### Motivation

<table>
<thead>
<tr>
<th>Combinatorics</th>
<th>Permutations</th>
<th>Binary Trees</th>
<th>Binary Sequences</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Algebra</strong></td>
<td>Malvenuto-Reutenauer algebra</td>
<td>Loday-Ronco algebra</td>
<td>Solomon algebra</td>
</tr>
<tr>
<td>FQSym = vect $hF_{\tau} \mid \tau \in \mathcal{S}$</td>
<td>$\text{PBT} = \text{vect } h\text{P}_{T} \mid T \in \text{BT}$</td>
<td>Rec = vect $hX_{\eta} \mid \eta \in \pm \ast i$</td>
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</tr>
<tr>
<td>$F_{\tau} \cdot F_{\tau^0} = \sum_{\sigma \in \tau \sqcup \tau^0} F_{\sigma}$</td>
<td>$\text{P}<em>{T} \cdot \text{P}</em>{T^0} = \sum_{T^0 \leq T \leq T^\square} \text{P}_{T^0}$</td>
<td>$X_{\eta} \cdot X_{\eta^0} = X_{\eta+\eta^0} + X_{\eta-\eta^0}$</td>
<td></td>
</tr>
<tr>
<td>$4 F_{\sigma} = \sum_{\tau \ast \tau^0} F_{\tau} \otimes F_{\tau^0}$</td>
<td>$4 \text{F}<em>{\gamma} = \sum</em>{\gamma \text{ cut}} B(T, \gamma) \otimes A(T, \gamma)$</td>
<td>$4 X_{\eta} = \sum_{\gamma \text{ cut}} B(\eta, \gamma) \otimes A(\eta, \gamma)$</td>
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</tbody>
</table>

### Geometry
$k$-TWISTS

$k = 0$  
$k = 1$  
$k = 2$  
$k = 3$

trapezoidal shape of height $n$ and width $k$
(k, n)-twist = pipe dream in the trapezoidal shape of height n and width k
\(k\)-TWISTS

\((k, n)\)-twist = pipe dream in the trapezoidal shape of height \(n\) and width \(k\)

contact graph of a twist \(T\) = vertices are pipes of \(T\) and arcs are elbows of \(T\)
1-TWISTS AND TRIANGULATIONS

Correspondence

- Elbow in row $i$ and column $j$ $\leftrightarrow$ diagonal $[i,j]$ of the $(n+2)$-gon
- $(1,n)$-twist $T$ $\leftrightarrow$ triangulation $T^*$ of the $(n+2)$-gon
- $p$th relevant pipe of $T$ $\leftrightarrow$ $p$th triangle of $T^*$
- Contact graph of $T$ $\leftrightarrow$ dual binary tree of $T^*$
- Elbow flips in $T$ $\leftrightarrow$ diagonal flips in $T^*$

Woo. Catalan numbers and Schubert Polynomials for $w = 1(n+1)\ldots 2$. Unpub 2004
$k$-TWISTS AND $k$-TRIANGULATIONS

Correspondence

- elbow in row $i$ and column $j$ ↔ diagonal $[i, j]$ of the $(n + 2k)$-gon
- $(k, n)$-twist $T$ ↔ $k$-triangulation $T^*$ of the $(n + 2k)$-gon
- $p$th relevant pipe of $T$ ↔ $p$th $k$-star of $T^*$
- contact graph of $T$ ↔ dual graph of $T^*$
- elbow flips in $T$ ↔ diagonal flips in $T^*$

NUMEROLOGY

**THM.** The \((k,n)\)-twists are counted by \(\det(C_{n+2k-i-j})_{i,j\in[k]}\), where \(C_m = \frac{1}{m+1}\binom{2m}{m}\).

