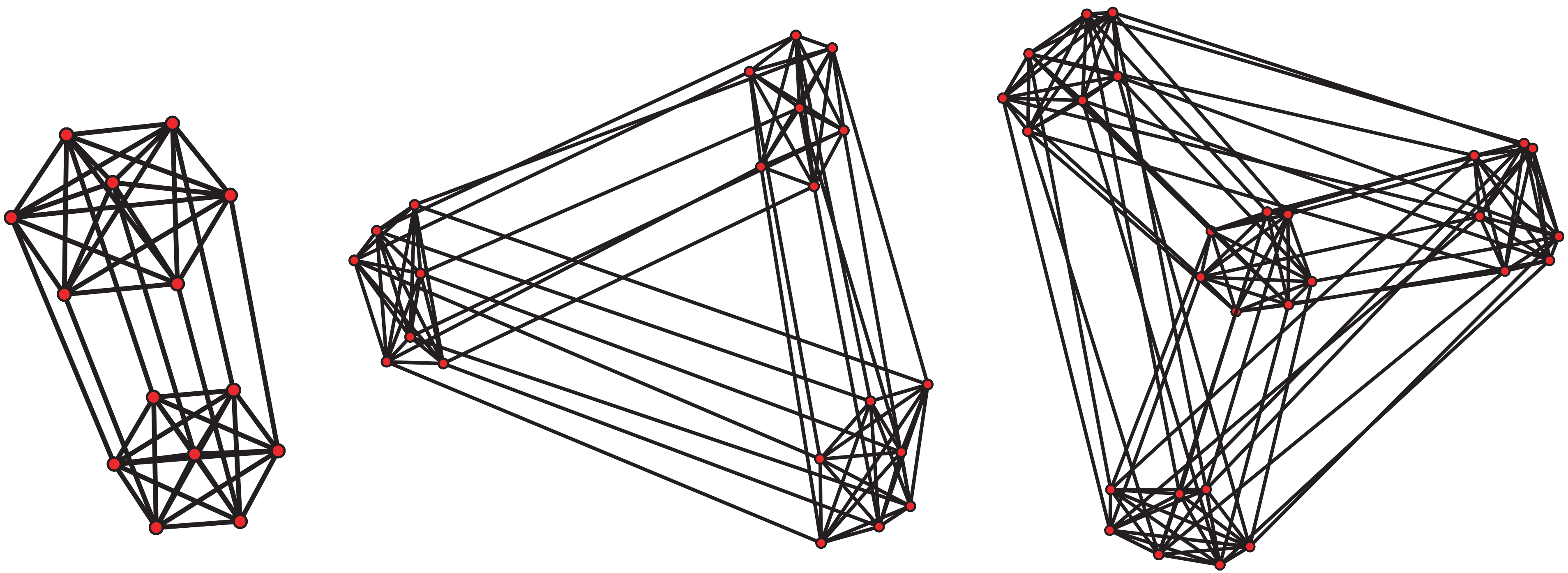
Polytopality of Cartesian products of graphs

Julian Pfeifle
Francisco Santos

CARTESIAN PRODUCTS OF GRAPHS

Cartesian product of polytopes: $P \times Q := \{(p,q) \mid p \in P, q \in Q\}$.

Cartesian product of graphs: $\begin{cases} V(G \times H) := V(G) \times V(H), \\ E(G \times H) := (V(G) \times E(H)) \cup (E(G) \times V(H)). \end{cases}$



REMARK. graph of $P \times Q$ = (graph of P) $\times$ (graph of Q).

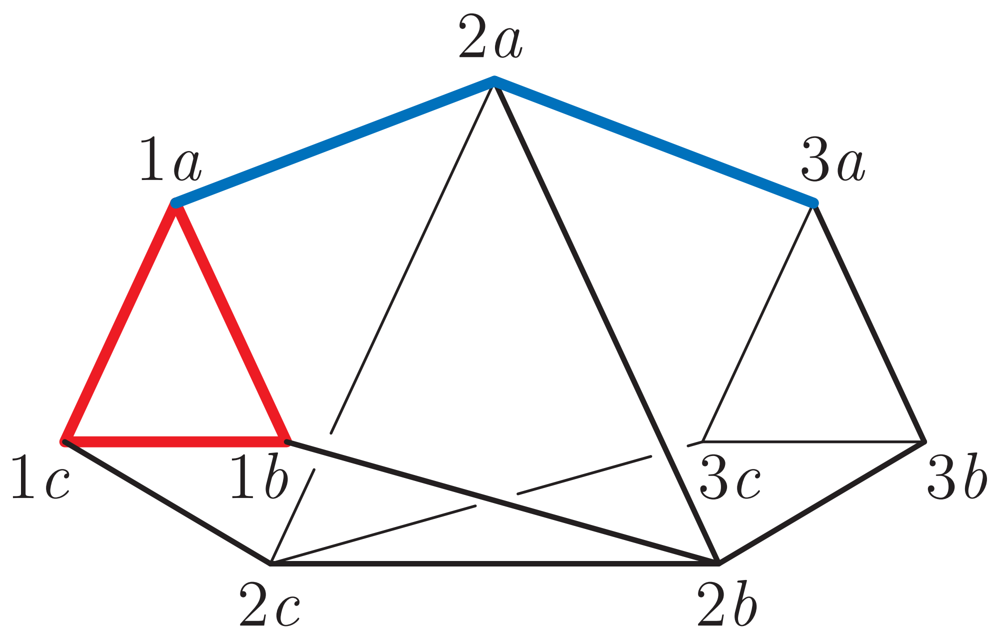
PROBLEM. Does the polytopality of $G \times H$ imply that of G and H?

