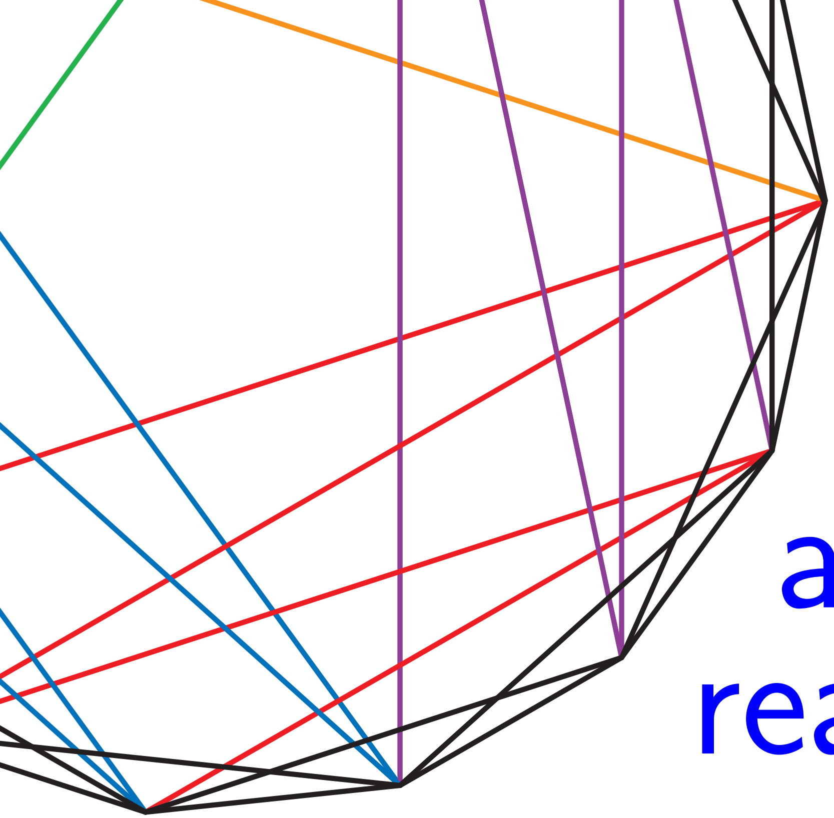
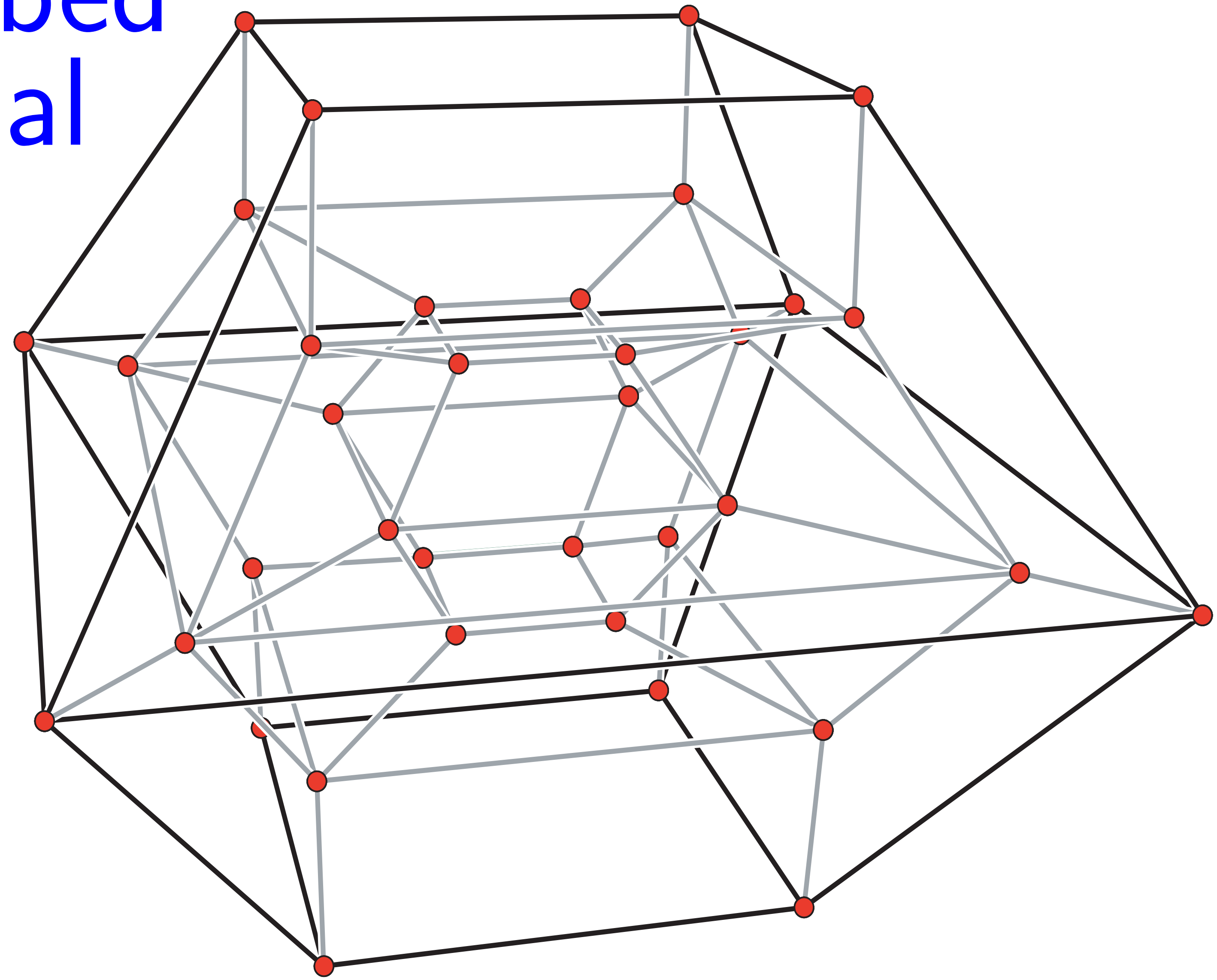


Multitriangulations, pseudotriangulations and some problems of realization of polytopes

Vincent PILAUD

Introduction: Polytopes and spheres with prescribed combinatorial structure

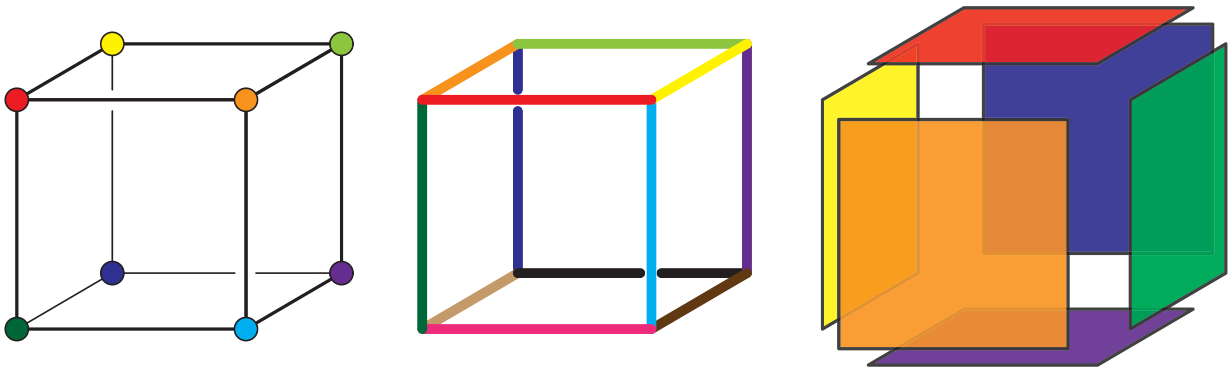


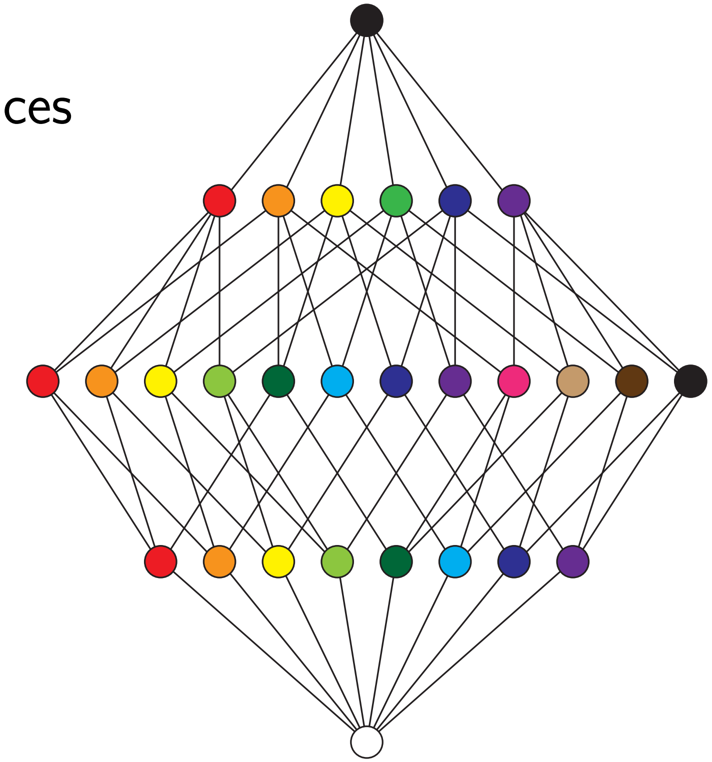
COMBINATORICS OF POLYTOPES
POLYTOPES FROM COMBINATORICS

polytope $=$ convex hull of a finite set of $\mathbb{R}^d$
$=$ bounded intersection of finitely many half-spaces

face $=$ intersection with a supporting hyperplane

face lattice $=$ all the faces with their inclusion relations





Given a set of points, determine the face lattice of its convex hull.

Given (part of) a face lattice, is there a polytope which realizes it?
In which dimension(s)?

POLYTOPES WITH PRESCRIBED COMBINATORICS

Given (part of) a face lattice, is there a polytope which realizes it?

For example, which graphs are polytopal?

THEOREM. (Steinitz) Graphs of 3-polytopes $=$ planar and 3-connected graphs.

Realizability questions are interesting for two kinds of structures:

1. lattices coming from combinatorial structures: for example, transformation graphs on combinatorial objects (permutohedron, associahedron, ...).

   $\Rightarrow$ understanding of the combinatorial objects.

2. lattices derived from operations on other lattices: Cartesian product, ΔY, ...

   $\Rightarrow$ understanding of polytopes.

