Multitriangulations, pseudotriangulations and some problems of realization of polytopes

Vincent PILAUD
Introduction: Polytopes and spheres with prescribed combinatorial structure
**polytope** = convex hull of a finite set of $\mathbb{R}^d$

= bounded intersection of finitely many half-spaces

**face** = intersection with a supporting hyperplane

**face lattice** = all the faces with their inclusion relations

---

Given a set of points, determine the face lattice of its convex hull.

Given (part of) a face lattice, is there a **polytope which realizes it**? In which dimension(s)?
Given (part of) a face lattice, is there a polytope which realizes it?

For example, which graphs are polytopal?

**THEOREM.** (Steinitz) Graphs of 3-polytopes = planar and 3-connected graphs.

Realizability questions are interesting for two kinds of structures:

1. lattices coming from combinatorial structures: for example, transformation graphs on combinatorial objects (permutohedron, associahedron, . . .).
   \[\Rightarrow \text{understanding of the combinatorial objects}.\]

2. lattices derived from operations on other lattices: Cartesian product, \(\Delta Y\), . . .
   \[\Rightarrow \text{understanding of polytopes}.\]

\[
\text{cell complex} \quad \rightarrow \quad \text{topological sphere} \quad \rightarrow \quad \text{matroid polytope} \quad \rightarrow \quad \text{polytope}
\]
MULTITRIANGULATIONS

1. Introduction
2. Stars in multitriangulations
3. Multipseudotriangulations
4. Three open problems: bijective counting, rigidity, multiassociahedron
   A. Two enumeration algorithms

POLYTOPALITY OF PRODUCTS

5. Introduction
6. Cartesian products of non-polytopal graphs
7. Prodsimplicial neighborly polytopes
CONTENTS

MULTITRIANGULATIONS

1. Introduction
2. Stars in multitriangulations
3. Multipseudotriangulations
4. Three open problems: bijective counting, rigidity, multiassociahedron
   A. Two enumeration algorithms

POLYTOPALITY OF PRODUCTS

5. Introduction
6. Cartesian products of non-polytopal graphs
7. Prodsimplicial neighborly polytopes
Multitriangulations

V. P. & F. Santos, Multitriangulations as complexes of star polygons, 2009.
DEFINITION

\( k \geq 1 \) and \( n \geq 2k + 1 \) two fixed integers.

`\(-\)crossing` = set of `\` mutually crossing diagonals of the convex \( n \)-gon.

\[ \textbf{\textit{k}}\text{-triangulation} = \text{maximal} \ (k + 1)\text{-crossing-free set of diagonals of the} \ n\text{-gon}. \]

\[ \begin{array}{c}
\begin{tikzpicture}
  \draw[very thick] (0,0) -- (1,0) -- (1.5,1) -- (1,2) -- (0,2) -- (-1,2) -- (-1,1) -- (-1.5,0) -- cycle;
  \draw[thick] (0,0) -- (1.5,1) -- (1,2) -- (0,2) -- (-1,2) -- (-1.5,1) -- (-1,0) -- cycle;
\end{tikzpicture}
\end{array} \]

J. Jonsson, Generalized triangulations and diagonal-free subsets of stack polyominoes, 2005.
**STARS IN MULTITRIANGULATIONS**

\[ \text{k-star} = \text{star polygon with vertices } s_0, s_1, \ldots, s_{2k} \text{ cyclically ordered and edges } [s_0, s_k], [s_1, s_{1+k}], \ldots, [s_k, s_{2k}], [s_{k+1}, s_0], \ldots, [s_{2k}, s_{k-1}]. \]
THEOREM. In a $k$-triangulation $T$,

(i) a $k$-relevant diagonal belongs to exactly two $k$-stars of $T$,
(ii) a $k$-boundary diagonal belongs to exactly one $k$-star of $T$,
(iii) a $k$-irrelevant diagonal does not belong to any $k$-star of $T$.

V. P. & F. Santos, Multitriangulations as complexes of star polygons, 2009.
COMMON BISECTORS

THEOREM. T a k-triangulation of the n-gon. Every pair of k-stars of T have a unique common bisector. Reciprocally, any diagonal not in T is the common bisector of a unique pair of k-stars of T.

