polytope = convex hull of a finite set of $\mathbb{R}^d$
  = bounded intersection of finitely many half-spaces

face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations

Given a set of points, determine the face lattice of its convex hull.

Given a lattice, is there a polytope which realizes it?
Polytopes of dimension $\geq 4$

Polytopes of dimension $3 \leftrightarrow$ planar 3-connected graphs

Various open conjectures in dimension 4:

Hirsch conjecture
- diameter $\leq \#\text{facets} - \text{dimension}$ (Santos)
- complexity of the simplex algorithm

$3^d$ Conjecture (Kalai)

$f$-vecteur shape (Barany, Ziegler)

“Our main limits in understanding the combinatorial structure of polytopes still lie in our ability to raise the good questions and in the lack of examples, methods of constructing them, and means of classifying them.”

\( \Pi_n = \text{conv} \{ (\sigma(1), \ldots, \sigma(n))^T \mid \sigma \in \mathfrak{S}_n \} \)

\( \partial \Pi_n = \text{refinement poset on ordered partitions of } [n] \)

\( \Sigma(P) = \text{conv} \{ \sum_{p \in P} \text{vol}(T, p)e_p \mid T \text{ triang. } P \} \)

\( \partial \Sigma(P) = \text{refinement poset on regular polyhedral subdivisions of } P \)

Triangulations are non-regular
J. Humphreys, Reflection groups and Coxeter groups, 1990.
$W = \text{finite Coxeter group}$
$W = \text{finite Coxeter group}$

Coxeter fan
$W = \text{finite Coxeter group}$

Coxeter fan

fundamental chamber
$W =$ finite Coxeter group
Coxeter fan
fundamental chamber
$S =$ simple reflections
\( W \) = finite Coxeter group
Coxeter fan
fundamental chamber
\( S \) = simple reflections
\( \Delta = \{ \alpha_s \mid s \in S \} \) = simple roots
$W =$ finite Coxeter group
Coxeter fan
fundamental chamber
$S =$ simple reflections
$\Delta = \{\alpha_s \mid s \in S\} =$ simple roots
$\Phi = W(\Delta) =$ root system
$W = \text{finite Coxeter group}$

Coxeter fan

fundamental chamber

$S = \text{simple reflections}$

$\Delta = \{\alpha_s \mid s \in S\} = \text{simple roots}$

$\Phi = W(\Delta) = \text{root system}$

$\Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}[\Delta] = \text{positive roots}$
$W = \text{finite Coxeter group}$

Coxeter fan

fundamental chamber

$S = \text{simple reflections}$

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$\Phi = W(\Delta) = \text{root system}$

$\Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}[\Delta] = \text{positive roots}$

$\nabla = \{\omega_s \mid s \in S\} = \text{fundamental weights}$
\( W = \) finite Coxeter group

Coxeter fan

fundamental chamber

\( S = \) simple reflections

\( \Delta = \{ \alpha_s \mid s \in S \} = \) simple roots

\( \Phi = W(\Delta) = \) root system

\( \Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}[\Delta] = \) positive roots

\( \nabla = \{ \omega_s \mid s \in S \} = \) fundamental weights
$W$ = finite Coxeter group
Coxeter fan
fundamental chamber
$S$ = simple reflections
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permutahedron
\( W = \) finite Coxeter group
Coxeter fan
fundamental chamber
\( S = \) simple reflections
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\( \Phi^+ = \Phi \cap \mathbb{R}_{\geq 0}[\Delta] = \) positive roots
\( \nabla = \{ \omega_s \mid s \in S \} = \) fundamental weights
permutahedron
weak order = \( u \leq w \iff \exists v \in W, uv = w \) and \( \ell(u) + \ell(v) = \ell(w) \)
EXAMPLES: TYPE $A$ AND $B$

TYPE $A_n = \text{symmetric group } \mathfrak{S}_{n+1}$

$S = \{(i, i+1) \mid i \in [n]\}$
$\Delta = \{e_{i+1} - e_i \mid i \in [n]\}$
roots = $\{e_i - e_j \mid i, j \in [n+1]\}$
$\nabla = \{\sum_{j > i} e_j \mid i \in [n]\}$