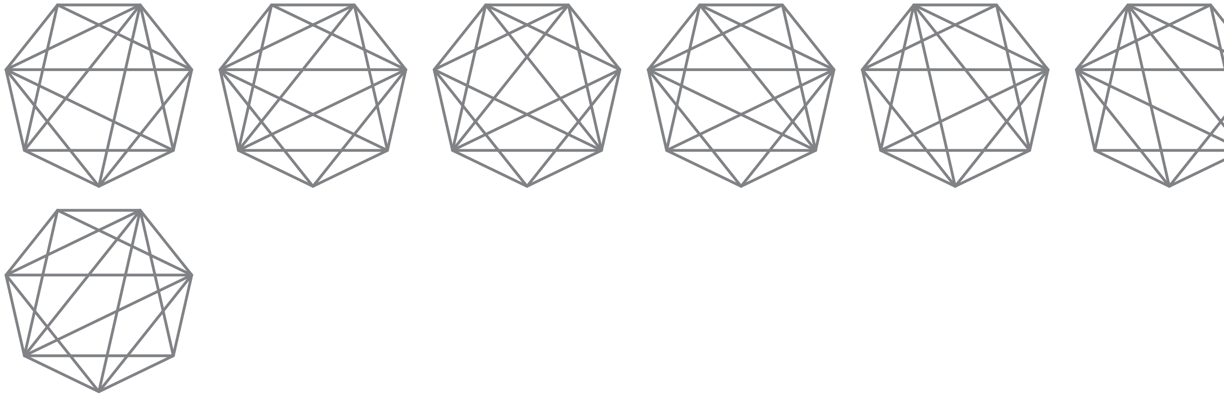
cell complex $\longrightarrow$ topological sphere $\longrightarrow$ matroid polytope $\longrightarrow$ polytope

CONTENTS

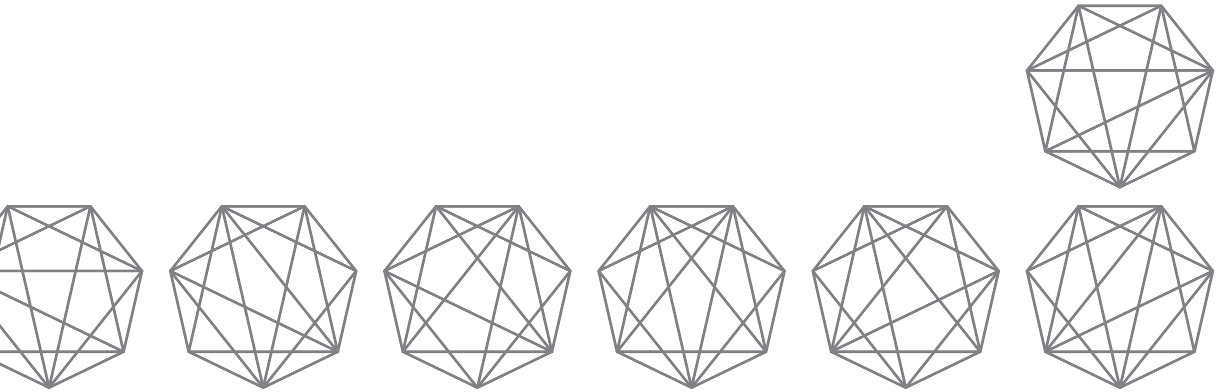
MULTITRIANGULATIONS

POLYTOPALITY OF PRODUCTS

CONTENTS



Multitriangulations



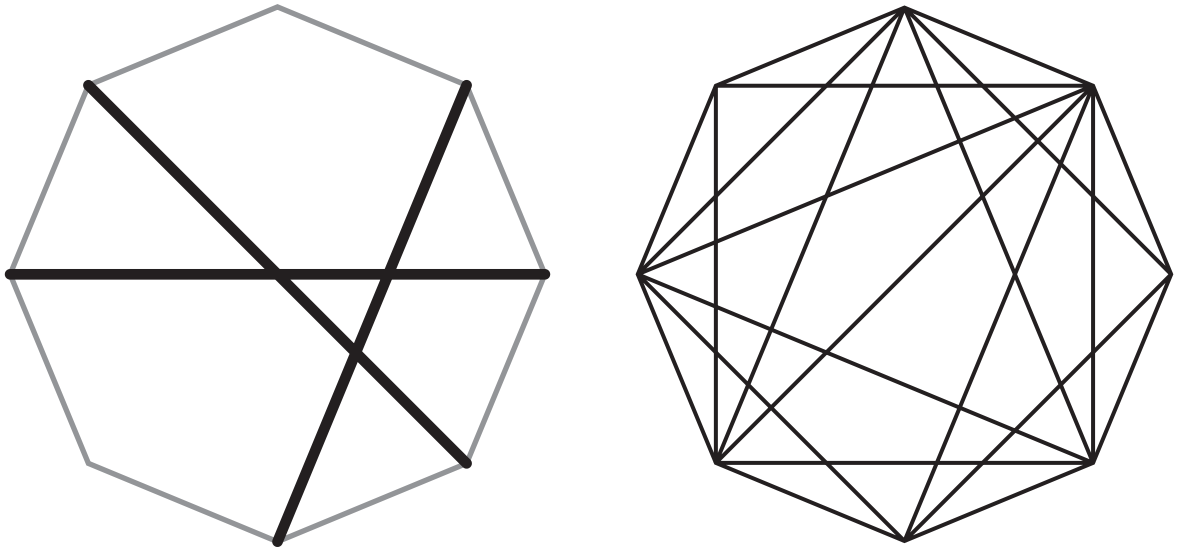
V. P. & F. Santos, Multitriangulations as complexes of star polygons, 2009.

DEFINITION

$k \geq 1$ and $n \geq 2k+1$ two fixed integers.

ℓ-crossing $=$ set of ℓ mutually crossing diagonals of the convex n-gon.

> k-triangulation $=$ maximal $(k+1)$-crossing-free set of diagonals of the n-gon.



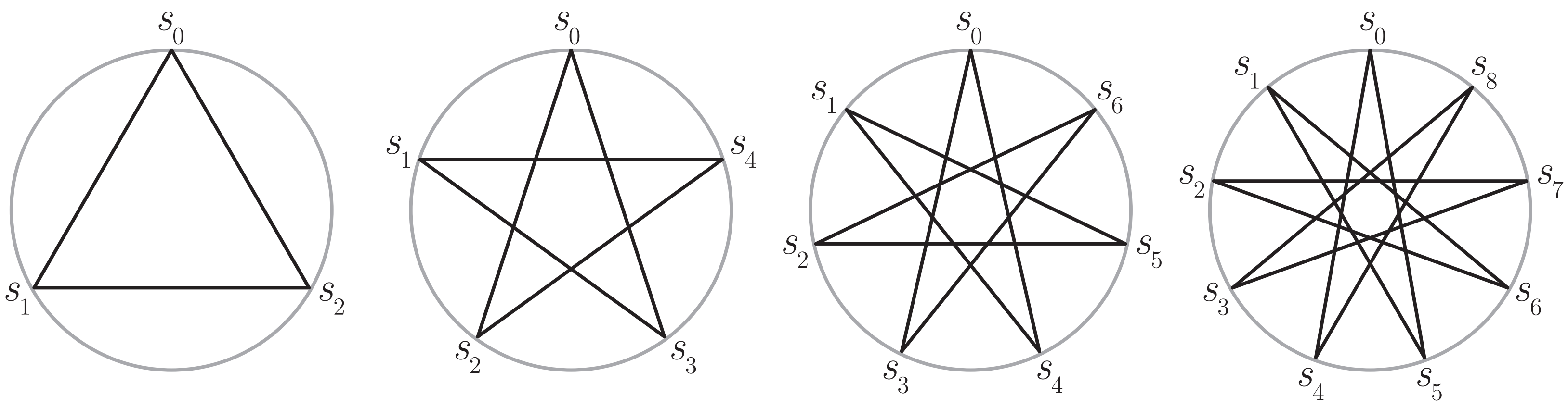
V. Capoyleas & J. Pach, A Turán-type theorem on chords of a convex polygon, 1992.

T. Nakamigawa, A generalization of diagonal flips in a convex polygon, 2000.

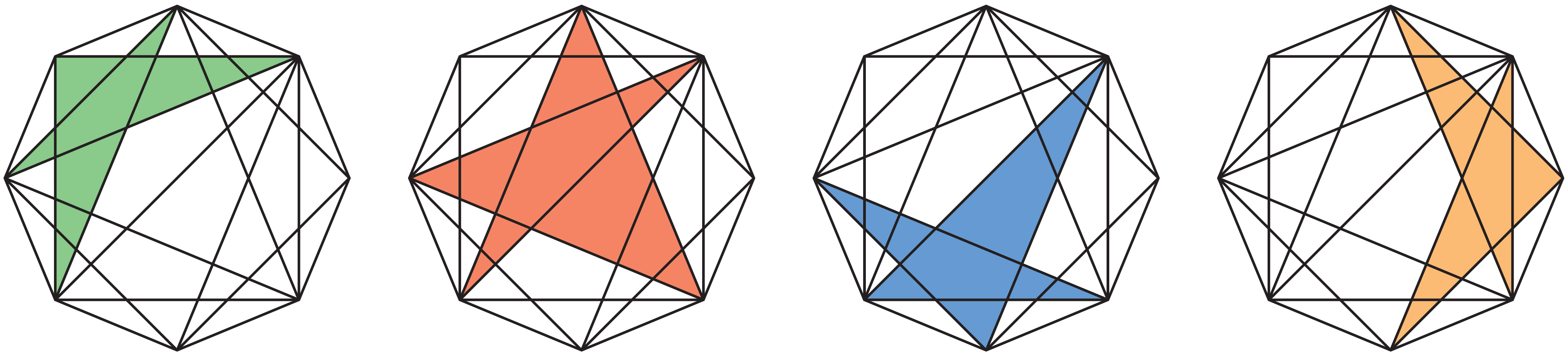
A. Dress, J. Koolen & V. Moulton, On line arrangements in the hyperbolic plane, 2002.

J. Jonsson, Generalized triangulations and diagonal-free subsets of stack polyominoes, 2005.

STARS IN MULTITRIANGULATIONS



k-star $=$ star polygon with vertices $s_0, s_1, \ldots, s_{2k}$ cyclically ordered and edges $[s_0, s_k], [s_1, s_{1+k}], \ldots, [s_k, s_{2k}], [s_{k+1}, s_0], \ldots, [s_{2k}, s_{k-1}]$.



