polytope = convex hull of a finite set of $\mathbb{R}^d$
= bounded intersection of finitely many half-spaces

face = intersection with a supporting hyperplane
face lattice = all the faces with their inclusion relations

Given a set of points, determine the face lattice of its convex hull.

Given a lattice, is there a polytope which realizes it?
One of these graphs is the 1-skeleton of a polytope. Can you guess which?
One of these graphs is the 1-skeleton of a polytope. Can you guess which?
Polytopes of dimension $3 \iff$ planar 3-connected graphs

Various open conjectures in dimension $4$:  

- Hirsch conjecture  
  \[ \text{diameter} \leq \#\text{facets} - \text{dimension} \]  
  \text{(Santos)}  
  \text{complexity of the simplex algorithm}

- $3^d$ Conjecture \text{(Kalai)}

- $f$-vecteur shape \text{(Barany, Ziegler)}

“Our main limits in understanding the combinatorial structure of polytopes still lie in our ability to raise the good questions and in the lack of examples, methods of constructing them, and means of classifying them.”

MATCHING POLYTOPE

$G = (V, E)$ finite graph, $\omega : E \mapsto \mathbb{R}$ weight function

Matching polytope of $G = \text{conv}\{1_M \mid M \subset E \text{ matching of } G\}$

Maximum-weight matching

$$\max \left\{ \sum_{e \in M} \omega(e) \bigg| M \subset E \text{ matching of } G \right\} = \max \left\{ \omega^T x \mid x \in \text{MP}(G) \right\}$$

If $G$ is bipartite, the matching polytope is defined by the inequalities

$$x_e \geq 0 \quad \forall e \in E$$
$$\sum_{e \ni v} x_e \leq 1 \quad \forall v \in V$$

otherwise, we have to add the inequalities

$$\sum_{e \in U} x_e \leq \left\lfloor \frac{1}{2}|U| \right\rfloor \quad \forall U \subset V, \ |U| \text{odd}$$
$\Pi_n = \text{conv}\{(\sigma(1), \ldots, \sigma(n))^T \mid \sigma \in \mathfrak{S}_n\} = \sum_{i<j}[e_i, e_j]$

$\partial \Pi_n = \text{refinement poset on ordered partitions of } [n]$
\[ \partial A_n = \text{reverse inclusion poset on non-crossing sets of diagonals of the } n\text{-gon} \]
$P$ point set

Regular subdivision of $P = \text{projection of the lower envelope of a lifting of } P$
SECONDARY POLYTOPE

$P$ point set

Regular subdivision of $P = projection of the lower envelope of a lifting of $P$
$P$ point set

Regular subdivision of $P = \text{projection of the lower envelope of a lifting of } P$
SECONDARY POLYTOPE

\[ \Sigma(P) = \text{conv}\left\{ \sum_{p \in P} \text{vol}(T, p)e_p \mid T \text{ triang. } P \right\} \]

\[ \partial \Sigma(P) = \text{refinement poset on regular polyhedral subdivisions of } P \]
...TRIANGULATIONS...TRIANGULATIONS...TRIANGULATIONS...TRIANGULATIONS
THREE GEOMETRIC STRUCTURES

Triangulations

Pseudotriangulations

Multitriangulations

\[ k = 2 \]

**triangulation** = maximal crossing-free set of edges,

**pseudotriangulation** = maximal crossing-free pointed set of edges,

**\( k \)-triangulation** = maximal \((k + 1)\)-crossing-free set of edges,
THREE GEOMETRIC STRUCTURES

- **Triangulations**: maximal crossing-free set of edges.
- **Pseudotriangulations**: maximal crossing-free pointed set of edges.
- **Multitriangulations**: maximal \((k + 1)\)-crossing-free set of edges, for \(k = 2\).
**THREE GEOMETRIC STRUCTURES**

- **Triangulations**
  - triangulation = maximal crossing-free set of edges,
  - = decomposition into triangles.