COROLLARY. Any k-triangulation of the n-gon contains exactly $n - 2k$ k-stars and $k(2n - 2k - 1)$ diagonals.
THEOREM. Let $e$ be a $k$-relevant diagonal of a $k$-triangulation $T$, let $R$ and $S$ be the two $k$-stars of $T$ containing $e$, and let $f$ be the common bisector of $R$ and $S$. Then $T \triangle \{e, f\}$ is the only $k$-triangulation other than $T$ containing $T \setminus \{e\}$.

THEOREM. The graph of flips is connected, regular of degree $k(n - 2k - 1)$, and its diameter is at most $2k(n - 2k - 1)$. 

THEOREM. Let $e$ be a $k$-relevant diagonal of a $k$-triangulation $T$, let $R$ and $S$ be the two $k$-stars of $T$ containing $e$, and let $f$ be the common bisector of $R$ and $S$. Then $T \triangle \{e, f\}$ is the only $k$-triangulation other than $T$ containing $T \setminus \{e\}$.

THEOREM. The graph of flips is connected, regular of degree $k(n - 2k - 1)$, and its diameter is at most $2k(n - 2k - 1)$.
k ≥ 1 and n ≥ 2k + 1 two fixed integers.

`-crossing = set of ` mutually crossing diagonals of the convex n-gon.

k-relevant diagonal = at least k vertices on each side
                   = diagonals which may appear in a (k + 1)-crossing.

\[ \Delta_{n,k} = \text{simplicial complex of } (k+1)\text{-crossing-free sets}
                   \text{ of } k\text{-relevant diagonals of the convex } n\text{-gon.} \]

**THEOREM.** \( \Delta_{n,k} \) is a topological sphere of dimension \( k(n - 2k - 1) - 1 \).


**QUESTION.** Is \( \Delta_{n,k} \) the boundary complex of a simplicial \( k(n - 2k - 1) \)-polytope?
\[ k = 1 \] Maximal elements of \( \Delta_{n,1} \) = triangulations of the \( n \)-gon.
\( \Delta_{n,1} \) = boundary complex of the dual of the \((n - 3)\)-dimensional associahedron.

L.J. Billera, P. Filliman & B. Sturmfels,
Maximal elements of $\Delta_{n,1} = \text{triangulations of the } n\text{-gon.}$

$\Delta_{n,1} = \text{boundary complex of the dual of the } (n-3)\text{-dimensional associahedron.}$

OTHER EXAMPLES

\[ k = 1 \]
Maximal elements of \( \Delta_{n,1} \) = triangulations of the \( n \)-gon.
\( \Delta_{n,1} \) = boundary complex of the dual of the \((n - 3)\)-dimensional associahedron.

\[ n = 2k + 1 \]
\( \Delta_{2k+1,k} \) = single \( k \)-triangulation.
\(k = 1\) \(\Delta_{n,1}\) maximal elements of \(\Delta_{n,1}\) = triangulations of the \(n\)-gon.

\(\Delta_{n,1}\) = boundary complex of the dual of the \((n - 3)\)-dimensional associahedron.

\(n = 2k + 1\) \(\Delta_{2k+1,k}\) single \(k\)-triangulation.

\(n = 2k + 2\) \(\Delta_{2k+2,k}\) boundary complex of the \(k\)-simplex.
OTHER EXAMPLES

$k = 1$ Maximal elements of $\Delta_{n,1} = \text{triangulations of the } n\text{-gon.}$
$\Delta_{n,1} = \text{boundary complex of the dual of the } (n - 3)\text{-dimensional associahedron.}$

$n = 2k + 1$ $\Delta_{2k+1,k} = \text{single } k\text{-triangulation.}$

$n = 2k + 2$ $\Delta_{2k+2,k} = \text{boundary complex of the } k\text{-simplex.}$
OTHER EXAMPLES

\( k = 1 \)  Maximal elements of \( \Delta_{n,1} \) = triangulations of the \( n \)-gon.
\[ \Delta_{n,1} = \text{boundary complex of the dual of the } (n-3)\text{-dimensional associahedron.} \]

\( n = 2k + 1 \)  \( \Delta_{2k+1,k} \) = single \( k \)-triangulation.

\( n = 2k + 2 \)  \( \Delta_{2k+2,k} \) = boundary complex of the \( k \)-simplex.

\( n = 2k + 3 \)  \( \Delta_{2k+3,k} \) = boundary complex of the cyclic polytope of dimension \( 2k \) with \( 2k + 3 \) vertices.
OTHER EXAMPLES

\( k = 1 \)  Maximal elements of \( \Delta_{n,1} \) = triangulations of the \( n \)-gon. 
\( \Delta_{n,1} \) = boundary complex of the dual of the \((n - 3)\)-dimensional associahedron.

