MANY GEOMETRIC REALIZATIONS OF GRAPH ASSOCIAHEDRA

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POLYTOPES & COMBINATORICS
simplicial complex = collection of subsets of $X$ downward closed

exm:

$X = [n] \cup [n]$

$\Delta = \{ I \subseteq X \mid \forall i \in [n], \{ i, i \} \not\subseteq I \}$
**FANS**

**polyhedral cone** = positive span of a finite set of $\mathbb{R}^d$
= intersection of finitely many linear half-spaces

**fan** = collection of polyhedral cones closed by faces
and where any two cones intersect along a face

**simplicial fan** = maximal cones generated by $d$ rays
**POLYTOPES**

polytope = convex hull of a finite set of $\mathbb{R}^d$

= bounded intersection of finitely many affine half-spaces

face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations

simple polytope = facets in general position = each vertex incident to $d$ facets
P polytope, F face of P

normal cone of F = positive span of the outer normal vectors of the facets containing F

normal fan of P = \{ normal cone of F | F face of P \}

simple polytope \implies simplicial fan \implies simplicial complex
Permutohedron \( \text{Perm}(n) \)

\[
\text{Perm}(n) = \text{conv}\{(1), \ldots, (n+1)\} \in \Sigma_{n+1}
\]

\[
= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subseteq [n+1]} \left\{ x \in \mathbb{R}^{n+1} \mid \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}
\]
Permutohedron $Perm(n)$

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connections to
- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group
**Permutohedron** \( \text{Perm}(n) \)

\[
\begin{align*}
\text{Permutohedron } & \text{Perm}(n) \\
= & \text{conv} \{ (1), \ldots, (n+1) \mid \sigma \in \Sigma_{n+1} \} \\
= & \mathbb{H} \cap \bigcap_{\emptyset \neq J \subseteq [n+1]} \left\{ x \in \mathbb{R}^{n+1} \left| \sum_{j \in J} x_j \geq \binom{|J| + 1}{2} \right. \right\}
\end{align*}
\]

**Connections to**
- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group

**k-faces of** \( \text{Perm}(n) \)

\[
\equiv \text{surjections from } [n+1] \text{ to } [n+1 - k]
\]
Permutohedron $\text{Perm}(n)$

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connections to

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$k$-faces of $\text{Perm}(n)$

$$\equiv$$ surjections from $[n+1]$ to $[n+1-k]$.

$$\equiv$$ ordered partitions of $[n+1]$ into $n+1-k$ parts.
Permutohedron \( \text{Perm}(n) \)

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\begin{aligned}
\text{Perm}(n) &= \operatorname{conv}\{(1, \ldots, n+1) \mid \sigma \in \Sigma_{n+1}\} \\
&= \mathbb{H} \cap \bigcap_{\emptyset \neq J \subseteq [n+1]} \left\{ x \in \mathbb{R}^{n+1} \left| \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right. \right\} 
\end{aligned}
\]

connections to

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\( k \)-faces of \( \text{Perm}(n) \)

\[
\begin{aligned}
k\text{-faces of } \text{Perm}(n) &\equiv \text{surjections from } [n+1] \\
&\text{to } [n+1-k] \\
&\equiv \text{ordered partitions of } [n+1] \\
&\text{into } n+1-k \text{ parts} \\
&\equiv \text{collections of } n-k \text{ nested subsets of } [n+1]
\end{aligned}
\]
COXETER ARRANGEMENT
**Associahedron** = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex \((n + 3)\)-gon, ordered by reverse inclusion.

- Vertices ↔ triangulations
- Edges ↔ flips
- Faces ↔ dissections
- Vertices ↔ binary trees
- Edges ↔ rotations
- Faces ↔ Schröder trees
**VARIOUS ASSOCIAHEDRA**

**Associahedron** = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex \((n+3)\)-gon, ordered by reverse inclusion

Tamari ('51) — Stasheff ('63) — Haimann ('84) — Lee ('89) —

...— Gel'fand-Kapranov-Zelevinski ('94) — ...— Chapoton-Fomin-Zelevinsky ('02) — ...— Loday ('04) — ...