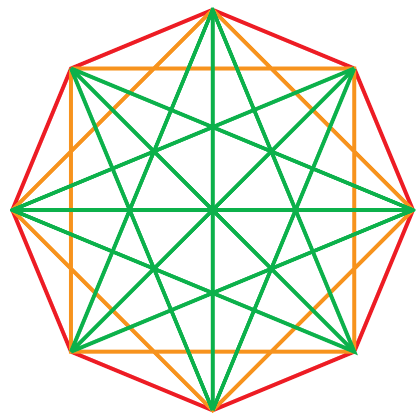
COMPLEXES OF STARS

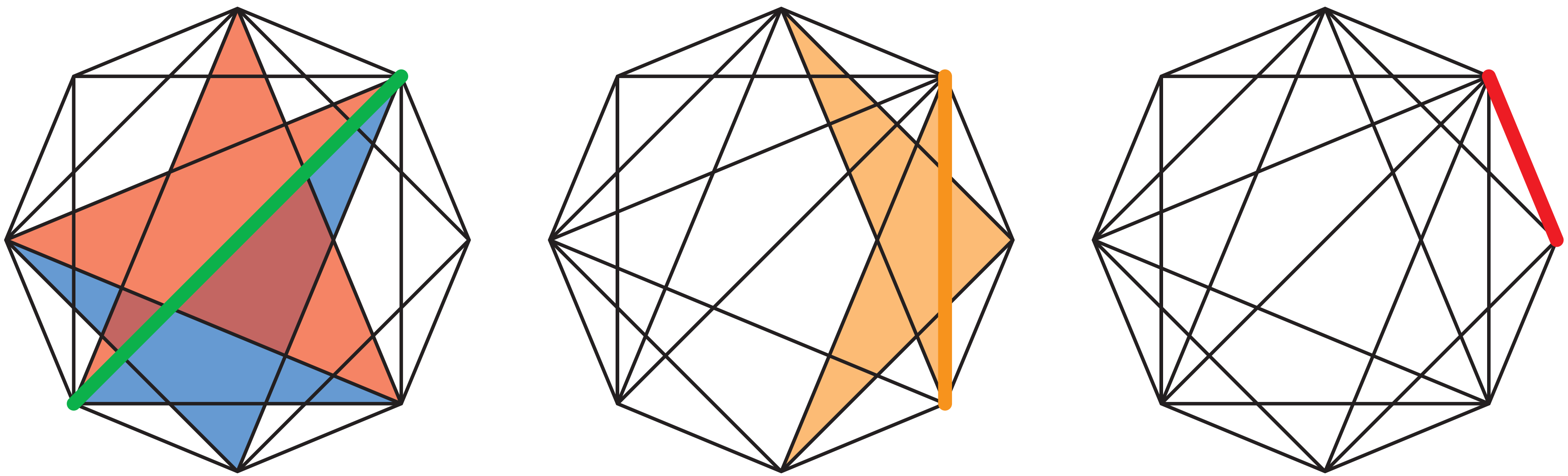
THEOREM. In a k-triangulation T,

(i) a k-relevant diagonal belongs to exactly two k-stars of T,

(ii) a k-boundary diagonal belongs to exactly one k-star of T,

(iii) a k-irrelevant diagonal does not belong to any k-star of T.

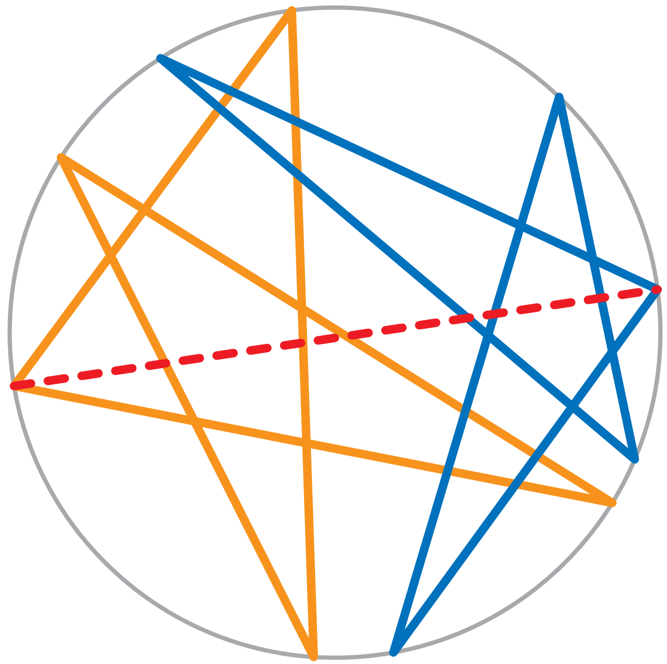




V. P. & F. Santos, Multitriangulations as complexes of star polygons, 2009.

COMMON BISECTORS

THEOREM. T a k-triangulation of the n-gon. Every pair of k-stars of T have a unique common bisector. Reciprocally, any diagonal not in T is the common bisector of a unique pair of k-stars of T.

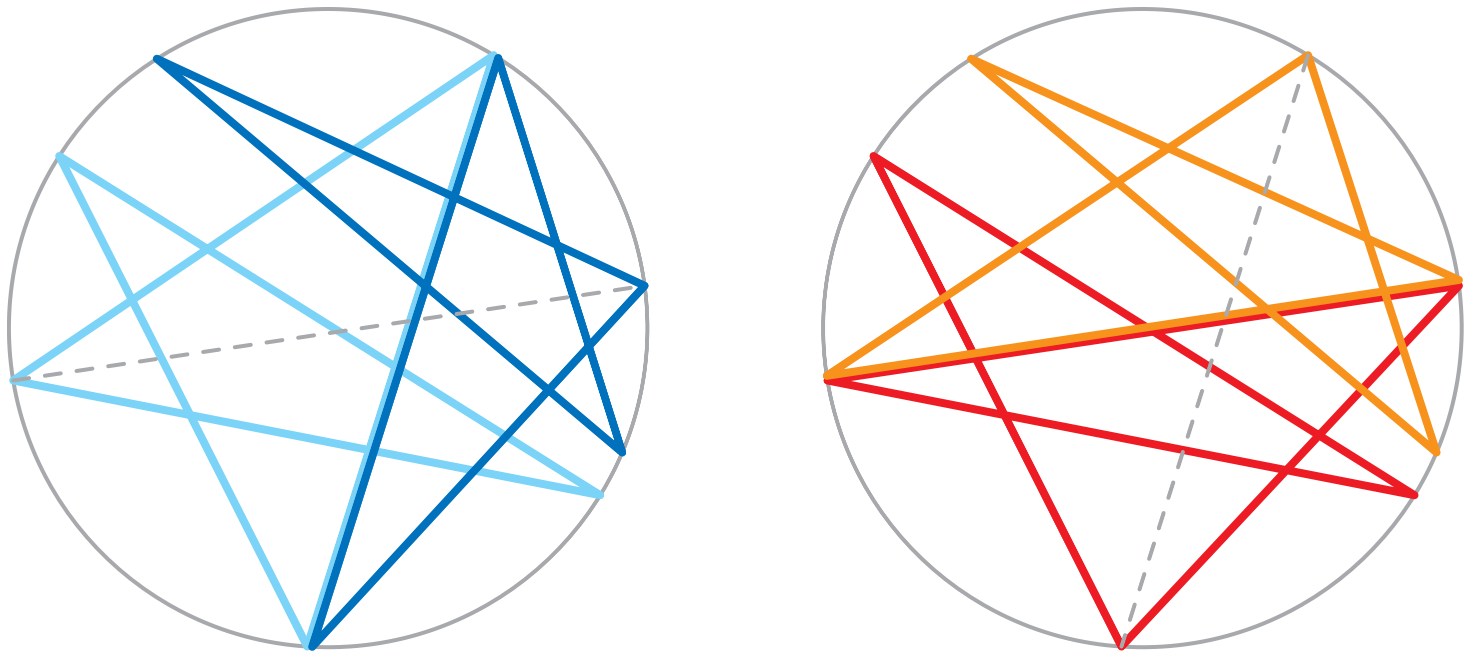


COROLLARY. Any k-triangulation of the n-gon contains exactly

$n - 2k$ k-stars and $k(2n - 2k - 1)$ diagonals.

THE GRAPH OF FLIPS

THEOREM. Let e be a k-relevant diagonal of a k-triangulation T, let R and S be the two k-stars of T containing e, and let f be the common bisector of R and S.
Then $T \triangle \{e; f\}$ is the only k-triangulation other than T containing $T \smallsetminus \{e\}$.



THEOREM. The graph of flips is connected, regular of degree $k(n-2k-1)$, and its diameter is at most $2k(n-2k-1)$.

THE (POLYTOPAL?) SIMPLICIAL COMPLEX $\Delta_{n;k}$

$k \geq 1$ and $n \geq 2k+1$ two fixed integers.

ℓ-crossing $=$ set of ℓ mutually crossing diagonals of the convex n-gon.

k-relevant diagonal $=$ at least k vertices on each side
$=$ diagonals which may appear in a $(k+1)$-crossing.

$\Delta_{n;k}$ $=$ simplicial complex of $(k+1)$-crossing-free sets of k-relevant diagonals of the convex n-gon.

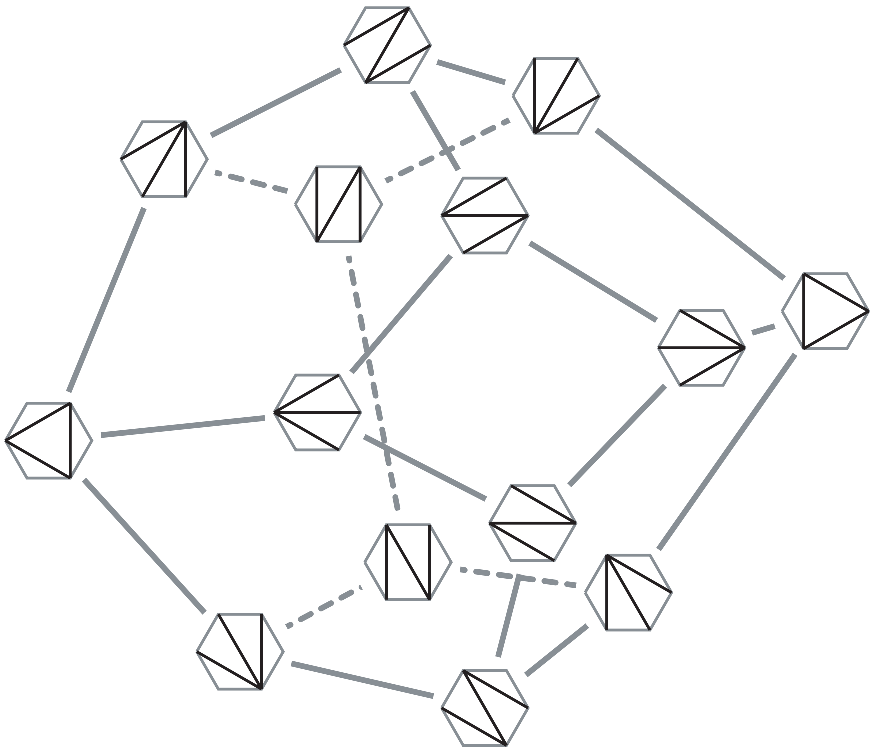
THEOREM. $\Delta_{n;k}$ is a topological sphere of dimension $k(n-2k-1)-1$.