TYPE $B_n = \text{semidirect product } \mathfrak{S}_n \rtimes (\mathbb{Z}_2)^n$

$S = \{(i, i+1) \mid i \in [n-1]\} \cup \{\chi\}$
$\Delta = \{e_{i+1} - e_i \mid i \in [n-1]\} \cup \{e_1\}$
roots = $\{\pm e_i \pm e_j \mid i, j \in [n]\} \cup \{\pm e_i \mid i \in [n]\}$
$\nabla = \{\sum_{j \geq i} e_j \mid i \in [n]\}$
(\(W, S\)) a finite Coxeter system, \(Q = q_1q_2 \cdots q_m\) a word on \(S\), \(\rho\) an element of \(W\).

Subword complex \(S(Q, \rho)\) = simplicial complex of subsets of positions of \(Q\) whose complement contains a reduced expression of \(\rho\).


\[W = S_3\]
\[S = \{(1 2), (2 3)\} = \{a, b\}\]
\[Q = ababa\]
\[\rho = aba = bab\]

The subword complex is either a sphere (when the Demazure product of \(Q\) is \(\rho\)) or a ball.

**QUESTION.** Are all spherical subword complexes polytopal?
Classical situation of type $A$:

- Coxeter group $W = S_{n+1}$
- simple system $S = \{\tau_i \mid i \in [n]\}$, where $\tau_i = (i \ i + 1)$
- word $Q = q_1q_2 \cdots q_m$ on $S$
- $w$ element of $W$

The subword complex can be interpreted with a primitive sorting network:

- $N_Q$ formed by $n + 1$ levels and $m$ commutators
- facets of $S(Q, w) \leftrightarrow$ pseudoline arrangements on $N_Q$
**FLIPS**

\[\text{flip} = \text{exchange a contact with the corresponding crossing}\]
flip = exchange a contact with the corresponding crossing
Type $A$ subword complexes give combinatorial models for:

- Triangulations of convex polygons
- Multitriangulations of convex polygons
- Pseudotriangulations of point sets in general position
- Pseudotriangulations of sets of disjoint convex bodies

THREE GEOMETRIC STRUCTURES

Triangulations

Pseudotriangulations

Multitriangulations

\( k = 2 \)

**triangulation** = maximal crossing-free set of edges

**pseudotriangulation** = maximal crossing-free pointed set of edges

\( k \)-**triangulation** = maximal \((k + 1)\)-crossing-free set of edges
THREE GEOMETRIC STRUCTURES

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$

triangulation = maximal crossing-free set of edges

pseudotriangulation = maximal crossing-free pointed set of edges

$k$-triangulation = maximal $(k + 1)$-crossing-free set of edges
THREE GEOMETRIC STRUCTURES

Triangulations

- triangulation = maximal crossing-free set of edges
- decomposition into triangles

Pseudotriangulations

- pseudotriangulation = maximal crossing-free pointed set of edges
- decomposition into pseudotriangles

Multitriangulations

- $k$-triangulation = maximal $(k + 1)$-crossing-free set of edges
- decomposition into $k$-stars

VP & F. Santos, Multitriangulations as complexes of star polygons, 2009.
THREE GEOMETRIC STRUCTURES

Triangulations | Pseudotriangulations | Multitriangulations

flip = exchange an internal edge with the common bisector of the two adjacent cells
THREE GEOMETRIC STRUCTURES

Triangulations

Pseudotriangulations

Multitriangulations

associahedron $\leftrightarrow$ crossing-free sets of internal edges
pseudotriangulations polytope $\leftrightarrow$ pointed crossing-free sets of internal edges
multiassociahedron $\leftrightarrow$ $(k + 1)$-crossing-free sets of $k$-internal edges
VP & M. Pocchiola, Multitriangulations, pseudotriangulations and primitive sorting networks, 2012.
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
Duality