Jonsson. Generalized triangulations and diagonal-free subsets of stack polyominoes. 2005


Stump. A new perspective on \(k\)-triangulations. 2011

**QU.** What is the number of acyclic \((k,n)\)-twists?

<table>
<thead>
<tr>
<th>(k) (\setminus) (n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<th>7</th>
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<tr>
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</table>
\textbf{$k$-TWIST INSERTION}

Input: a permutation $\tau = \tau_1 \cdots \tau_n$

Algo: Insert pipes one by one (from right to left) \textbf{as northwest as possible}

Output: an acyclic $(k, n)$-twist $\text{ins}^k(\tau)$

Exm: Insertion of $\tau = 31542$
$k$-TWIST INSERTION

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**Exm:** Insertion of $\tau = 31542$

**THM.** $\text{ins}^k$ is a surjection from permutations of $[n]$ to acyclic $(k, n)$-twists.  
fiber of a $(k, n)$-twist $T = \text{linear extensions}$ of its contact graph $T^\#$.

**Exm:** insertion in binary search trees
**DEF.** \( k\)-twist congruence = equivalence relation \( \equiv^k \) on \( S_n \) defined as the transitive closure of the rewriting rule

\[
UacV_1b_1V_2b_2\cdots V_kb_kW \equiv^k UcaV_1b_1V_2b_2\cdots V_kb_kW \quad \text{if} \ a < b_i < c \ \text{for all} \ i \in [k].
\]

**PROP.** For any \( \tau, \tau^0 \in S_n \), we have \( \tau \equiv^k \tau^0 \iff \ins^k(\tau) = \ins^k(\tau^0) \).
DEF. Order congruence = equivalence relation $\equiv$ on a poset $P$ such that:

(i) Every equivalence class under $\equiv$ is an interval of $P$.

(ii) The projection $\pi_\downarrow: P \to P$ (resp. $\pi_\uparrow: P \to P$), which maps an element of $P$ to the minimal (resp. maximal) element of its equivalence class, is order preserving.

poset quotient $= X \leq Y$ in $P/\equiv \iff \exists x \in X, y \in Y$ such that $x \leq y$ in $P$.

If moreover $P$ is a lattice, $\equiv$ is automatically a lattice congruence, compatible with meets and joins: $x \equiv x^0$ and $y \equiv y^0 \Rightarrow x \land y \equiv x^0 \land y^0$ and $x \lor y \equiv x^0 \lor y^0$.

lattice quotient $= X \land Y$ and $X \lor Y$ are the congruence classes of $x \land y$ and $x \lor y$ for arbitrary representatives $x \in X$ and $y \in Y$.

THM. The $k$-twist congruence is a lattice quotient of the weak order.
**INCREASING FLIP LATTICE**

flip in a $k$-twist = exchange an elbow with the unique crossing between its two pipes  
increasing flip = the elbow is southwest of the crossing  
increasing flip order = transitive closure of the increasing flip graph
INCREASING FLIP LATTICE
flip in a $k$-twist = exchange an elbow with the unique crossing between its two pipes
increasing flip = the elbow is southwest of the crossing
increasing flip order = transitive closure of the increasing flip graph

PROP. The increasing flip order on acyclic $k$-twists is isomorphic to:

- the quotient lattice of the weak order by the $k$-twist congruence $\equiv^k$,
- the sublattice of the weak order induced by the permutations of $\mathfrak{S}_n$ avoiding the pattern $1(k + 2) - (\sigma_1 + 1) - \cdots - (\sigma_k + 1)$ for all $\sigma \in \mathfrak{S}_k$. 
$G^k(n) = \text{graph with vertex set } [n] \text{ and edge set } \{ \{i, j\} \in [n]^2 \mid i < j \leq i + k \}$

number of acyclic orientations of $G^k(n) = \begin{cases} n! & \text{if } n \leq k \\ k! (k + 1)^{n-k} & \text{if } n \geq k \end{cases}$

$k$-recoils scheme of $\tau \in S_n = \text{acyclic orientation } \text{rec}^k(\tau) \text{ of } G^k(n) \text{ with edge } i \to j \text{ for all } i, j \in [n] \text{ such that } |i - j| \leq k \text{ and } \tau^{-1}(i) < \tau^{-1}(j)$
$k$-RECOILS