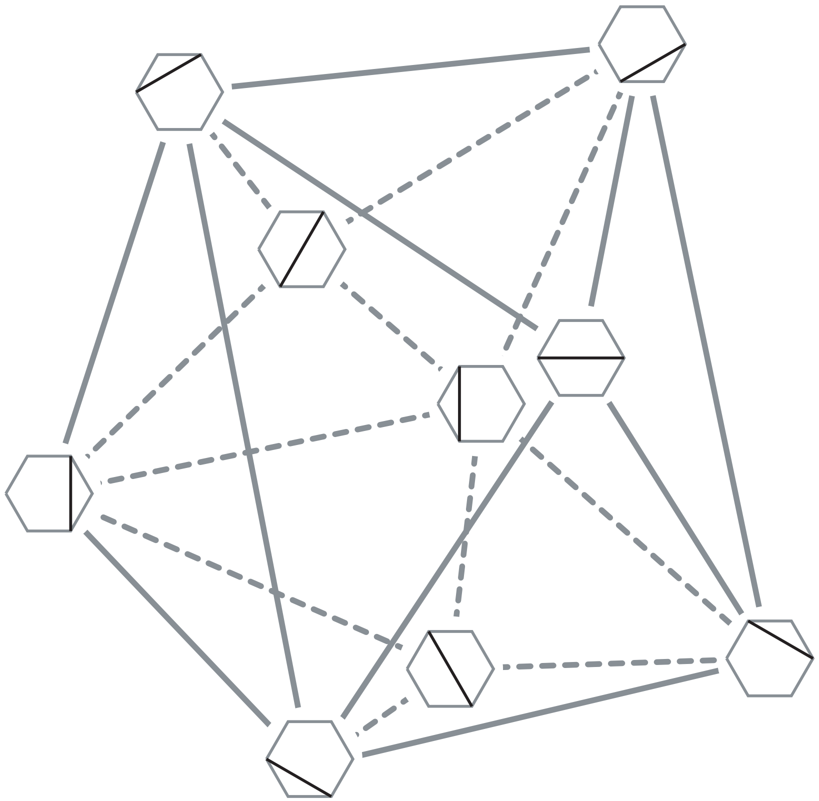
J. Jonsson, Generalized triangulations of the n-gon, 2003.

QUESTION. Is $\Delta_{n;k}$ the boundary complex of a simplicial $k(n-2k-1)$-polytope?

ASSOCIAHEDRON

$\boxed{k=1}$ Maximal elements of $\Delta_{n;1}$ = triangulations of the n-gon.
$\Delta_{n;1}$ = boundary complex of the dual of the $(n-3)$-dimensional associahedron.





L.J. Billera, P. Filliman & B. Sturmfels,
Constructions and complexity of secondary polytopes, 1990.

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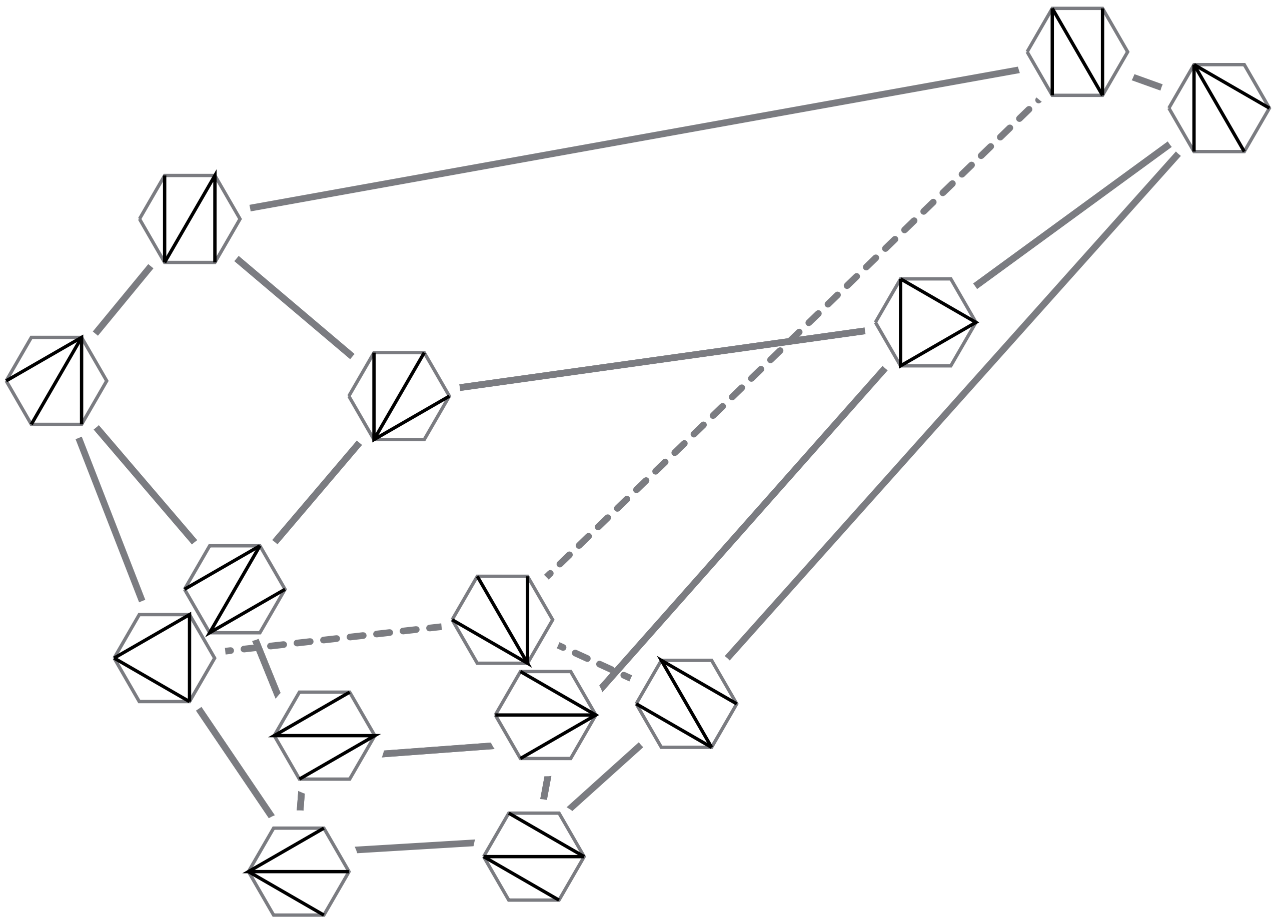

J.-L. Loday, Realization of the Stasheff polytope, 2004.

OTHER EXAMPLES

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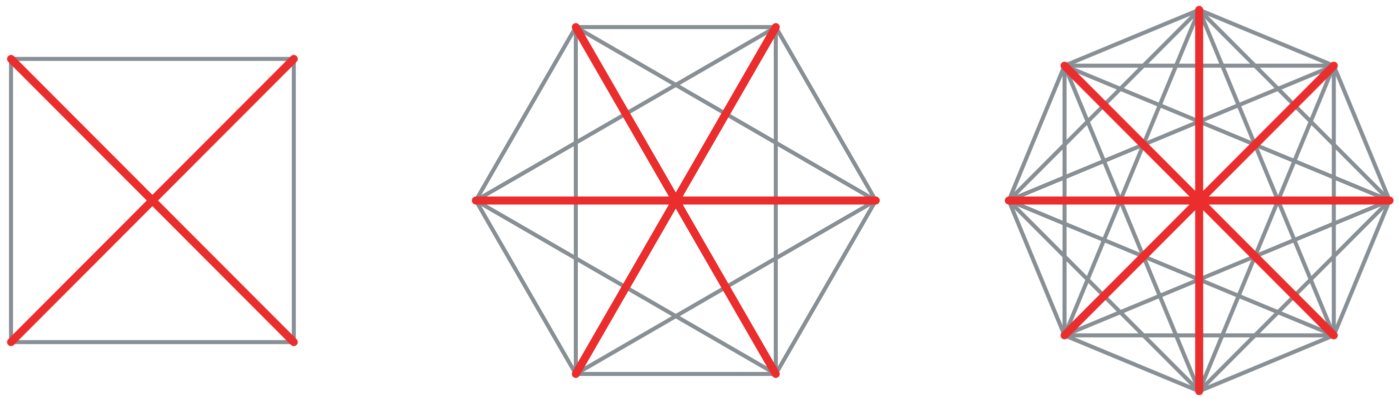

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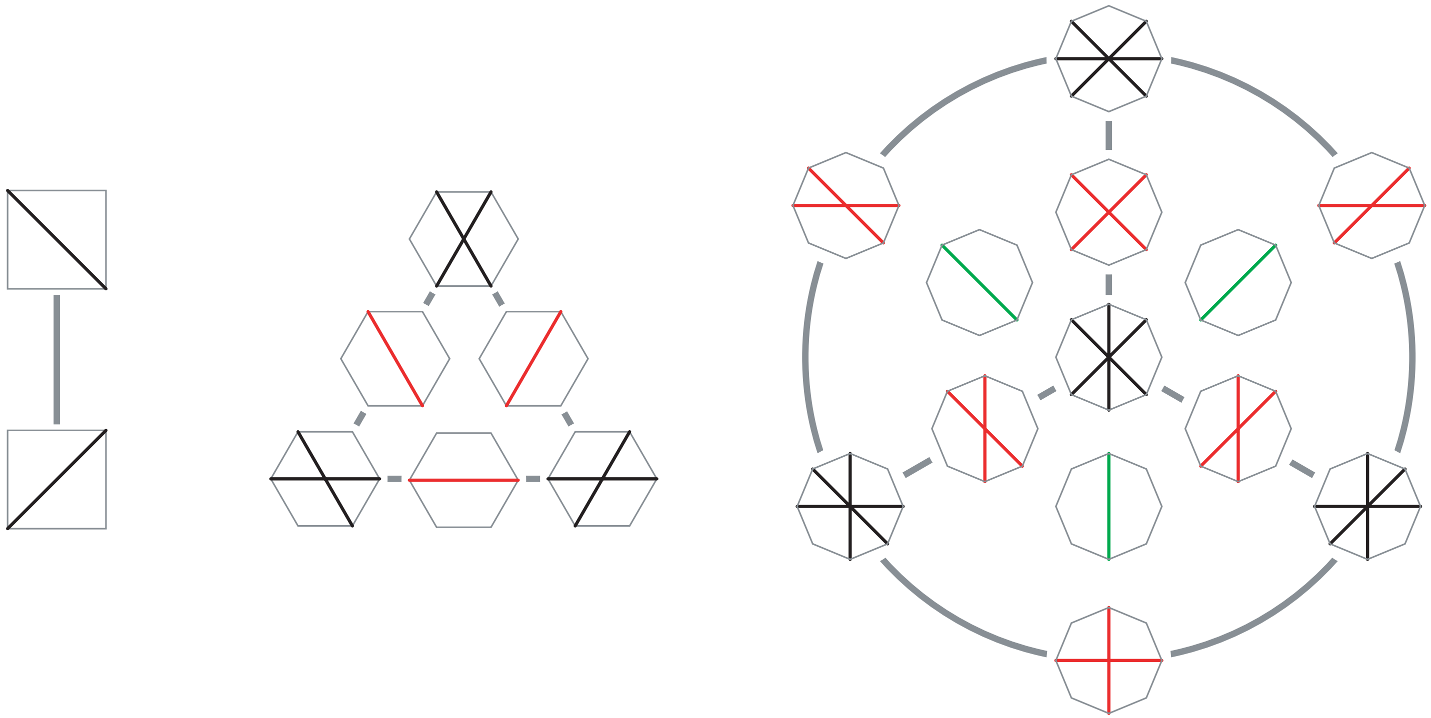

