The greedy flip tree of a subword complex

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REDUCED EXPRESSIONS & SUBWORD COMPLEXES
$\mathfrak{S}_n$ = symmetric group
$S = \{\tau_i \mid i \in n - 1\}$ set of simple transpositions $\tau_i = (i \ i + 1)$

$\rho$ permutation of $\mathfrak{S}_n$
reduced expression of $\rho$ = minimal length expression $\rho = s_1 \cdots s_\ell$ with $s_i \in S$

Count and enumerate reduced expressions of $\rho$

Example. $\rho = [4, 1, 3, 2] = \tau_2 \tau_3 \tau_2 \tau_1 = \tau_3 \tau_2 \tau_3 \tau_1 = \tau_3 \tau_2 \tau_1 \tau_3$
\(\mathfrak{S}_n = \text{symmetric group}\)
\(S = \{\tau_i \mid i \in n - 1\}\) set of simple transpositions \(\tau_i = (i \ i + 1)\)

\(\rho\) permutation of \(\mathfrak{S}_n\)

reduced expression of \(\rho\) = minimal length expression \(\rho = s_1 \cdots s_\ell\) with \(s_i \in S\)

Count and enumerate reduced expressions of \(\rho\)

\[
\# \text{ reduced expressions of } \omega = \frac{(n)!}{1^{n-1}2^{n-2} \cdots (2n-3)!}
\]

Stanley.  
On the number of reduced decompositions of elements of Coxeter groups. 1984

Edelmann & Greene.  
Combinatorial correspondences for Young tableaux, balanced tableaux, and maximal chains in the Bruhat order of \(\mathfrak{S}_n\). 1984
REDUCED EXPRESSIONS AS SUBWORDS

\[ \mathfrak{S}_n = \text{symmetric group} \]
\[ S = \{ \tau_i \mid i \in n - 1 \} \text{ set of simple transpositions } \tau_i = (i \ i + 1) \]
\[ \rho \text{ permutation of } \mathfrak{S}_n \]
\[ Q = q_1 q_2 \cdots q_m \text{ word on the alphabet } S \]

**Enumerate subwords of** \( Q \) **which are reduced expressions for** \( \rho \)

**Example.** \( \rho = [4, 1, 3, 2] = \tau_2 \tau_3 \tau_2 \tau_1 = \tau_3 \tau_2 \tau_3 \tau_1 = \tau_3 \tau_2 \tau_1 \tau_3 \)
\[ Q = \tau_2 \tau_3 \tau_1 \tau_3 \tau_2 \tau_1 \tau_2 \tau_3 \tau_1 \]

**Possible subwords:**
\[ \tau_2 \tau_3 \cdots \tau_2 \tau_1 \cdots \rightarrow 34789 \]
\[ \tau_2 \tau_3 \cdots \cdots \tau_2 \tau_1 \rightarrow 34568 \]
\[ \cdot \tau_3 \cdots \tau_2 \cdot \tau_3 \tau_1 \rightarrow 13467 \]
\[ \cdot \tau_3 \cdots \tau_2 \tau_1 \cdot \tau_3 \cdot \rightarrow 13479 \]

**etc**
\[ \mathfrak{S}_n = \text{symmetric group} \]
\[ S = \{\tau_i \mid i \in n - 1\} \text{ set of simple transpositions } \tau_i = (i \ i + 1) \]
\[ \rho \text{ permutation of } \mathfrak{S}_n \]
\[ Q = q_1 q_2 \cdots q_m \text{ word on the alphabet } S \]

Enumerate subwords of \( Q \) which are reduced expressions for \( \rho \)

Example. \( \rho = [4, 1, 3, 2] = \tau_2 \tau_3 \tau_2 \tau_1 = \tau_3 \tau_2 \tau_3 \tau_1 = \tau_3 \tau_2 \tau_1 \tau_3 \)
\[ Q = \tau_2 \tau_3 \tau_1 \tau_3 \tau_2 \tau_1 \tau_2 \tau_3 \tau_1 \]
$W$ = finite Coxeter group
$S$ = simple system of generators for $W$
$\rho$ element of $W$
$Q = q_1 q_2 \cdots q_m$ word on the alphabet $S$