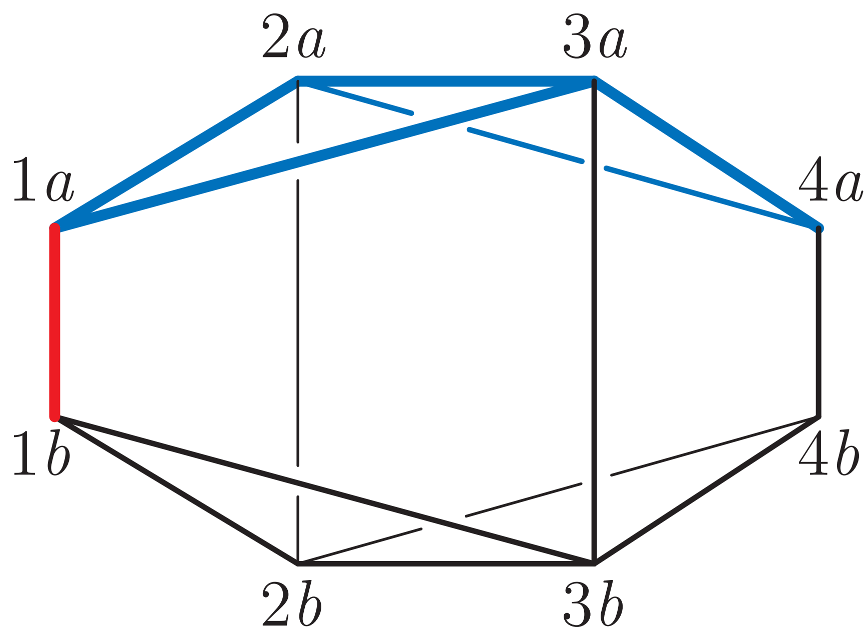
POLYTOPALITY AND CARTESIAN PRODUCTS

PROBLEM. Does the polytopality of $G \times H$ imply that of G and H?

THEOREM. $G \times H$ simply polytopal $\iff$ G and H simply polytopal.

THEOREM. The product of a d-polytopal graph by the graph of a regular subdivision of an e-polytope is $(d+e)$-polytopal.



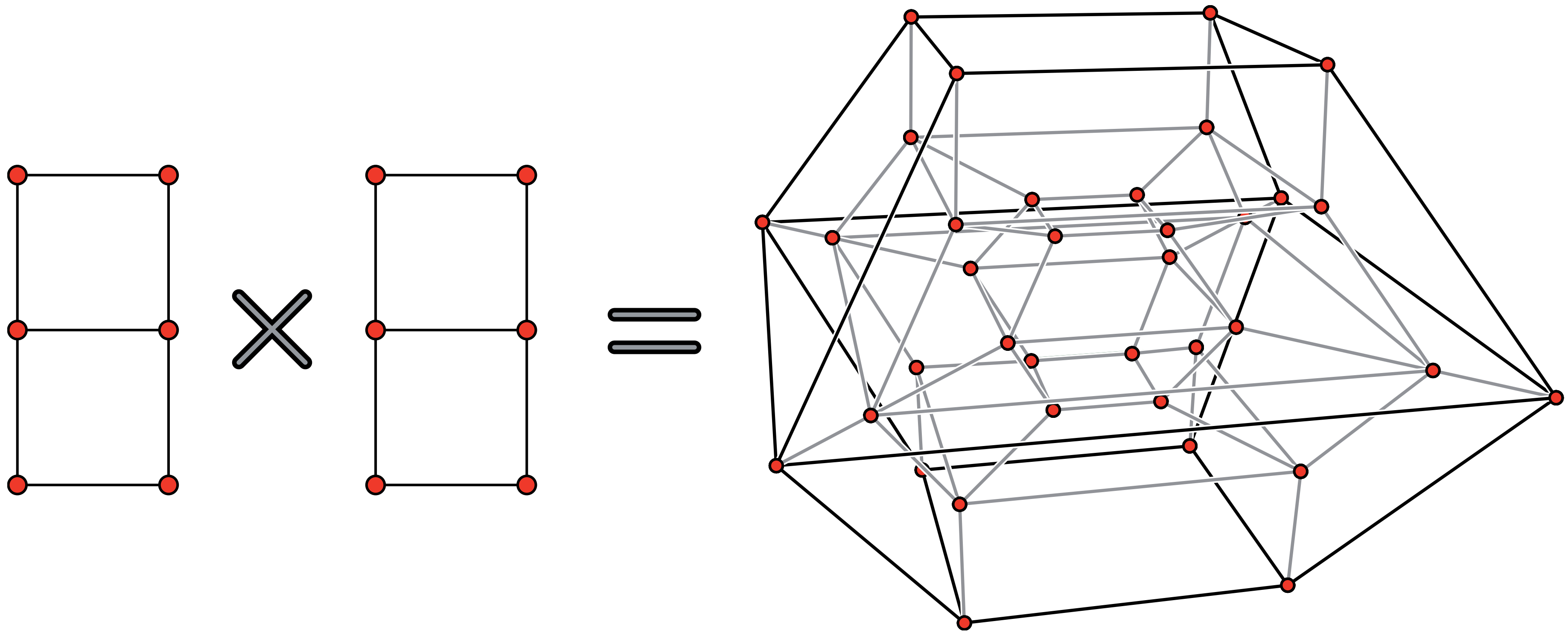


J. Pfeifle, V. P. & F. Santos, On polytopality of Cartesian products of graphs, 2010

POLYTOPALITY AND CARTESIAN PRODUCTS

THEOREM. The product of a d-polytopal graph by the graph of a regular subdivision of an e-polytope is $(d+e)$-polytopal.