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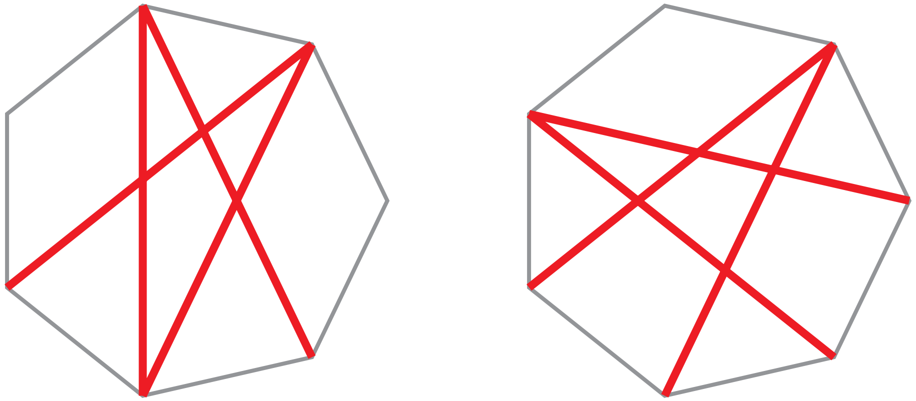
$\boxed{n = 2k + 2}$ $\Delta_{2k+2;k}$ = boundary complex of the k-simplex.

$\boxed{n = 2k + 3}$ $\Delta_{2k+3;k}$ = boundary complex of the cyclic polytope of dimension $2k$ with $2k + 3$ vertices.



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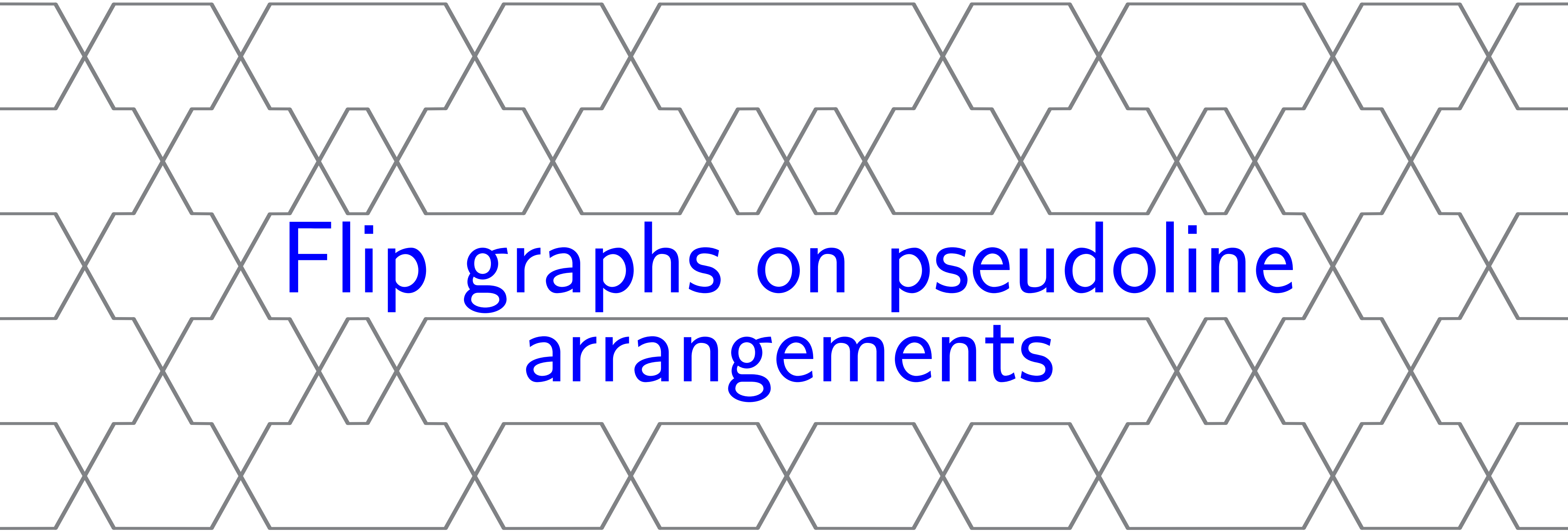
$\boxed{n = 8 \ \& \ k = 2}$

f-vector of $\Delta_{8;2} = (12; 66; 192; 306; 252; 84)$

THEOREM. The space of symmetric realizations of $\Delta_{8;2}$ has dimension 4.

J. Bokowski & V. P., On symmetric realizations of the simplicial complex of 3-crossing-free sets of diagonals of the octagon, 2009.

Flip graphs on pseudoline arrangements



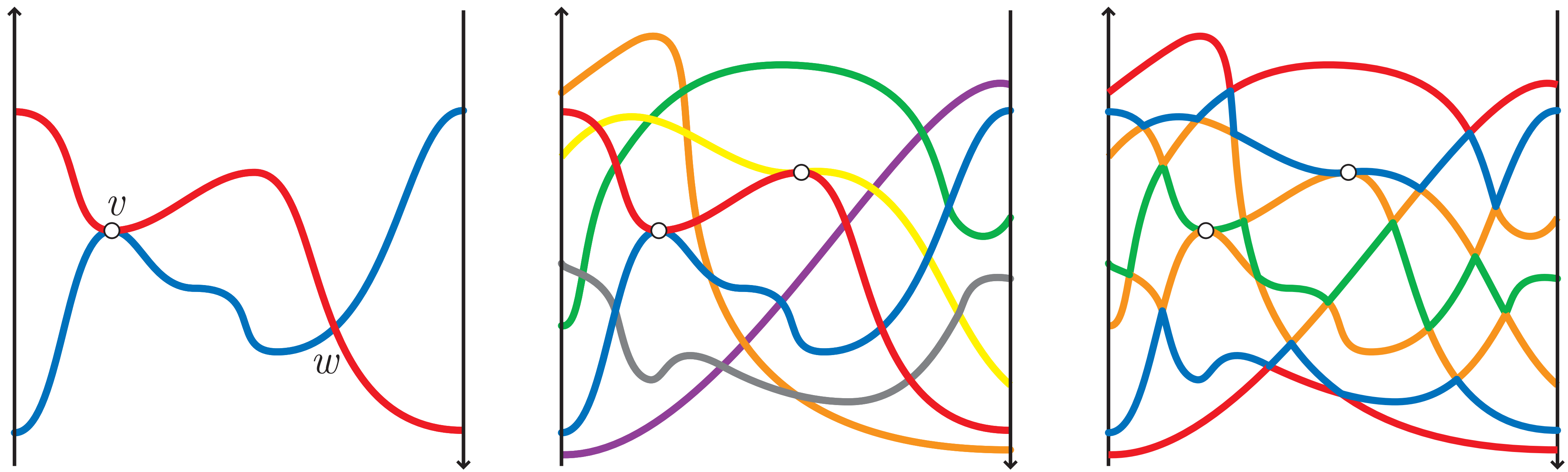
V. P. & M. Pocchiola, Multipseudotriangulations, 2010.

PSEUDOLINE ARRANGEMENTS

Mobius strip $= \mathbb{R}^2$=$(x;y) \sim (x + \ \ ; -y)$.

pseudoline $=$ non-separating simple closed curve in the Möbius strip.

pseudoline arrangement $=$ finite set of pseudolines such that any two of them have exactly one crossing point and possibly some contact points.

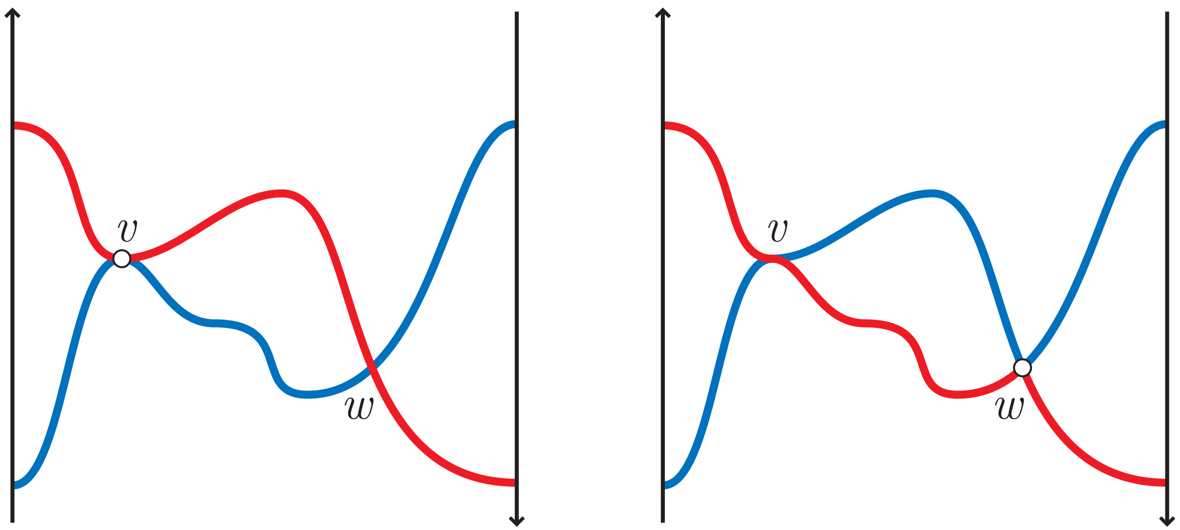


support $=$ union of pseudolines

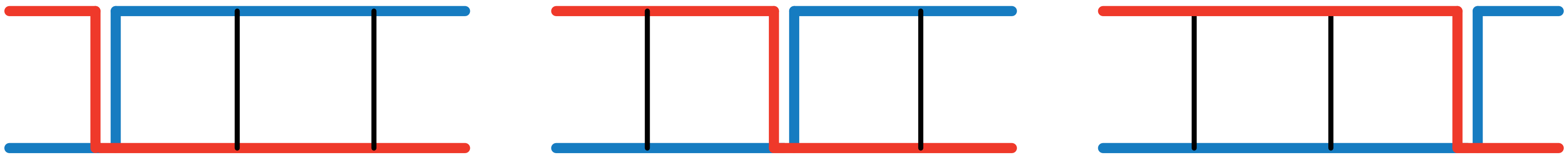
levels $=$ layers of the arrangement

FLIP GRAPHS

Flip $=$ exchange a contact point between two pseudolines with their crossing point.
$G(\mathcal{S})$ $=$ the flip graph on all pseudoline arrangements supported by a given support $\mathcal{S}$.