Enumerate subwords of $Q$ which are reduced expressions for $\rho$
\[ \mathfrak{S}_n = \text{symmetric group} \]

\[ S = \{ \tau_i \mid i \in n - 1 \} \text{ set of simple transpositions } \tau_i = (i \ i + 1) \]

\[ \rho \text{ permutation of } \mathfrak{S}_n \]

\[ Q = q_1 q_2 \cdots q_m \text{ word on the alphabet } S \]

**Subword complex** \( SC(Q, \rho) = \text{simplicial complex with} \)

- vertices = \([m]\) = positions in the word \(Q\)
- facets = \(\mathcal{F}(Q, \rho) = \text{complements in } [m] \text{ of position sets of reduced expressions of } \rho \text{ in } Q\)

flip = two subwords of \( Q \) which differ at precisely two positions
The flip graph is connected

GOAL: Find a natural spanning tree of the flip graph
INDUCTIVE STRUCTURE

\[ Q = q_1 q_2 \cdots q_{m-1} q_m \quad \text{and} \quad Q_\perp = q_1 q_2 \cdots q_{m-1} \]

\[ \mathcal{F}(Q, \rho) = \text{facets of } \mathcal{SC}(Q, \rho) = \text{complements of reduced expressions of } \rho \text{ in } Q \]

\[ \mathcal{F}(Q, \rho) = \mathcal{F}(Q_\perp, \rho q_m) \sqcup (\mathcal{F}(Q_\perp, \rho) \star m) \]
\[ \mathcal{F}(Q, \rho) = \mathcal{F}(Q_+, \rho q_m) \sqcup (\mathcal{F}(Q_-, \rho) \star m) \]
\( Q = q_1 q_2 \cdots q_{m-1} q_m \) and \( Q^{-} = q_1 q_2 \cdots q_{m-1} \)

\[ F(Q, \rho) = \text{facets of } SC(Q, \rho) = \text{complements of reduced expressions of } \rho \text{ in } Q \]

\[
F(Q, \rho) = \begin{cases} 
F(Q^{-}, \rho q_m) & \text{if } \rho \not\prec Q^{-} \\
F(Q^{-}, \rho) \star m & \text{if } \ell(\rho q_m) > \ell(\rho) \\
F(Q^{-}, \rho q_m) \sqcup (F(Q^{-}, \rho) \star m) & \text{otherwise}
\end{cases}
\]

⇒ Inductive enumeration of \( F(Q, \rho) \) with complexity \( O(m^2n) \) per facet
COMBINATORIAL MODELS FOR GEOMETRIC GRAPHS
bijection between

• triangulations of a convex $(n + 2)$-gon

• subwords of the odd-even word $Q = \left( \prod_{i \in \left[ \frac{n}{2} \right]} \tau_{2i+1} \cdot \prod_{i \in \left[ \frac{n}{2} \right]} \tau_{2i} \right)^{\frac{n}{2}}$

which are reduced expressions for the longest element $w_\circ = [n, n-1, \ldots, 2, 1]$
FLIP IN TRIANGULATIONS

[Diagram showing triangulations with labels S, R, U, V, W, T.
A sequence of flips is illustrated with corresponding changes in the diagram structure.]
The flip graph is the 1-skeleton of the associahedron
COMBINATORIAL MODELS FOR GEOMETRIC GRAPHS

- triangulations of convex polygons,
- multitriangulations of convex polygons,
- pseudotriangulations of point sets in general position,
- pseudotriangulations of sets of disjoint convex bodies.


