The greedy flip tree of a subword complex

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REDUCED EXPRESSIONS & SUBWORD COMPLEXES
$S_n$ = symmetric group
$S = \{\tau_i \mid i \in n - 1\}$ set of simple transpositions $\tau_i = (i \ i + 1)$

$\rho$ permutation of $S_n$
Reduced expression of $\rho = \text{minimal length expression } \rho = s_1 \cdots s_\ell$ with $s_i \in S$

Count and enumerate reduced expressions of $\rho$

Example. $\rho = [4, 1, 3, 2] = \tau_2 \ \tau_3 \ \tau_2 \ \tau_1 = \tau_3 \ \tau_2 \ \tau_3 \ \tau_1 = \tau_3 \ \tau_2 \ \tau_1 \ \tau_3$
\[ S_n = \text{symmetric group} \]
\[ S = \{ \tau_i \mid i \in n - 1 \} \text{ set of simple transpositions } \tau_i = (i \ i + 1) \]
\[ \rho \text{ permutation of } S_n \]
\[ \text{reduced expression of } \rho = \text{minimal length expression } \rho = s_1 \cdots s_\ell \text{ with } s_i \in S \]

Count and enumerate reduced expressions of \( \rho \)

\[
\# \text{ reduced expressions of } w_\circ = \frac{(n)!}{1^{n-1}2^{n-2} \cdots (2n - 3)^1}
\]

Stanley.
On the number of reduced decompositions of elements of Coxeter groups. 1984

Edelmann & Greene.
Combinatorial correspondences for Young tableaux, balanced tableaux, and maximal chains in the Bruhat order of \( S_n \). 1984
REDUCED EXPRESSIONS AS SUBWORDS

\( \mathfrak{S}_n = \) symmetric group
\( S = \{\tau_i \mid i \in n - 1\} \) set of simple transpositions \( \tau_i = (i \ i + 1) \)
\( \rho \) permutation of \( \mathfrak{S}_n \)
\( Q = q_1 q_2 \cdots q_m \) word on the alphabet \( S \)

Enumerate subwords of \( Q \) which are reduced expressions for \( \rho \)

Example. \( \rho = [4, 1, 3, 2] = \tau_2 \tau_3 \tau_2 \tau_1 = \tau_3 \tau_2 \tau_3 \tau_1 = \tau_3 \tau_2 \tau_1 \tau_3 \)
\( Q = \tau_2 \tau_3 \tau_1 \tau_3 \tau_2 \tau_1 \tau_2 \tau_3 \tau_1 \)

Possible subwords:

\[
\begin{align*}
\tau_2 \tau_3 \cdots \tau_2 \tau_1 & \cdots \rightarrow 34789 \\
\tau_2 \tau_3 \cdots \cdots \tau_2 \tau_1 & \rightarrow 34568 \\
\cdot \tau_3 \cdots \tau_2 \cdots \tau_3 \tau_1 & \rightarrow 13467 \\
\cdot \tau_3 \cdots \tau_2 \tau_1 \cdot \tau_3 \cdot & \rightarrow 13479 \\
\text{etc}
\end{align*}
\]
REDUCED EXPRESSIONS AS SUBWORDS

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\( Q = \tau_2 \tau_3 \tau_1 \tau_3 \tau_2 \tau_1 \tau_2 \tau_3 \tau_1 \)

\[
\begin{array}{cccccccc}
4 & & & & & & & 2 \\
3 & & & & & & & 3 \\
2 & & & & & & & 1 \\
1 & & & & & & & 4
\end{array}
\]
\( W \) = finite Coxeter group
\( S \) = simple system of generators for \( W \)
\( \rho \) element of \( W \)
\( Q = q_1 q_2 \cdots q_m \) word on the alphabet \( S \)

Enumerate subwords of \( Q \) which are reduced expressions for \( \rho \)
\( \mathfrak{S}_n = \text{symmetric group} \)
\( S = \{ \tau_i \mid i \in n - 1 \} \) set of simple transpositions \( \tau_i = (i \ i + 1) \)
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\( Q = q_1 q_2 \cdots q_m \) word on the alphabet \( S \)