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

Triangulations  Pseudotriangulations  Multitriangulations

$\begin{array}{c}
5 & 6 \\
7 & 8 \\
1 & 4 \\
3 & 2 \\
\end{array}$

$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

Triangulations  Pseudotriangulations  Multitriangulations

$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

k = 2

VP & M. Pocchiola, Multitriangulations, pseudotriangulations and primitive sorting networks, 2012.
Type $B$ subword complexes give models for centrally symmetric triangulations:
Type $B$ subword complexes give models for centrally symmetric triangulations:

For a facet $I$ of $S(Q, \rho)$ and a position $k \in [m]$, define the root $r(I,k) = Q_{[k-1]\setminus I}(\alpha_{q_k})$, where $Q_{[k-1]\setminus I}$ is the product of all reflections $q_j$ for $j$ from $1$ to $k-1$ but not in $I$.

The root function of the facet $I$ is $r(I, \cdot) : [m] \longrightarrow \Phi$

The root configuration of $I$ is $R(I) = \{r(I,i) \mid i \in I\}$
**PROPOSITION.** The root function encodes flips in subword complexes:

1. The map $r(I, \cdot)$ is a bijection from the complement of $I$ to $\text{inv}(\rho)$.

2. If $I$ and $J$ are two adjacent facets of $S(Q)$ with $I \setminus i = J \setminus j$, then $j$ is the unique position in the complement of $I$ such that $r(I, i) = \pm r(I, j)$.

3. In the situation of 2, the root function of $J$ is obtained from that of $I$ by

   $$r(J, k) = \begin{cases} 
   s_{r(I,i)}(r(I, k)) & \text{if } \min(i, j) < k \leq \max(i, j), \\
   r(I, k) & \text{otherwise.}
   \end{cases}$$

---

$\mathcal{N}$ a sorting network with $n + 1$ levels
$\mathcal{N}$ a sorting network with $n + 1$ levels

A pseudoline arrangement supported by $\mathcal{N}$ $\mapsto$ brick vector $B(\Lambda) \in \mathbb{R}^{n+1}$
\( \mathcal{N} \) a sorting network with \( n + 1 \) levels

\( \Lambda \) pseudoline arrangement supported by \( \mathcal{N} \) \quad \longmapsto \quad \text{brick vector} \ B(\Lambda) \in \mathbb{R}^{n+1}

\( B(\Lambda)_j = \) number of bricks of \( \mathcal{N} \) below the \( j \)th pseudoline of \( \Lambda \)
A sorting network with \( n + 1 \) levels

\( \Lambda \) pseudoline arrangement supported by \( N \) 

\[
B(\Lambda)_j = \text{number of bricks of } N \text{ below the } j\text{th pseudoline of } \Lambda
\]
$\mathcal{N}$ a sorting network with $n + 1$ levels

\( \Lambda \) pseudoline arrangement supported by \( \mathcal{N} \) \( \quad \mapsto \quad \) brick vector \( B(\Lambda) \in \mathbb{R}^{n+1} \)

\( B(\Lambda)_j = \) number of bricks of \( \mathcal{N} \) below the \( j \)th pseudoline of \( \Lambda \)
A sorting network with \( n + 1 \) levels

\( \Lambda \) pseudoline arrangement supported by \( \mathcal{N} \)    \( \rightarrow \)          brick vector  \( B(\Lambda) \in \mathbb{R}^{n+1} \)

\( B(\Lambda)_j = \) number of bricks of \( \mathcal{N} \) below the \( j \)th pseudoline of \( \Lambda \)
$\mathcal{N}$ a sorting network with $n + 1$ levels

$\Lambda$ pseudoline arrangement supported by $\mathcal{N} \quad \mapsto \quad$ brick vector $B(\Lambda) \in \mathbb{R}^{n+1}$

$B(\Lambda)_j =$ number of bricks of $\mathcal{N}$ below the $j$th pseudoline of $\Lambda$

Brick polytope $\mathcal{B}(\mathcal{N}) = \text{conv} \{ B(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N} \}$
BRICK POLYTOPE

B({2,3,5}) = (1,6,3,8)
B({2,3,9}) = (1,6,5,6)
B({2,5,6}) = (2,6,2,8)
B({3,4,5}) = (1,7,3,7)
B({3,4,9}) = (1,7,5,5)
B({2,6,7}) = (3,6,2,7)
B({2,7,9}) = (3,6,3,6)
B({4,6,7}) = (3,7,2,6)
B({4,7,9}) = (3,7,3,5)
B({4,5,6}) = (2,7,2,7)
(\(W, S\)) a finite Coxeter system, \(Q = q_1 q_2 \cdots q_m\) a word on \(S\), \(w_\circ\) longest element of \(W\).