— Ceballos-Santos-Ziegler ('11)
THREE FAMILIES OF REALIZATIONS

SECONDARY POLYTOPE

Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)

LODAY’S ASSOCIAHEDRON

Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

CHAP.-FOM.-ZEL.’S ASSOCIAHEDRON

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(Pictures by CFZ)
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Hopf algebra
Cluster algebras

Cluster algebras
GRAPH ASSOCIATEDRA
G graph on ground set V

Tube of G = connected induced subgraph of G

Compatible tubes = nested, or disjoint and non-adjacent

Tubing on G = collection of pairwise compatible tubes of G

Nested complex $\mathcal{N}(G) =$ simplicial complex of tubings on G

= clique complex of the compatibility relation on tubes

G-associahedron = polytopal realization of the nested complex on G

Carr-Devadoss, Coxeter complexes and graph associahedra (’06)
EXM: NESTED COMPLEX
EXM: GRAPH ASSOCIAHEDRON
SPECIAL GRAPH ASSOCIAHEDRA

Path associahedron
= associahedron

Cycle associahedron
= cyclohedron

Complete graph associahedron
= permutahedron
COMPATIBILITY FANS FOR GRAPHICAL NESTED COMPLEXES

Thibault Manneville & VP

arXiv:1501.07152
COMPATIBILITY FANS FOR ASSOCIAHEDRA

\( T^\circ \) an initial triangulation, \( ' \) two internal diagonals

compatibility degree between \( \parallel \) and \( ' \)

\[
( \parallel ' ) = \begin{cases} 
-1 & \text{if } = ' \\
0 & \text{if } and \ ' \text{ do not cross} \\
1 & \text{if } and \ ' \text{ cross}
\end{cases}
\]

compatibility vector of \( w \) wrt \( T^\circ \):

\[
d(T^\circ, w) = \left[ ( \parallel ' ) \right]_{\circ \in T^\circ}
\]

compatibility fan wrt \( T^\circ \)

\[
\mathcal{D}(T^\circ) = \{ \mathbb{R}_{\geq 0} d(T^\circ, D) \mid D \text{ dissection} \}
\]

Fomin-Zelevinsky, Y-Systems and generalized associahedra ('03)
Fomin-Zelevinsky, Cluster algebras II: Finite type classification ('03)
Chapoton-Fomin-Zelevinsky, Polytopal realizations of generalized associahedra ('02)
Ceballos-Santos-Ziegler, Many non-equivalent realizations of the associahedron ('11)
Different initial triangulations $T^\circ$ yield different realizations.

**THM.** For any initial triangulation $T^\circ$, the cones \( \{ \mathbb{R}_{\geq 0} \, d(T^\circ, D) \mid D \text{ dissection} \} \) form a complete simplicial fan. Moreover, this fan is always polytopal.

Ceballos-Santos-Ziegler, Many non-equivalent realizations of the associahedron (’11)
COMPATIBILITY FANS FOR GRAPHICAL NESTED COMPLEXES

T\(^{\circ}\) an initial maximal tubing on \(G\)

t, t' two tubes of \(G\)

compatibility degree between \(t\) and \(t'\)

\[
(t \parallel t') = \begin{cases} 
-1 & \text{if } t = t' \\
0 & \text{if } t, t' \text{ are compatible} \\
|\{\text{neighbors of } t \text{ in } t' \setminus t\}| & \text{otherwise}
\end{cases}
\]

compatibility vector of \(t\) wrt \(T^{\circ}\):

\[
d(T^{\circ}, t) = [(t \parallel t)]_{t^{\circ} \in T^{\circ}}
\]

**THM.** For any initial maximal tubing \(T^{\circ}\) on \(G\),
the collection of cones

\[
\mathcal{D}(G, T^{\circ}) = \{\mathbb{R}_{\geq 0} d(T^{\circ}, T) \mid T \text{ tubing on } G\}
\]
forms a complete simplicial fan, called compatibility fan of \(G\).

Manneville-P., Compatibility fans for graphical nested complexes
THM. When none of the connected components of $G$ is a spider,

$$\#	ext{ linear isomorphism classes of compatibility fans of } G$$

$$= \# \text{ orbits of maximal tubings on } G \text{ under graph automorphisms of } G.$$
QU. Are all compatibility fans polytopal?
Are all compatibility fans polytopal?

Polytopality of a complete simplicial fan $\iff$ Feasibility of a linear program

Exm: We check that the compatibility fan on the complete graph $K_7$ is polytopal by solving a linear program on 126 variables and 17,640 inequalities.
Are all compatibility fans polytopal?

