UNEXPECTED DIAGONALS

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arXiv:2308.12119
arXiv:2303.10986

\( P \) polytope in \( \mathbb{R}^d \)

**diagonal of** \( P = \delta : P \rightarrow P \times P \)

\[ p \mapsto (p, p) \]
$\mathbb{P}$ polytope in $\mathbb{R}^d$

diagonal of $\mathbb{P} = \delta : \mathbb{P} \to \mathbb{P} \times \mathbb{P}$
\[ p \mapsto (p, p) \]

cellular approximation of the diagonal of $\mathbb{P} = \text{map } \mathbb{P} \to \mathbb{P} \times \mathbb{P}$ s.t.
- its image is a union of faces of $\mathbb{P} \times \mathbb{P}$
- it agrees with $\delta$ on the vertices of $\mathbb{P}$
- it is homotopic to $\delta$
**DIAGONALS OF POLYTOPES**

\[ P \text{ polytope in } \mathbb{R}^d \]

**diagonal of** \( P = \delta : P \to P \times P \)

\[ p \mapsto (p, p) \]

**cellular approximation of the diagonal of** \( P = \) map \( P \to P \times P \) s.t.

- its image is a union of faces of \( P \times P \)
- it agrees with \( \delta \) on the vertices of \( P \)
- it is homotopic to \( \delta \)
any vertex of the fiber polytope

\[ \sum \left( \begin{array}{c}
\mathbb{P} \times \mathbb{P} \\
\downarrow \\
\mathbb{P} \\
\end{array} , \begin{array}{c}
(p, q) \\
\frac{p+q}{2}
\end{array} \right) \]

gives a cellular approximation of the diagonal of \( \mathbb{P} \)
projecting back on \( \mathbb{P} \), we see it as a polyhedral subdivision of \( \mathbb{P} \)
the vertex of the fiber polytope selected by \((-v, v)\)

\[
\sum \left( \frac{p \times p}{p + q} \right)
\]

gives a cellular approximation of the diagonal of \(P\) projecting back on \(P\), we see it as a polyhedral subdivision \(\Delta_{P,v}\) of \(P\)
THM. combinatorics of the diagonal $\Delta_{P,v}$ of $P$ 
\[
\cong 
\]
common refinement of two copies of the normal fan of $P$ translated by $v$

Laplante-Anfossi '22
Faces of $\Delta_{P,v} \subseteq \text{pairs } (F, G)$ such that $\max_v(F) \leq \min_v(G)$

When these are exactly the faces, it is called “magical formula”

This is the case for simplices, cubes, associahedra, but not permutahedra (see later)
THM. Faces of $\Delta_{P,v} \subseteq$ pairs $(F, G)$ such that $\max_v(F) \leq \min_v(G)$

When these are exactly the faces, it is called “magical formula”

This is the case for simplices, cubes, associahedra, but not permutahedra (see later)

\[
f_k(\Delta_{\text{Simplex}(n)}) = (k + 1) \binom{n + 1}{k + 2}
\]

\[
f_k(\Delta_{\text{Cube}(n)}) = \binom{n}{k} 2^k 3^{n-k}
\]

[OEIS, A127717]  
[OEIS, A038220]
PERMUTAHEDRON & ASSOCIAHEDRON
weak order = permutations of $[n]$ ordered by paths of simple transpositions

Tamari lattice = binary trees on $[n]$ ordered by paths of right rotations
weak order = permutations of \([n]\) ordered by paths of simple transpositions

Tamari lattice = binary trees on \([n]\) ordered by paths of right rotations
**LATTICES: WEAK ORDER & TAMARI LATTICE**

Weak order = permutations of \([n]\) ordered by paths of simple transpositions

Tamari lattice = binary trees on \([n]\) ordered by paths of right rotations

Sylvester congruence = equivalence classes are sets of linear extensions of binary trees

= equivalence classes are fibers of BST insertion

= rewriting rule \(UacVbW \equiv_{\text{sylv}} UcaVbW\) with \(a < b < c\)

Quotient lattice = lattice on classes with \(X \leq Y \iff \exists x \in X, y \in Y, x \leq y\)
**weak order** = permutations of $[n]$
ordered by paths of simple transpositions

**Tamari lattice** = binary trees on $[n]$
ordered by paths of right rotations

**sylvester congruence** = equivalence classes are sets of linear extensions of binary trees
= equivalence classes are fibers of BST insertion
= rewriting rule $UacVbW \equiv_{sylv} UcaVbW$ with $a < b < c$

**quotient lattice** = lattice on classes with $X \leq Y \iff \exists x \in X, y \in Y, x \leq y$
**FANS: BRAID FAN & SYLVESTER FAN**

\[
\text{braid fan} = \mathcal{C}(\sigma) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \} \\
\text{sylvester fan} = \mathcal{C}(T) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_i \leq x_j \text{ if } i \to j \text{ in } T \}
\]
FANS: BRAID FAN & SYLVESTER FAN

\begin{align*}
\text{braid fan} & = \mathcal{C}(\sigma) = \left\{ x \in \mathbb{R}^n \mid x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \right\} \\
\text{sylvester fan} & = \mathcal{C}(T) = \left\{ x \in \mathbb{R}^n \mid x_i \leq x_j \text{ if } i \to j \text{ in } T \right\}
\end{align*}
braid fan = \( \mathcal{C}(\sigma) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \} \)

sylvester fan = \( \mathcal{C}(T) = \{ \mathbf{x} \in \mathbb{R}^n \mid x_i \leq x_j \text{ if } i \to j \text{ in } T \} \)

quotient fan = \( \mathcal{C}(T) \) is obtained by gluing \( \mathcal{C}(\sigma) \) for all linear extensions \( \sigma \) of \( T \)
**Polytopes: Permutahedron & Associahedron**