EXAMPLE. The product of two domino graphs is polytopal.

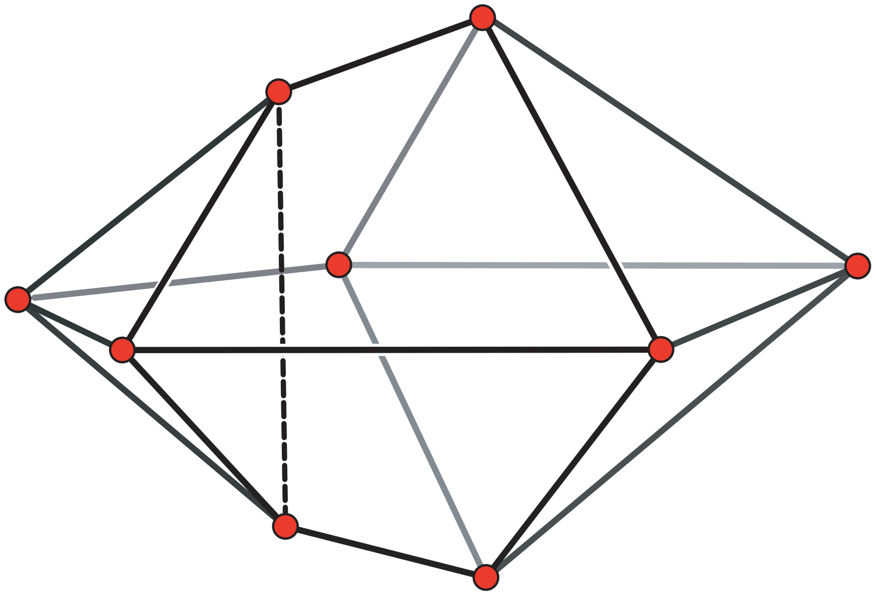


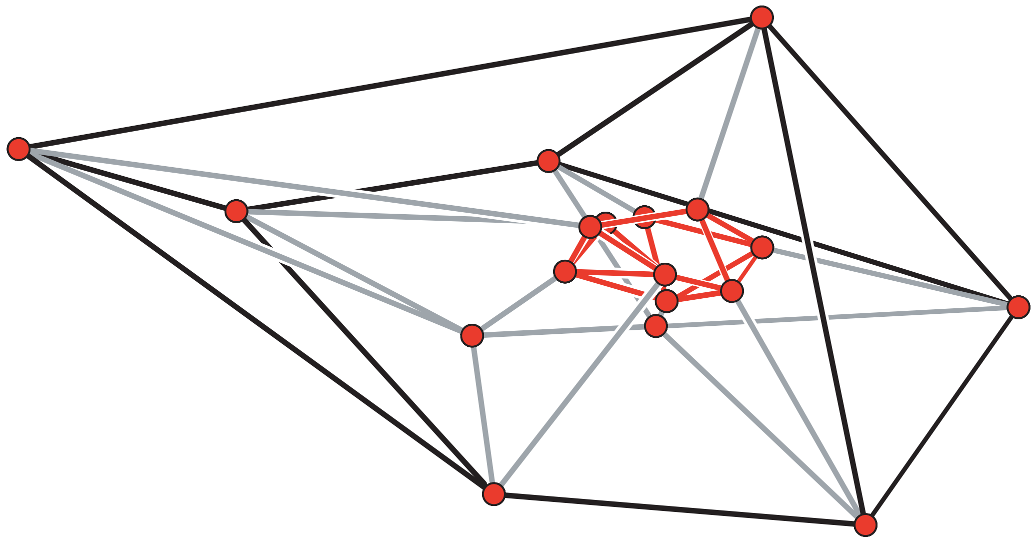
J. Pfeifle, V. P. & F. Santos, On polytopality of Cartesian products of graphs, 2010

POLYTOPALITY AND CARTESIAN PRODUCTS

THEOREM. The product of a d-polytopal graph by the graph of a regular subdivision of an e-polytope is $(d+e)$-polytopal.

EXAMPLE. Polytopal product of regular non-polytopal graphs.