\( \text{n} = 2k + 1 \)  \( \Delta_{2k+1,k} \) = single \( k \)-triangulation.

\( \text{n} = 2k + 2 \)  \( \Delta_{2k+2,k} \) = boundary complex of the \( k \)-simplex.

\( \text{n} = 2k + 3 \)  \( \Delta_{2k+3,k} \) = boundary complex of the cyclic polytope of dimension \( 2k \) with \( 2k + 3 \) vertices.

\( \text{n} = 8 \) & \( k = 2 \)
\( f \)-vector of \( \Delta_{8,2} \) = \((12, 66, 192, 306, 252, 84)\)

**THEOREM.** The space of symmetric realizations of \( \Delta_{8,2} \) has dimension 4.

Flip graphs on pseudoline arrangements

V. P. & M. Pocchiola, Multipseudotriangulations, 2010.
Mobius strip = $\mathbb{R}^2 / (x, y) \sim (x + \pi, -y)$.

pseudoline = non-separating simple closed curve in the Möbius strip.

pseudoline arrangement = finite set of pseudolines such that any two of them have exactly one crossing point and possibly some contact points.

support = union of pseudolines

levels = layers of the arrangement
Flip = exchange a contact point between two pseudolines with their crossing point. \( G(S) = \) the flip graph on all pseudoline arrangements supported by a given support \( S \).

**EXAMPLE.** \( S = \) support with 2 levels and \( p \) intersection points. Then \( G(S') = \) complete graph \( K_p \).
line space of the Euclidean plane $= \mathbb{R}^2/((\theta, d) \sim (\theta + \pi, -d) = \text{M"obius strip.}$
$V_n$ vertices of the convex n-gon  
$V_n^*$ dual pseudoline arrangement of $V_n$
**THEOREM.** \( T \subset \binom{V_n}{2} \) k-triangulation of \( V_n \) \iff \( T^* \) covers \( V_n^* \) minus its first k levels.
A pseudotriangulation of a finite point set $P$ is:

(i) a maximal crossing-free pointed subset of $\binom{P}{2}$,

(ii) a pointed subset of $\binom{P}{2}$ that decomposes the convex hull of $P$ into pseudotriangles.

M. Pocchiola & G. Vegter, Topologically sweeping visibility complexes via pseudotriangulations, 1996.
I. Streinu, Pseudo-triangulations, rigidity and motion planning, 2005.
A pseudotriangulation of a finite point set $P$ is:

(i) a maximal crossing-free pointed subset of $\binom{P}{2}$,
(ii) a pointed subset of $\binom{P}{2}$ that decomposes the convex hull of $P$ into pseudotriangles.

PROPERTIES.

(i) A pseudotriangulation of $P$ has exactly $2|P| - 3$ edges.
(ii) Two pseudotriangles have a unique common tangent.
(iii) $T$ pseudotriangulation of $P$; $e$ internal edge of $T$; $f$ common tangent between the two pseudotriangles of $T$ containing $e$ $\Rightarrow$ $T \triangle \{e, f\}$ pseudotriangulation of $P$.
(iv) The graph of flips is polytopal.

M. Pocchiola & G. Vegter, Topologically sweeping visibility complexes via pseudotriangulations, 1996.
I. Streinu, Pseudo-triangulations, rigidity and motion planning, 2005.
G. Rote, F. Santos, & I. Streinu,
DUALITY AND PSEUDOTRIANGULATIONS

**P** point set in general position

**P**\(^*\) dual pseudoline arrangement of **P**
### Duality and Pseudotriangulations

<table>
<thead>
<tr>
<th>P point set in general position</th>
<th>P* dual pseudoline arrangement of P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Δ pseudotriangle</td>
<td>Δ* = {tangent to Δ} dual pseudoline of Δ</td>
</tr>
<tr>
<td>T pseudotriangulation of P</td>
<td>T* = {Δ*</td>
</tr>
</tbody>
</table>

**Theorem:** \( T \subset \binom{P}{2} \) pseudotriangulation of P \( \iff \) \( T^* \) covers \( P^* \) minus its first level.
**k-pseudotriangulation** of a point set \( P \) in general position in the plane = set \( T \) of edges of \( \binom{P}{2} \) which corresponds via duality to the contact points of a pseudoline arrangement \( T^* \) supported by \( P^* \) minus its first \( k \) levels.
**MULTIPSEUDOTRIANGULATIONS**


definition of a point set $P$ in general position in the plane =
set $T$ of edges of $\binom{P}{2}$ which corresponds via duality to the contact points of a
pseudoline arrangement $T^*$ supported by $P^*$ minus its first $k$ levels.