**Permutahedron** $\text{Perm}(n)$

- $\text{conv} \left\{ [\sigma^{-1}(i)]_{i \in [n]} \mid \sigma \in \mathfrak{S}_n \right\}$
- $H \cap \bigcap_{\emptyset \neq J \subset [n]} H_J$

Where $H_J = \left\{ x \in \mathbb{R}^n \mid \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$

**Associahedron** $\text{Asso}(n)$

- $\text{conv} \left\{ [\ell(T, i) \cdot r(T, i)]_{i \in [n]} \mid T \text{ binary tree} \right\}$
- $H \cap \bigcap_{1 \leq i < j \leq n} H_{[i,j]}$

Stasheff (’63) 
Shnider – Sternberg (’93) 
Loday (’04)
**POLYTOPES: PERMUTAHEDRON & ASSOCIAHEDRON**

**Permutahedron** $\text{Perm}(n)$

\[
\text{Perm}(n) = \text{conv} \left\{ [\sigma^{-1}(i)]_{i \in [n]} \mid \sigma \in \mathfrak{S}_n \right\} \\
= H \cap \bigcap_{\emptyset \neq J \subseteq [n]} H_J
\]

where $H_J = \left\{ x \in \mathbb{R}^n \mid \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$

**Associahedron** $\text{Asso}(n)$

\[
\text{Asso}(n) = \text{conv} \left\{ [\ell(T, i) \cdot r(T, i)]_{i \in [n]} \mid T \text{ binary tree} \right\} \\
= H \cap \bigcap_{1 \leq i < j \leq n} H_{[i,j]}
\]

Stasheff (’63)  
Shnider – Sternberg (’93)  
Loday (’04)
permutahedron $\Perm(n)$
\[
= \text{conv} \left\{ [\sigma^{-1}(i)]_{i \in [n]} \mid \sigma \in \mathfrak{S}_n \right\}
\]
\[= \mathbb{H} \cap \bigcap_{\emptyset \neq J \subsetneq [n]} \mathbb{H}_J\]
where $\mathbb{H}_J = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$

associahedron $\Asso(n)$
\[
= \text{conv} \left\{ \ell(T, i) \cdot r(T, i)_{i \in [n]} \mid T \text{ binary tree} \right\}
\]
\[= \mathbb{H} \cap \bigcap_{1 \leq i < j \leq n} \mathbb{H}_{[i,j]}\]
Stasheff ('63)
Shnider – Sternberg ('93)
Loday ('04)
LATTICES – FANS – POLYTOPES

permutahedron $\mathbb{Perm}(n)$

$\rightarrow$ braid fan

$\rightarrow$ weak order on permutations

associahedron $\mathbb{Asso}(n)$

$\rightarrow$ Sylvester fan

$\rightarrow$ Tamari lattice on binary trees
$F$-VECTOR OF DIAGONALS

Saneblidze – Umble ’04
Markl – Shnider ’06
Loday ’11

Masuda – Thomas – Tonks – Vallette ’21
Laplante-Anfossi ’22
\[ f_k = \sum_{F \leq G} \prod_{i \in [2]} \prod_{p \in G_i} (\#F_i[p] - 1)! \]

\[ f_0 = [x^n] \exp \left( \sum_m \frac{x^m}{m(m+1)} \binom{2m}{m} \right) \]

\[ f_{n-1} = 2(n + 1)^{n-2} \]

\[ f_k = \frac{2}{(3n + 1)(3n + 2)} \binom{n - 1}{k} \binom{4n + 1 - k}{n + 1} \]
DIAGONAL OF THE ASSOCIAHEDRON

arXiv:2303.10986

with
Alin BOSTAN (INRIA)
Frédéric CHYZAK (INRIA)
$\text{Tam}(n) = \text{Tamari lattice on binary trees with } n \text{ nodes}$
NUMBER OF TAMARI INTERVALS

Tam\((n)\) = Tamari lattice on binary trees with \(n\) nodes

**THM.** For any \(n \geq 1\),

\[
\#\{S \leq T \in \text{Tam}(n)\} = \frac{2}{(3n + 1)(3n + 2)} \binom{4n + 1}{n + 1}
\]

1, 3, 13, 68, 399, 2530, 16965, ... [OEIS A000260]
Tam(n) = Tamari lattice on binary trees with \( n \) nodes
\( \text{des}(T) \) = number of binary trees covered by \( T \)
\( \text{asc}(T) \) = number of binary trees covering \( T \)
**FIRST REFINED FORMULA ON TAMARI INTERVALS**

Tam$(n) =$ Tamari lattice on binary trees with $n$ nodes

des$(T) =$ number of binary trees covered by $T$

asc$(T) =$ number of binary trees covering $T$

**THM.** For any $n, k \geq 1$,

\[
\# \{ S \leq T \in \text{Tam}(n) \mid \text{des}(S) + \text{asc}(T) = k \} = \frac{2}{n(n+1)} \binom{n+1}{k+2} \binom{3n}{k}
\]

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CANONICAL COMPLEX OF THE TAMARI LATTICE

\((L, \leq, \land, \lor)\) lattice

**join semidistributive** \iff \(x \lor y = x \lor z\) implies \(x \lor (y \land z) = x \lor y\) for all \(x, y, z \in L\)

\iff any \(x \in L\) admits a canonical join representation \(x = \lor J\)

canonical join complex = simplicial complex of canonical join representations

= a simplex \(J\) for each element \(\lor J\) of \(L\)
(\(L, \leq, \land, \lor\)) lattice

\textbf{meet semidistributive} \iff x \land y = x \land z \text{ implies } x \land (y \lor z) = x \land y \text{ for all } x, y, z \in L

\iff \text{any } x \in L \text{ admits a canonical meet representation } x = \bigwedge M

canonical meet complex = simplicial complex of canonical meet representations

= a simplex \(M\) for each element \(\bigwedge M\) of \(L\)
CANONICAL COMPLEX OF THE TAMARI LATTICE