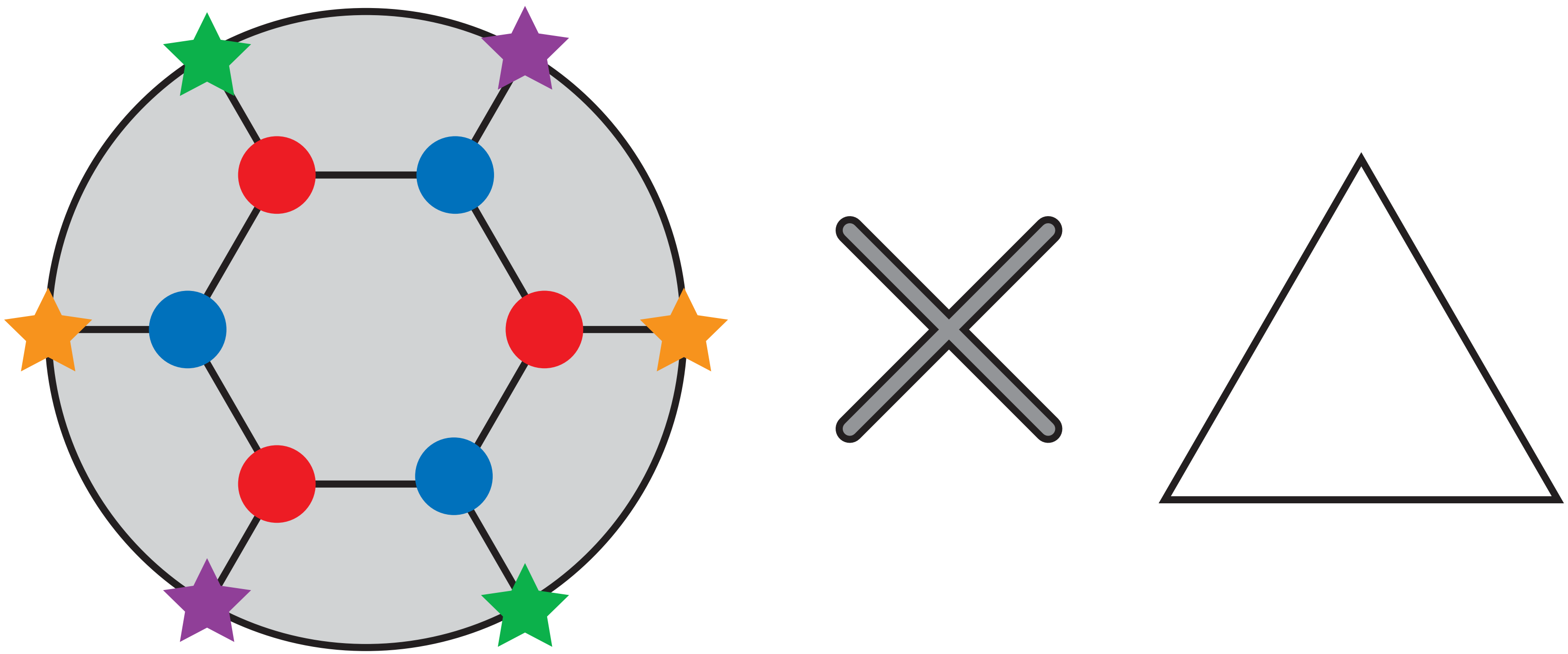


J. Pfeifle, V. P. & F. Santos, On polytopality of Cartesian products of graphs, 2010

SOME CHALLENGING EXAMPLES

THEOREM. The graph $K_{n,n} \times K_2$ is not polytopal for $n \geq 3$.

THEOREM. There is a unique combinatorial 3-dimensional manifold whose graph is $K_{3,3} \times K_3$. It is homeomorphic to $\mathbb{RP}^2 \times \mathbb{S}^1$.



A. Guedes de Oliveira, E. Kim, M. Noy, A. Padrol, J. Pfeifle & V. P.

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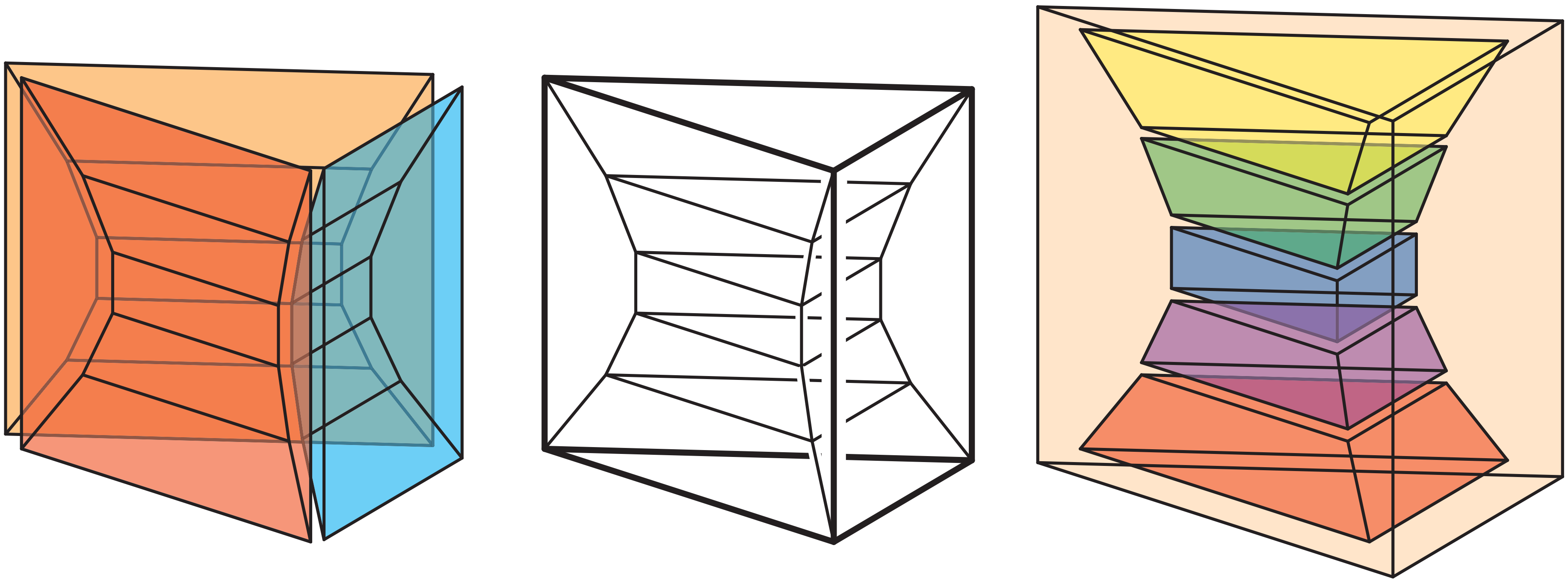

