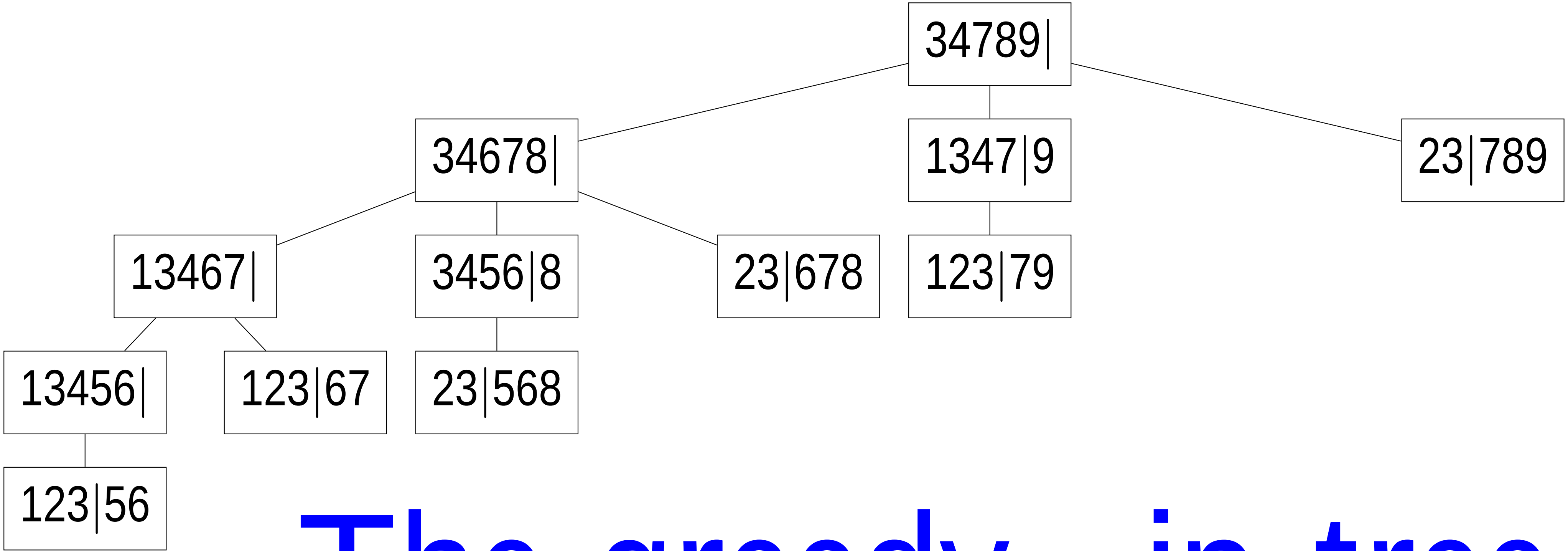


The greedy ip tree of a subword complex

Vincent PILAUD
CNRS - LIX, Ecole Polytechnique

REDUCED EXPRESSIONS & SUBWORD COMPLEXES

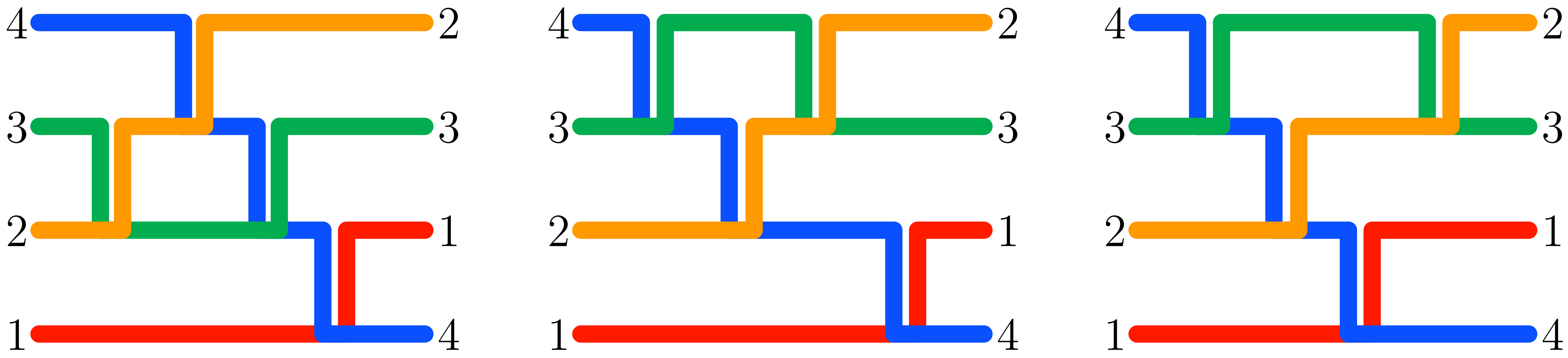
REDUCED EXPRESSIONS

$\mathfrak{S}_n$ = symmetric group
$S = \{\tau_i \mid i \in n-1\}$ set of simple transpositions $\tau_i = (i \;\; i+1)$

ρ permutation of $\mathfrak{S}_n$
reduced expression of ρ = minimal length expression $\rho = s_1 \cdots s_\ell$ with $s_i \in S$

Count and enumerate reduced expressions of ρ

Example. $\rho = [4, 1, 3, 2] = \tau_2\ \tau_3\ \tau_2\ \tau_1 = \tau_3\ \tau_2\ \tau_3\ \tau_1 = \tau_3\ \tau_2\ \tau_1\ \tau_3$



REDUCED EXPRESSIONS

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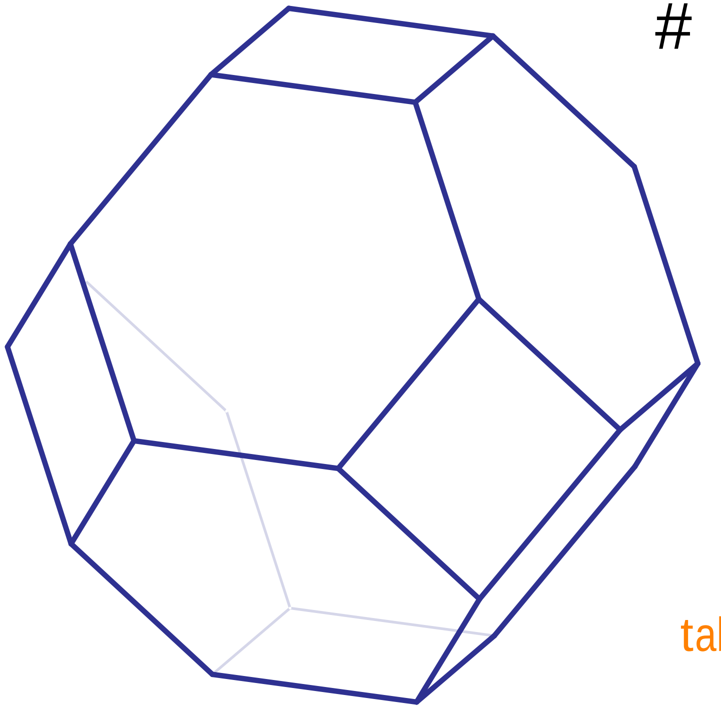

\# reduced expressions of $w_\circ$ =

$$\frac{\binom{n}{2}!}{1^{n-1}2^{n-2}\cdots(2n-3)^1}$$

Stanley.
On the number of reduced decompositions
of elements of Coxeter groups. 1984

Edelmann & Greene.
Combinatorial correspondences for Young tableaux, balanced
tableaux, and maximal chains in the Bruhat order of $\mathfrak{S}_n$. 1984

REDUCED EXPRESSIONS AS SUBWORDS

$\mathfrak{S}_n$ = symmetric group
$S = \{\tau_i \mid i \in n-1\}$ set of simple transpositions $\tau_i = (i \;\; i+1)$
ρ permutation of $\mathfrak{S}_n$
Q $= q_1\, q_2 \cdots q_m$ word on the alphabet S

> Enumerate subwords of Q which are reduced expressions for ρ

Example. $\rho = [4, 1, 3, 2] = \tau_2\ \tau_3\ \tau_2\ \tau_1 = \tau_3\ \tau_2\ \tau_3\ \tau_1 = \tau_3\ \tau_2\ \tau_1\ \tau_3$
Q $= \tau_2\ \tau_3\ \tau_1\ \tau_3\ \tau_2\ \tau_1\ \tau_2\ \tau_3\ \tau_1$

Possible subwords:

$$\begin{array}{ccccccccccc}
\tau_2 & \tau_3 & \cdot & \cdot & \tau_2 & \tau_1 & \cdot & \cdot & \cdot & \longrightarrow & 34789 \\
\tau_2 & \tau_3 & \cdot & \cdot & \cdot & \cdot & \tau_2 & \cdot & \tau_1 & \longrightarrow & 34568 \\
\cdot & \tau_3 & \cdot & \cdot & \tau_2 & \cdot & \cdot & \tau_3 & \tau_1 & \longrightarrow & 13467 \\
\cdot & \tau_3 & \cdot & \cdot & \tau_2 & \tau_1 & \cdot & \tau_3 & \cdot & \longrightarrow & 13479
\end{array}$$