Polytopality of a complete simplicial fan $\iff$ Feasibility of a linear program

Exm: We check that the compatibility fan on the complete graph $K_7$ is polytopal by solving a linear program on 126 variables and 17,640 inequalities

$\implies$ All compatibility fans on complete graphs of $\leq 7$ vertices are polytopal...
$\implies$ All compatibility fans on graphs of $\leq 4$ vertices are polytopal...
**POLYTOPALITY?**

**QU.** Are all compatibility fans polytopal?

Polytopality of a complete simplicial fan $\iff$ Feasibility of a linear program

Exm: We check that the compatibility fan on the complete graph $K_7$ is polytopal by solving a linear program on $126$ variables and $17,640$ inequalities

$\implies$ All compatibility fans on complete graphs of $\leq 7$ vertices are polytopal...

$\implies$ All compatibility fans on graphs of $\leq 4$ vertices are polytopal...

To go further, we need to understand better the linear dependences between the compatibility vectors of the tubes involved in a flip

**THM.** All compatibility fans on the paths and cycles are polytopal

Ceballos-Santos-Ziegler, Many non-equivalent realizations of the associahedron (´11)
Manneville-P., Compatibility fans for graphical nested complexes
QU. Are all compatibility fans polytopal?

Remarkable realizations of the stellohedra
Are all compatibility fans polytopal?

Remarkable realizations of the stellohedra

Convex hull of the orbits under coordinate permutations of the set \( \{ \sum_{i>k}^{} i \cdot e_i \mid 0 \leq k \leq n \} \)
SIGNED TREE ASSOCIAHEDRA
Loday’s Associahedron

\[ \text{Asso}(n) := \text{conv} \{ L(T) \mid T \text{ binary tree} \} = \mathbb{H} \cap \bigcap_{1 \leq i \leq j \leq n+1} H^{\geq}(i,j) \]

\[ L(T) := \left\lfloor (T, i) \cdot r(T, i) \right\rfloor_{i \in [n+1]} \]

\[ H^{\geq}(i,j) := \left\{ x \in \mathbb{R}^{n+1} \mid \sum_{i \leq k \leq j} x_k \geq \left( j - i + \frac{2}{2} \right) \right\} \]

Loday, Realization of the Stasheff polytope (’04)
LODAY’S ASSOCIAHEDRON

\[ \text{Asso}(n) := \text{conv} \{ \mathbf{L}(T) \mid T \text{ binary tree} \} = \mathbb{H} \cap \bigcap_{1 \leq i \leq j \leq n+1} H^{\geq}(i, j) \]

\[ \mathbf{L}(T) := \left[ \gamma(T, i) \cdot r(T, i) \right]_{i \in [n+1]} \]
spine of a tubing $T =$ Hasse diagram of the inclusion poset of $T$

tube $t$ of the tubing $T \iff$ node $s(t)$ of the spine $S$ labeled by $t \smallsetminus \bigcup \{t' \mid t' \in T, t' \subsetneq t\}$

tube $t(s) := \bigcup \{s' \mid s' \leq s \text{ in } S\}$ of the tubing $T \iff$ node $s$ of the spine $S$

$S$ spine on $G \iff$ for each node $s$ of $S$ with children $s_1 \cdots s_k$, the tubes $t(s_1) \cdots t(s_k)$ lie in distinct connected components of $G[t(s) \smallsetminus s]$. 
spine of a tubing $T =$ Hasse diagram of the inclusion poset of $T$

tube $t$ of the tubing $T$ $\mapsto$ node $s(t)$ of the spine $S$ labeled by $t \setminus \bigcup \{ t' \mid t' \in T, t' \subsetneq t \}$

tube $t(s) := \bigcup \{ s' \mid s' \leq s \text{ in } S \}$ $\mapsto$ node $s$ of the spine $S$

$S$ spine on $G$ $\iff$ for each node $s$ of $S$ with children $s_1 \cdots s_k$, the tubes $t(s_1) \cdots t(s_k)$ lie in distinct connected components of $G[t(s) \setminus s]$
**THM.** The collection of cones \( \{ x \in \mathbb{R}^n \mid x_i < x_j \text{ for all } i \to j \text{ in } T \} \) forms a complete simplicial fan, called the **nested fan** of \( G \). This fan is always polytopal.