$(L, \leq, \land, \lor)$ lattice

\textbf{semidistributive} $\iff$ join semidistributive and meet semidistributive

$\iff$ any $x \in L$ admits canonical representations $x = \lor J = \land M$

\textbf{canonical complex} = simplicial complex of canonical representations

$= \text{a simplex } J \sqcup M \text{ for each interval } \lor J \leq \land M \text{ in } L$
CANONICAL COMPLEX OF THE TAMARI LATTICE

$(L, \leq, \land, \lor)$ lattice

semidistributive $\iff$ join semidistributive and meet semidistributive
$\iff$ any $x \in L$ admits canonical representations $x = \lor J = \land M$

canonical complex = simplicial complex of canonical representations
$= a$ simplex $J \sqcup M$ for each interval $\lor J \leq \land M$ in $L$
THM. For any $n, k \geq 1$,

$$f_k(\mathcal{CC}_n) = \# \{S \leq T \in \text{Tam}(n) \mid \text{des}(S) + \text{asc}(T) = k\} = \frac{2}{n(n+1)} \binom{n+1}{k+2} \binom{3n}{k}$$

Bostan – Chyzak – P.'23+
1 + 12 + 33 + 22 = 68
SECOND REFINED FORMULA ON TAMARI INTERVALS

\[ \text{Tam}(n) = \text{Tamari lattice on binary trees with } n \text{ nodes} \]

\[ \text{des}(T) = \text{number of binary trees covered by } T \]

\[ \text{asc}(T) = \text{number of binary trees covering } T \]

**THM.** For any \( n, k \geq 1 \),

\[
\sum_{S \leq T \in \text{Tam}(n)} \binom{\text{des}(S) + \text{asc}(T)}{k} = \frac{2}{(3n + 1)(3n + 2)} \binom{n - 1}{k} \binom{4n + 1 - k}{n + 1}
\]

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$\Delta_{\text{asso}}(n) = \text{diagonal of } (n - 1)\text{-dimensional associahedron}$
$\Delta_{\text{asso}(n)} = \text{diagonal of } (n - 1)\text{-dimensional associahedron}$
\[ \Delta_{\text{Asso}(n)} = \text{diagonal of } (n - 1)\text{-dimensional associahedron} \]

**THM. (Magical formula)**

\[
\begin{align*}
  \text{k-faces of } \Delta_{\text{Asso}(n)} & \iff (F, G) \text{ faces of } \text{Asso}(n) \text{ with} \\
  \dim(F) + \dim(G) &= k \text{ and } \max(F) \leq \min(G)
\end{align*}
\]

Masuda – Thomas – Tonks – Vallette ’21
\[ \Delta_{\text{asso}}(n) = \text{diagonal of } (n - 1)\text{-dimensional associahedron} \]

**THM.** For any \( n, k \geq 1 \),

\[
f_k(\Delta_{\text{asso}}(n)) = \sum_{S \leq T \in \text{Tam}(n)} \binom{\text{des}(S) + \text{asc}(T)}{k} = \frac{2}{(3n + 1)(3n + 2)} \binom{n - 1}{k} \binom{4n + 1 - k}{n + 1}
\]

Bostan – Chyzak – P. '23
CONNECTION BETWEEN THE TWO FORMULAS

**THM.** For any \( n, k \geq 1 \),

\[
f_k(\mathcal{C}_n) = \# \{ S \leq T \in \text{Tam}(n) \mid \text{des}(S) + \text{asc}(T) = k \} = \frac{2}{n(n+1)} \binom{n+1}{k+2} \binom{3n}{k}
\]

**THM.** For any \( n, k \geq 1 \),

\[
f_k(\Delta_{\text{Asso}}(n)) = \sum_{S \leq T \in \text{Tam}(n)} \binom{\text{des}(S) + \text{asc}(T)}{k} = \frac{2}{(3n+1)(3n+2)} \binom{n-1}{k} \binom{4n+1-k}{n+1}
\]
CONNECTION BETWEEN THE TWO FORMULAS

**THM.** For any $n, k \geq 1$,

\[ f_k(\mathbb{C}C_n) = \# \{ S \leq T \in \text{Tam}(n) \mid \text{des}(S) + \text{asc}(T) = k \} = \frac{2}{n(n+1)} \binom{n+1}{k+2} \binom{3n}{k} \]

**THM.** For any $n, k \geq 1$,

\[ f_k(\Delta_{\text{asso}}(n)) = \sum_{S \leq T \in \text{Tam}(n)} \binom{\text{des}(S) + \text{asc}(T)}{k} = \frac{2}{(3n+1)(3n+2)} \binom{n-1}{k} \binom{4n+1-k}{n+1} \]

Second formula follows from the first since ...

**THM.** For any $n, k, r \in \mathbb{N}$,

\[ \sum_{\ell=k}^{n-1} \binom{n+1}{\ell+2} \binom{r}{\ell} \binom{\ell}{k} = \frac{n(n+1)}{(r+1)(r+2)} \binom{n-1}{k} \binom{r+n+1-k}{n+1} \]

... by application of Chu – Vandermonde equality
\[ n(T) = \text{number of nodes of } T \]
\[ \ell(T) = \text{number of bounded edges on the left branch of } T \]

\[ \mathcal{A}(u, v, t, z) := \sum_{S \leq T} u^{\ell(S)} v^{\ell(T)} t^{n(S)} z^{\text{des}(S) + \text{asc}(T)} \]
QUADRATIC EQUATION

\[ n(T) = \text{number of nodes of } T \]
\[ \ell(T) = \text{number of bounded edges on the left branch of } T \]

\[
\mathbb{A}(u, v, t, z) := \sum_{S \leq T} u^{\ell(S)} v^{\ell(T)} t^{n(S)} z^{\text{des}(S) + \text{asc}(T)}
\]

We want to compute

\[
A := A(t, z) := \sum_{S \leq T} t^{n(S)} z^{\text{des}(S) + \text{asc}(T)} = \mathbb{A}(1, 1, t, z)
\]

we will use \( u \) and \( v \) as catalytic variables ...
QUADRATIC EQUATION

\[ n(T) = \text{number of nodes of } T \]
\[ \ell(T) = \text{number of bounded edges on the left branch of } T \]

\[
\mathbb{A}(u, v, t, z) := \sum_{S \leq T} u^{\ell(S)} v^{\ell(T)} t^{n(S)} z^{\text{des}(S) + \text{asc}(T)}
\]

We want to compute

\[
A := A(t, z) := \sum_{S \leq T} t^{n(S)} z^{\text{des}(S) + \text{asc}(T)} = \mathbb{A}(1, 1, t, z)
\]

we will use \( u \) and \( v \) as catalytic variables ...