$k$-recoil congruence $=\text{equivalence relation } \approx^k \text{ on } \mathfrak{S}_n \text{ defined as the transitive closure of the rewriting rule } UijV \approx^k UjiV \text{ if } i + k < j.$

**PROP.** For any $\tau, \tau^0 \in \mathfrak{S}_n$, we have $\tau \approx^k \tau^0 \iff \operatorname{rec}^k(\tau) = \operatorname{rec}^k(\tau^0)$.


**THM.** The $k$-recoil congruence is a lattice quotient of the weak order.
The maps $\text{ins}^k$, $\text{can}^k$, and $\text{rec}^k$ define a commutative diagram of lattice homomorphisms:

\[ \mathcal{S}_n \quad \xrightarrow{\text{rec}^k} \quad \text{AO}^k(n) \]
\[ \text{ins}^k \quad \xrightarrow{} \quad \text{AT}^k(n) \quad \xrightarrow{\text{can}^k} \]
ALGEBRA
For $n, n^0 \in \mathbb{N}$, consider the set of perms of $\mathcal{G}_{n+n^0}$ with at most one descent, at position $n$:

$$\mathcal{G}^{(n,n^0)} := \{\tau \in \mathcal{G}_{n+n^0} \mid \tau(1) < \cdots < \tau(n) \text{ and } \tau(n+1) < \cdots < \tau(n+n^0)\}$$

For $\tau \in \mathcal{G}_n$ and $\tau^0 \in \mathcal{G}_{n^0}$, define

- shifted concatenation $\tau \bar{\tau}^0 = [\tau(1), \ldots, \tau(n), \tau^0(1)+n, \ldots, \tau^0(n^0)+n] \in \mathcal{G}_{n+n^0}$
- shifted shuffle product $\tau \lll \tau^0 = \{(\tau \bar{\tau}^0) \circ \pi^{-1} \mid \pi \in \mathcal{G}^{(n,n^0)}\} \subset \mathcal{G}_{n+n^0}$
- convolution product $\tau \ast \tau^0 = \{\pi \circ (\tau \bar{\tau}^0) \mid \pi \in \mathcal{G}^{(n,n^0)}\} \subset \mathcal{G}_{n+n^0}$

Exm: $12 \lll 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}$

$12 \ast 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}$
**DEF.** Combinatorial Hopf Algebra = combinatorial vector space $B$ endowed with

- **product** $\cdot : B \otimes B \to B$
- **coproduct** $\Delta : B \to B \otimes B$

which are “compatible”, ie.

$$
\begin{array}{ccc}
B \otimes B & \cdot & B \\
\downarrow 4 \otimes 4 & & \downarrow 4 \\
B \otimes B \otimes B \otimes B & I \otimes \text{swap} \otimes I & B \otimes B \otimes B \otimes B
\end{array}
$$

**Malvenuto-Reutenauer algebra** = Hopf algebra $\text{FQSym}$ with basis $(F_{\tau})_{\tau \in S}$ and where

$$
F_{\tau} \cdot F_{\tau^0} = \sum_{\sigma \in \tau \shuffle \tau^0} F_{\sigma} \quad \text{and} \quad 4F_{\sigma} = \sum_{\tau \in \tau \times \tau^0} F_{\tau} \otimes F_{\tau^0}
$$

**HOPF SUBALGEBRA**

$k$-Twist algebra = vector subspace $\text{Twist}^k$ of $\text{FQSym}$ generated by

$$P_T := \sum_{\tau \in \mathcal{S} \text{ins}^k(\tau) = T} F_\tau = \sum_{\tau \in \mathcal{T}(T^\#)} F_\tau,$$

for all acyclic $k$-twists $T$.