GREEDY FLIP ALGORITHM
increasing flip = flip from $I$ to $J$ with $I \setminus i = J \setminus j$ and $i < j$

The increasing flip graph is acyclic, connected, and has a unique sink

greedy facet $G(Q, \rho) =$ unique sink of the increasing flip graph

$= \text{lexicographically maximal facet of } SC(Q, \rho)$
The greedy facet $G(Q, \rho)$ can be constructed inductively from $G(\varepsilon, e) = \emptyset$ using the following formulas:

$$G(Q, \rho) = \begin{cases} G(Q\downarrow, \rho) \cup m & \text{if } \rho \prec Q\downarrow \\ G(Q\downarrow, \rho q_m) & \text{otherwise} \end{cases}$$

$$G(Q, \rho) = \begin{cases} G(Q\uparrow, q_1 \rho) \rightarrow & \text{if } \ell(q_1 \rho) < \ell(\rho) \\ 1 \cup G(Q\uparrow, \rho) \rightarrow & \text{otherwise} \end{cases}$$

where $Q\downarrow = q_1 q_2 \cdots q_{m-1}$, $Q\uparrow = q_2 \cdots q_{m-1} q_m$ and $X\rightarrow = \{x + 1 \mid x \in X\}$
If $m$ is a flippable element of $G(Q, \rho)$, then $G(Q^{-1}, \rho q_m)$ is obtained from $G(Q, \rho)$ flipping $m$. 
\( \mathcal{F}(Q, \rho) = \mathcal{F}(Q_{\perp}, \rho q_m) \sqcup (\mathcal{F}(Q_{\perp}, \rho) \star m) \)
\[ G(Q, \rho) = G(Q_{\perp}, \rho q_{m}) \sqcup (G(Q_{\perp}, \rho) \ast m) \sqcup \{ \text{arc from } G(Q_{\perp}, \rho q_{m}) \text{ to } G(Q, \rho) = G(Q_{\perp}, \rho) \cup m \} \]
Inductive structure of the facets $\mathcal{F}(Q, \rho)$ of the subword complex $\mathcal{S}C(Q, \rho)$:

$$
\mathcal{F}(Q, \rho) = \begin{cases} 
\mathcal{F}(Q\upharpoonright, \rho q_m) & \text{if } \rho \not\prec Q\upharpoonright \\
\mathcal{F}(Q\upharpoonright, \rho) \ast m & \text{if } \ell(\rho q_m) > \ell(\rho) \\
\mathcal{F}(Q\upharpoonright, \rho q_m) \sqcup \big( \mathcal{F}(Q\upharpoonright, \rho) \ast m \big) & \text{otherwise}
\end{cases}
$$

Inductive definition of the greedy flip tree $\mathcal{G}(Q, \rho)$:

$$
\mathcal{G}(Q, \rho) = \begin{cases} 
\mathcal{G}(Q\upharpoonright, \rho q_m) & \text{if } \rho \not\prec Q\upharpoonright \\
\mathcal{G}(Q\upharpoonright, \rho) \ast m & \text{if } \ell(\rho q_m) > \ell(\rho) \\
\mathcal{G}(Q\upharpoonright, \rho q_m) \sqcup \big( \mathcal{G}(Q\upharpoonright, \rho) \ast m \big) \\
\sqcup \{ \text{arc from } \mathcal{G}(Q\upharpoonright, \rho q_m) \text{ to } \mathcal{G}(Q, \rho) = \mathcal{G}(Q\upharpoonright, \rho) \cup m \} & \text{otherwise}
\end{cases}
$$
$g(I) = \text{greedy index of a facet } I \in \mathcal{F}(Q, \rho) = \text{last position } x \in [m] \text{ such that } I \cap [x] = G(q_1 \cdots q_x, \sigma_{[x]} \setminus I)$

If $I, J \in \mathcal{F}(Q, \rho)$ with $I \setminus i = J \setminus j$ and $i < j \leq g(J)$, then $g(I) = j - 1$
The greedy flip tree $\mathcal{G}(Q, \rho)$ has

- nodes = $\mathcal{F}(Q, \rho)$ = complements of reduced expressions of $\rho$ in $Q$
- arcs = flip $(I, J)$ such that $I \setminus i = J \setminus j$ with $i < j \leq g(J)$. 
Greedy Flip Algorithm = Depth first search generation on the greedy flip tree
Preorder traversal provides an iterator on the reduced expressions of $\rho$ in $Q$

Working space in $O(mn)$
Running time in $O(m^2n)$ per facet $\rightarrow$ similar to the inductive algorithm

Implemented in Sage (Stump’s combinat patch on subword complexes)
Experimental time comparison to generate the $k$-triangulations of the $n$-gon:
Thank you