**PROP.** The generating functions \( A_u := \mathbb{A}(u, 1, t, z) \) and \( A_1 := \mathbb{A}(1, 1, t, z) \) satisfy the quadratic functional equation

\[
(u - 1)A_u = t(u - 1 + u(u + z - 1)A_u - zA_1)(1 + uzA_u)
\]
GRAFTING DECOMPOSITIONS

$S \setminus T =$ binary tree obtained by grafting $S$ on the leftmost leaf of $T$

$S = S_0 \setminus S_1 \ldots \setminus S_k$ grafting decomposition

\[ \begin{array}{ccc}
\text{red} & = & \text{green} \\
\text{blue} & = & \text{green} \\
\end{array} \]

**LEM.** If $S = S_0 \setminus S_1 \ldots \setminus S_k$ and $T = T_0 \setminus T_1 \ldots \setminus T_k$ are s.t. $n(S_i) = n(T_i)$ for all $i \in [k]$, then

$S \leq T \iff S_i \leq T_i$ for all $i \in [k]$  

Chapoton '07

\[ \begin{array}{ccc}
\text{red} & = & \text{green} \\
\text{blue} & = & \text{green} \\
\end{array} \]

**LEM.** If $S \leq T$, then we can write $S = S_0 \setminus S_1 \ldots \setminus S_\ell$ and $T = T_0 \setminus T_1 \ldots \setminus T_\ell$ where

$\ell = \ell(T)$ and $n(S_i) = n(T_i)$ for all $i \in [\ell]$  

Chapoton '07

$\ell(T) =$ number of bounded edges on the left branch of $T$
\[ n(T) = \text{number of nodes of } T \]
\[ \ell(T) = \text{number of bounded edges on the left branch of } T \]

\[
\mathbb{A}(u, v, t, z) := \sum_{S \leq T} u^{\ell(S)} n(S) z^{\text{des}(S) + \text{asc}(T)}
\]

Consider

\[
A_u(t, z) := \mathbb{A}(u, 1, t, z)
\]

and

\[
A_u^0(t, z) := \mathbb{A}(u, 0, t, z)
\]

= all Tamari intervals

= indecomposable intervals
QUADRATIC EQUATION

$A_u = A_u(t, z) = \text{all Tamari intervals}$

$A^\circ_u = A^\circ_u(t, z) = \text{indecomposable intervals}$

$$\sum_{S \leq T} u^{\ell(S)} t^{n(S)} z^{\text{des}(S) + \text{asc}(T)}$$

Chapoton '07
\[ A_u = A_u(t, z) = \text{all Tamari intervals} \]
\[ A_u^\circ = A_u^\circ(t, z) = \text{indecomposable intervals} \]

1. all intervals \( = \) indecomposable intervals \( \setminus (\varepsilon + \text{all intervals}) \)

\[ A_u \quad = \quad A_u^\circ \quad (1 + uzA_u) \]
\[ A_u = A_u(t, z) = \text{all Tamari intervals} \]
\[ A_u^\circ = A_u^\circ(t, z) = \text{indecomposable intervals} \]

1. all intervals = indecomposable intervals \( \setminus (\varepsilon + \text{all intervals}) \)
   \[ A_u = A_u^\circ (1 + uzA_u) \]

2. from any Tamari interval \((S, T)\) where \(S = S_0/S_1/\ldots/S_\ell(S)\), we can construct \(\ell(S) + 2\) indecomposable Tamari intervals \((S'_k, T')\) for \(0 \leq k \leq \ell(S) + 1\), where
   \[ S'_k = (S_0/\ldots/S_{k-1})/Y\setminus(S_k/\ldots/S_\ell(S)) \quad \text{and} \quad T' = Y\setminus T \]

... and all indecomposable intervals are obtained this way

\[ A_u^\circ = t \left( 1 + \frac{zuA_u - A_1}{u - 1} + uA_u \right) \]

Chapoton '07
QUADRATIC EQUATION

\[ A_u = A_u(t, z) = \text{all Tamari intervals} \]
\[ A_u^\circ = A_u^\circ(t, z) = \text{indecomposable intervals} \]

1. \[ A_u = A_u^\circ(1 + uzA_u) \]

2. \[ A_u^\circ = t \left( 1 + z \frac{uA_u - A_1}{u - 1} + uA_u \right) \]

**PROP.** The generating functions \( A_u := A(u, 1, t, z) \) and \( A_1 := A(1, 1, t, z) \) satisfy the quadratic functional equation

\[ (u - 1)A_u = t \left( u - 1 + u(u + z - 1)A_u - zA_1 \right) \left( 1 + uzA_u \right) \]
QUADRATIC METHOD

**PROP.** The generating functions $A_u := A(u, 1, t, z)$ and $A_1 := A(1, 1, t, z)$ satisfy the quadratic functional equation

$$(u - 1)A_u = t\left(u - 1 + u(u + z - 1)A_u - zA_1\right)(1 + uzA_u)$$

Quadratic equation with a catalytic variable... quadratic method
The discriminant of this quadratic polynomial must have multiple roots, hence, its own discriminant vanishes