**Exm:**

\[
P_{\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}} = \sum_{\tau \in \mathcal{S}_5} F_\tau \quad P_{\begin{array}{cc}
1 & 3 \\
2 & 4 \\
5 & 6 \\
\end{array}} = F_{13542} + F_{15342} + F_{31542} + F_{51342} + F_{35142} + F_{53142} + F_{35412} + F_{53412}
\]

\[
P_{\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}} = F_{31542} \quad P_{\begin{array}{cc}
1 & 2 \\
3 & 4 \\
5 & 6 \\
\end{array}} = F_{31542} \quad P_{\begin{array}{cc}
1 & 2 \\
3 & 4 \\
5 & 6 \\
\end{array}} = F_{12345}.
\]

**THEO.** $\text{Twist}^k$ is a subalgebra of $\text{FQSym}$


**GAME:** Explain the product and coproduct directly on the $k$-twists...
**PROP.** For \( T \in \mathcal{AT}^k(n) \) and \( T^0 \in \mathcal{AT}^k(n^0) \) acyclic \( k \)-twists, \( \mathbb{P}_T \cdot \mathbb{P}_{T^0} = \sum_S \mathbb{P}_S \), where \( S \) runs over the interval between \( T \setminus T^0 \) and \( T/T^0 \) in the \((k, n + n^0)\)-twist lattice.
GEOMETRY
**Permutahedron**

$\text{Perm}^k(n) = \text{conv} \{ (\tau(1), \ldots, \tau(n)) \mid \tau \in S_n \}$

\[
= \mathbb{H} \cap \bigcap_{\emptyset \neq J \subseteq [n]} \left\{ x \in \mathbb{R}^n \mid \sum_{j \in J} x_j \geq \binom{|J| + 1}{2} \right\}
\]

\[
= 1 + \sum_{1 \leq i < j \leq n} [e_i, e_j]
\]

connections to
- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group
brick vector of a \((k, n)\)-twist \(T = \text{vector } b(T) \in \mathbb{R}^n\) with \(b(T)_i = \text{number of boxes below the } i\text{th pipe of } T\)

Brick polytope

\(\text{Brick}^k(n) = \text{conv } \{ b(T) \mid T \text{ (}k, n\text{)-twist} \}\)

Vertices \(\leftrightarrow\) acyclic \((k, n)\)-twists
Facets \(\leftrightarrow\) 0/1-seqs with no subseqs \(10^\ell 1\) for \(\ell \geq k\)

connections to
- Loday associahedron
- incidence cones of binary trees
- Tamari lattice
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with \(b(T)_i = \text{number of boxes below the } i\text{th pipe of } T\)

Brick polytope
\[
\text{Brick}^k(n) = \text{conv } \{b(T) \mid \text{T } (k,n)\text{-twist} \}
\]

Vertices \(\leftrightarrow\) acyclic \((k,n)\)-twists
Facets \(\leftrightarrow\) 0/1-seqs with no subseqs 10\(\ell\)1 for \(\ell \geq k\)

connections to
- Loday associahedron
- incidence cones of binary trees
- Tamari lattice
Zonotope

\[ Zono^k(n) = \sum_{|i-j| \leq k} [e_i, e_j] \]

Vertices ↔ acyclic orientations of \( G^k(n) \)
connections to
• matroids and oriented matroids
• hyperplane arrangements
Zonotope

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MATROIOCHKA POLYTOPES

Permutahedron $\text{Perm}^k(n)$
MATRIOCHKA POLYTOPES

Permutahedron $\text{Perm}^k(n) \subset \text{Brick polytope} \ Brick^k(n)$
MATRIOCHKA POLYTOPIES

Permutahedron $\text{Perm}^k(n) \subset$ Brick polytope $\text{Brick}^k(n) \subset$ Zonotope $\text{Zono}^k(n)$
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$\text{Permutahedron } \text{Perm}^k(n) \subset \text{Brick polytope } \text{Brick}^k(n) \subset \text{Zonotope } \text{Zono}^k(n)$