etc

REDUCED EXPRESSIONS AS SUBWORDS

$\mathfrak{S}_n$ = symmetric group
$S = \{\tau_i \mid i \in n-1\}$ set of simple transpositions $\tau_i = (i \;\; i+1)$
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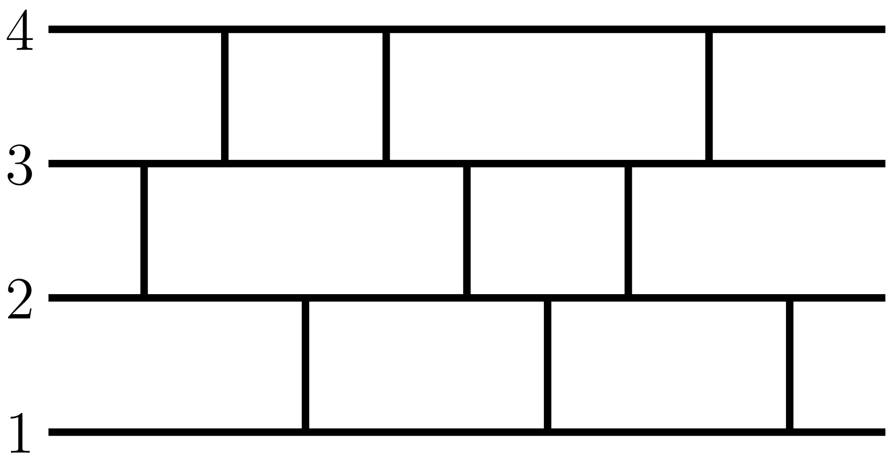
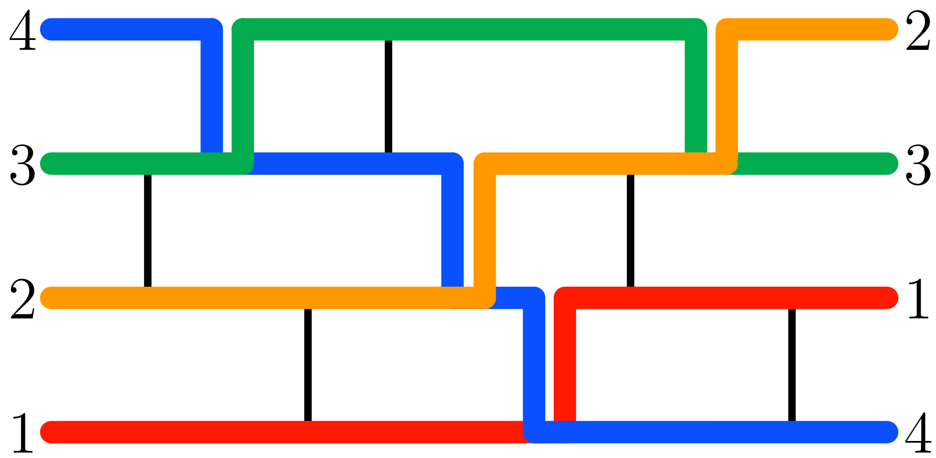
Q = $\tau_2\ \tau_3\ \tau_1\ \tau_3\ \tau_2\ \tau_1\ \tau_2\ \tau_3\ \tau_1$

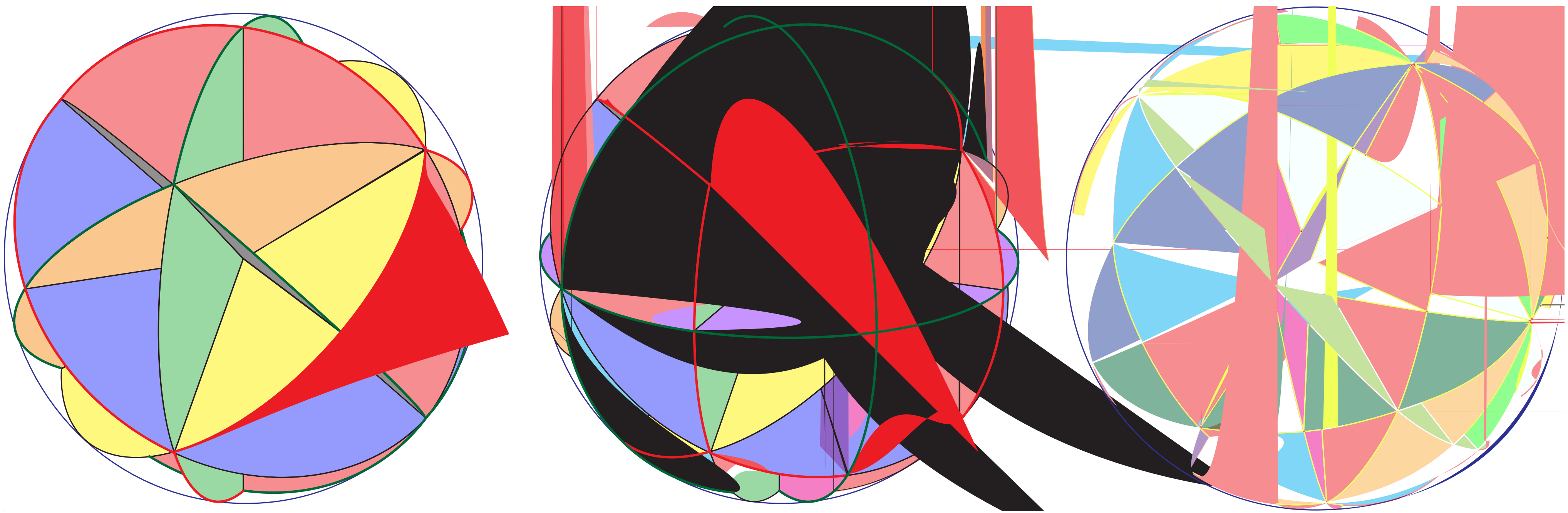




GENERALIZATION TO COXETER GROUPS

W = nite Coxeter group
S = simple system of generators for W
ρ element of W
Q = $q_1\, q_2 \cdots q_m$ word on the alphabet S

Enumerate subwords of Q which are reduced expressions for ρ



SUBWORD COMPLEX

$\mathfrak{S}_n$ = symmetric group
$S = \{\tau_i \mid i \in n-1\}$ set of simple transpositions $\tau_i = (i \;\; i+1)$
ρ permutation of $\mathfrak{S}_n$
Q = $q_1\, q_2 \cdots q_m$ word on the alphabet S

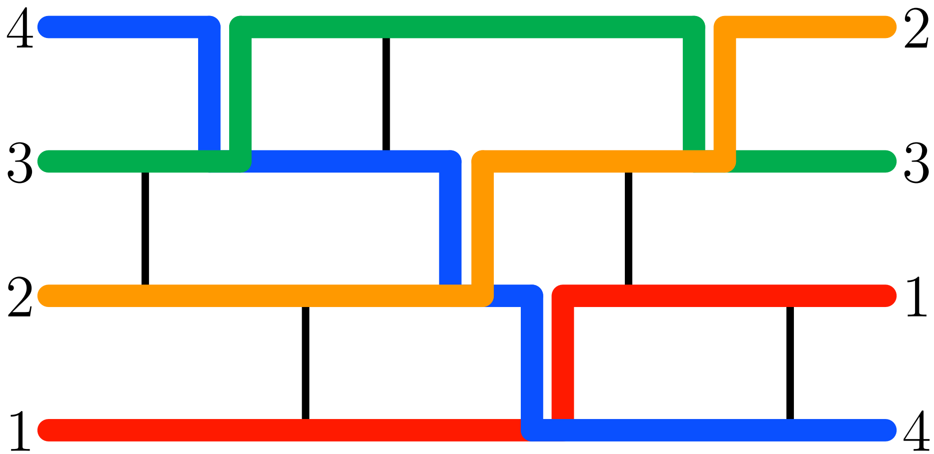
Subword complex $\mathcal{SC}(\mathrm{Q}, \rho)$ = simplicial complex with

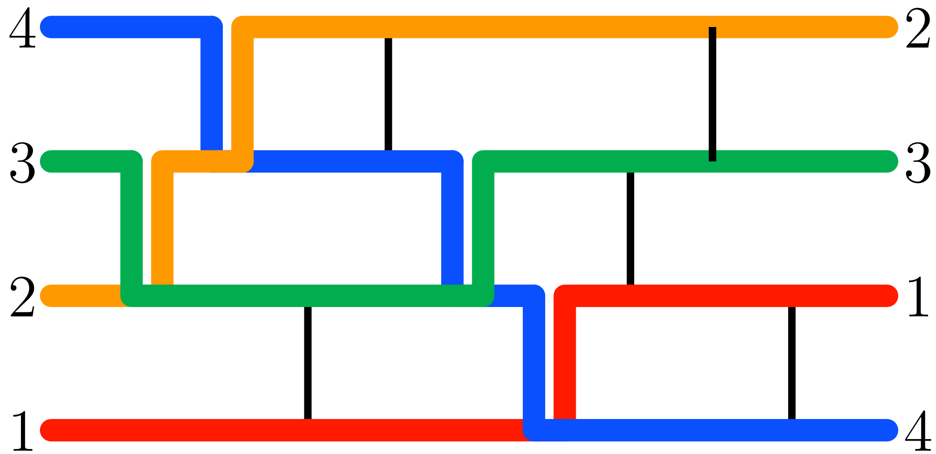
- vertices = $[m]$ = positions in the word Q
- facets = $\mathcal{F}(\mathrm{Q}, \rho)$ = complements in $[m]$ of position sets of reduced expressions of ρ in Q

Knutson & Miller. Subword complexes in Coxeter groups. 2004.