**CORO.** The generating function $A = A(t, z)$ is a root of the polynomial

$$t^3 z^6 X^4$$

$$+ t^2 z^4 (t z^2 + 6 t z - 3 t + 3) X^3$$

$$+ t z^2 (6 t^2 z^3 + 9 t^2 z^2 - 12 t^2 z + 2 t z^2 + 3 t^2 - 6 t z + 21 t + 3) X^2$$

$$+ (12 t^3 z^4 - 4 t^3 z^3 - 9 t^3 z^2 - 10 t^2 z^3 + 6 t^3 z + 26 t^2 z^2$$

$$- t^3 + 6 t^2 z + t z^2 + 3 t^2 - 12 t z - 3 t + 1) X$$

$$+ t(8 t^2 z^3 - 12 t^2 z^2 + 6 t^2 z - t z^2 - t^2 + 10 t z + 2 t - 1)$$
The generating function $A = A(t, z)$ is a root of the polynomial

\[t^3 z^6 X^4 + t^2 z^4 (tz^2 + 6tz - 3t + 3) X^3 \]
\[+ t z^2 (6t^2 z^3 + 9t^2 z^2 - 12t^2 z + 2t z^2 + 3t^2 - 6tz + 21t + 3) X^2 \]
\[+ (12t^3 z^4 - 4t^3 z^3 - 9t^3 z^2 - 10t^2 z^3 + 6t^3 z + 26t^2 z^2 \]
\[- t^3 + 6t^2 z + tz^2 + 3t^2 - 12tz - 3t + 1) X \]
\[+ t(8t^2 z^3 - 12t^2 z^2 + 6t^2 z - tz^2 - t^2 + 10tz + 2t - 1) \]

Reparametrize by

\[t = \frac{s}{(s + 1)(sz + 1)^3} \quad X = s - zs^2 - zs^3 \]

The generating function $A = A(t, z)$ can be written

\[A = S - zs^2 - zs^3 \quad \text{where} \quad t = \frac{S}{(S + 1)(Sz + 1)^3} \]
LAGRANGE INVERSION

**CORO.** The generating function $A = A(t, z)$ can be written

$$A = S - zS^2 - zS^3 \quad \text{where} \quad t = \frac{S}{(S+1)(Sz+1)^3}$$

**THM.** (Lagrange inversion) If $S = t \psi(S)$, then $[t^n] S^r = \frac{r}{n} [s^{n-r}] \phi(s)^n$ for any $r \geq 1$

Here $\phi(s) := (s + 1)(sz + 1)^3$

Hence $[s^a] \phi(s)^n = [s^a](s + 1)^n(sz + 1)^3n = \sum_{i+j=a} \binom{n}{i} \binom{3n}{j} z^j$

Hence $[t^n z^k] S^r = \frac{r}{n} [s^{n-r} z^k] \phi(s)^n = \frac{r}{n} \binom{n}{n-r-k} \binom{3n}{k} = \frac{r}{n} \binom{n}{k+r} \binom{3n}{k}$

Finally,

$$[t^n z^k] A = [t^n z^k] S - [t^n z^{k-1}] S^2 - [t^n z^{k-1}] S^3 = \frac{2}{n(n+1)} \binom{3n}{k} \binom{n+1}{k+2}$$
Tam\( (n) \) = Tamari lattice on binary trees with \( n \) nodes

**THM.** For any \( n \geq 1 \),

\[
\#\{S \leq T \in \text{Tam}(n)\} = \frac{2}{(3n+1)(3n+2)} \binom{4n+1}{n+1}
\]

Chapoton '07

Also counts rooted 3-connected planar triangulations with \( 2n + 2 \) faces

Tutte
BIJECTIONS TO PLANAR TRIANGULATIONS

Tam\( (n) \) = Tamari lattice on binary trees with \( n \) nodes

**THM.** For any \( n \geq 1 \),

\[
\# \{ S \leq T \in \text{Tam}(n) \} = \frac{2}{(3n+1)(3n+2)} \binom{4n+1}{n+1}
\]

Also counts rooted 3-connected planar triangulations with \( 2n + 2 \) faces
BIJECTIONS TO PLANAR TRIANGULATIONS

Tam(n) = Tamari lattice on binary trees with n nodes

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\]

Also counts rooted 3-connected planar triangulations with \( 2n + 2 \) faces

Bernardi – Bonichon, '09
$M$ planar triangulation with external vertices $v_0, v_1, v_3$
$n$ internal nodes, $3n$ internal edges, $2n + 1$ internal triangles

**Schnyder wood** = color (with 0, 1, 2) and orient the internal edges s.t.
- the edges colored $i$ form a spanning tree oriented towards $v_i$
- each vertex satisfies the vertex rule:
$M$ planar triangulation with external vertices $v_0, v_1, v_3$

$n$ internal nodes, $3n$ internal edges, $2n + 1$ internal triangles

**Schnyder wood** = color (with 0, 1, 2) and orient the internal edges s.t.
- the edges colored $i$ form a spanning tree oriented towards $v_i$
- each vertex satisfies the vertex rule:

Used for graph drawing and representations:
A planar triangulation with external vertices $v_0, v_1, v_3$

$n$ internal nodes, $3n$ internal edges, $2n + 1$ internal triangles

**Schnyder wood** = color (with 0, 1, 2) and orient the internal edges s.t.
- the edges colored $i$ form a spanning tree oriented towards $v_i$
- each vertex satisfies the **vertex rule**:

**THM.** The Schnyder woods of a planar triangulation form a lattice structure under reorientations of clockwise essential cycles

**CORO.** Any planar triangulation admits a unique Schnyder wood with no clockwise cycle

---

Ossona de Mendez '94
Propp '97
Felsner '04
binary trees $S \leq T$ with $n$ nodes

Dyck paths $\mu \leq \nu$ with semilength $n$

planar triangulations with $n$ internal vertices
BERNARDI – BONICHON BIJECTION

binary trees \( S \leq T \) with \( n \) nodes

descents of \( S \) ascents of \( T \)