**PROPOSITION.** $P \cup \{q\}$ point set in general position. $T$ a $k$-pseudotriangulation of $P$.

$k$-depth of $q$ in $P = \sum_{\lambda \in T^*} \text{winding number of } S(\lambda)$.
EXAMPLES OF APPLICATIONS

**ENUMERATION**  ...  greedy pseudoline arrangements

$\Rightarrow$ enumeration algorithm for pseudoline arrangements covering a given support.

**CHARACTERIZATION THEOREM**

**THEOREM.** A set $\Sigma$ of $k$-stars of the $n$-gon such that:
(i) any $k$-relevant edge is contained in zero or two $k$-stars of $\Sigma$, one on each side, and
(ii) any $k$-boundary edge is contained in exactly one $k$-star of $\Sigma$,
is the set of $k$-stars of a $k$-triangulation of the $n$-gon.

**LOWER BOUND THEOREM**  ...  for $d$-polytopes with $d + 3$ vertices.
The brick polytope

Loday’s associahedron $\Omega(n) = \text{conv}\{\omega(T) \mid T \text{ triangulation of the } n\text{-gon}\}.$

LODAY’S ASSOCIAHEDRON REVISITED

T triangulation of the \( n \)-gon \( \mapsto \) vector \( \omega(T) \in \mathbb{R}^{n-2} \).

Loday’s associahedron \( \Omega(n) = \text{conv}\{\omega(T) \mid T \text{ triangulation of the } n\text{-gon}\} \).

LODAY’S ASSOCIAHEDRON REVISITED

T triangulation of the n-gon $\rightarrow$ vector $\omega(T) \in \mathbb{R}^{n-2}$.

Loday’s associahedron $\Omega(n) = \text{conv}\{\omega(T) \mid T \text{ triangulation of the } n\text{-gon}\}$. 

Loday’s associahedron $\Omega(n) = \text{conv}\{\omega(T) \mid T \text{ triangulation of the } n\text{-gon}\}$.
THEOREM. $\Omega(n)$ is a realization of the $(n - 3)$-dimensional associahedron.

$S =$ sorting network $=$ support of pseudoline arrangements.

$\Lambda$ pseudoline arrangement supported by $S \mapsto$ vector $\omega(\Lambda) \in \mathbb{R}^m$.

Brick polytope $\Omega(S) = \text{conv}\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $S\}$. 
$S =$ sorting network $=$ support of pseudoline arrangements.

$\Lambda$ pseudoline arrangement supported by $S$ $\mapsto$ vector $\omega(\Lambda) \in \mathbb{R}^m$.

Brick polytope $\Omega(S) = \text{conv}\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $S\}$. 

$\omega_j(\Lambda) =$ number of bricks above the $j$th pseudoline of $\Lambda$. 

---

### THE BRICK POLYTOPE

- $S =$ sorting network $=$ support of pseudoline arrangements.
- $\Lambda$ pseudoline arrangement supported by $S$ $\mapsto$ vector $\omega(\Lambda) \in \mathbb{R}^m$.
- Brick polytope $\Omega(S) = \text{conv}\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $S\}$. 
- $\omega_j(\Lambda) =$ number of bricks above the $j$th pseudoline of $\Lambda$. 

---

**Diagram:** A visual representation of the brick polytope with 5 levels, each level having blocks representing the number of bricks above each pseudoline.
THE BRICK POLYTOPE

\( S = \text{sorting network} = \text{support of pseudoline arrangements.} \)

\( \Lambda \) pseudoline arrangement supported by \( S \) \( \iff \) vector \( \omega(\Lambda) \in \mathbb{R}^m \).