\(S(Q) = S(Q, w_\circ)\) spherical subword complex.

To a facet \(I\) of \(S(Q)\) and a position \(k \in [m]\), associate a weight \(w(I, k) = Q_{[k-1]} \backslash I(\omega_{q_k})\), where \(Q_{[k-1]} \backslash I\) is the product of all reflections \(q_j\) for \(j\) from 1 to \(k-1\) but not in \(I\).

The **brick vector** of \(I\) is the vector \(B(I) = \sum_{k \in [m]} w(I, k)\).

The **brick polytope** is the convex polytope \(B(Q) = \text{conv} \{ B(I) \mid I \text{ facet of } S(Q) \}\).

In type \(A\), \(w(I, k) = \) characteristic vector of the pseudolines passing above the \(k\)th brick.

\(B(I) = (\text{number of bricks below the } j\text{th pseudoline of } I)_{j \in [n+1]}\)
If $\Lambda$ and $\Lambda'$ are two pseudoline arrangements supported by $\mathcal{N}$ and related by a flip between their $i$th and $j$th pseudolines, then $B(\Lambda) - B(\Lambda') \in \mathbb{N}_{>0} (e_j - e_i)$.

**THEOREM.** The cone of the brick polytope $B(Q)$ at the brick vector $B(I)$ is generated by $-R(I)$, for any facet $I$ of $S(Q)$. 
The **brick polytope** is the convex polytope $\mathcal{B}(Q) = \text{conv} \{ B(I) \mid I \text{ facet of } S(Q) \}$.

**Theorem.** The polar of the brick polytope $\mathcal{B}(Q)$ realizes the subword complex $S(Q)$ if and only if $Q$ is such that $R(I)$ is linearly independent, for $I$ facet of $S(Q)$.

**Theorem.** If $Q$ is realizing, the cone of the brick polytope $\mathcal{B}(Q)$ at the brick vector $B(I)$ is generated by $-R(I)$, for any facet $I$ of $S(Q)$.

**Theorem.** If $Q$ is realizing, the Coxeter fan refines the normal fan of the brick polytope. More precisely,

$$\text{normal cone of } B(I) \text{ in } \mathcal{B}(Q) = \bigcup_{w \in W} w(\Phi^+) \text{ (fundamental cone)}.$$
THEOREM. If \( Q \) is realizing, the Coxeter fan refines the normal fan of the brick polytope.
REMEMBER THE RIGHT WEAK ORDER
$I, J$ two adjacent facets of $\mathcal{S}(Q)$, with $I \setminus i = J \setminus j$.
The flip from $I$ to $J$ is increasing if $i < j$. 
INCREASING FLIP GRAPH

\[
B(\{2,3,5\}) = (1,6,3,8) \\
B(\{2,3,9\}) = (1,6,5,6) \\
B(\{2,5,6\}) = (2,6,2,8) \\
B(\{3,4,5\}) = (1,7,3,7) \\
B(\{3,4,9\}) = (1,7,5,5) \\
B(\{2,6,7\}) = (3,6,2,7)
\]
INCREASING FLIP GRAPH

$I, J$ two adjacent facets of $S(Q)$, with $I \setminus i = J \setminus j$.
The flip from $I$ to $J$ is increasing if $i < j$.

**THEOREM.** Assume that $Q$ is realizing. Then $I$ is covered by $J$ in increasing flip order iff there exists $w_I, w_J \in W$ with $R(I) \subset w_I(\Phi^+)$, $R(J) \subset w_J(\Phi^+)$ and $w_I$ is covered by $w_J$ in weak order.