- **Pseudotriangulations**
  - pseudotriangulation = maximal crossing-free pointed set of edges,
  - = decomposition into pseudotriangles.

- **Multitriangulations**
  - \(k\)-triangulation = maximal \((k + 1)\)-crossing-free set of edges,
  - = decomposition into \(k\)-stars.

VP & F. Santos, Multitriangulations as complexes of star polygons, 2009.
Three Geometric Structures

- **Triangulations**
- **Pseudotriangulations**
- **Multitriangulations**

**Flip** = exchange an internal edge with the common bisector of the two adjacent cells.

$k = 2$
THREE GEOMETRIC STRUCTURES

Triangulations  Pseudotriangulations  Multitriangulations

associahedron $\leftrightarrow$ crossing-free sets of internal edges.
pseudotriangulations polytope $\leftrightarrow$ pointed crossing-free sets of internal edges.
multiassociahedron $\leftrightarrow$ $(k + 1)$-crossing-free sets of $k$-internal edges.
DUALITY

Triangulations Pseudotriangulations Multitriangulations

$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
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Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

| Triangulations | Pseudotriangulations | Multitriangulations |

$k = 2$
**Duality**

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$k = 2$
DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

$k = 2$
DUALITY

Triangulations

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\( k = 2 \)
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Triangulations

Pseudotriangulations

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$k = 2$
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DUALITY

Triangulations

Pseudotriangulations

Multitriangulations

\[ k = 2 \]
SORTING NETWORKS

Bubble sort  Insertion sort  Even-odd sorting

network $\mathcal{N} = n$ horizontal levels and $m$ vertical commutators.
bricks of $\mathcal{N} =$ bounded cells.
network $\mathcal{N} = n$ horizontal levels and $m$ vertical commutators.

bricks of $\mathcal{N} =$ bounded cells.

pseudoline = $x$-monotone path which starts at a level $l$ and ends at the level $n + 1 - l$.

crossing = contact =

pseudoline arrangement (with contacts) = $n$ pseudolines supported by $\mathcal{N}$ which have pairwise exactly one crossing, eventually some contacts, and no other intersection.
Contact graph $\Lambda^#$ of a pseudoline arrangement $\Lambda =$

- a node for each pseudoline of $\Lambda$, and
- an arc for each contact point of $\Lambda$ oriented from top to bottom.
**FLIPS**

**THEOREM.** Let $\mathcal{N}$ be a sorting network with $n$ levels and $m$ commutators. The graph of flips $G(\mathcal{N})$ is $(m - \binom{n}{2})$-regular and connected.

**QUESTION.** Is $G(\mathcal{N})$ the graph of a simple $(m - \binom{n}{2})$-dimensional polytope?
A pseudoline arrangement supported by $\mathcal{N}$ $\rightarrow$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$. $\omega(\Lambda)_j =$ number of bricks of $\mathcal{N}$ below the $j$th pseudoline of $\Lambda$.

Brick polytope $\Omega(\mathcal{N}) = \text{conv}\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$.
A pseudoline arrangement supported by $\mathcal{N}$ \rightleftharpoons brick vector $\omega(\Lambda) \in \mathbb{R}^n$.

$\omega(\Lambda)_j =$ number of bricks of $\mathcal{N}$ below the $j$th pseudoline of $\Lambda$.

Brick polytope $\Omega(\mathcal{N}) = \text{conv} \{ \omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N} \}$. 

---

BRICK POLYTOPE
A pseudoline arrangement supported by \( \mathcal{N} \) \( \rightarrow \) brick vector \( \omega(\Lambda) \in \mathbb{R}^n \).

\[ \omega(\Lambda)_j = \text{number of bricks of } \mathcal{N} \text{ below the } j\text{th pseudoline of } \Lambda. \]

Brick polytope \( \Omega(\mathcal{N}) = \text{conv}\{\omega(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\} \).
A pseudoline arrangement supported by $\mathcal{N}$ $\rightarrow$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$.