Brick polytope \( \Omega(S) = \text{conv}\{\omega(\Lambda) | \text{\( \Lambda \) pseudoline arrangement supported by \( S \)}\}. \)

\[ \omega(\Lambda)_j = \text{number of bricks above the } j\text{th pseudoline of } \Lambda. \]
$S = \text{sorting network} = \text{support of pseudoline arrangements.}$

$\Lambda$ pseudoline arrangement supported by $S \mapsto \text{vector } \omega(\Lambda) \in \mathbb{R}^m.$

Brick polytope $\Omega(S) = \text{conv}\{\omega(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } S\}.$

$\omega(\Lambda)_j = \text{number of bricks above the } j\text{th pseudoline of } \Lambda.$
$S = \text{sorting network} = \text{support of pseudoline arrangements.}$

$\Lambda \text{ pseudoline arrangement supported by } S \mapsto \text{ vector } \omega(\Lambda) \in \mathbb{R}^m.$

Brick polytope $\Omega(S) = \text{conv}\{\omega(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } S\}.$

$\omega(\Lambda)_j = \text{number of bricks above the } j \text{th pseudoline of } \Lambda.$
$S = \text{sorting network} = \text{support of pseudoline arrangements.}$

$\Lambda$ pseudoline arrangement supported by $S \iff \text{vector } \omega(\Lambda) \in \mathbb{R}^m.$

Brick polytope $\Omega(S) = \text{conv}\{\omega(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } S\}.$

$\omega(\Lambda)_j = \text{number of bricks above the } j\text{th pseudoline of } \Lambda.$
$S$ = sorting network = support of pseudoline arrangements.

A pseudoline arrangement supported by $S \mapsto$ vector $\omega(\Lambda) \in \mathbb{R}^m$.

Brick polytope $\Omega(S) = \text{conv}\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $S\}$.

$\omega(\Lambda)_j =$ number of bricks above the $j$th pseudoline of $\Lambda$. 

\[
\begin{array}{cccccc}
5 & 4 & 3 & 2 & 1 & 7 \\
10 & 14 & 7 & 7 & & \\
\end{array}
\]
LEMMA. \( \Lambda \) and \( \Lambda' \) related by a flip between their \( i \)th and \( j \)th pseudolines

\[ \Rightarrow \omega(\Lambda) - \omega(\Lambda') = \lambda(e_i - e_j). \]
THE INCIDENCE CONE OF A MULTIGRAPH

G oriented (multi)graph $\mapsto$ Incidence configuration $I(G) = \{e_i - e_j \mid (i, j) \in G\}$, $\mapsto$ Incidence cone $C(G) = \text{cone generated by } I(G)$.

REMARK. circuits in $I(G) \leftrightarrow$ simple cycles in $G$, cocircuits in $I(G) \leftrightarrow$ minimal cuts in $G$, (and signs correspond to the orientations of the edges).

REMARK. $C(G)$ is pointed $\leftrightarrow$ $G$ is acyclic. facets of $C(G) \leftrightarrow$ complements of the minimal directed cuts of $G$. 
Contact graph $\Lambda^\#$ of a pseudoline arrangement $\Lambda =$

- a node for each pseudoline of $\Lambda$, and
- an arc for each contact point of $\Lambda$ oriented from top to bottom.
Contact graph $\Lambda^\#$ of a pseudoline arrangement $\Lambda =$
- a node for each pseudoline of $\Lambda$, and
- an arc for each contact point of $\Lambda$ oriented from top to bottom.

**THEOREM.** Cone of $\Omega(S)$ at $\omega(\Lambda) =$ incidence cone $C(\Lambda^\#)$.

**COROLLARY.** $\omega(\Lambda)$ vertex of $\Omega(S) \iff \Lambda^\#$ acyclic.

**COROLLARY.** Normal vectors of $\Omega(S) =$ characteristic vectors of sinks of directed cuts of acyclic contact graphs of pseudoline arrangements supported by $S$. 
TRIANGULATIONS AND MULTITRIANGULATIONS

**TRIANGULATIONS**

1. Up to translation, Loday’s associahedron = Brick polytope of $V_n^*$ minus its first level.
2. Contact graph = dual tree ⇒ each triangulation appears as a simple vertex.
3. Normal vectors of facets of $\Omega(n) = \{0^{i-1}11^{j-i-1}0^{n-j} | [i,j] \text{ internal diagonal}\}$.

**MULTITRIANGULATIONS**

1. Not all $k$-triangulations appear as vertices of $\Omega(V_n^{*k})$, and not all vertices are simple.
2. Normal vectors of facets of $\Omega(V_n^{*k}) = 0/1$-sequences of length $n - 2k$, distinct from $0^{n-2k}$ and $1^{n-2k}$ and not containing $10^r1$, for $r \geq k$. 