Dyck paths \( \mu \leq \nu \) double falls of \( \mu \) valleys of \( \nu \)

with semilength \( n \)

planar triangulations intermediate red vertices intermediate blue vertices

with \( n \) internal vertices

Bernardi – Bonichon, ’09
THM. The generating function $F := F(u, v, w) := \sum_{S \leq T} u^{S} v^{v} w^{w}$ is given by

$$wvF = uU + vV + wUV - \frac{UV}{(1 + U)(1 + V)}$$

where the series $U := U(u, v, w)$ and $V := V(u, v, w)$ satisfy the system

$$U = (v + wU)(1 + U)(1 + V)^{2}$$
$$V = (u + wV)(1 + V)(1 + U)^{2}$$

Fusy – Humbert '19
CORO. The function $A := A(t, z) := \sum_{S \leq T} t^n(S) z^{\text{des}(S)} + \text{asc}(T) = tF(tz, tz, t)$ is given by

$$tz^2 A = 2tzS + tS^2 - \frac{S^2}{(1 + S)^2}$$

where the series $S := S(t, z)$ satisfies

$$S = t(z + S')(1 + S')^3$$

... and Lagrange inversion again (thanks to Éric Fusy)
$T$ binary tree with $n$ nodes, labeled in inorder and oriented towards its root.

**canopy of** $T = \text{vector } \text{can}(T) \in \{-, +\}^{n-1}$ with $\text{can}(T)_i = -$  
  $\iff$ $(j+1)$st leaf of $T$ is a right leaf  
  $\iff$ there is an oriented path joining its $j$th node to its $(j+1)$st node  
  $\iff$ the $j$th node of $T$ has an empty right subtree  
  $\iff$ the $(j+1)$st node of $T$ has a non-empty left subtree  
  $\iff$ the cone corresponding to $T$ is located in the halfspace $x_j \leq x_{j+1}$
$T$ binary tree with $n$ nodes, labeled in inorder and oriented towards its root.

**canopy of** $T = \text{vector } \text{can}(T) \in \{-, +\}^{n-1}$ with $\text{can}(T)_i = -$  
$\iff$ the $j$th node of $T$ has an empty right subtree  
$\iff$ the $(j+1)$st node of $T$ has a non-empty left subtree

**LEM.** \[ \text{asc}(T) = \# \{i \mid \text{can}(T)_i = -\} \quad \text{and} \quad \text{des}(T) = \# \{i \mid \text{can}(T)_i = +\} \]

**LEM.** If $S \leq T$, then  
\begin{itemize}
  \item $\text{can}(S) \leq \text{can}(T)$ componentwise  
  \item $\text{des}(S) = \# \{i \mid \text{can}(S)_i = \text{can}(T)_i = +\}$ and $\text{asc}(S) = \# \{i \mid \text{can}(S)_i = \text{can}(T)_i = -\}$
\end{itemize}

**CORO.** \[ \text{des}(S) + \text{asc}(T) = \#\text{canopy agreements between } S \text{ and } T \]
FANG – FUSY – NADEAU BIJECTION

\[ \sum \text{meandres} \]

\[ u \leftrightarrow v \leftrightarrow w \]

Diagram of meander with labeled elements.
\[ \sum_{\text{meandres}} \left( \left( \begin{array}{c} \updownarrow \\ \downarrow \end{array} \right) + \left( \begin{array}{c} \updownarrow \\ \downarrow \end{array} \right) + \left( \begin{array}{c} \updownarrow \\ \downarrow \end{array} \right) \right) - 1 \right) \begin{array}{c} u \\ v \\ w \end{array} \]
\[ \sum_{\text{meandres}} (\underline{+} + \underline{+} + \underline{-} - 1) uvw = \sum_{\text{cyan half-meanders}} uvw \cdot \sum_{\text{orange half-meanders}} uvw \]
\[ \sum \text{meandres} \left( u \varepsilon v \varepsilon w - 1 \right) = \text{CHM}(u, v, w) \cdot \text{OHM}(u, v, w) \]
\[
\sum_{\text{meandres}} \left( \frac{1}{1 + \frac{1}{1 - \text{CHM}}} \right)^2 \left( u + \frac{w}{1 - \text{OHM}} \right) = \text{CHM}(u, v, w) \cdot \text{OHM}(u, v, w)
\]
FANG – FUSY – NADEAU BIJECTION

\[ \sum_{\text{meandres}} (u + v + w - 1) u v w = \text{CHM}(u, v, w) \cdot \text{OHM}(u, v, w) \]

\[
\begin{aligned}
\text{CHM} &= \frac{1}{(1 - \text{CHM})^2} \left(u + \frac{w \text{OHM}}{1 - \text{OHM}}\right) \\
\text{and} \quad \text{OHM} &= \frac{1}{(1 - \text{OHM})^2} \left(v + \frac{w \text{CHM}}{1 - \text{CHM}}\right)
\end{aligned}
\]
\[
\sum_{\text{meandres}} \frac{1}{\varphi} \left( 1 - \frac{1}{\varphi} - 1 \right) (t z)^{\frac{1}{2}} (t z)^{\frac{1}{2}} t = \text{HM}(t, z)^2
\]

where

\[
\text{HM} = \frac{t}{(1 - \text{HM})^2} \left( z + \frac{\text{HM}}{1 - \text{HM}} \right)
\]
\[
\sum_{\text{meandres}} (z + \frac{1}{z} + \frac{1}{z} - 1)(tz)^{\frac{1}{2}}(tz)^{\frac{1}{2}}t = \text{HM}(t, z)^2
\]

where
\[
\text{HM} = \frac{t}{(1 - \text{HM})^2}(z + \frac{\text{HM}}{1 - \text{HM}})
\]