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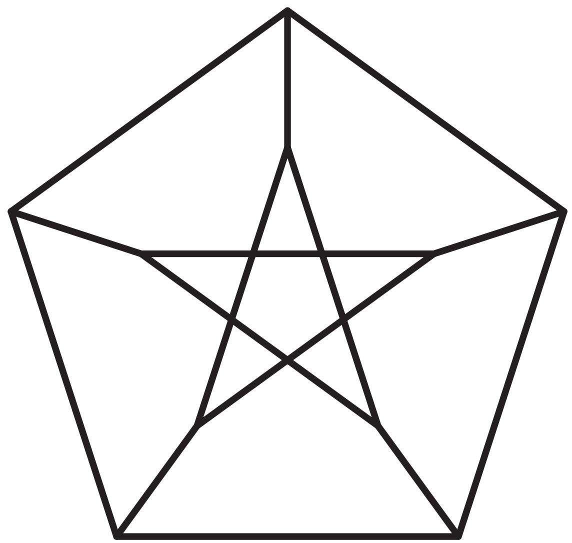
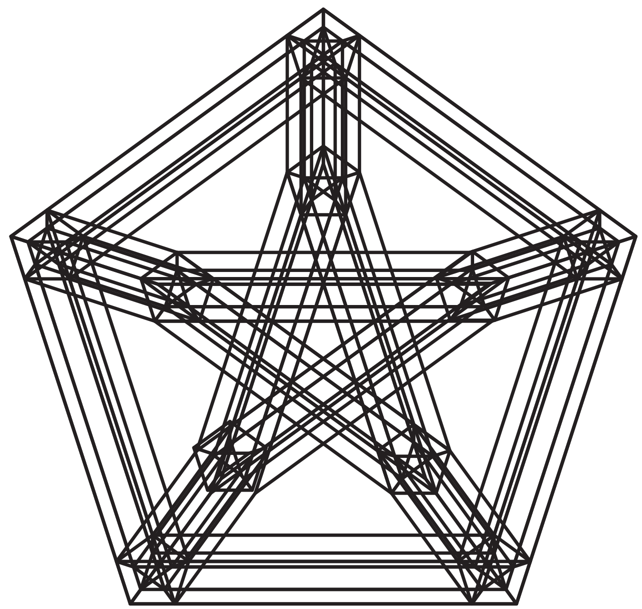
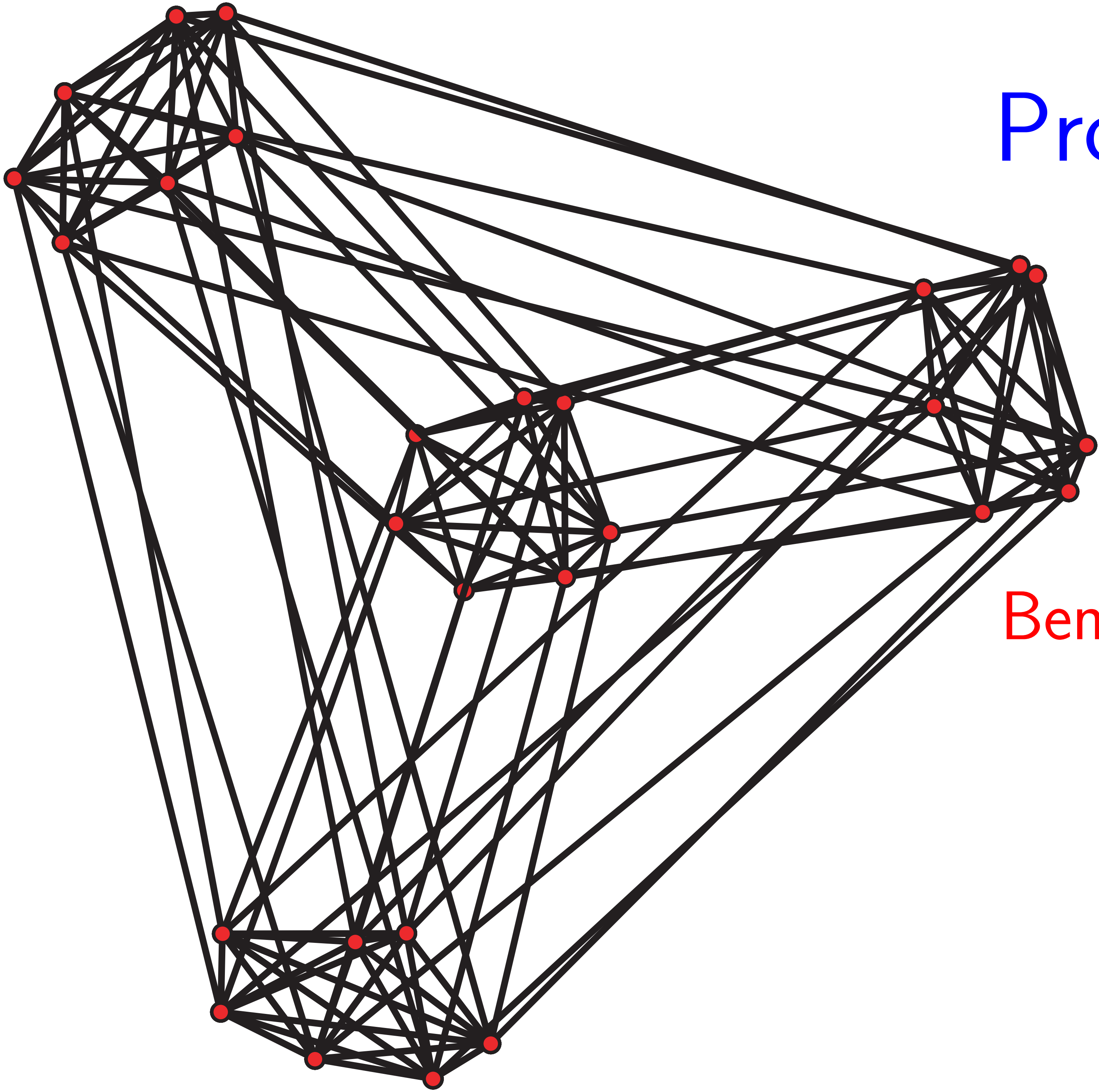
PROBLEM. Is the product of two Petersen graphs the graph of a polytope?