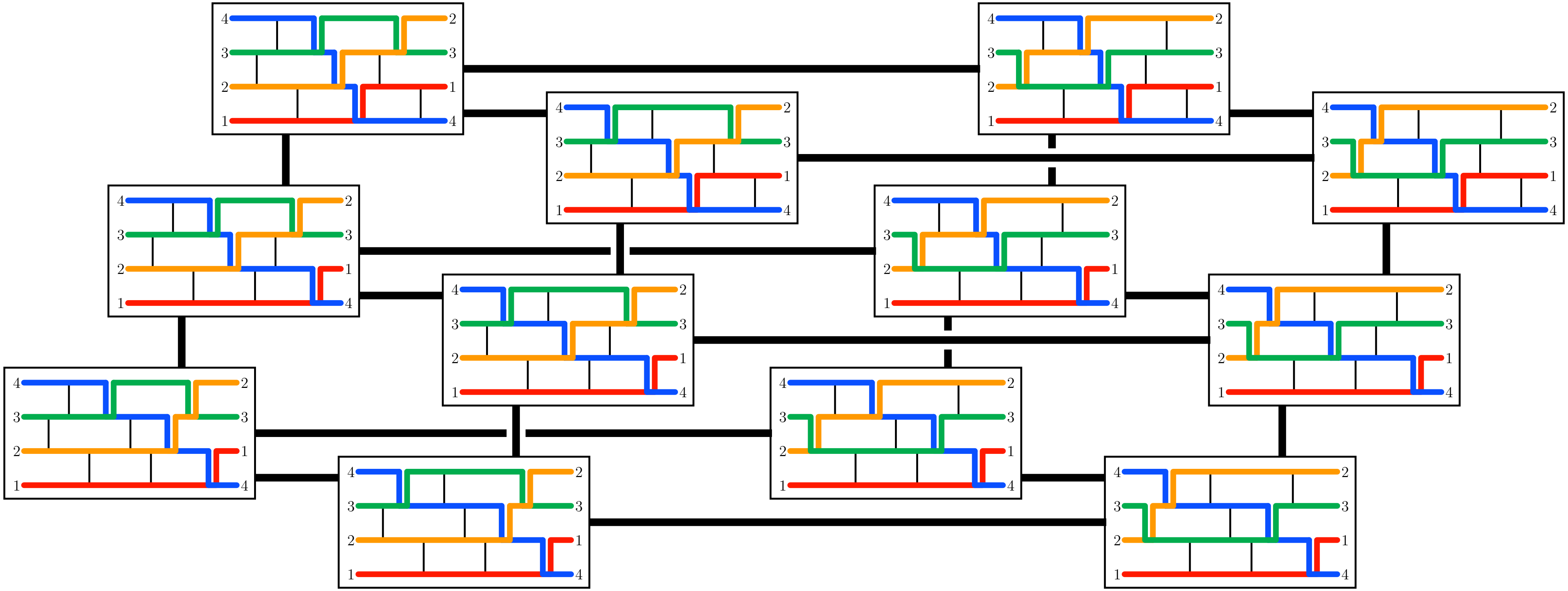
FLIP

ip = two subwords of Q which di er at precisely two positions





FLIP



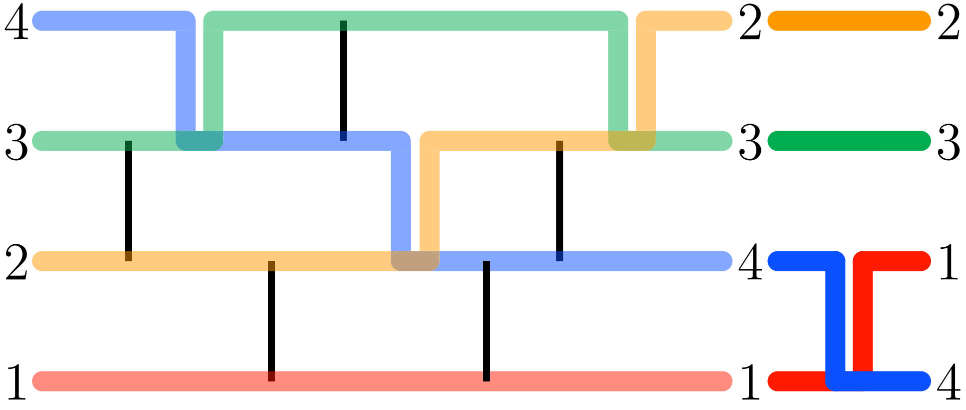
The ip graph is connected

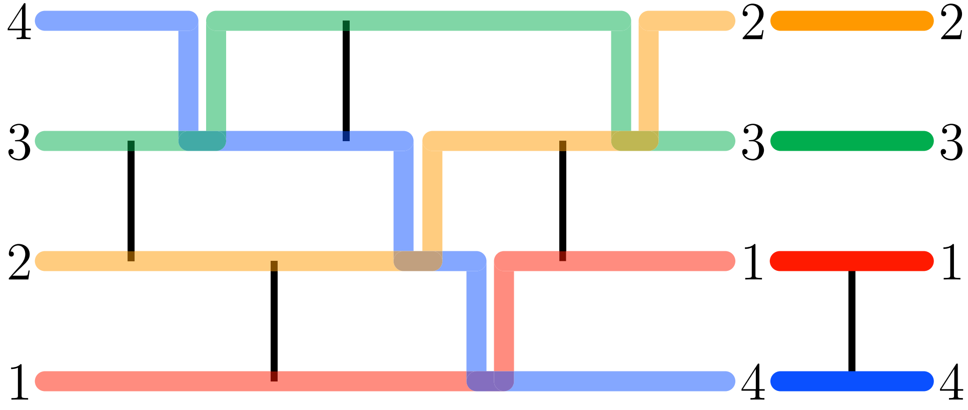
GOAL: Find a natural spanning tree of the ip graph

INDUCTIVE STRUCTURE

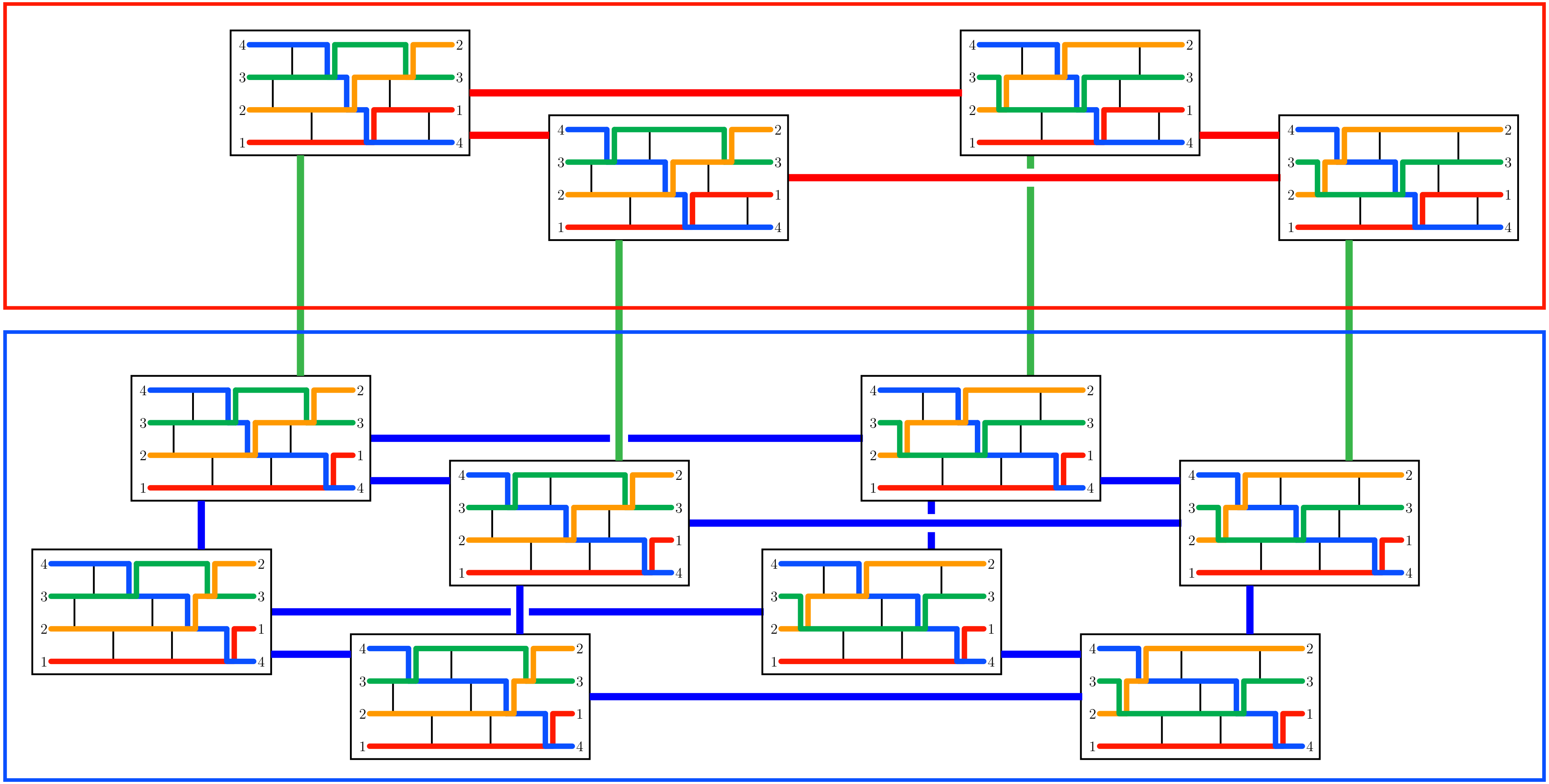
Q = $q_1\, q_2 \cdots q_{m-1}\, q_m$ and $\mathrm{Q}_{\dashv} = q_1\, q_2 \cdots q_{m-1}$
$\mathcal{F}(\mathrm{Q}, \rho)$ = facets of $\mathcal{SC}(\mathrm{Q}, \rho)$ = complements of reduced expressions of ρ in Q

$$\mathcal{F}(\mathrm{Q}, \rho) \;=\; \mathcal{F}(\mathrm{Q}_{\dashv}, \rho q_m) \sqcup \big(\mathcal{F}(\mathrm{Q}_{\dashv}, \rho) \star m\big)$$





INDUCTIVE STRUCTURE



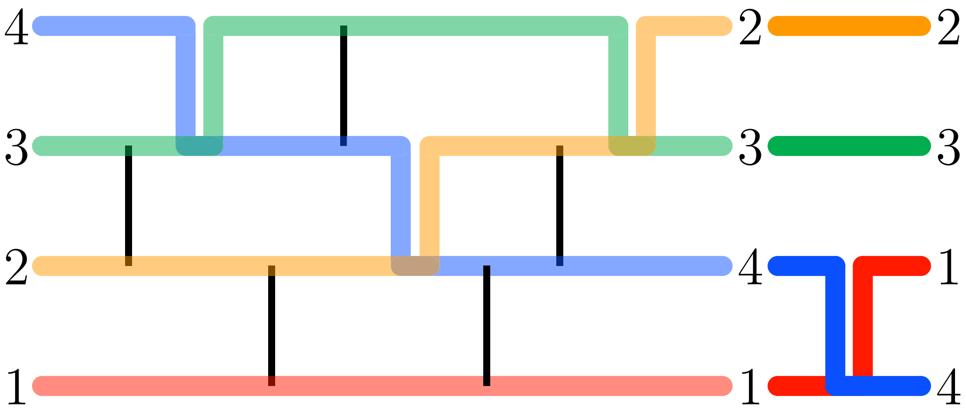
$$\mathcal{F}(\mathsf{Q},\rho) \;=\; \mathcal{F}(\mathsf{Q}_{\dashv},\rho q_m) \;\sqcup\; \big(\mathcal{F}(\mathsf{Q}_{\dashv},\rho) \star m\big)$$