$k$-triangulation of the $n$-gon. Then:

$$(T^*)^\# = \text{contact graph of the dual pseudoline arrangement of } T$$

$= \text{dual graph of } T \text{ as complex of } k\text{-stars}.$$
Open problems and perspectives
OPEN PROBLEMS AND PERSPECTIVES

1. Multi Dyck paths

**THEOREM.** The number of $k$-triangulations of the $n$-gon is

$$\det(C_{n-i-j})_{1 \leq i,j \leq k} = \begin{vmatrix} C_{n-2} & C_{n-3} & \ldots & C_{n-k} & C_{n-k-1} \\ C_{n-3} & C_{n-4} & \ldots & C_{n-k} & C_{n-k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-k-1} & C_{n-k-2} & \ldots & C_{n-2k+1} & C_{n-2k} \end{vmatrix},$$

where $C_m = \frac{1}{m+1} \binom{2m}{m}$ is the $m$th Catalan number.

J. Jonsson, Generalized triangulations and diagonal-free subsets of stack polyominoes, 2005.

**PROBLEM.** Find an explicit bijection

$k$-triangulations $\leftrightarrow$ $k$-Dyck paths.

S. Elizalde, A bijection between 2-triangulations and pairs of non-crossing Dyck paths, 2006.
C. Nicolas, Another bijection between 2-triangulations and pairs of non-crossing Dyck paths, 2009.
OPEN PROBLEMS AND PERSPECTIVES

1. Multi Dyck paths
2. Pseudotriangulations and multipseudotriangulations in higher dimension

- multipseudotriangulations of 2-dimensional point sets
  - Positivity of the $j$-depth for all $j$
  - Lower Bound Theorem for $d$-polytopes with $d + 3$ vertices


**PROBLEM.** Define (multi)pseudotriangulations in higher dimension.
OPEN PROBLEMS AND PERSPECTIVES

1. Multi Dyck paths
2. Pseudotriangulations and multipseudotriangulations in higher dimension
3. Polytopality of flip graphs

\[ \Delta(S) = \text{simplicial complex whose maximal simplices are the sets of contact points of pseudoline arrangements supported by } S. \]

**PROBLEM.** Is \( \Delta(S) \) the boundary complex of a polytope?

Remark:
- Multitriangulations are universal.
- First open case: pseudotriangulations of non-realizable pseudoline arrangements.
Thank you
Questions

You will ask about that, right?
DIAMETER OF $G_{n,k}$

**PROPOSITION.** The diameter $\delta_{h,k}$ of the graph of flips on $k$-triangulations of the $n$-gon is bounded by

$$2 \left\lfloor \frac{n}{2} \right\rfloor \left( k + \frac{1}{2} \right) - k(2k + 3) \leq \delta_{h,k} \leq 2k(n - 4k - 1),$$

when $n > 4k^2(2k + 1)$.

Diameter for little values of $n$ and $k$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{h,1}$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>12</td>
<td>15</td>
<td>16</td>
<td>18</td>
<td>20</td>
<td>22</td>
<td>24</td>
<td>26</td>
</tr>
<tr>
<td>$\delta_{h,2}$</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>8</td>
<td>11</td>
<td>14</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\delta_{h,3}$</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
MAPS ON SURFACES
Let $T$ be a $k$-triangulation of the $n$-gon. Then:

$$(T^*)\# = \text{contact graph of the dual pseudoline arrangement of } T$$

$$= \text{dual graph of } T \text{ as complex of } k\text{-stars}.$$
fundamental group of the flip graph $G_{n,k} \longleftrightarrow$ mapping class group of the surface $S_{n,k}$
$\mathcal{P} = $ projective plane = disk with antipodal boundary points identified.

pseudoline = non-separating simple closed curve
double pseudoline = separating simple closed curve

double pseudoline arrangement = finite set of double pseudolines such that any two of them cross in exactly four points, transversally at these points and induce a cell decomposition of $\mathcal{P}$. 
ENUMERATION OF DOUBLE PSEUDOLINE ARRANGEMENTS
**ENUMERATION OF DOUBLE PSEUDOLINE ARRANGEMENTS**

Number of arrangements with \( n \) pseudolines and \( m \) double pseudolines:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>1</td>
<td>13</td>
<td>6570</td>
<td>181403533</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>626</td>
<td>4822394</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>48</td>
<td>86715</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>5</td>
<td>1329</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>25</td>
<td>80253</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>302</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>9194</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>11</td>
<td>556298</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>135</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>4382</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>312356</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
USE SYMMETRY