Carr-Devadoss, Coxeter complexes and graph associahedra ('06)

Postnikov, Permutohedra, associahedra, and beyond ('09)

Zelevinsky, Nested complexes and their polyhedral realizations ('06)
for an arbitrary signature $\sigma \in \pm^{n+1}$,

$$\text{Asso}(\sigma) := \text{conv} \{ \text{HL}(T) \mid T \text{ $\sigma$-Cambrian tree} \}$$

with $\text{HL}(T)_j := \begin{cases} 
(T, j) \cdot r(T, j) & \text{if } \sigma(j) = - \\
(n + 2 - (T, j) \cdot r(T, j) & \text{if } \sigma(j) = + 
\end{cases}$

Hohlweg-Lange, Realizations of the associahedron and cyclohedron ('07)
Lange-P., Using spines to revisit a construction of the associahedron ('13+)
for an arbitrary signature $\sigma \in \pm^{n+1}$,

$$\text{Asso}(\sigma) := \text{conv} \{ \text{HL}(T) \mid T \text{-Cambrian tree} \}$$

with $\text{HL}(T)_j := \begin{cases} (T,j) \cdot r(T,j) & \text{if } \sigma(j) = -1 \\ n + 2 - (T,j) \cdot r(T,j) & \text{if } \sigma(j) = +1 \end{cases}$

Hohlweg-Lange, Realizations of the associahedron and cyclohedron ('07)
Lange-P., Using spines to revisit a construction of the associahedron ('13+)

- Asso(n) obtained by deleting inequalities in the facet description of the permutahedron
- normal cone of $\text{HL}(T)$ in $\text{Asso}(\sigma) = \{ x \in \mathbb{H} \mid x_i < x_j \text{ for all } i \rightarrow j \text{ in } T \}$

Rule for Cambrian trees
Signed spine on signed trees

Signed spine on $T = \text{directed and labeled tree } S$ st

(i) the labels of the nodes of $S$ form a partition of the signed ground set $V$

(ii) at a node of $S$ labeled by $U = U^- \sqcup U^+$, the source label sets of the different incoming arcs are subsets of distinct connected components of $T \setminus U^-$, while the sink label sets of the different outgoing arcs are subsets of distinct connected components of $T \setminus U^+$
Signed spine complex $S(T) = \text{simplicial complex whose inclusion poset is isomorphic to the poset of edge contractions on the signed spines of } T$
For $S$ spine on $T$, define $C(S) := \{ x \in \mathbb{H} \mid x_u \leq x_v, \text{ for all arcs } u \to v \text{ in } S \}$.
THM. The spine fan $\mathcal{F}(T)$ is the normal fan of the signed tree associahedron $\text{Asso}(T)$, defined equivalently as

(i) the convex hull of the points

$$a(S)_v = \begin{cases} \{ \pi(S) : v \in \pi(S) \text{ and } r_v \leq v \} & \text{if } v \in V^- \\ |V| + 1 - \{ \pi(S) : v \in \pi(S) \text{ and } r_v \leq v \} & \text{if } v \in V^+ \end{cases}$$

for all maximal signed spines $S \in \mathcal{S}(T)$

(ii) the intersection of the hyperplane $H$ with the half-spaces

$$H^\leq(B) := \left\{ x \in \mathbb{R}^V \left| \sum_{v \in B} x_v \geq \binom{|B| + 1}{2} \right. \right\}$$

for all signed building blocks $B \in \mathcal{B}(T)$
The signed tree associahedron $\text{Asso}(T)$ is sandwiched between the permutahedron $\text{Perm}(V)$ and the parallelepiped $\text{Para}(T)$

$$
\sum_{u \neq v \in V} [e_u, e_v] = \text{Perm}(T) \subset \text{Asso}(T) \subset \text{Para}(T) = \sum_{u-v \in T} (u - v) \cdot [e_u, e_v]
$$
WHAT SHOULD I TAKE HOME FROM THIS TALK?
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(Pictures by CFZ)
TAKE HOME YOUR ASSOCIAHEDRA!

SECONDARY POLYTOPE

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CHAP.-FOM.-ZEL.’S ASSOCIAHEDRON
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Compatibility fans for graphical nested complexes
arXiv:1501.07152

VP
Signed tree associahedra
arXiv:1309.5222

THANK YOU
SECONDARY POLYTOPE

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