Lagrange inversion again:
\[
[t^n z^k] \text{HM}^2 = \frac{2}{n} [s^{n-2} z^k] \frac{1}{(1 - s)^{2n}} (z + \frac{s}{1 - s})^n = \frac{2}{n} \binom{n}{k} [s^{n-k}] \frac{s^{n-k}}{(1 - s)^{3n-k}}
\]
\[
= \frac{2}{n} \binom{n}{k} [s^{k-2}] \frac{1}{(1 - s)^{3n-k}} = \frac{2}{n} \binom{n}{k} \left( \frac{3n - 3}{k - 2} \right)
\]
\[
\sum_{\text{meandres}} \left( \frac{1}{t} + \frac{1}{z} + \frac{1}{t} - 1 \right) (tz)^{\frac{1}{2}} (tz)^{\frac{1}{2}} t = \text{HM}(t, z)^2
\]

where
\[
\text{HM} = \frac{t}{(1 - \text{HM})^2} \left( z + \frac{\text{HM}}{1 - \text{HM}} \right)
\]

Lagrange inversion again:
\[
[t^n z^k] \text{HM}^2 = \frac{2}{n} \left[ s^{n-2} z^k \right] \frac{1}{(1 - s)^{2n}} \left( z + \frac{s}{1 - s} \right)^n = \frac{2}{n} \binom{n}{k} \left[ s^{n-2} \right] \frac{s^{n-k}}{(1 - s)^{3n-k}}
\]
\[
= \frac{2}{n} \binom{n}{k} \left[ s^{k-2} \right] \frac{1}{(1 - s)^{3n-k}} = \frac{2}{n} \binom{n}{k} \binom{3n - 3}{k - 2}
\]

Hence
\[
[t^n z^k] A(t, z) = \frac{1}{n + 1} \left[ t^{n+1} z^{k+2} \right] \text{HM}^2 = \frac{2}{n(n + 1)} \binom{n + 1}{k + 2} \binom{3n}{k}
\]
DIAGONAL OF THE PERMUTAHEDRON

with

Bérénice DELCROIX-OGER (Univ. Montpellier)
Matthieu JOSUAT-VERGES (CNRS & Univ. Paris Cité)
Guillaume LAPLANTE-ANFOSSI (Univ. Melbourne)
Kurt STOECKL (Univ. Melbourne)
\[ \Delta_{\text{Perm}(n)} = \text{diagonal of } (n - 1)\text{-dimensional permutahedron} \]

**THM.** \( k \)-faces of \( \Delta_{\text{Perm}(n)} \) \( \iff \) \( (\mu, \nu) \) ordered partitions of \([n]\) such that

\[
\forall (I, J) \in D(n), \ \exists k \in [n], \ #\mu_{[k]} \cap I > #\mu_{[k]} \cap J
\]

or \( \exists \ell \in [n], \ #\mu_{[\ell]} \cap I < #\mu_{[\ell]} \cap J \)

where \( D(n) := \{(I, J) \mid I, J \subseteq [n], \ #I = #J, \ I \cap J = \emptyset, \ \min(I \cup J) \in I\} \)

Laplante-Anfossi '22
\( \Delta_{\text{Perm}(n)} = \text{diagonal of } (n - 1)\text{-dimensional permutahedron} \)

**PROP.** \( B^2_n = \text{two generically translated copies of the braid arrangement} \)

\[ f_k\left( \Delta_{\text{Perm}(n)} \right) = f_{n-k-1}\left( B^2_n \right) \]
flat poset $Fl(A)$ of an hyperplane arrangement $A = \\text{reverse inclusion poset on nonempty intersections of hyperplanes of } A$
FLAT POSET & ZASLAVSKY’S THEOREM

flat poset $\text{Fl}(\mathcal{A})$ of an hyperplane arrangement $\mathcal{A} =$ reverse inclusion poset on nonempty intersections of hyperplanes of $\mathcal{A}$

\begin{align*}
\text{EXM.} & \quad \text{flat poset of braid arrangement } \mathcal{B}_n \\
\text{refinement poset on partitions of } [n] & \quad \left\{ \begin{array}{c}
\mathbf{x} \in \mathbb{R}^n \\
x_i = x_j \text{ for all } i, j \text{ in the same part of } \pi
\end{array} \right\}
\end{align*}

$\uparrow$

$\downarrow$

$\pi$
flat poset $Fl(\mathcal{A})$ of an hyperplane arrangement $\mathcal{A}$ = reverse inclusion poset on nonempty intersections of hyperplanes of $\mathcal{A}$

Möbius function $\mu$ of a poset: $\mu(x, x) = 1$ and $\sum_{x \leq y \leq z} \mu(x, y) = 0$ for all $x < z$
The flat poset $\text{Fl}(\mathcal{A})$ of an hyperplane arrangement $\mathcal{A}$ is defined as the reverse inclusion poset on nonempty intersections of hyperplanes of $\mathcal{A}$.

The Möbius function $\mu$ of a poset: $\mu(x, x) = 1$ and $\sum_{x \leq y \leq z} \mu(x, y) = 0$ for all $x < z$. 
flat poset $Fl(A)$ of an hyperplane arrangement $A =$ reverse inclusion poset on nonempty intersections of hyperplanes of $A$

Möbius function $\mu$ of a poset: $\mu(x, x) = 1$ and $\sum_{x \leq y \leq z} \mu(x, y) = 0$ for all $x < z$

Möbius polynomial $\mu_A(x, y) = \sum_{F \leq G} \mu(F, G) x^{\dim(F)} y^{\dim(G)}$

**THM.** $f_A(x) = \mu_A(-x, -1)$ and $b_A(x) = \mu_A(-x, 1)$

Zaslavsky '75
$\mathcal{B}_n^\ell = \text{union of } \ell \text{ generically translated copies of the braid arrangement}$
\( \mathcal{B}^\ell_n = \) union of \( \ell \) generically translated copies of the braid arrangement

\((\ell, n) \text{ partition forest} = \)
\( \ell \)-tuple of partitions of \([n]\) whose intersection hypergraph is a forest