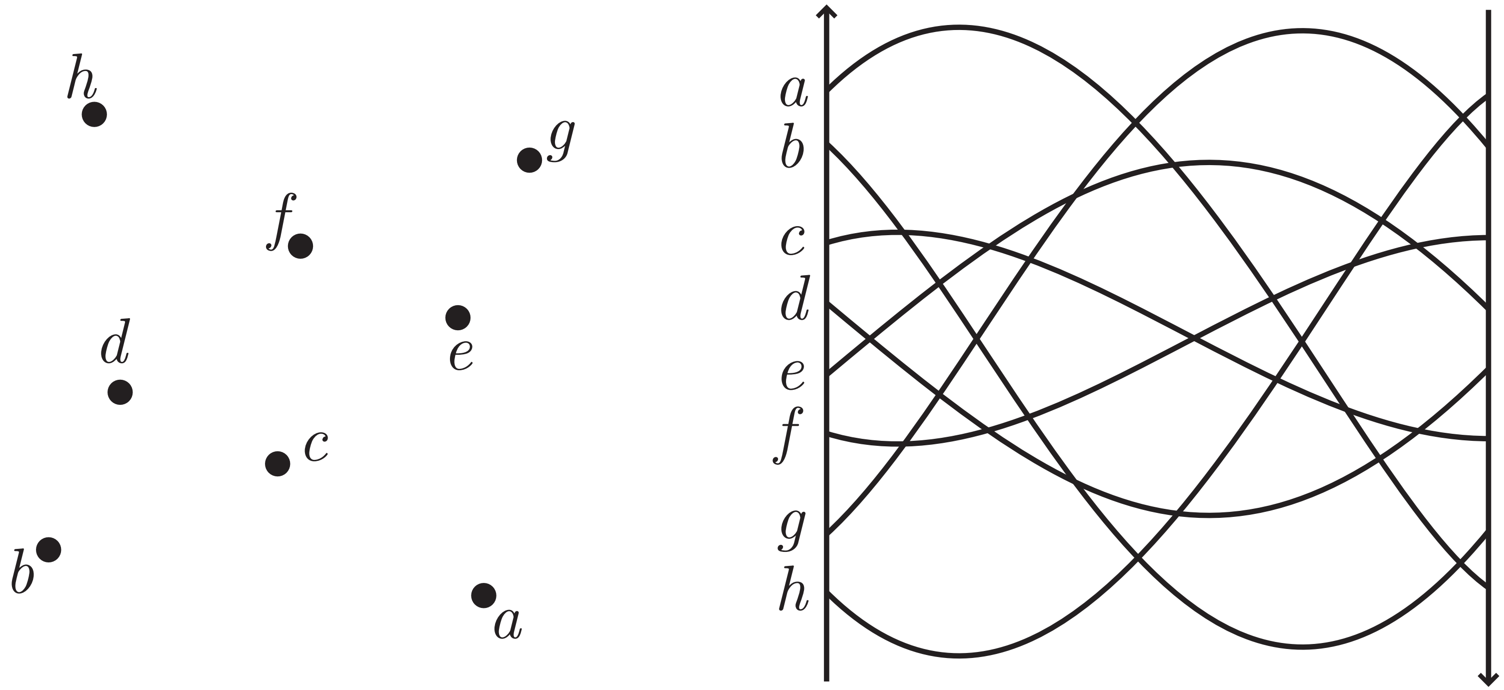


EXAMPLE. $\mathcal{S}$ $=$ support with 2 levels and p intersection points.
Then $G(\mathcal{S})$ $=$ complete graph K_p.



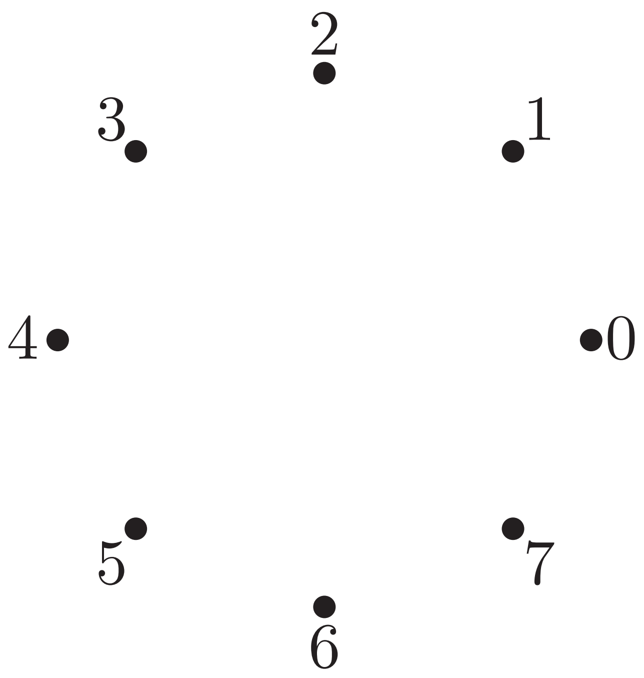
DUALITY

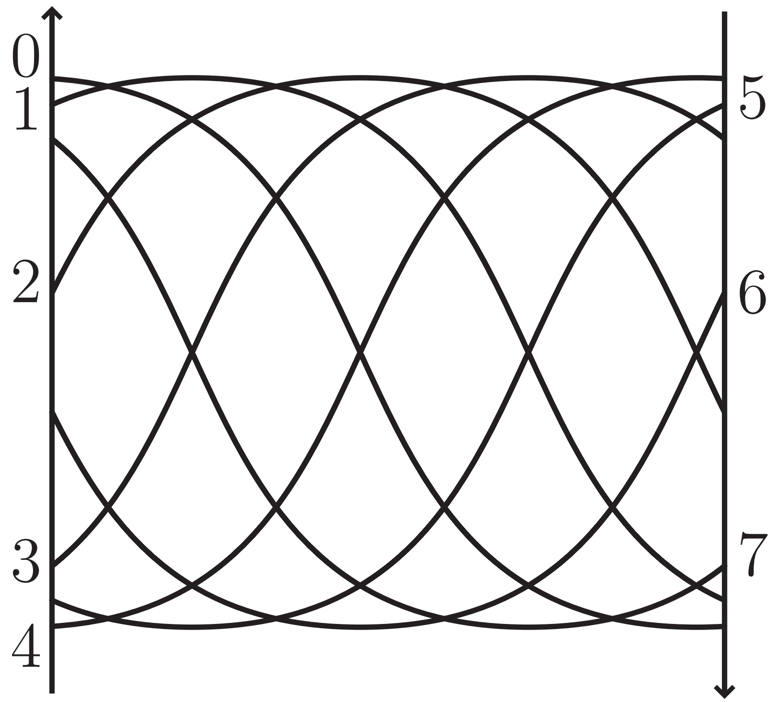
line space of the Euclidean plane $= \mathbb{R}^2$=(; d) $\sim$ ($+$; $-d$) $=$ Möbius strip.



p point of the plane	$p^* = \{$lines passing through p$\}$ dual pseudoline
P point set in general position	$P^* = \{p^* \mid p \in P\}$ dual pseudoline arrangement

DUALITY AND MULTITRIANGULATIONS

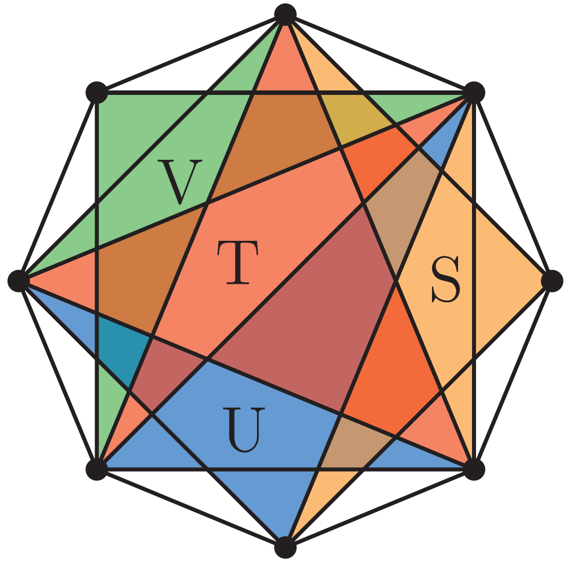


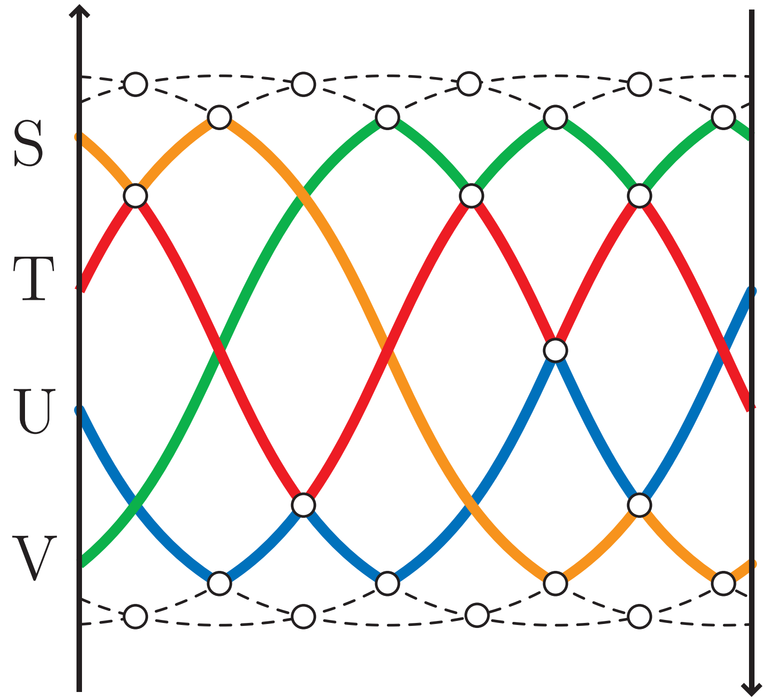


V_n vertices of the convex n-gon

V_n^* dual pseudoline arrangement of V_n

DUALITY AND MULTITRIANGULATIONS





V_n vertices of the convex n-gon	V_n^* dual pseudoline arrangement of V_n
S k-star of a k-triangulation T	$S^* = \{\text{bisectors of } S\}$ dual pseudoline of S
T k-triangulation of V_n	$T^* = \{S^* \mid S \text{ } k\text{-star of } T\}$ dual pseudoline arrangement of T

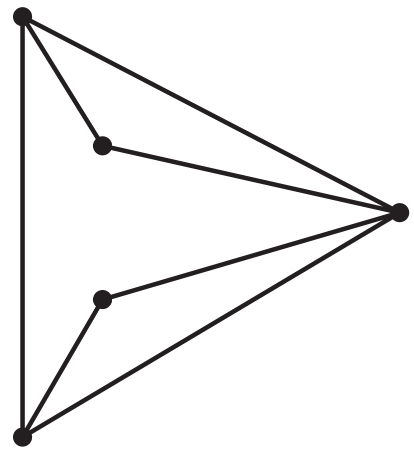
THEOREM. $T \subset \binom{V_n}{2}$ k-triangulation of V_n $\Leftrightarrow$ T^* covers V_n^* minus its first k levels.