$\text{Brick}^1(n)$

$\text{Zono}^1(n)$
MATRIOCHKA POLYTOPES

Permutahedron $\text{Perm}^k(n) \subset\text{Brick polytope} \text{Brick}^k(n) \subset\text{Zonotope} \text{Zono}^k(n)$

$\text{Brick}^1(n) \cap \text{Brick}^2(n)$

$\text{Zono}^1(n) \cap \text{Zono}^2(n)$
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$\text{Brick}^1(n) \cap \text{Brick}^2(n) \cap \text{Brick}^3(n) \cap \cdots \cap \text{Perm}^k(n)$

$\text{Zono}^1(n) \cap \text{Zono}^2(n) \cap \text{Zono}^3(n) \cap \cdots \cap \text{Perm}^k(n)$
NORMAL CONES

For a poset $\triangleleft$, define $C(\triangleleft) = \{x \in \mathbb{H} \mid x_i \leq x_j \text{ for all } i \triangleleft j \text{ in } T\}$.

PROP. The cones form complete simplicial fans:
- $\{C(\tau) \mid \tau \in \mathcal{S}_n\}$ = braid fan = normal fan of the permutahedron $\text{Perm}^k(n)$,
- $\{C(T) \mid T \in \text{AT}^k(n)\}$ = brick fan = normal fan of the brick polytope $\text{Brick}^k(n)$,
- $\{C(O) \mid O \in \text{AO}^k(n)\}$ = boolean fan = normal fan of the zonotope $\text{Zono}^k(n)$.

PROP. The insertion map $\text{ins}^k : \mathcal{S}_n \rightarrow \text{AT}^k(n)$, the $k$-canopy $\text{can}^k : \text{AT}^k(n) \rightarrow \text{AO}^k(n)$ and the $k$-recoil map $\text{rec}^k : \mathcal{S}_n \rightarrow \text{AO}^k(n)$ are characterized by:
- $T = \text{ins}^k(\tau) \iff C(T) \subseteq C(\tau) \iff C(T) \supseteq C(\tau)$,
- $O = \text{can}^k(T) \iff C(O) \subseteq C(T) \iff C(O) \supseteq C(T)$,
- $O = \text{rec}^k(\tau) \iff C(O) \subseteq C(\tau) \iff C(O) \supseteq C(\tau)$. 
Oriented in the direction $\sum_{i\in[n]} (n + 1 - 2i) e_i$, their graphs are Hasse diagrams of lattices:

- **permutahedron** $\text{Perm}^k(n)$
- **brick polytope** $\text{Brick}^k(n)$
- **zonotope** $\text{Zono}^k(n)$

- **weak order on $S_n$**
- **increasing flip lattice on acyclic $(k, n)$-twists**
- **boolean lattice on acyclic orientations of $G^k(n)$**
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weak order on \( \mathcal{S}_n \)  
increasing flip lattice on acyclic \( (k, n) \)-twists  
boolean lattice on acyclic orientations of \( G^k(n) \)
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zonotope $\text{Zono}^k(n)$

weak order on $S_n$  
increasing flip lattice on acyclic $(k, n)$-twists  
boolean lattice on acyclic orientations of $G^k(n)$
THREE EXTENSIONS
$k \in \mathbb{N}$ and $\varepsilon \in \pm^n$, define a shape $\text{Sh}_\varepsilon^k$ formed by four monotone lattices paths:

(i) **enter path**: from $(|\varepsilon|_+, 0)$ to $(0, |\varepsilon|_-)$ with $p$th step north if $\varepsilon_p = -$ and west if $\varepsilon_p = +$,

(ii) **exit path**: from $(|\varepsilon|_+ + k, n + k)$ to $(n + k, |\varepsilon|_- + k)$ with $p$th step east if $\varepsilon_p = -$ and south if $\varepsilon_p = +$,