**Subword complex** \( \mathcal{SC}(Q, \rho) = \text{simplicial complex with} \)
- vertices = \( [m] = \text{positions in the word } Q \)
- facets = \( \mathcal{F}(Q, \rho) = \text{complements in } [m] \text{ of position sets of reduced expressions of } \rho \text{ in } Q \)

\(\text{flip} = \text{two subwords of } Q \text{ which differ at precisely two positions}\)
The flip graph is connected

GOAL: Find a natural spanning tree of the flip graph
$Q = q_1 q_2 \cdots q_{m-1} q_m$ and $Q^\perp = q_1 q_2 \cdots q_{m-1}$

$\mathcal{F}(Q, \rho) = \text{facets of } \mathcal{S}\mathcal{C}(Q, \rho) = \text{complements of reduced expressions of } \rho \text{ in } Q$

\[
\mathcal{F}(Q, \rho) = \mathcal{F}(Q^\perp, \rho q_m) \sqcup (\mathcal{F}(Q^\perp, \rho) \ast m)
\]
\[ \mathcal{F}(Q, \rho) = \mathcal{F}(Q_\perp, \rho q_m) \sqcup (\mathcal{F}(Q_\perp, \rho) \ast m) \]
\( Q = q_1 q_2 \cdots q_{m-1} q_m \) and \( Q^\perp = q_1 q_2 \cdots q_{m-1} \)

\( \mathcal{F}(Q, \rho) = \) facets of \( \mathcal{SC}(Q, \rho) = \) complements of reduced expressions of \( \rho \) in \( Q \)

\[
\mathcal{F}(Q, \rho) = \begin{cases} 
\mathcal{F}(Q^\perp, \rho q_m) \\
\mathcal{F}(Q^\perp, \rho) \star m \\
\mathcal{F}(Q^\perp, \rho q_m) \sqcup (\mathcal{F}(Q^\perp, \rho) \star m) 
\end{cases}
\]

\[
\text{if } \rho \not\preceq Q^\perp \\
\text{if } \ell(\rho q_m) > \ell(\rho) \\
\text{otherwise}
\]

\[\Rightarrow \text{ Inductive enumeration of } \mathcal{F}(Q, \rho) \text{ with complexity } O(m^2 n) \text{ per facet}\]
COMBINATORIAL MODELS FOR GEOMETRIC GRAPHS
bijection between

- triangulations of a convex \((n + 2)\)-gon
- subwords of the odd-even word \(Q = \left( \prod_{i \in \left[ \frac{n}{2} \right]} \tau_{2i+1} \cdot \prod_{i \in \left[ \frac{n}{2} \right]} \tau_{2i} \right)^{\frac{n}{2}}\)

which are reduced expressions for the longest element \(w_\circ = [n, n-1, \ldots, 2, 1]\)
FLIP IN TRIANGULATIONS
The flip graph is the 1-skeleton of the associahedron
COMBINATORIAL MODELS FOR GEOMETRIC GRAPHS

triangulations of convex polygons,

multitriangulations of convex polygons,

pseudotriangulations of point sets in general position,

pseudotriangulations of sets of disjoint convex bodies.


GREEDY FLIP ALGORITHM
increasing flip = flip from $I$ to $J$ with $I \setminus i = J \setminus j$ and $i < j$