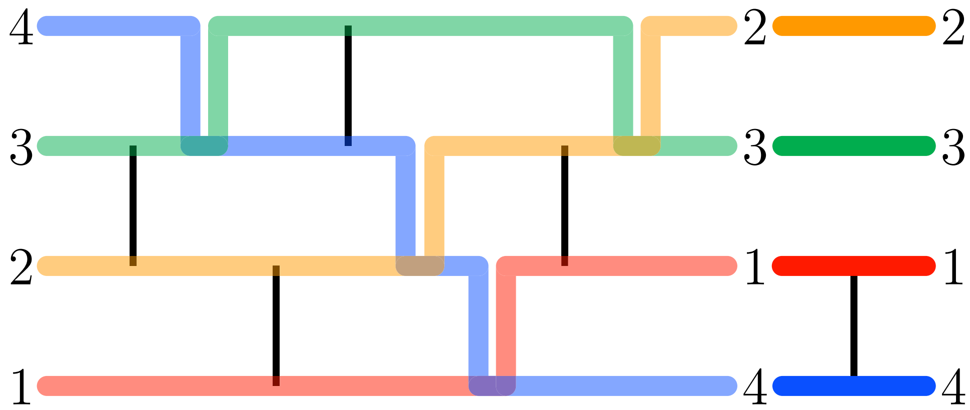
INDUCTIVE STRUCTURE

$\mathrm{Q} = q_1\, q_2 \cdots q_{m-1}\, q_m$ and $\mathrm{Q}_{\dashv} = q_1\, q_2 \cdots q_{m-1}$

$\mathcal{F}(\mathrm{Q}, \rho)$ = facets of $\mathcal{SC}(\mathrm{Q}, \rho)$ = complements of reduced expressions of ρ in Q

$$\mathcal{F}(\mathrm{Q}, \rho) = \begin{cases} \mathcal{F}(\mathrm{Q}_{\dashv}, \rho q_m) & \text{if } \rho \not\prec \mathrm{Q}_{\dashv} \\ \mathcal{F}(\mathrm{Q}_{\dashv}, \rho) \star m & \text{if } \ell(\rho q_m) > \ell(\rho) \\ \mathcal{F}(\mathrm{Q}_{\dashv}, \rho q_m) \sqcup \big(\mathcal{F}(\mathrm{Q}_{\dashv}, \rho) \star m\big) & \text{otherwise} \end{cases}$$



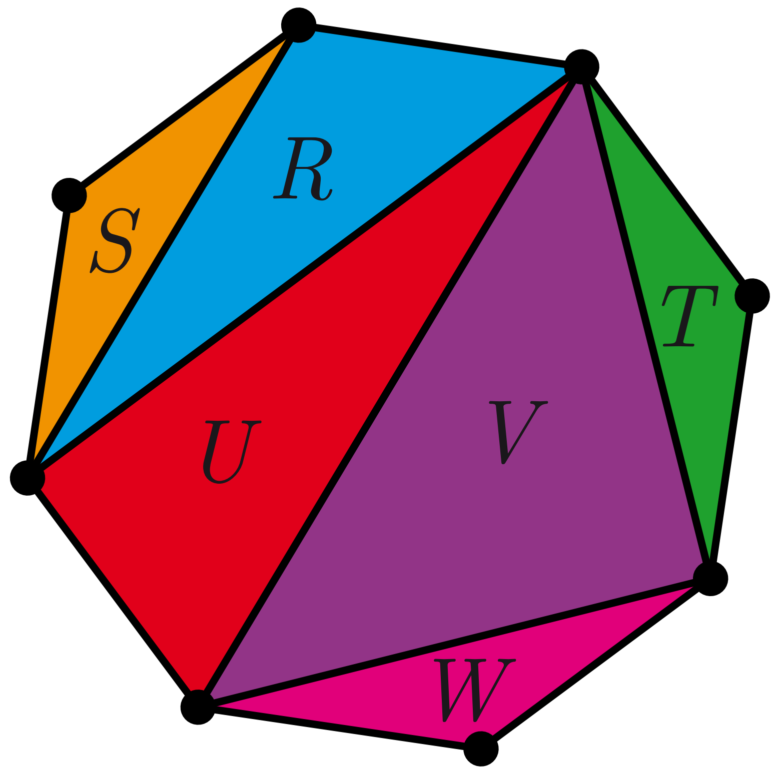


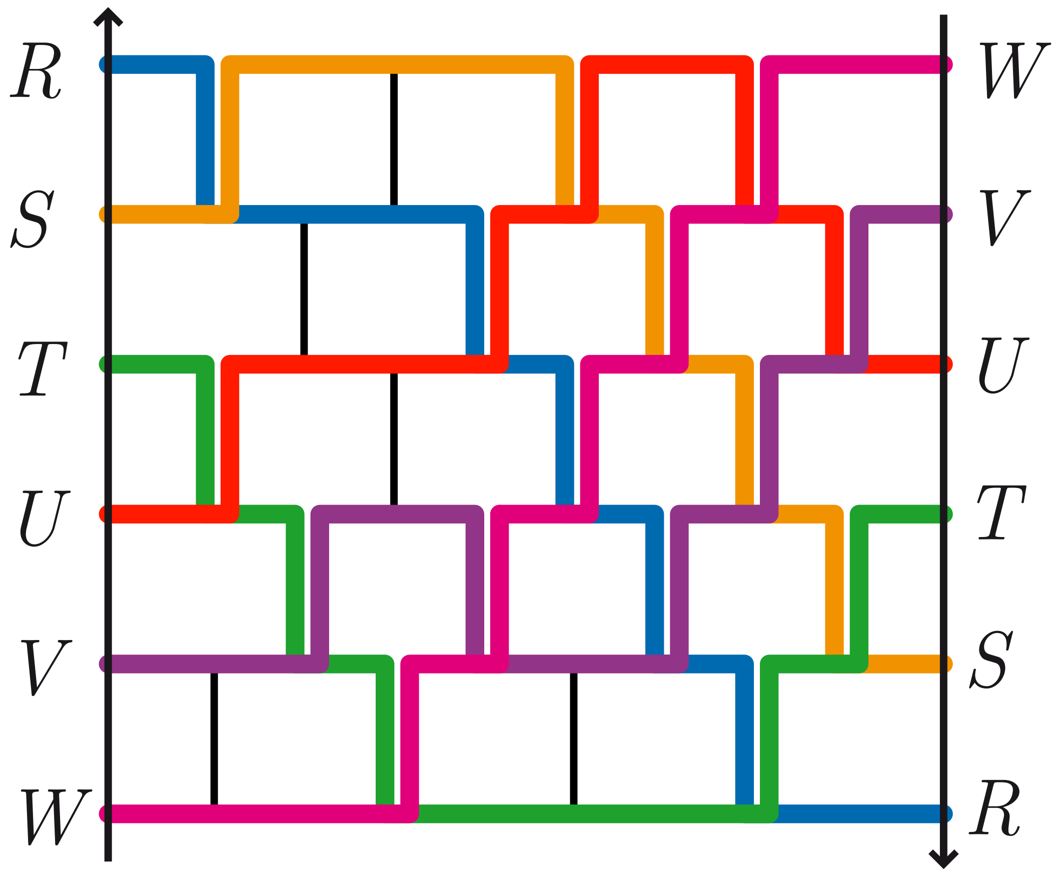
$\Rightarrow$ Inductive enumeration of $\mathcal{F}(\mathrm{Q}, \rho)$ with complexity $O(m^2 n)$ per facet

COMBINATORIAL MODELS FOR GEOMETRIC GRAPHS

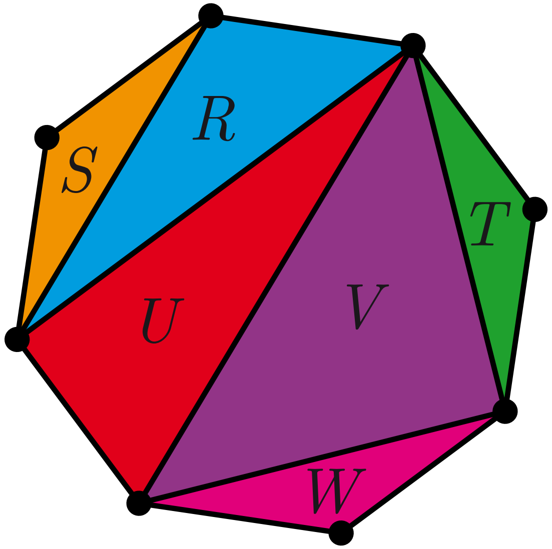
bijection between

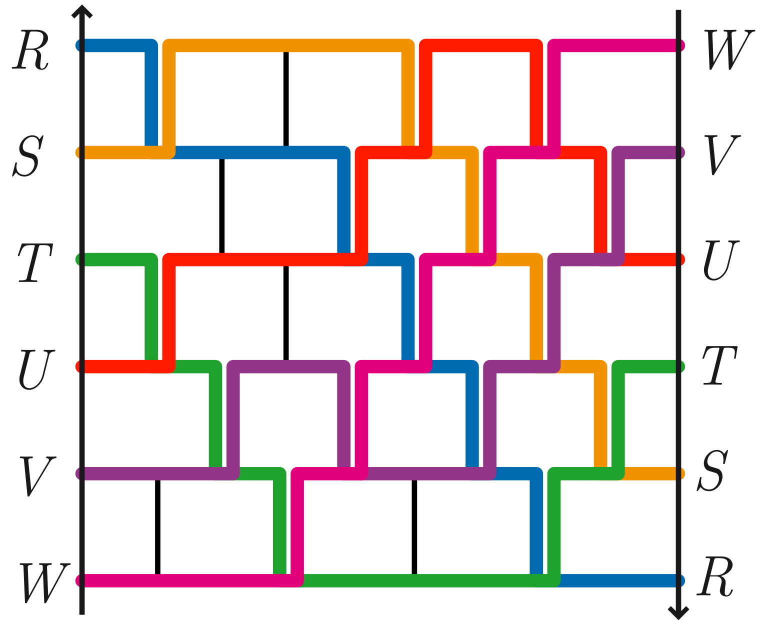
- triangulations of a convex (n + 2)-gon
- subwords of the odd-even word Q = $\left(\prod_{i\in[\frac{n}{2}]}\tau_{2i+1}\cdot\prod_{i\in[\frac{n}{2}]}\tau_{2i}\right)^{\frac{n}{2}}$
  which are reduced expressions for the longest element $w_\circ = [n, n{-}1, \dots, 2, 1]$