This polytope could have dimension 4 or 5.



Prodsimplicial Neighborly Polytopes

Benjamin Matschke
Julian Pfeifle

PRODSIMPLICIAL NEIGHBORLY POLYTOPES

$k \geq 0$ and $\underline{n} := (n_1, ..., n_r)$.

A polytope is $(k, \underline{n})$-prodsimplicial-neighborly if its k-skeleton is combinatorially equivalent to that of the product of simplices $\triangle_{\underline{n}} := \triangle_{n_1} \times \cdots \times \triangle_{n_r}$.

EXAMPLE.

(i) neighborly polytopes arise when $r = 1$.
For example, the cyclic polytope $C_{2k+2}(n+1)$ is (k, n)-PSN.

(ii) neighborly cubical polytopes arise when $\underline{n} = (1, 1, \ldots, 1)$.

M. Joswig & G. Ziegler, Neighborly cubical polytopes, 2000

PROBLEM. What is the minimal dimension of a $(k, \underline{n})$-PSN polytope?

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PROBLEM. What is the minimal dimension of a (k, n)-PSN polytope?

A $(k, \underline{n})$-PSN polytope is $(k, \underline{n})$-projected-prodsimplicial-neighborly if it is a projection of a polytope combinatorially equivalent to $\triangle_{\underline{n}}$.

PROBLEM. What is the minimal dimension of a (k, n)-PPSN polytope?

PRODUCT OF CYCLIC POLYTOPES

$C_d(n) := \mathrm{conv}\,\{\mu_d(t_i) \mid i \in [n]\}$ the d-dimensional cyclic polytope with n vertices, where $\mu_d(t) = (t, t^2, \ldots, t^d)^T$ and $t_1, t_2, \ldots, t_n \in \mathbb{R}$ distinct.

PROPOSITION. Any subset of at most $\left\lfloor \frac{d}{2} \right\rfloor$ vertices of $C_d(n)$ forms a face of $C_d(n)$.

$F \subset [n]$ defines a facet of $C_d(n) \iff |F| = d$ and all inner blocs are even.
The normal vector of this facet is given by the coefficients of the polytope

$$\prod_{i \in F}(t - t_i) = \sum_{i=1}^{d} \gamma_i(F) t^i = \begin{pmatrix} \gamma_1(F) \\ \vdots \\ \gamma_d(F) \end{pmatrix} \cdot \begin{pmatrix} t^1 \\ \vdots \\ t^d \end{pmatrix} + \gamma_0(F).$$

PROPOSITION. Let $k \geq 0$ and $\underline{n} := (n_1, \ldots, n_r)$. Let $I := \{i \in [n] \mid n_i \geq 2k+3\}$. The product

$$\prod_{i \in I} C_{2k+2}(n_i + 1) \times \prod_{i \notin I} \triangle_{n_i}$$

is a $(k, \underline{n})$-PPSN polytope of dimension $(2k+2)|I| + \sum_{i \notin I} n_i \leq (2k+2)r$.

PROPOSITION. Let $k \geq 0$ and $\underline{n} := (n_1, ..., n_r)$. Define

$$v_{a_1,\dots,a_r} := \begin{pmatrix} \sum_{i\in[r]} a_i \\ \sum_{i\in[r]} a_i^2 \\ \vdots \\ \sum_{i\in[r]} a_i^{2k+2r} \end{pmatrix} \in \mathbb{R}^{2k+2r}.$$

For any pairwise disjoint index sets $I_1, \dots, I_r \subset \mathbb{R}$, with $|I_i| = n_i$ for all $i \in [r]$, the polytope $\operatorname{conv}\{v_{a_1,\dots,a_r} \mid (a_1, \dots, a_r) \in I_1 \times \cdots \times I_r\} \subset \mathbb{R}^{2k+2r}$ is a $(k, \underline{n})$-PPSN $(2k + 2r)$-dimensional polytope.

PROPOSITION. Let $k \geq 0$ and $\underline{n} := (n_1, ..., n_r)$. Define

$$w_{a_1,\dots,a_r} := \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \sum_{i\in[r]} a_i^2 \\ \vdots \\ \sum_{i\in[r]} a_i^{2k+2} \end{pmatrix} \in \mathbb{R}^{2k+r+1}.$$

There exists pairwise disjoint index sets $I_1, \dots, I_r \subset \mathbb{R}$, with $|I_i| = n_i$ for all $i \in [r]$, such that the polytope $\operatorname{conv}\{w_{a_1,\dots,a_r} \mid (a_1, \dots, a_r) \in I_1 \times \cdots \times I_r\} \subset \mathbb{R}^{2k+r+1}$ is a $(k, \underline{n})$-PPSN $(2k+r+1)$-dimensional polytope.

B. Matschke, J. Pfeifle & V. P., Prodsimplicial neighborly polytopes, 2010

PRESERVING FACES UNDER PROJECTIONS

$n > d$.

$\pi : \mathbb{R}^n \to \mathbb{R}^d$ the orthogonal projection on the first d coordinates.