\( \mathbb{D}_n = \text{dihedral group} = \text{isometries of the regular } n\text{-gon} \)

Natural action of \( \mathbb{D}_n \) on \( \Delta_{n,k} \):

\[
\mathbb{D}_n \times \Delta_{n,k} \to \Delta_{n,k}
\]

\[
(\rho, E) \mapsto \rho E = \{ \rho e \mid e \in E \}
\]

DECOMPOSE INTO TWO STEPS

1. From face lattice to oriented matroids
   
   Find all possible symmetric oriented matroids realizing \( \Delta_{n,k} \)

2. From oriented matroids to polytopes
   
   Deduce the space of symmetric realizations of \( \Delta_{n,k} \)
SYMMETRY
\[ \Delta \text{ a simplicial complex with an action of a group } G \]
\[ P \subset \mathbb{R}^d \text{ a realization of } \Delta \text{ symmetric under } G, \text{ and } V \text{ its vertex set} \]

\[ \sigma : \begin{bmatrix} V^{d+1} \\ (v_0, v_1, \ldots, v_d) \end{bmatrix} \rightarrow \{-1, 0, +1\} \]

orientation of the simplex spanned by \( v_0, v_1, \ldots, v_d \) = \text{sign det} \begin{pmatrix} v_0 & v_1 & \cdots & v_d \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \]
\( \Delta \) a simplicial complex with an action of a group \( G \)

\( P \subset \mathbb{R}^d \) a realization of \( \Delta \) symmetric under \( G \), and \( V \) its vertex set

\[
\sigma : \begin{bmatrix} V^{d+1} \\
( v_0, v_1, \ldots, v_d ) \end{bmatrix} \rightarrow \{ -1, 0, +1 \}
\]

orientation of the simplex spanned by \( v_0, v_1, \ldots, v_d \)

satisfies the relations:

(i) **Alternating relations**
\[ \Delta \text{ a simplicial complex with an action of a group } G \]
\[ \mathcal{P} \subset \mathbb{R}^d \text{ a realization of } \Delta \text{ symmetric under } G, \text{ and } V \text{ its vertex set} \]

\[ \sigma : \begin{bmatrix} V^{d+1} \\ (v_0, v_1, \ldots, v_d) \end{bmatrix} \rightarrow \{-1, 0, +1\} \]
orientation of the simplex spanned by \( v_0, v_1, \ldots, v_d \)

\[ \Rightarrow \text{sign } \det \begin{pmatrix} v_0 & v_1 & \cdots & v_d \\ 1 & 1 & 1 & 1 \end{pmatrix} \]

satisfies the relations:

(i) Alternating relations

(ii) Grassmann-Plucker relations
\[ \Delta \text{ a simplicial complex with an action of a group } G \]
\[ P \subset \mathbb{R}^d \text{ a realization of } \Delta \text{ symmetric under } G, \text{ and } V \text{ its vertex set} \]

\[ \sigma : \begin{bmatrix} V^{d+1} \\ (v_0, v_1, \ldots, v_d) \end{bmatrix} \rightarrow \{-1, 0, +1\} \]
orientation of the simplex spanned by \( v_0, v_1, \ldots, v_d \)

\[ = \text{sign det} \begin{pmatrix} v_0 & v_1 & \cdots & v_d \\ 1 & 1 & 1 & 1 \end{pmatrix} \]
satisfies the relations:

(i) Alternating relations
(ii) Grassmann-Plucker relations
(iii) Necessary simplex orientations
A simplicial complex with an action of a group \( G \)

\( P \subset \mathbb{R}^d \) a realization of \( \Delta \) symmetric under \( G \), and \( V \) its vertex set

\[
\sigma : \begin{bmatrix} V^{d+1} \\ (v_0, v_1, \ldots, v_d) \end{bmatrix} \rightarrow \{ -1, 0, +1 \}
\]

orientation of the simplex spanned by \( v_0, v_1, \ldots, v_d \)

\[
= \text{sign} \det \begin{pmatrix} v_0 & v_1 & \ldots & v_d \\ 1 & 1 & 1 & 1 \end{pmatrix}
\]

satisfies the relations:

(i) Alternating relations

(ii) Grassmann-Plucker relations

(iii) Necessary simplex orientations

(iv) Symmetry
Problem. For a given oriented matroid, find a matrix representing it or a proof that such a matrix is impossible to find.