PSEUDOTRIANGULATIONS

A pseudotriangulation of a finite point set P is:

(i) a maximal crossing-free pointed subset of $\binom{P}{2}$,

(ii) a pointed subset of $\binom{P}{2}$ that decomposes the convex hull of P into pseudotriangles.



M. Pocchiola & G. Vegter, Topologically sweeping visibility complexes via pseudotriangulations, 1996.

I. Streinu, Pseudo-triangulations, rigidity and motion planning, 2005.

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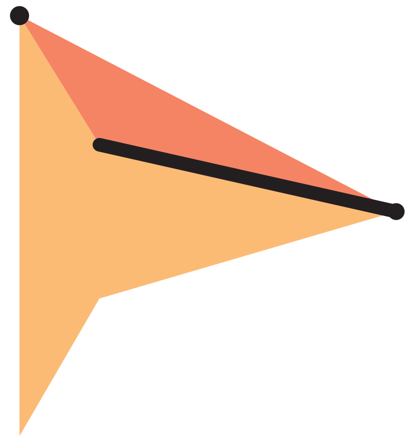


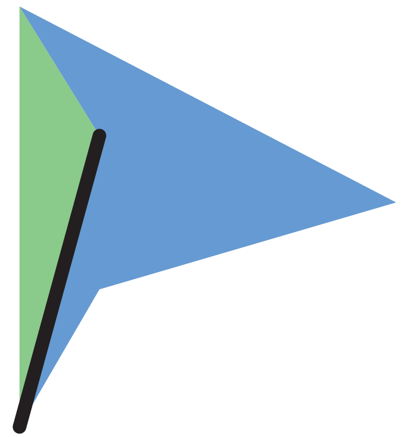
PROPERTIES.

(i) A pseudotriangulation of P has exactly $2|P| - 3$ edges.

(ii) Two pseudotriangles have a unique common tangent.

(iii) T pseudotriangulation of P; e internal edge of T; f common tangent between the two pseudotriangles of T containing $e \Rightarrow T \triangle \{e; f\}$ pseudotriangulation of P.

(iv) The graph of flips is polytopal.



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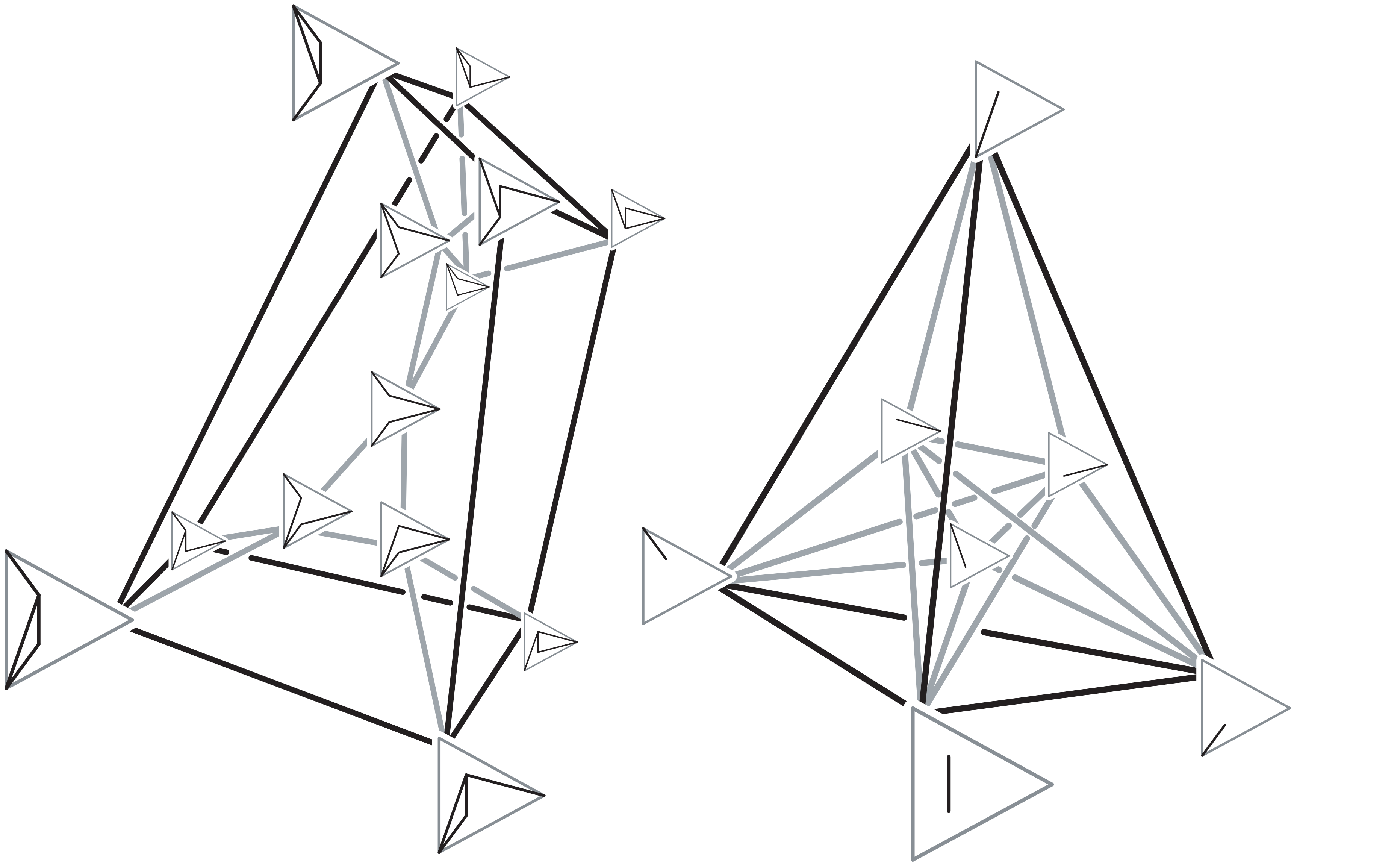
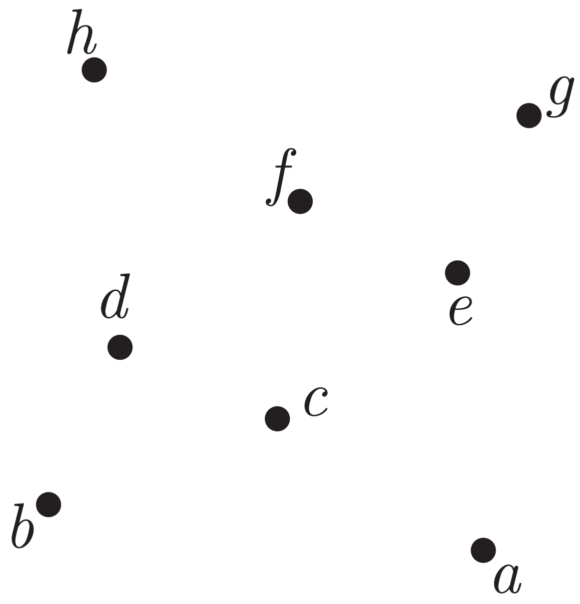
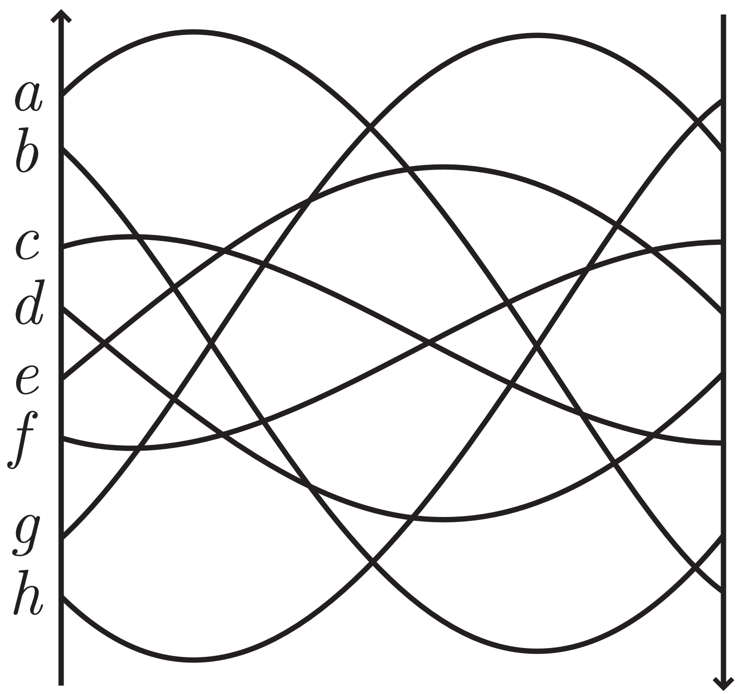
PSEUDOTRIANGULATIONS