The increasing flip graph is acyclic, connected, and has a unique sink

greedy facet $G(Q, \rho) =$ unique sink of the increasing flip graph
  $= \text{lexicographically maximal facet of } SC(Q, \rho)$
The greedy facet \( G(Q, \rho) \) can be constructed inductively from \( G(\varepsilon, e) = \emptyset \) using the following formulas:

\[
G(Q, \rho) = \begin{cases} 
G(Q_{\downarrow}, \rho) \cup m & \text{if } \rho \prec Q_{\downarrow} \\
G(Q_{\downarrow}, \rho q_m) & \text{otherwise}
\end{cases}
\]

\[
G(Q, \rho) = \begin{cases} 
G(Q_{\leftarrow}, q_1\rho) \rightarrow & \text{if } \ell(q_1\rho) < \ell(\rho) \\
1 \cup G(Q_{\leftarrow}, \rho) \rightarrow & \text{otherwise}
\end{cases}
\]

where \( Q_{\downarrow} = q_1 q_2 \cdots q_{m-1} \), \( Q_{\leftarrow} = q_2 \cdots q_{m-1} q_m \) and \( X\rightarrow = \{x + 1 \mid x \in X\} \)
If $m$ is a flippable element of $G(Q, \rho)$, then $G(Q^{-1}, \rho q_m)$ is obtained from $G(Q, \rho)$ flipping $m$. 
\[ \mathcal{F}(Q, \rho) = \mathcal{F}(Q_{\downarrow}, \rho q_m) \sqcup (\mathcal{F}(Q_{\downarrow}, \rho) \star m) \]
\[ G(Q, \rho) = G(Q_{\perp}, \rho q_m) \sqcup (G(Q_{\perp}, \rho) \ast m) \sqcup \left\{ \text{arc from } G(Q_{\perp}, \rho q_m) \to G(Q, \rho) = G(Q_{\perp}, \rho) \cup m \right\} \]
**Inductive definition of the facets $F(Q, \rho)$ of the subword complex $SC(Q, \rho)$:**

$$F(Q, \rho) = \begin{cases} 
F(Q_{\perp}, \rho q_m) & \text{if } \rho \not\lesssim Q_{\perp} \\
F(Q_{\perp}, \rho) \ast m & \text{if } \ell(\rho q_m) > \ell(\rho) \\
F(Q_{\perp}, \rho q_m) \sqcup (F(Q_{\perp}, \rho) \ast m) & \text{otherwise}
\end{cases}$$

**Inductive definition of the greedy flip tree $G(Q, \rho)$:**

$$G(Q, \rho) = \begin{cases} 
G(Q_{\perp}, \rho q_m) & \text{if } \rho \not\lesssim Q_{\perp} \\
G(Q_{\perp}, \rho) \ast m & \text{if } \ell(\rho q_m) > \ell(\rho) \\
G(Q_{\perp}, \rho q_m) \sqcup (G(Q_{\perp}, \rho) \ast m) \sqcup \{ \text{arc from } G(Q_{\perp}, \rho q_m) \text{ to } G(Q, \rho) = G(Q_{\perp}, \rho) \cup m \} & \text{otherwise}
\end{cases}$$
**GREEDY FLIP TREE — DIRECT DEFINITION**

\[ g(I) = \text{greedy index of a facet} \ I \in \mathcal{F}(Q, \rho) = \]

last position \( x \in [m] \) such that \( I \cap [x] = G(q_1 \cdots q_x, \sigma_{[x]} \setminus I) \)

If \( I, J \in \mathcal{F}(Q, \rho) \) with \( I \setminus i = J \setminus j \) and \( i < j \leq g(J) \), then \( g(I) = j - 1 \)
The greedy flip tree $G(Q, \rho)$ has
- nodes = $F(Q, \rho)$ = complements of reduced expressions of $\rho$ in $Q$
- arcs = flip $(I, J)$ such that $I \setminus i = J \setminus j$ with $i < j \leq g(J)$. 
Greedy Flip Algorithm = Depth first search generation on the greedy flip tree. Preorder traversal provides an iterator on the reduced expressions of $\rho$ in $Q$.

Working space in $O(mn)$
Running time in $O(m^2n)$ per facet $\rightarrow$ similar to the inductive algorithm

Implemented in Sage (Stump’s combinat patch on subword complexes)
Experimental time comparison to generate the $k$-triangulations of the $n$-gon:
Thank you