$\omega(\Lambda)_j =$ number of bricks of $\mathcal{N}$ below the $j$th pseudoline of $\Lambda$.

Brick polytope $\Omega(\mathcal{N}) = \text{conv}\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$.
A pseudoline arrangement supported by $\mathcal{N} \; \mapsto \; \text{brick vector } \omega(\Lambda) \in \mathbb{R}^n$.

$\omega(\Lambda)_j = \text{number of bricks of } \mathcal{N} \text{ below the } j\text{th pseudoline of } \Lambda$.

Brick polytope $\Omega(\mathcal{N}) = \text{conv}\{\omega(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\}$. 
A pseudoline arrangement supported by \( \mathcal{N} \) $\mapsto$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$. \\
$\omega(\Lambda)_j =$ number of bricks of \( \mathcal{N} \) below the \( j \)th pseudoline of \( \Lambda \).

Brick polytope $\Omega(\mathcal{N}) = \text{conv}\{\omega(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\}$. 
A pseudoline arrangement supported by $\mathcal{N}$ maps to a brick vector $\omega(\Lambda) \in \mathbb{R}^n$.

$\omega(\Lambda)_j$ is the number of bricks of $\mathcal{N}$ below the $j$th pseudoline of $\Lambda$.

**Remark.** The brick polytope is not full-dimensional:

$$\Omega(\mathcal{N}) \subset \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \left| \sum_{i=1}^{n} x_i = \sum_{b \text{ brick of } \mathcal{N}} \text{depth}(b) \right. \right\}.$$
\( \mathcal{X}_m = \) network with two levels and \( m \) commutators.

Graph of flips \( G(\mathcal{X}_m) = \) complete graph \( K_m \).

Brick polytope \( \Omega(\mathcal{X}_m) = \text{conv} \left\{ \binom{m - i}{i - 1} \mid i \in [m] \right\} = \left[ \binom{m - 1}{0}, \binom{0}{m - 1} \right] \).
REMARK. If $\Lambda$ and $\Lambda'$ are two pseudoline arrangements supported by $\mathcal{N}$ and related by a flip between their $i$th and $j$th pseudolines, then $\omega(\Lambda) - \omega(\Lambda') \in \mathbb{N}_0 (e_j - e_i)$. 

BRICK VECTORS AND FLIPS
# INCIDENCE CONE OF A DIRECTED MULTIGRAPH

<table>
<thead>
<tr>
<th>$G$ directed (multi)graph</th>
<th>$\iff$ Incidence configuration $I(G') = {e_j - e_i \mid (i, j) \in G}$,</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$\iff$ Incidence cone $C(G') = \text{cone generated by } I(G)$</td>
</tr>
</tbody>
</table>

**REMARK.**

- independent sets in $I(G)$ $\iff$ forests in $G$,
- spanning sets of $\langle 1 \mid x \rangle = 0$ $\iff$ connected spanning subgraphs of $G$,
- basis of $\langle 1 \mid x \rangle = 0$ $\iff$ spanning trees of $G$,
- circuits in $I(G)$ $\iff$ simple cycles in $G$,
- cocircuits in $I(G)$ $\iff$ minimal cuts in $G$,
- and signs correspond to the orientations of the edges.

**REMARK.** $H$ subgraph of $G$. Then $I(H)$ forms a $k$-face of $C(G)$ $\iff$ $H$ has $n - k$ connected components and $G/H$ is acyclic. In particular:

- $C(G)$ is pointed $\iff$ $G$ is acyclic,
- facets of $C(G)$ $\iff$ complements of the minimal directed cuts of $G$. 
**Contact graph** $\Lambda#$ of a pseudoline arrangement $\Lambda =$

- a node for each pseudoline of $\Lambda$, and
- an arc for each contact point of $\Lambda$ oriented from top to bottom.