In other words, the oriented graph of the brick polytope is a quotient of the oriented graph of the permutohedron.
THEOREM. If $Q$ is realizing, the Hasse diagram of the increasing flip order is a quotient of the Hasse diagram of the weak order.
THEOREM. If $Q$ is realizing, the brick polytope $B(Q)$ is the Minkowski sum of the polytopes $B(Q, k) = \text{conv} \{ w(I, k) \mid I \text{ facet of } S(Q) \}$. In other words,

$$B(Q) = \text{conv}_I \sum_k w(I, k) = \sum_k \text{conv}_I w(I, k) = \sum_k B(Q, k).$$
S. Stella, Polyhedral models for generalized associahedra via Coxeter elements, 2011.
VP & C. Stump, Brick polytopes of spherical subword complexes, 2012\textsuperscript{+}.
C. Hohlweg, Permutahedra and associahedra, 2013.
cluster algebra = commutative ring generated by distinguished cluster variables grouped into overlapping clusters

clusters computed by a mutation process:

- cluster seed = algebraic data $\{x_1, \ldots, x_n\}$, combinatorical data $B$ (matrix or quiver)
- cluster mutation = $(\{x_1, \ldots, x_k, \ldots, x_n\}, B) \xrightarrow{\mu_k} (\{x_1, \ldots, x'_k, \ldots, x_n, \mu_k(B)\})$

$$x_k \cdot x'_k = \prod_{i, b_{ik} > 0} x_i^{b_{ik}} + \prod_{i, b_{ik} < 0} x_i^{-b_{ik}}$$

$$(\mu_k(B))_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} + |b_{ik}| \cdot b_{kj} & \text{if } k \notin \{i, j\} \text{ and } b_{ik} \cdot b_{kj} > 0 \\ b_{ij} & \text{otherwise} \end{cases}$$

cluster complex = simplicial complex w/ vertices = cluster variables & facets = clusters

**THEOREM.** (Laurent phenomenon)
All cluster variables are Laurent polynomials in the variables of the initial cluster seed.


**THEOREM.** (Classification)
Finite type cluster algebras are classified by the Cartan-Killing classification for crystallographic root systems.


In fact, for a root system $\Phi$, there is a bijection

- cluster variables $\leftrightarrow \Phi_{\leq -1} = \Phi^+ \cup -\Delta$
- $y = \frac{F(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}} \leftrightarrow \beta = d_1 \alpha_1 + \cdots + d_n \alpha_n$
- cluster $\leftrightarrow c$-cluster
- cluster complex $\leftrightarrow c$-cluster complex
THEOREM. The cluster complex is polytopal.

C. Hohlweg, Permutahedra and associahedra, 2013.
New approach to the combinatorics and geometry of the cluster complex:

**THEOREM.** The subword complex $S(c_{\text{w.o.c}})$ is isomorphic to the cluster complex.


Cluster variables $\longleftrightarrow \Phi_{\geq -1} = \Phi^+ \cup -\Delta \longleftrightarrow$ position in $c_{\text{w.o.c}}$

$y = \frac{F(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}} \longleftrightarrow \beta = d_1\alpha_1 + \cdots + d_n\alpha_n \longleftrightarrow \begin{cases} i \quad \text{if } \beta = -\alpha_{c_i} \\ j \quad \text{if } \beta = r([n], j) \end{cases}$

Cluster $\longleftrightarrow$ c-cluster $\longleftrightarrow$ facet of $S(c_{\text{w.o.c}})$

Cluster complex $\longleftrightarrow$ c-cluster complex $\longleftrightarrow$ subword complex $S(c_{\text{w.o.c}})$

**THEOREM.** The brick polytope $B(c_{\text{w.o.c}})$ realizes the subword complex $S(c_{\text{w.o.c}})$.

**THEOREM.** The brick polytope $B(c_{\text{w.o.c}})$ is a translate of the known realizations of the generalized associahedron.
FURTHER PROPERTIES OF GENERALIZED ASSOCIAHEDRA

**CAMBRIAN LATTICES & FANS.** The graph of the associahedron $\text{Asso}_c(W)$, oriented from $e$ to $w_0$ is the Hasse diagram of the $c$-Cambrian lattice.