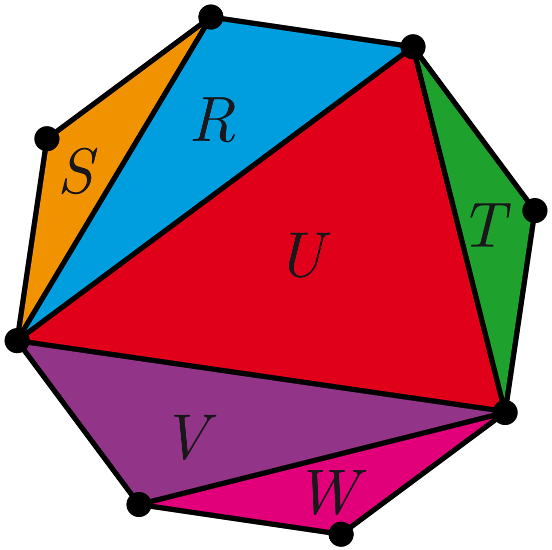


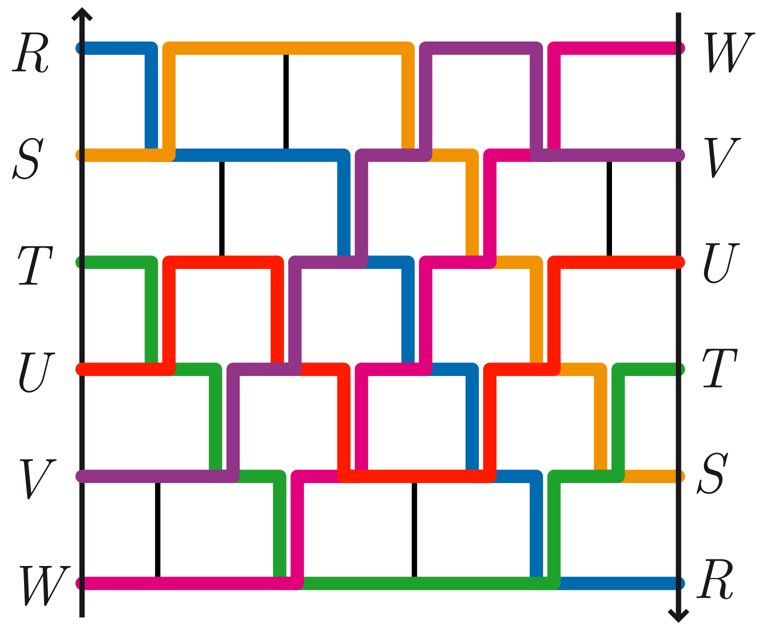


FLIP IN TRIANGULATIONS



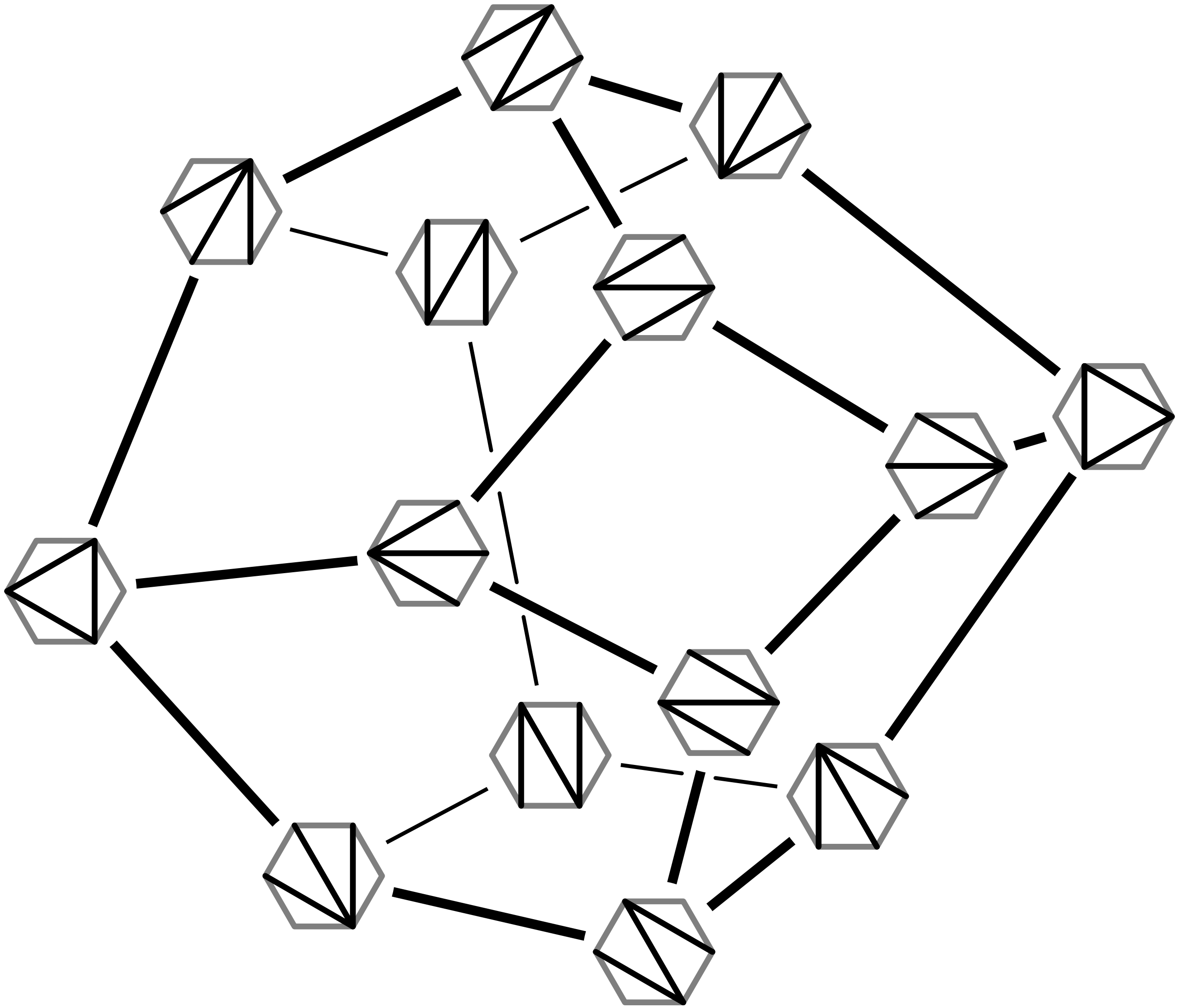






ASSOCIAHEDRON

The ip graph is the 1-skeleton of the associahedron



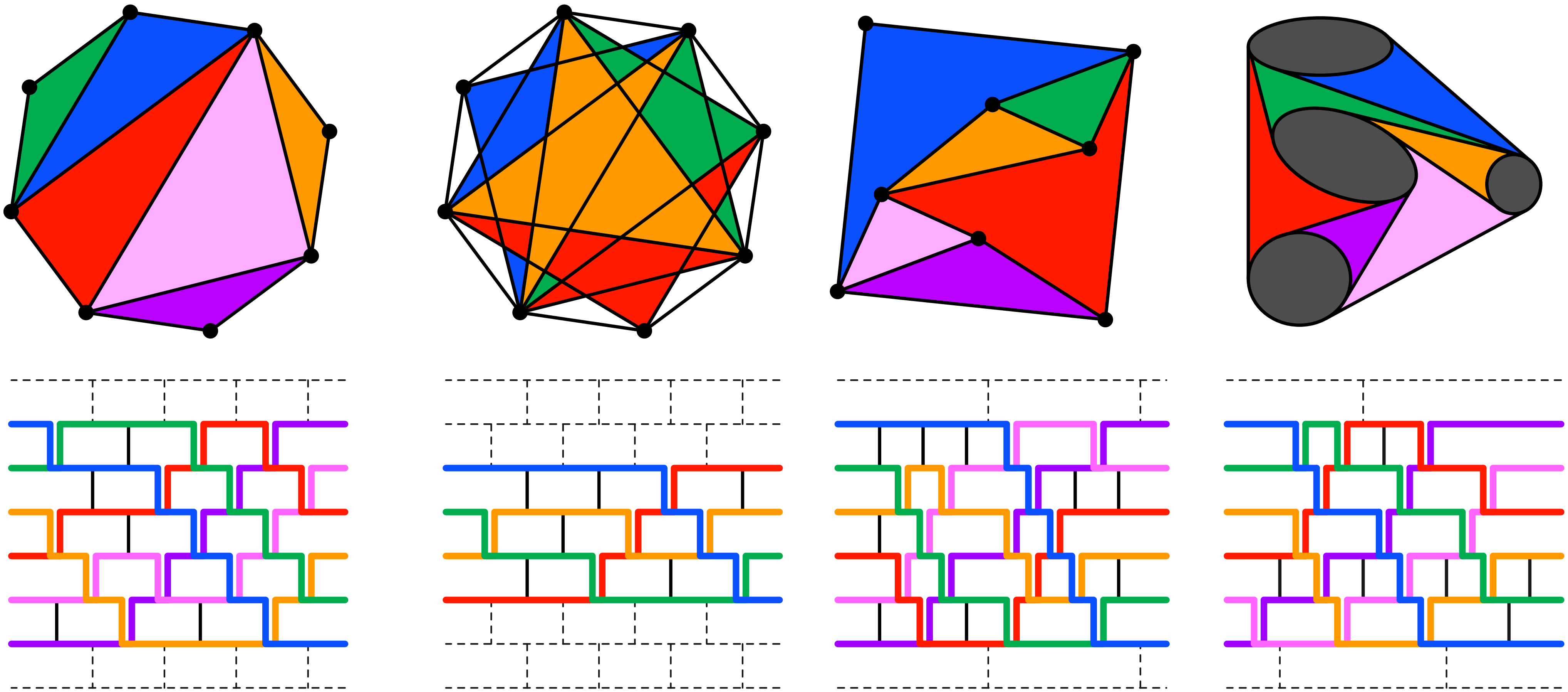
COMBINATORIAL MODELS FOR GEOMETRIC GRAPHS

triangulations of convex polygons,

multitriangulations of convex polygons,

pseudotriangulations of point sets in general position,

pseudotriangulations of sets of disjoint convex bodies.