**Theorem.** The cone of the brick polytope $\Omega(\mathcal{N})$ at the brick vector $\omega(\Lambda)$ is the incidence cone $C(\Lambda#) = \text{cone} \{ e_j - e_i \mid (i, j) \in \Lambda# \}$ of the contact graph of $\Lambda$. 
THEOREM. The cone of the brick polytope $\Omega(N)$ at the brick vector $\omega(\Lambda)$ is the incidence cone $C(\Lambda\#)$ of the contact graph of $\Lambda$:

$$\text{cone}\{\omega(\Lambda') - \omega(\Lambda) \mid \Lambda' \text{ supported by } N\} = \text{cone}\{e_j - e_i \mid (i,j) \in \Lambda\#\}.$$ 

VERTICES OF $\Omega(N)$

The brick vector $\omega(\Lambda)$ is a vertex of $\Omega(N) \iff$ the contact graph $\Lambda\#$ is acyclic.

GRAPH OF $\Omega(N)$

The graph of the brick polytope is a subgraph of $G(N)$ whose vertices are the pseudoline arrangements with acyclic contact graphs.

FACETS OF $\Omega(N)$

The facets of $\Omega(N)$ correspond to the minimal directed cuts of the contact graphs of the pseudoline arrangements supported by $N$. 

**Reduced network** = network with \( n \) levels and \( \binom{n}{2} \) commutators. It supports only one pseudoline arrangement.

**Duplicated network** \( \Pi = \) network with \( n \) levels and \( 2\binom{n}{2} \) commutators obtained by duplicating each commutator of a reduced network.

Any pseudoline arrangement supported by \( \Pi \) has one contact and one crossing among each pair of duplicated commutators.
Graph of flips $G(\Pi) = \binom{n}{2}$-dimensional cube.
Any pseudoline arrangement supported by $\Pi$ has one contact and one crossing among each pair of duplicated commutators. $\Rightarrow$ The contact graph $\Lambda#$ is a tournament.

Vertices of $\Omega(\Pi) \iff$ acyclic tournaments $\iff$ permutations of $[n]$,
Facets of $\Omega(\Pi) \iff$ cuts in a tournament $\iff$ ordered bipartitions of $[n]$.

Brick polytope $\Omega(\Pi) = \text{permutahedron}$. 
Brick polytope $\Omega(\Pi) = \text{permutahedron}$. 
DUPLICATED NETWORKS: PERMUTAHEDRA
For $x \in \{a, b\}^{n-2}$, we define a reduced alternating network $\mathcal{N}_x$ and a polygon $\mathcal{P}_x$.

$\mathcal{N}_x$ is the dual pseudoline arrangement of the polygon $\mathcal{P}_x$. 
**THEOREM.** There is a duality between the pseudoline arrangements supported by $\mathcal{N}_x^1$ and the triangulations of the polygon $\mathcal{P}_x$.

\[
\begin{align*}
T \text{ triangulation of } \mathcal{P}_x & \iff T^* \text{ pseudoline arrangement supported by } \mathcal{N}_x^1 \\
\Delta \text{ triangle of } T & \iff \Delta^* \text{ pseudoline of } T^* \\
e \text{ common edge of } \Delta \text{ and } \Delta' & \iff e^* \text{ contact between } \Delta^* \text{ and } \Delta'^* \\
f \text{ common bissector of } \Delta \text{ and } \Delta' & \iff f^* \text{ crossing between } \Delta^* \text{ and } \Delta'^*
\end{align*}
\]

**COROLLARY.** (i) The graph of flips $G(\mathcal{N}_x^1)$ is (isomorphic to) the graph of flips $G(\mathcal{P}_x)$.

(ii) The contact graph $(T^*)^\#$ is (isomorphic to) the dual binary tree of $T$. 
THEOREM. For any word $x \in \{a, b\}^{n-2}$, the simplicial complex of crossing-free sets of internal diagonals of the convex $n$-gon $P_x$ is (isomorphic to) the boundary complex of the polar of the brick polytope $\Omega(\mathcal{N}_x^1)$.

REMARK. Up to translation, we obtain Hohlweg & Lange’s associahedra.

THANK YOU