(iii) **accordion paths**: the path $(NE)^{|\varepsilon|_++k}$ from $(0, |\varepsilon|_-)$ to $(|\varepsilon|_+ + k, n + k)$ and the path $(EN)^{|\varepsilon|_-+k}$ from $(|\varepsilon|_+, 0)$ to $(n + k, |\varepsilon|_- + k)$. 
$k = 0$           $k = 1$           $k = 2$           $k = 3$

Cambrian $(k, \varepsilon)$-twist = pipe dream in $\text{Sh}_\varepsilon^k$

contact graph of a twist $T$ = vertices are pipes of $T$ and arcs are elbows of $T$
CAMBRIANIZATION

Input: a signed permutation $\tau = \tau_1 \cdots \tau_n$

Algo: Insert pipes one by one (from right to left) as northwest as possible

Output: an acyclic Cambrian $(k, \varepsilon)$-twist $\text{ins}^k(\tau)$

Exm: Insertion of $\tau = \overline{31542}$
CAMBRIANIZATION

Input: a signed permutation \( \tau = \tau_1 \cdots \tau_n \)

Algo: Insert pipes one by one (from right to left) as northwest as possible

Output: an acyclic Cambrian \((k, \varepsilon)\)-twist \(\text{ins}^k(\tau)\)

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Exm: Insertion of $\tau = 31 \overline{54} \overline{2}$
permutahedron $\text{Perm}^k(n)$

brick polytope $\text{Brick}^k(\varepsilon)$

zonotope $\text{Zono}^k(n)$
CAMBRIANIZATION

permutahedron $\text{Perm}^k(n)$

brick polytope $\text{Brick}^k(\varepsilon)$

zonotope $\text{Zono}^k(n)$
permutahedron $\text{Perm}_k(n)$  
brick polytope $\text{Brick}_k(\varepsilon)$  
zonotope $\text{Zono}_k(n)$
TUPLIZATION

\[ E = [\varepsilon_1, \ldots, \varepsilon_\ell] \text{ an } \ell\text{-tuple of signatures} \]

\((k, E)\text{-twist tuple} = \text{ an } \ell\text{-tuple } [T_1, \ldots, T_\ell] \text{ where} \]

- \(T_i\) is a \((k, \varepsilon_i)\)-twist
- the union of the contact graphs \(T_1^\# \cup \cdots \cup T_\ell^\#\) is acyclic
TUPLIZATION
hypercippe = union of pipes whose common elbows are changed to crossings

(k, n)-hypertwist = collection of hyperpipes obtained from a (k, n)-twist T by merging subsets of pipes inducing connected subgraphs of T#
SCHRODERIZATION

Input: an ordered partition $\lambda = \lambda_1 \cdots \lambda_n$

Algo: Insert hyperpipes one by one (from right to left) as northwest as possible

Output: an acyclic $(k, n)$-hypertwist $\text{ins}^k(\lambda)$

Exm: Insertion of $\tau = 3\,|15|\,42$
**SCHRODERIZATION**

Input: an ordered partition \( \lambda = \lambda_1 \cdots \lambda_n \)

Algo: Insert hyperpipes one by one (from right to left) as northwest as possible

Output: an acyclic \((k, n)\)-hypertwist \(\text{ins}^k(\lambda)\)

Exm: Insertion of \(\tau = 3|15|42\)
SCHRODERIZATION

Input: an ordered partition $\lambda = \lambda_1 \cdots \lambda_n$

Algo: Insert hyperpipes one by one (from right to left) as northwest as possible

Output: an acyclic $(k, n)$-hypertwist $\operatorname{ins}^k(\lambda)$

Exm: Insertion of $\tau = 3|15|42$
Input: an ordered partition $\lambda = \lambda_1 \cdots \lambda_n$

Algo: Insert hyperpipes one by one (from right to left) as northwest as possible

Output: an acyclic $(k, n)$-hypertwist $\text{ins}^k(\lambda)$

Exm: Insertion of $\tau = 3|15|42$
THANK YOU