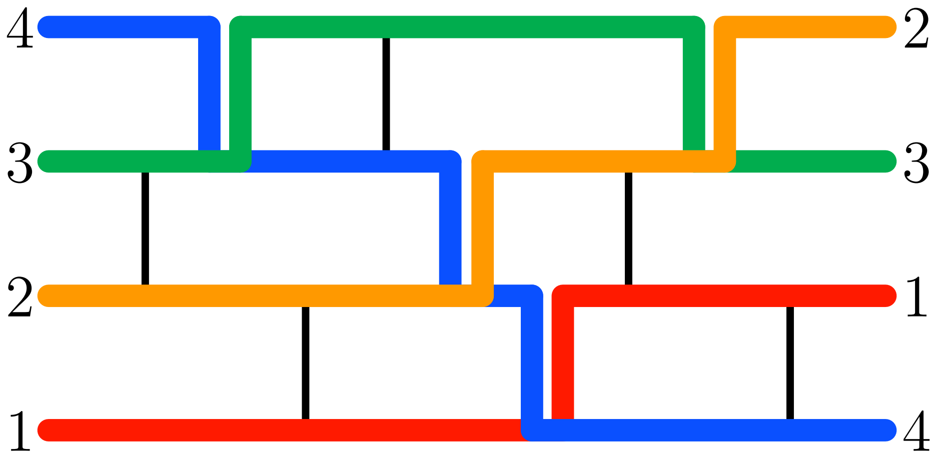


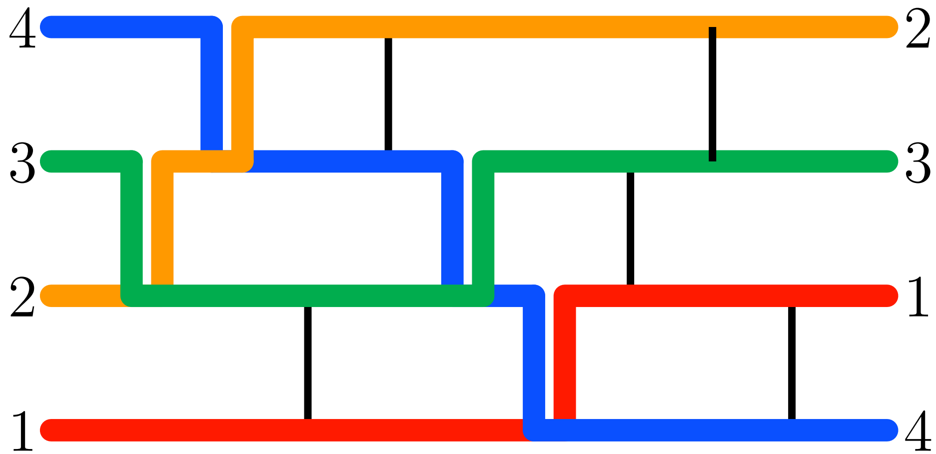
P. & Pocchiola, Pseudotriangulations, multitriangulations, and primitive sorting networks, 2012.

Stump, A new perspective on multitriangulations, 2011.

GREEDY FLIP ALGORITHM

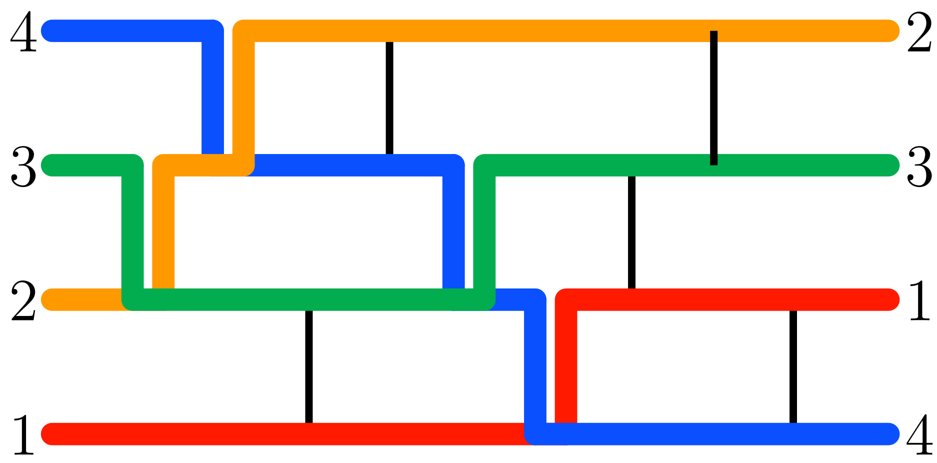
increasing ip = ip from I to J with $I \smallsetminus i = J \smallsetminus j$ and $i < j$





The increasing ip graph is acyclic, connected, and has a unique sink

greedy facet $G(Q, \rho)$ = unique sink of the increasing ip graph
= lexicographically maximal facet of $\mathcal{SC}(Q, \rho)$



The greedy facet $\mathsf{G}(\mathsf{Q}, \rho)$ can be constructed inductively from $\mathsf{G}(\varepsilon, e) = \emptyset$ using the following formulas:

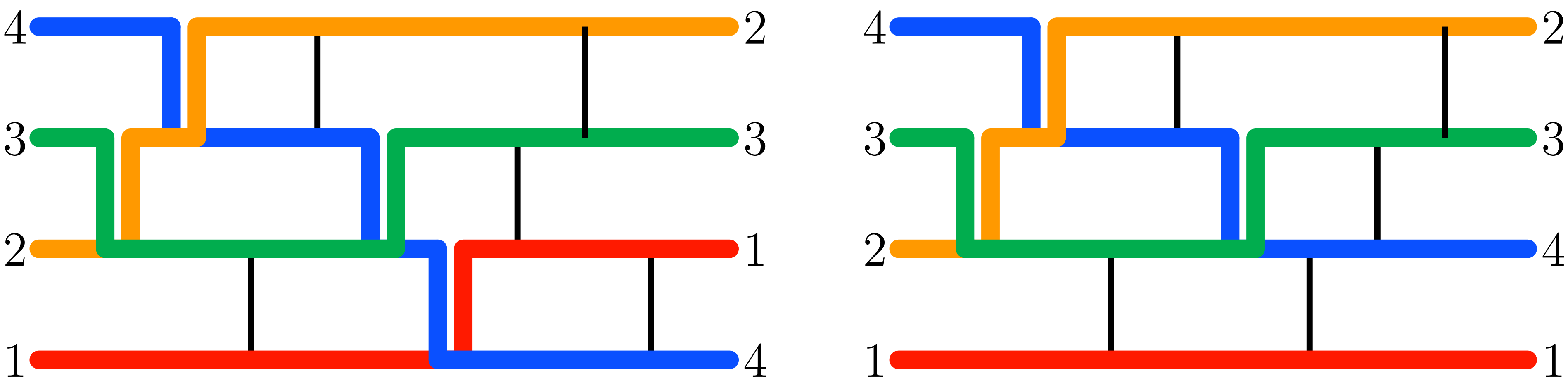
$$\mathsf{G}(\mathsf{Q}, \rho) = \begin{cases} \mathsf{G}(\mathsf{Q}_{\dashv}, \rho) \cup m & \text{if } \rho \prec \mathsf{Q}_{\dashv} \\ \mathsf{G}(\mathsf{Q}_{\dashv}, \rho q_m) & \text{otherwise} \end{cases}$$

$$\mathsf{G}(\mathsf{Q}, \rho) = \begin{cases} \mathsf{G}(\mathsf{Q}_{\vdash}, q_1\rho)^{\rightarrow} & \text{if } \ell(q_1\rho) < \ell(\rho) \\ 1 \cup \mathsf{G}(\mathsf{Q}_{\vdash}, \rho)^{\rightarrow} & \text{otherwise} \end{cases}$$