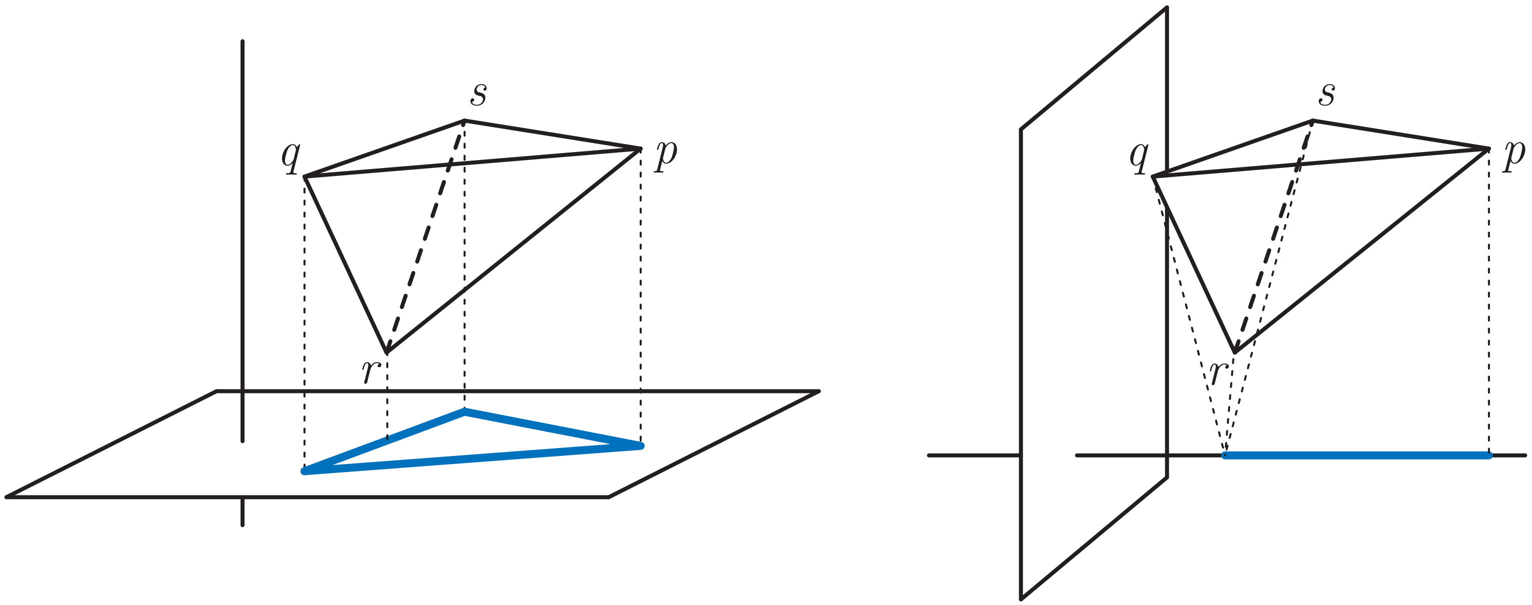
$\tau : \mathbb{R}^n \to \mathbb{R}^{n-d}$ the dual projection on the last $n - d$ coordinates.

A proper face F of a polytope P is strictly preserved under π if:

(i) $\pi(F)$ is a face of $\pi(P)$,

(ii) F and $\pi(F)$ are combinatorially isomorphic, and

(iii) $\pi^{-1}(\pi(F))$ equals F.



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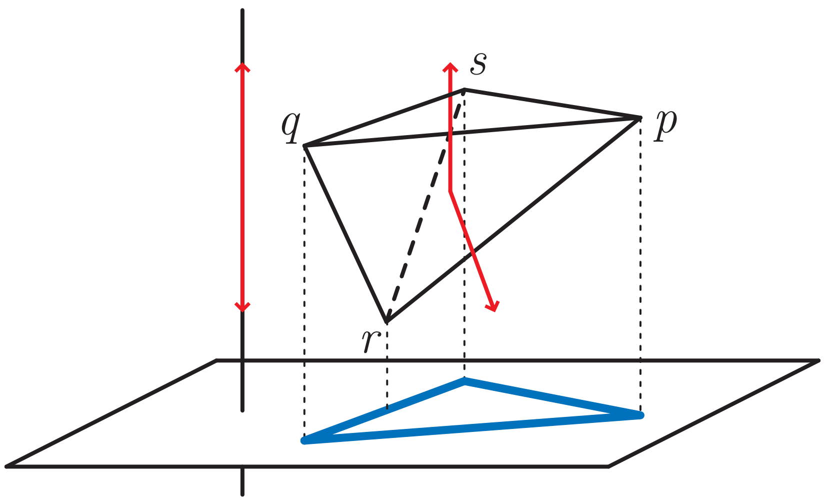
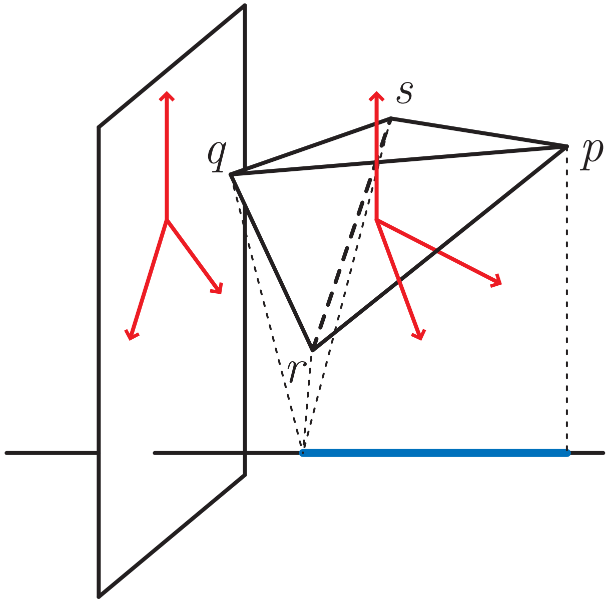
Let $F_1, \dots, F_m$ be the facets of P. Let f_i be the normal vector of F_i and $g_i = \tau(f_i)$.
For any face F of P, let $\phi(F) = \{i \in [m] \mid F \subset F_i\}$. In other words, $F = \cap_{i \in \phi(F)} F_i$.