**Prop.** Intersection poset of \( \mathcal{B}^\ell_n \) \( \leftrightarrow \) refinement poset on \((\ell, n) \) partition forests
\( \mathcal{B}_n^\ell = \text{union of } \ell \text{ generically translated copies of the braid arrangement} \)

\((\ell, n) \text{ partition forest} =\)
\(\ell\)-tuple of partitions of \([n]\) whose intersection hypergraph is a forest

**PROP.** Intersection poset of \( \mathcal{B}_n^\ell \) \( \longleftrightarrow \) refinement poset on \((\ell, n)\) partition forests
MÖBIUS POLYNOMIAL

$\mathbb{P}_p = \text{refinement poset on partitions of } [p]$

$\mathbb{PF}_n^\ell = \text{refinement poset on } (\ell, n) \text{ partition forests}$

**FACT 1.** The Möbius function of $\mathbb{P}_p$ is $\mu(\hat{0}, \hat{1}) = (-1)^{p-1}(p-1)!$

**FACT 2.** In $\mathbb{P}_p$, $[F, G] \simeq \prod_{p \in G} \mathbb{P}_p \#F[p]$ where $F[p] = \text{restriction of } F \text{ to } p$

**FACT 2.** $[F, G] \simeq \prod_{i \in [\ell]} [F_i, G_i]$ for $F = (F_1, \ldots, F_\ell)$ and $G = (G_1, \ldots, G_\ell)$ in $\mathbb{PF}_n^\ell$

**FACT 4.** Möbius is multiplicative $\mu_{P \times Q}((p, q), (p', q')) = \mu_P(p, p') \cdot \mu_Q(q, q')$
The Möbius function of $\mathbb{P}_p$ is $\mu(\hat{0}, \hat{1}) = (-1)^{p-1}(p-1)!$

In $\mathbb{P}_p$, $[F, G] \simeq \prod_{p \in G} \mathbb{P}^{{\#F}[p]}$ where $F[p] = \text{restriction of } F \text{ to } p$

for $F = (F_1, \ldots, F_\ell)$ and $G = (G_1, \ldots, G_\ell)$ in $\mathbb{PF}^{\ell}_n$

$\mu_{P \times Q}((p, q), (p', q')) = \mu_P(p, p') \cdot \mu_Q(q, q')$

$\mu_{B_n} = x^{n-1-\ell n} y^{n-1-\ell n} \sum_{F \leq G} \prod_{i \in [\ell]} x^{\#F_i} y^{\#G_i} \prod_{p \in G_i} (-1)^{\#F_i[p]-1} (\#F_i[p] - 1)!$
THM. \[
f_{B_n^\ell}(x) = x^{n-1-\ell n} \sum_{F \leq G} \prod_{i \in [\ell]} x^{\#F_i} \prod_{p \in G_i} (\#F_i[p] - 1)!
\]
**BOUNDLED FACE POLYNOMIAL**

**THM.**

\[
b_{B_{n}^\ell}(x) = (-1)^\ell x^{n-1-\ell n} \sum_{F \leq G} \prod_{i \in [\ell]} x^{\# F_i} \prod_{p \in G_i} - (\# F_i[p] - 1)!
\]

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\( \ell = 2 \)

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<td>684</td>
<td>702</td>
<td>243</td>
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\( \ell = 3 \)
**VERTICES**

**THM.** \( f_0(\mathcal{B}_n^\ell) = \#\{(\ell, n) \text{ partition trees}\} = \ell(n(\ell - 1) + 1)^{n-2} \)


<table>
<thead>
<tr>
<th>(n \backslash \ell)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<td>1</td>
<td>(\uparrow)</td>
<td>(\leftarrow 2(n + 1)^{n-2}) [OEIS, A007334]</td>
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</table>
\textbf{THM.} \, \, f_0(B^2_n) = \#\{(2, n) \text{ partition trees}\} = \#\text{spanning trees of } K_{n+1} \text{ with } 01

1, 2, 8, 50, 432, 4802, 65536, 1062882, 20000000, 428717762, \ldots

\textbf{[OEIS, A007334]}
\[
f_0(\mathcal{B}_n^2) = \# \{(2, n) \text{ partition trees} \} = \# \text{spanning trees of } K_{n+1} \cdot \frac{2}{n+1} = 2(n + 1)^{n-2}
\]

1, 2, 8, 50, 432, 4802, 65536, 1062882, 20000000, 428717762, \ldots

[OEIS, A007334]
**THM.**  \( f_{n-1}(B^\ell_n) = n! \left[z^n\right] \exp \left( \sum_{m \geq 1} \frac{F_{\ell,m} z^m}{m} \right) \) where \( F_{\ell,m} = \frac{1}{(\ell - 1)m + 1} \binom{\ell m}{m} \)

\[
\begin{array}{c|cccccc}
 n \backslash \ell & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
 1 & 1 & 1 & 1 & 1 & 1 & 1 \leftarrow 1 \\
 2 & 2 & 3 & 4 & 5 & 6 & 7 \leftarrow \ell + 1 \\
 3 & 6 & 17 & 34 & 57 & 86 & 121 \leftarrow 3\ell^2 + 2\ell + 1 \text{ [OEIS, A056109]} \\
 4 & 24 & 149 & 472 & 1089 & 2096 & 3589 \\
\end{array}
\]

\( n! \rightarrow \leftarrow \text{ [OEIS, A213507]} \)
THM. \( b_{n-1}(\mathcal{B}_n^\ell) = (n - 1)! [z^{n-1}] \exp \left( (\ell - 1) \sum_{m \geq 1} F_{\ell,m} z^m \right) \)

<table>
<thead>
<tr>
<th>( n \backslash \ell )</th>
<th>1</th>
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<th>3</th>
<th>4</th>
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0 \( \mapsto \leftarrow \) [OEIS, A251568]