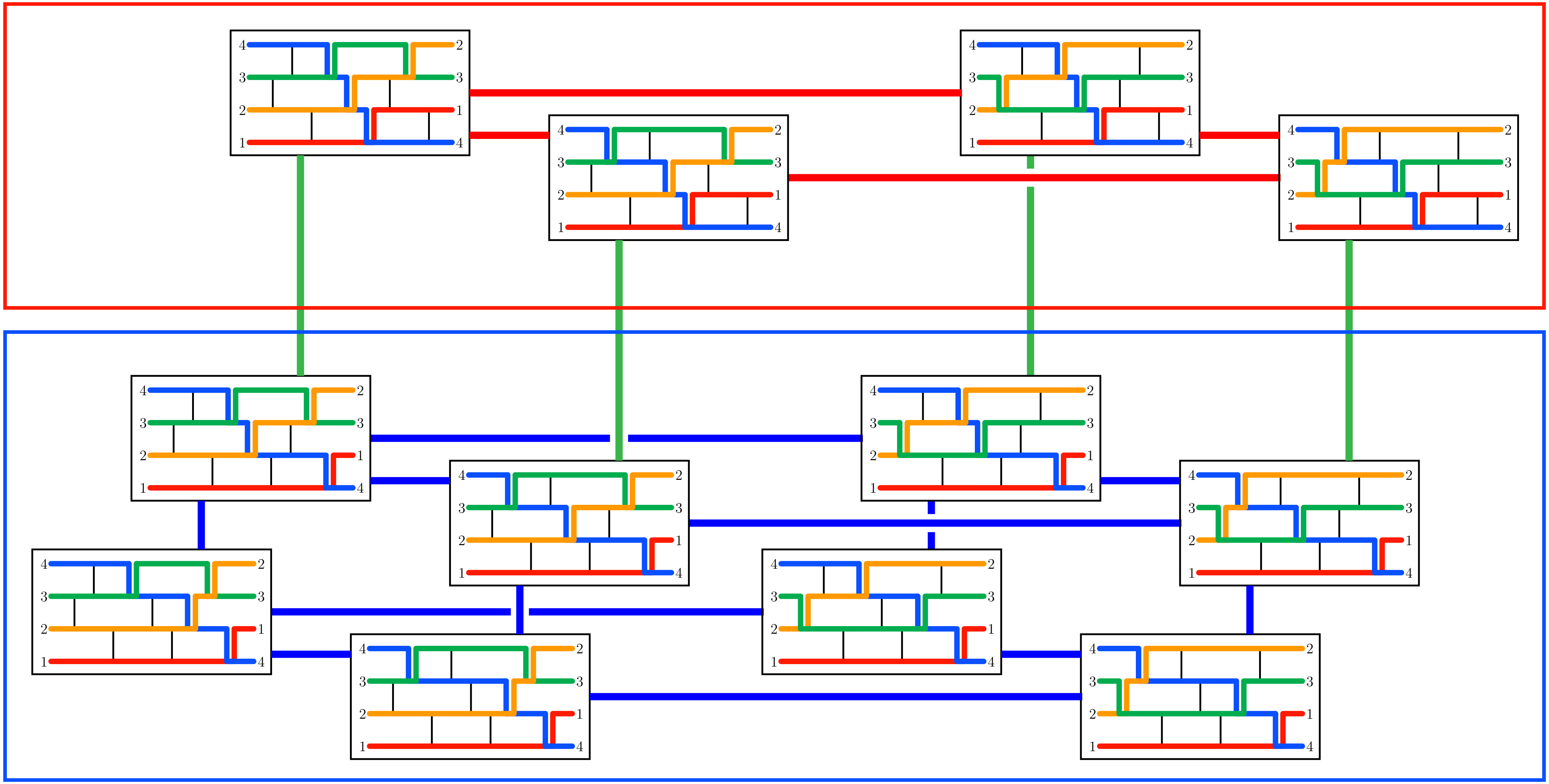
where $\mathsf{Q}_{\dashv} = q_1\, q_2 \cdots q_{m-1}$, $\mathsf{Q}_{\vdash} = q_2 \cdots q_{m-1}\, q_m$ and $X^{\rightarrow} = \{x + 1 \mid x \in X\}$

GREEDY FLIP PROPERTY

If m is a ippable element of G(Q, ρ),
then G(Q$_{\dashv}$, ρq_m) is obtained from G(Q, ρ) ipping m

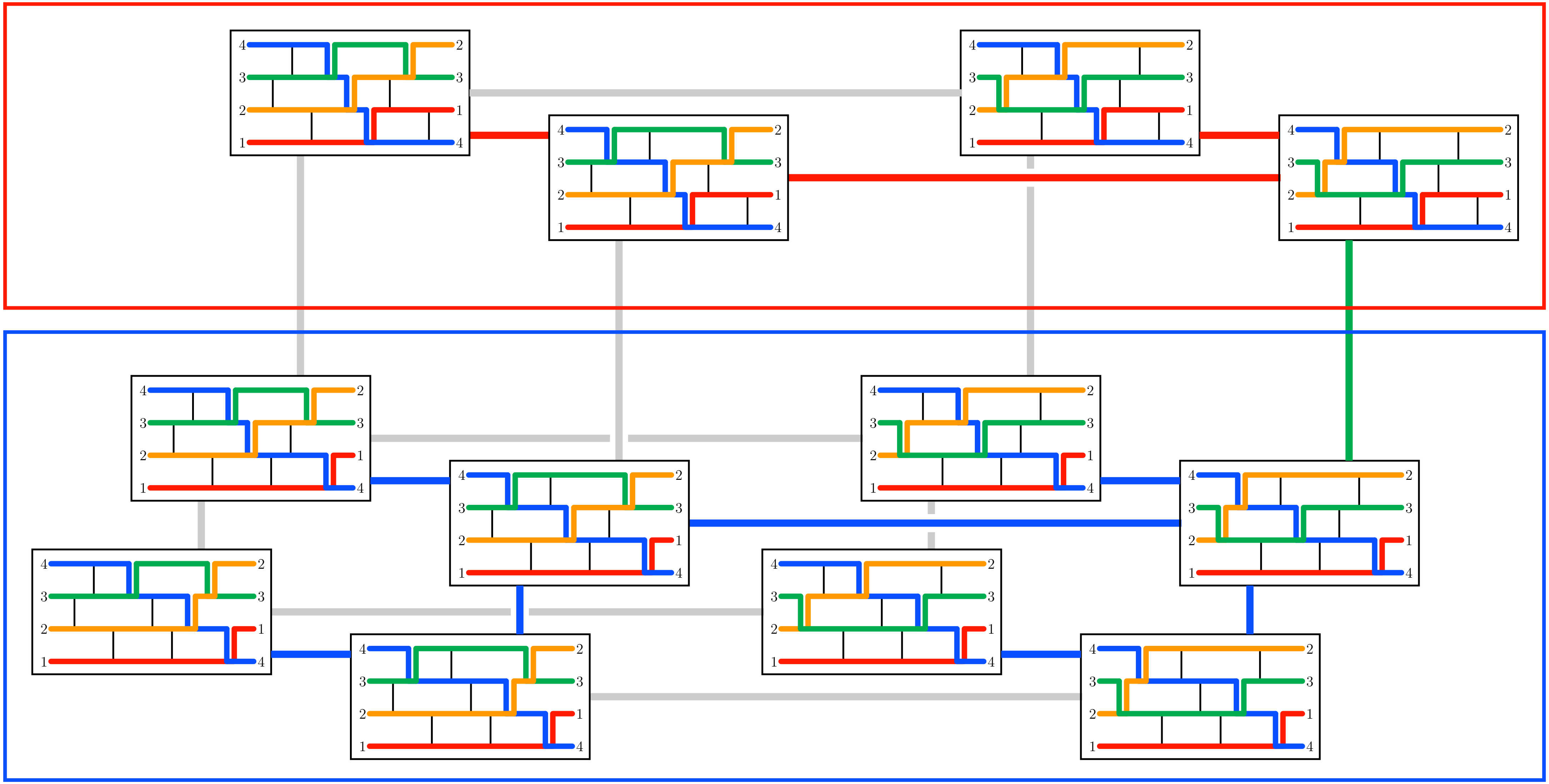


GREEDY FLIP TREE | INDUCTIVE DEFINITION



$$\mathcal{F}(\mathsf{Q}, \rho) \;=\; \mathcal{F}(\mathsf{Q}_{\dashv}, \rho q_m) \sqcup \big(\mathcal{F}(\mathsf{Q}_{\dashv}, \rho) \star m\big)$$

GREEDY FLIP TREE | INDUCTIVE DEFINITION



$$\mathcal{G}(\mathrm{Q}, \rho) = \mathcal{G}(\mathrm{Q}_{\dashv}, \rho q_m) \sqcup \left(\mathcal{G}(\mathrm{Q}_{\dashv}, \rho) \star m\right) \sqcup \left\{ \begin{array}{c} \text{arc from } \mathrm{G}(\mathrm{Q}_{\dashv}, \rho q_m) \text{ to} \\ \mathrm{G}(\mathrm{Q}, \rho) = \mathrm{G}(\mathrm{Q}_{\dashv}, \rho) \cup m \end{array} \right\}$$

Inductive structure of the facets $\mathcal{F}(\mathrm{Q},\rho)$ of the subword complex $\mathcal{SC}(\mathrm{Q},\rho)$:

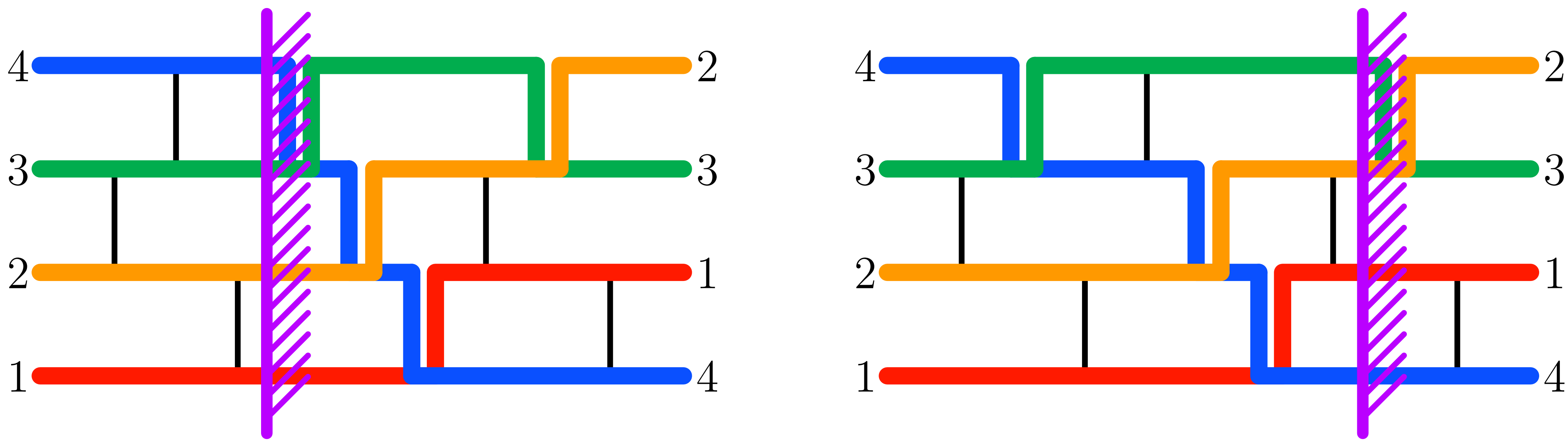
$$\mathcal{F}(\mathrm{Q},\rho) = \begin{cases} \mathcal{F}(\mathrm{Q}_{\dashv},\rho q_m) & \text{if } \rho \nprec \mathrm{Q}_{\dashv} \\ \mathcal{F}(\mathrm{Q}_{\dashv},\rho) \star m & \text{if } \ell(\rho q_m) > \ell(\rho) \\ \mathcal{F}(\mathrm{Q}_{\dashv},\rho q_m) \sqcup \big(\mathcal{F}(\mathrm{Q}_{\dashv},\rho) \star m\big) & \text{otherwise} \end{cases}$$