“On the one hand, there is a general algorithm to solve this problem. On the other hand, it is known that this algorithm from real algebraic geometry is far from applicable for our cases in the theory of oriented matroids.”

J. Bokowski, Computational Oriented Matroids, 2006

⇒ USE HEURISTICAL METHODS

Our heuristic is symmetry
THE SPACE OF SYMMETRIC REALIZATIONS OF $\Delta_{8,2}$

**Proposition.** The space of symmetric realizations of $\Delta_{8,2}$ has dimension 4.

**Example.** With some arbitrary values of the 4 parameters, we obtain a particular symmetric realization of $\Delta_{8,2}$:

$$
\mathbf{M} \approx \begin{pmatrix}
0.21 & 0 & 0 & -0.21 & 0.52 & -0.74 & 0.74 & -0.52 & 0 & 0 & 0 & 0 \\
-0.95 & 0.66 & 0 & -0.63 & 0.8 & -0.4 & -0.3 & 0.68 & 0 & 0 & 0 & 0 \\
-0.17 & 0.75 & -1 & 0.77 & -0.21 & -0.4 & 0.60 & -0.34 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\
0.55 & 0.55 & -0.55 & -0.55 & 0.5 & 0.7 & -0.7 & -0.4 & 1 & 0 & -1 & 0 \\
-0.55 & 0.55 & 0.55 & -0.4 & -0.7 & 0.7 & 0.5 & -0.55 & 0 & 1 & 0 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
$$

```plaintext
$> \text{polymake multiassociahedron82 F_VECTOR}$
$\text{F_VECTOR}$
$12 66 192 306 252 84$
```
POLYTOPALITY OF PRODUCTS OF NON-POLYTOPAL GRAPHS

Cartesian product of polytopes: \( P \times Q := \{(p, q) \mid p \in P, q \in Q\} \).

Cartesian product of graphs:
\[
\begin{align*}
V(G \times H) &:= V(G) \times V(H), \\
E(G \times H) &:= (V(G) \times E(H)) \cup (E(G) \times V(H)).
\end{align*}
\]

REMARK. graph of \( P \times Q = (\text{graph of } P) \times (\text{graph of } Q) \).

PROBLEM. Does the polytopality of \( P \times Q \) imply that of \( P \) and \( Q \)?
**POLYTOPALITY OF PRODUCTS OF NON-POLYTOPAL GRAPHS**

**PROBLEM.** Does the polytopality of $P \times Q$ imply that of $P$ and $Q$?

**THEOREM.** $G \times H$ simple polytopal $\iff$ $G$ and $H$ simply polytopal.

**THEOREM.** The product of a $d$-polytopal graph by the graph of a regular subdivision of an $e$-polytope is $(d+e)$-polytopal.

---

A polytope is \((k, \underline{n})\)-prodsimplicial-neighborly if its \(k\)-skeleton is combinatorially equivalent to that of the product of simplices \(\triangle \underline{n} := \triangle n_1 \times \cdots \times \triangle n_r\).

**EXAMPLE.**

(i) **neighborly** polytopes arise when \(r = 1\).
   For example, the cyclic polytope \(C_{2k+2}(n+1)\) is \((k, n)\)-PSN.

(ii) **neighborly cubical** polytopes arise when \(\underline{n} = (1, 1, \ldots, 1)\).


**PROBLEM.** What is the minimal dimension of a \((k, n)\)-PSN polytope?
CONSTRUCTIONS

(i) products of cyclic polytopes.
(ii) reflections of cyclic polytopes.
(iii) Minkowski sums of cyclic polytopes.
(iv) projections of deformed products of polytopes.

OBSTRUCTIONS

A \((k, n)\)-PSN polytope is \((k, n)\)-projected-prodsimplicial-neighborly if it is a projection of a polytope combinatorially equivalent to \(\Delta_n\).

Sanyal’s topological obstruction method:
Projection preserving the k-skeleton of \(\Delta_n\)
\[\mapsto\] simplicial complex embeddable in a certain dimension (Gale duality)
\[\mapsto\] topological obstruction (Sarkaria’s criterion).

2pm. Room 0C05. Thesis defense.

3pm. Room 0C08. Pot de thèse. Be careful, scientific program is not over yet...

6pm. Room 0C05. FRANCISCO SANTOS disproves the Hirsch Conjecture !!!!!

7pm. Room 0C08. Back to the pot. The scientific program is over now...