LEMMA. F face of P is strictly preserved $\iff$ $\{g_i \mid i \in \phi(F)\}$ is positively spanning.

N. Amenta & G. Ziegler, Deformed products and maximal shadows of polytopes, 1999
G. Ziegler, Projected products of polytopes, 2004





DEFORMED PRODUCTS

$P_1, \dots, P_r$ simple polytopes, with facet description:

$$P_i := \{x \in \mathbb{R}^{n_i} \mid A_i x \leq b_i\}\,, \text{ where } A_i \in \mathbb{R}^{m_i \times n_i} \text{ and } b_i \in \mathbb{R}^{m_i}.$$

The product $P := P_1 \times \cdots \times P_r$ has dimension $\sum_{i \in [r]} n_i$ and is defined by the $\sum_{i \in [r]} m_i$ inequalities:

$$\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_r \end{pmatrix} x \leq \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}.$$

THEOREM. (DEFORMED PRODUCT CONSTRUCTION)

For any matrix $A^{\sim} := \begin{pmatrix} A_1 & \star & \star \\ & \ddots & \star \\ & & A_r \end{pmatrix}$ obtained by arbitrarily changing the 0's above the diagonal blocs, there exists $b^{\sim}$ such that the polytope defined by $A^{\sim} x \leq b^{\sim}$ is combinatorially equivalent to $P_1 \times \cdots \times P_r$.

N. Amenta & G. Ziegler, Deformed products and maximal shadows of polytopes, 1999

PROJECTED DEFORMED PRODUCTS

IDEA. Use your freedom on the upper part of the matrix $A^\sim$ to obtain a polytope $P^\sim := \left\{x \in \mathbb{R}^{\sum n_i} \;\middle|\; A^\sim x \leq b^\sim\right\}$ such that:

(i) $P^\sim$ is a deformed product combinatorially equivalent to $P := P_1 \times \cdots \times P_r$; and

(ii) the projection of $P^\sim$ on the first d coordinates preserves its k-skeleton.

EXAMPLE. Let $P_1, \dots, P_r$ be r simple polytopes of respective dimension n_i and with m_i many facets. If $d = \sum_{i \in [t]} n_i$, then there exists a d-dimensional polytope whose k-skeleton is combinatorially equivalent to that of $P_1 \times \cdots \times P_r$ provided

$$k \leq \sum_{i \in [r]} n_i - \sum_{i \in [r]} m_i + \left\lfloor \frac{\sum_{i \in [t]} m_i - 1}{2} \right\rfloor .$$

For improvements, see

B. Matschke, J. Pfeifle & V. P., Prodsimplicial neighborly polytopes, 2010

SANYAL'S TOPOLOGICAL OBSTRUCTION METHOD

$n > d$.
$\pi : \mathbb{R}^n \to \mathbb{R}^d$ the orthogonal projection on the first d coordinates.
$\tau : \mathbb{R}^n \to \mathbb{R}^{n-d}$ the dual projection on the last $n-d$ coordinates.

Let P be a simple full-dimensional polytope whose vertices are strictly preserved by π.
Let $F_1, \dots, F_m$ be the facets of P. Let f_i be the normal vector of F_i and $g_i = \tau(f_i)$.
For any face F of P, let $\phi(F) = \{i \in [m] \mid F \subset F_i\}$. In other words, $F = \cap_{i\in\phi(F)} F_i$.

LEMMA. The vector configuration $\{g_i \mid i \in [m]\}$ is the Gale transform of the vertex set $\{a_i \mid i \in [m]\}$ of a $(m-n+d-1)$-dimensional (simplicial) polytope Q.

A face F of P is strictly preserved by π

$$\iff \{g_i \mid i \in \phi(F)\} \text{ is positively spanning}$$
$$\iff \{a_i \mid i \in [m] \smallsetminus \phi(F)\} \text{ is a face of } Q.$$

R. Sanyal, Topological obstructions for vertex numbers of Minkowski sums, 2009

SANYAL'S TOPOLOGICAL OBSTRUCTION METHOD

Projection preserving the k-skeleton of $\triangle_{\underline{n}}$

$\longmapsto$ simplicial complex embeddable in a certain dimension (Gale duality)

$\longmapsto$ topological obstruction (Sarkaria's criterion).

THEOREM. (Topological obstruction for low-dimensional skeleta)
Let $\underline{n} := (n_1, \dots, n_r)$ and $R := \{i \in [r] \mid n_i \geq 2\}$. If $0 \leq k \leq \sum_{i \in R} \left\lfloor \frac{n_i - 2}{2} \right\rfloor$, then the dimension of any $(k, \underline{n})$-PPSN polytope is at least $2k + |R| + 1$.

THEOREM. (Topological obstruction for high-dimensional skeleta)
Let $\underline{n} := (n_1, \dots, n_r)$. If $k \geq \left\lfloor \frac{1}{2} \sum_{i \in [r]} n_i \right\rfloor$, then any $(k, \underline{n})$-PPSN polytope is combinatorially equivalent to $\triangle_{\underline{n}}$.

B. Matschke, J. Pfeifle & V. P., Prodsimplicial neighborly polytopes, 2010

THANK YOU