The normal fan of the associahedron $\text{Asso}_c(W)$ is the $c$-Cambrian fan, obtained by coarsening the braid fan.

Reading & Speyer, Cambrian fans, 2009.

**DIAMETER.** The diameter of the $\text{type } A_n$ associahedron is $2n - 4$ for $n \geq 9$.

The diameter of the $\text{type } D_n$ associahedron is $2n - 2$ for all $n$.

All type $A_n, B/C_n, D_n, H_3, H_4, F_4, E_6$ associahedra fulfill the non-leaving face property: every geodesic connecting two vertices stays in the minimal face containing them.


**BARYCENTER.** The vertex barycenters of the permutahedron and associahedron coincide.

Evolution of the brick vector $B_N(\Lambda)$ under three operations:

1. Rotate: $B_{N^\triangleright}(\Lambda^\triangleright) - B_N(\Lambda) \in \omega_i + \mathbb{R}(e_{i+1} - e_i)$
2. Reflect: $B_{N^\uparrow}(\Lambda^\uparrow) = \#\{\text{bricks of } N\} \cdot 1 - (B_N(\Lambda))^\rightarrow$
3. Reverse: $B_{N^\leftarrow}(\Lambda^\leftarrow) = (B_N(\Lambda))^\rightarrow$
THREE OPERATIONS

Evolution of the translated brick vector $\vec{B}_c(\Lambda) = B_c(\Lambda) - \Omega_c$ under three operations:

1. Rotate: $\vec{B}_c(\Lambda^{\circ}) - \vec{B}_c(\Lambda) \in \mathbb{R}(e_{i+1} - e_i)$
2. Reflect: $\vec{B}_c(\Lambda^\dagger) = -\overline{(\vec{B}_c(\Lambda))}$
3. Reverse: $\vec{B}_c(\Lambda^{\dagger\dagger}) = \overline{(\vec{B}_c(\Lambda))}$
THREE OPERATIONS

Evolution of the translated brick vector $\bar{B}_c(\Lambda) = B_c(\Lambda) - \Omega_c$ under three operations:

1. Rotate: $\bar{B}_{c^\circ}(\Lambda^\circ) - \bar{B}_c(\Lambda) \in \mathbb{R}(e_{i+1} - e_i)$

All associahedra $\text{Assoc}_c$ have the same barycenter
Evolution of the translated brick vector $\bar{B}_c(\Lambda) = B_c(\Lambda) - \Omega_c$ under three operations:

1. Original:

2. Reflect: $\bar{B}_{c\uparrow}(\Lambda^{\uparrow}) = -(\bar{B}_c(\Lambda))^{\uparrow}$

3. Reverse: $\bar{B}_{c\leftarrow}(\Lambda^{\leftarrow}) = (\bar{B}_c(\Lambda))^{\leftarrow}$

The barycenter of the superposition of the vertices of $\text{Asso}_{c\uparrow}$ and $\text{Asso}_{c\leftarrow}$ is the origin.
THREE OPERATIONS

Evolution of the translated brick vector $\tilde{B}_c(\Lambda) = B_c(\Lambda) - \Omega_c$ under three operations:

original rotate reflect reverse

All associahedra $\text{Asso}_c$ have the same barycenter

The barycenter of the superposition of the vertices of $\text{Asso}_c^\uparrow$ and $\text{Asso}_c^\rightarrow$ is the origin

THEOREM. All associahedra $\text{Asso}_c$ have vertex barycenter at the origin

...and the same method works for fairly balanced and generalized associahedra.
**THEOREM.** For any finite Coxeter group $W$, any Coxeter element $c$, any fairly-balanced point $u$, the vertex barycenters of the generalized associahedron $\text{Asso}_c^u(W)$ and of the permutahedron $\text{Perm}^u(W)$ coincide.

The point $u$ is fairly balanced if $w_\circ(u) = -u$, where $w_\circ$ is the longest element in $W$. 
THANK YOU