Inductive de nition of the greedy ip tree $\mathcal{G}(\mathrm{Q},\rho)$:

$$\mathcal{G}(\mathrm{Q},\rho) = \begin{cases} \mathcal{G}(\mathrm{Q}_{\dashv},\rho q_m) & \text{if } \rho \nprec \mathrm{Q}_{\dashv} \\ \mathcal{G}(\mathrm{Q}_{\dashv},\rho) \star m & \text{if } \ell(\rho q_m) > \ell(\rho) \\ \mathcal{G}(\mathrm{Q}_{\dashv},\rho q_m) \sqcup \big(\mathcal{G}(\mathrm{Q}_{\dashv},\rho) \star m\big) & \\ \sqcup \{\text{arc from } \mathrm{G}(\mathrm{Q}_{\dashv},\rho q_m) \text{ to} & \text{otherwise} \\ \qquad \mathrm{G}(\mathrm{Q},\rho) = \mathrm{G}(\mathrm{Q}_{\dashv},\rho) \cup m\} & \end{cases}$$

GREEDY FLIP TREE | DIRECT DEFINITION

$\mathsf{g}(I)$ = greedy index of a facet $I \in \mathcal{F}(\mathsf{Q}, \rho)$ =
last position $x \in [m]$ such that $I \cap [x] = \mathsf{G}(q_1 \cdots q_x, \sigma_{[x] \smallsetminus I})$

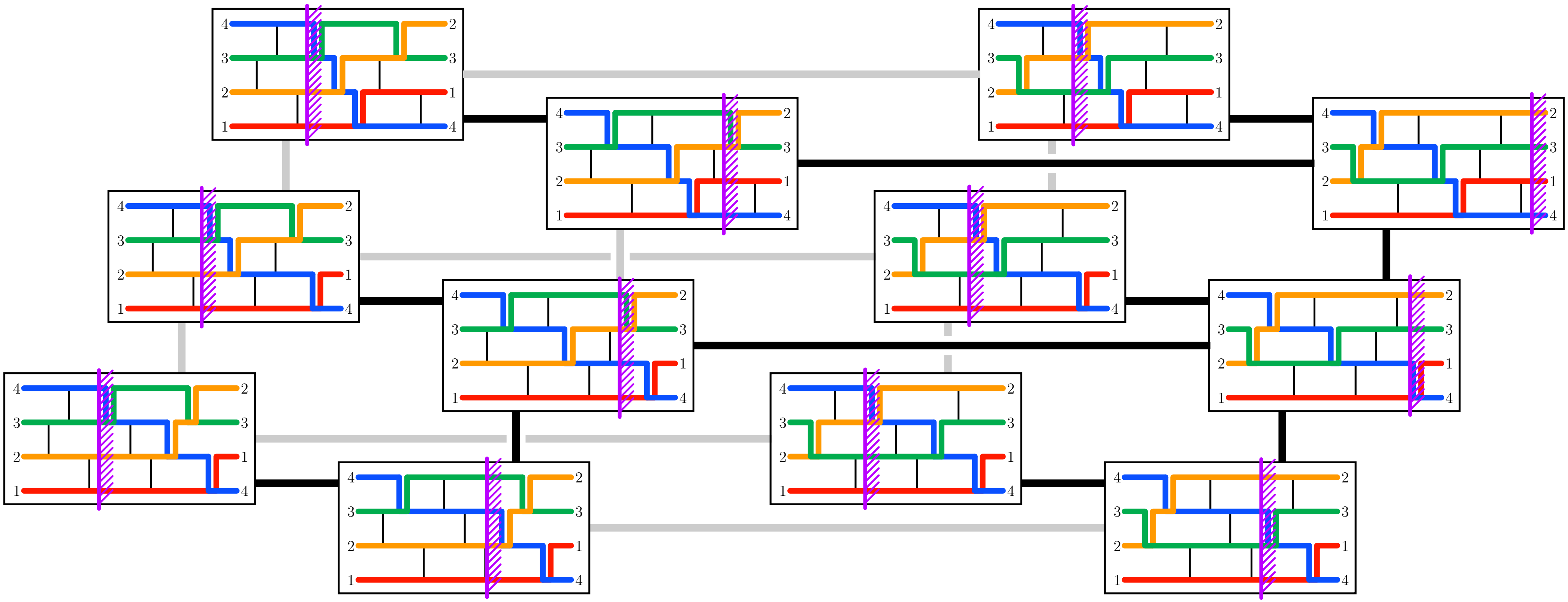


If $I, J \in \mathcal{F}(\mathsf{Q}, \rho)$ with $I \smallsetminus i = J \smallsetminus j$ and $i < j \leq \mathsf{g}(J)$, then $\mathsf{g}(I) = j - 1$

GREEDY FLIP TREE | DIRECT DEFINITION

The greedy ip tree $\mathcal{G}(\mathrm{Q}, \rho)$ has

- nodes = $\mathcal{F}(\mathrm{Q}, \rho)$ = complements of reduced expressions of ρ in Q
- arcs = ip (I, J) such that $I \smallsetminus i = J \smallsetminus j$ with $i < j \leq \mathrm{g}(J)$.



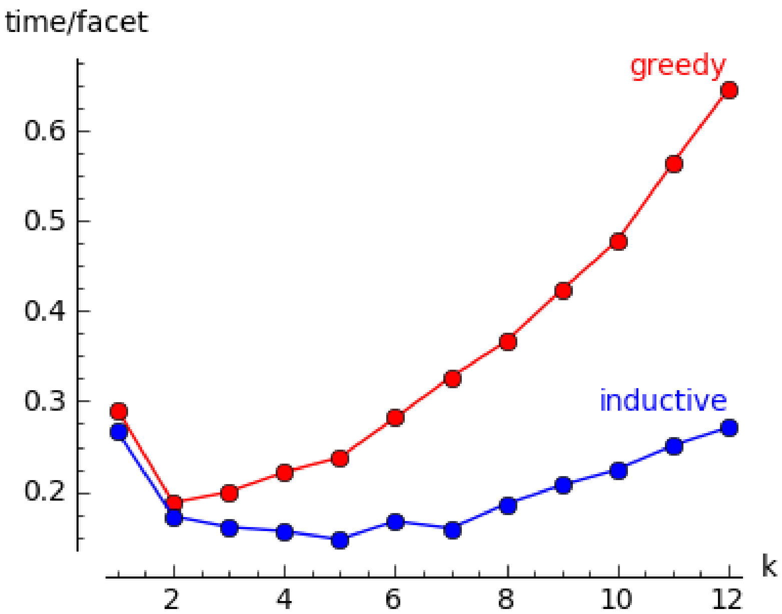
GREEDY FLIP ALGORITHM

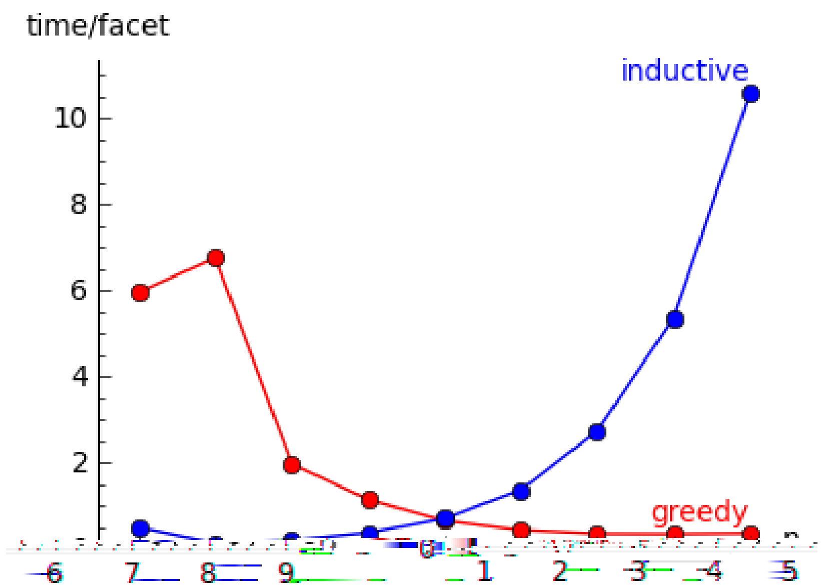
Greedy Flip Algorithm = Depth first search generation on the greedy flip tree

Preorder traversal provides an iterator on the reduced expressions of ρ in Q

Working space in $O(mn)$
Running time in $O(m^2n)$ per facet $\longrightarrow$ similar to the inductive algorithm

Implemented in Sage (Stump's combinat patch on subword complexes)
Experimental time comparison to generate the k-triangulations of the n-gon:





Thank you