G. Rote, F. Santos, & I. Streinu,
Expansive motions and the polytope of pointed pseudo-triangulations, 2003.

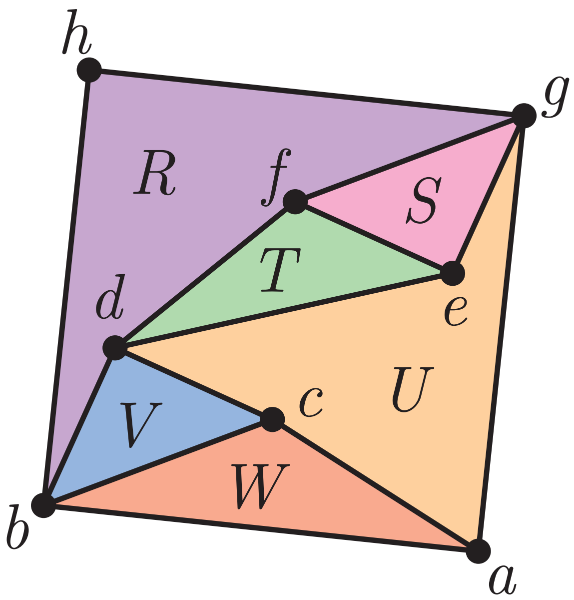
DUALITY AND PSEUDOTRIANGULATIONS

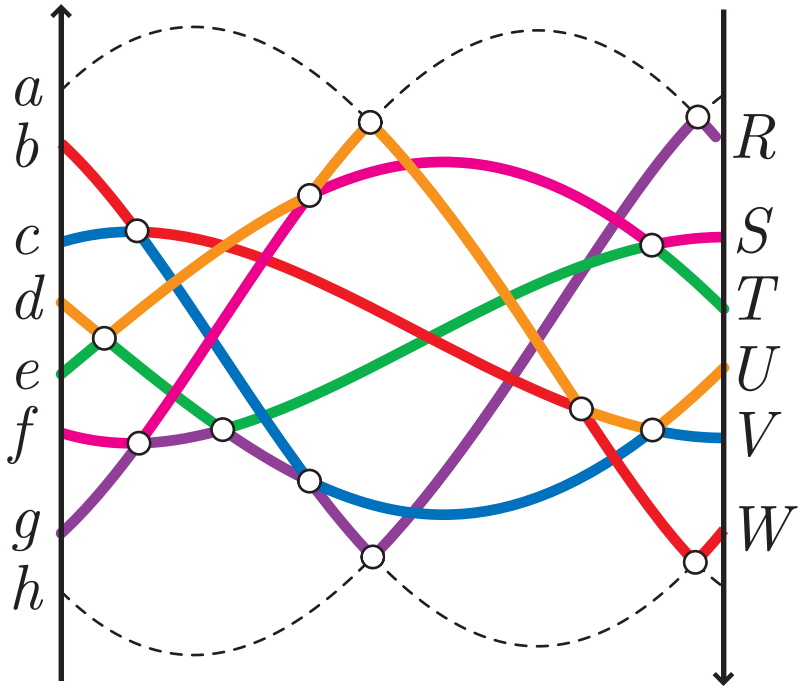




P point set in general position $\qquad$ P^* dual pseudoline arrangement of P

DUALITY AND PSEUDOTRIANGULATIONS





P point set in general position — P^* dual pseudoline arrangement of P

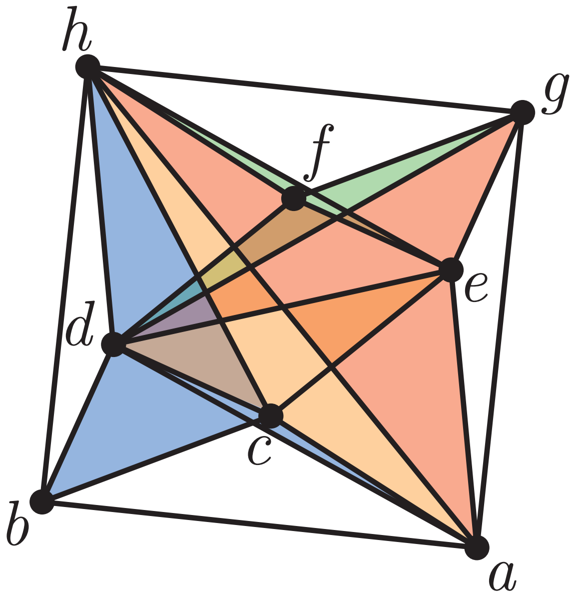
Δ pseudotriangle — $\Delta^* = \{\text{tangent to } \Delta\}$ dual pseudoline of Δ

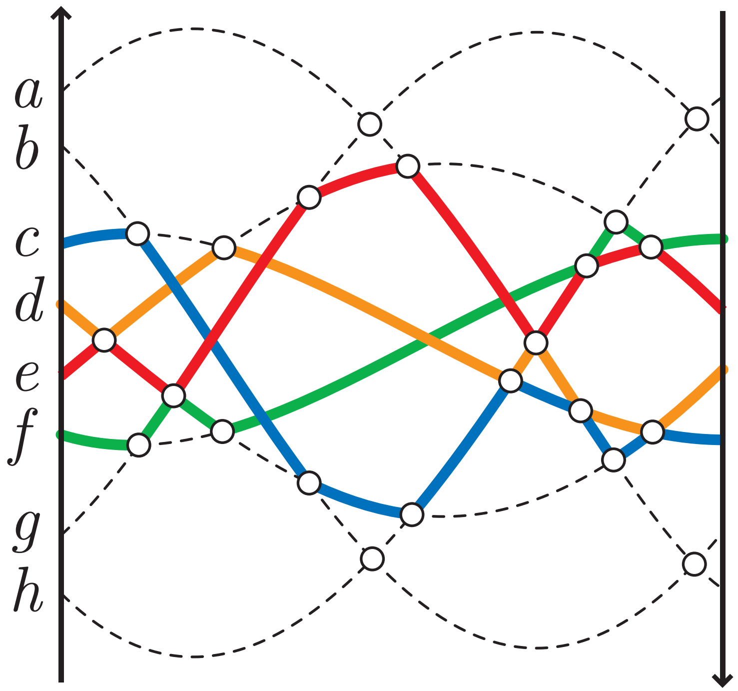
T pseudotriangulation of P — $T^* = \{\Delta^* \mid \Delta \text{ pseudotriangle of T}\}$ dual pseudoline arrangement of T

THEOREM. $T \subset \binom{P}{2}$ pseudotriangulation of P $\Leftrightarrow$ T^* covers P^* minus its first level.

MULTIPSEUDOTRIANGULATIONS

k-pseudotriangulation of a point set P in general position in the plane $=$ set T of edges of $\binom{P}{2}$ which corresponds via duality to the contact points of a pseudoline arrangement T^* supported by P^* minus its first k levels.





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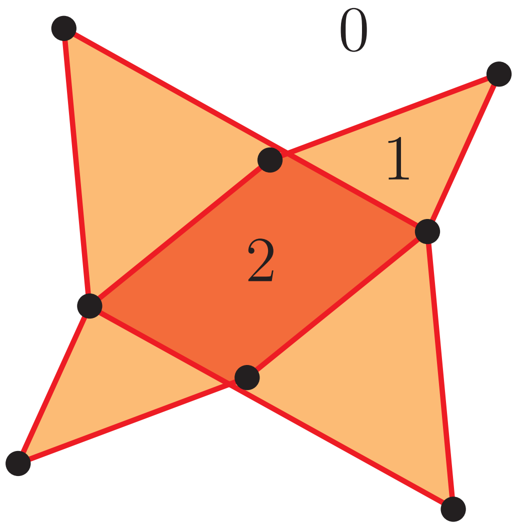
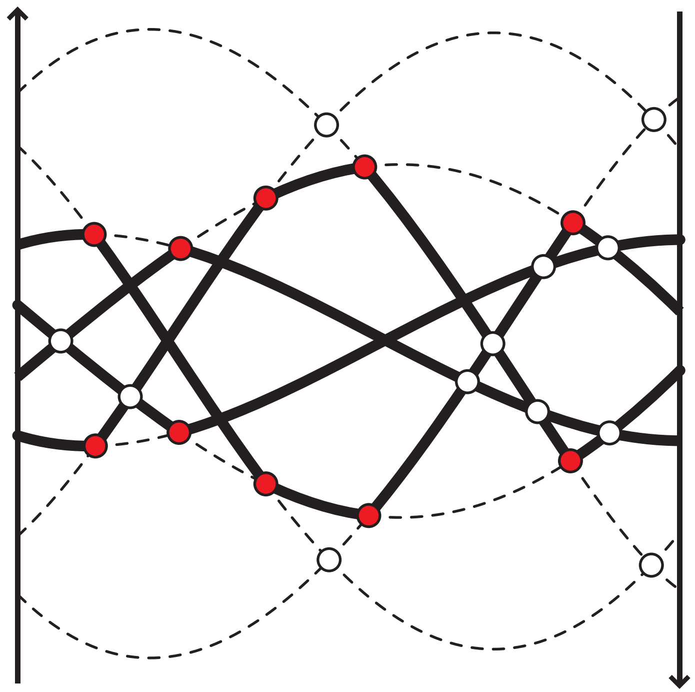


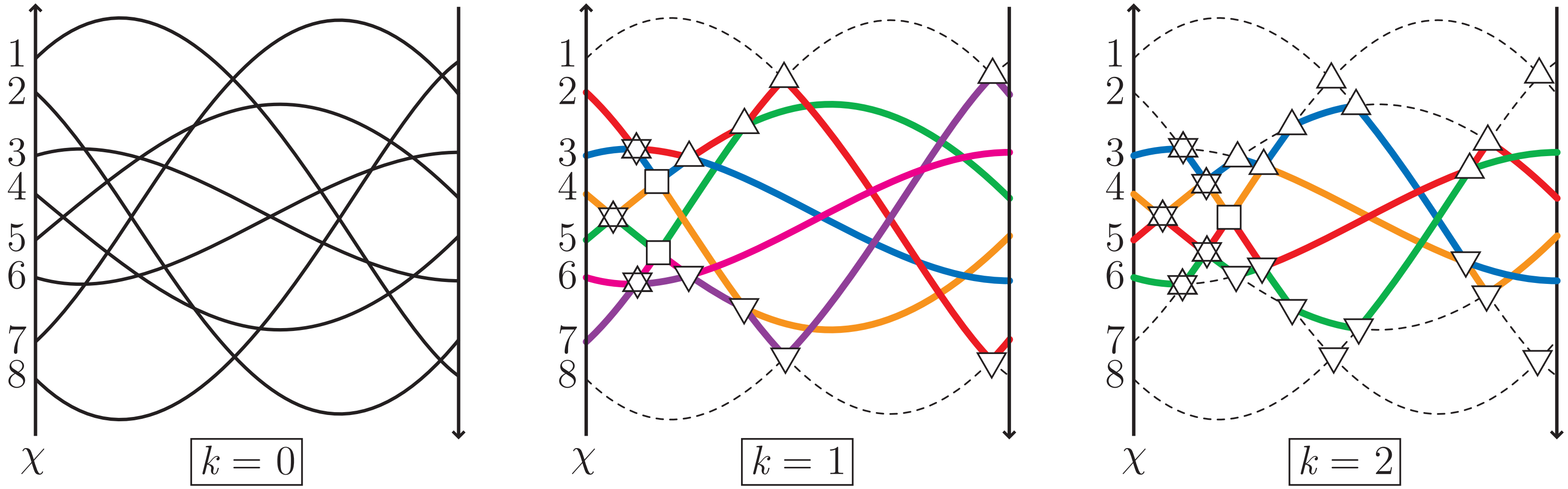


PROPOSITION. $P \cup \{q\}$ point set in general position. T a k-pseudotriangulation of P.

$$k\text{-depth of } q \text{ in } P = \sum_{\in T^*} \text{winding number of } S(\ ):$$

EXAMPLES OF APPLICATIONS

ENUMERATION ... greedy pseudoline arrangements



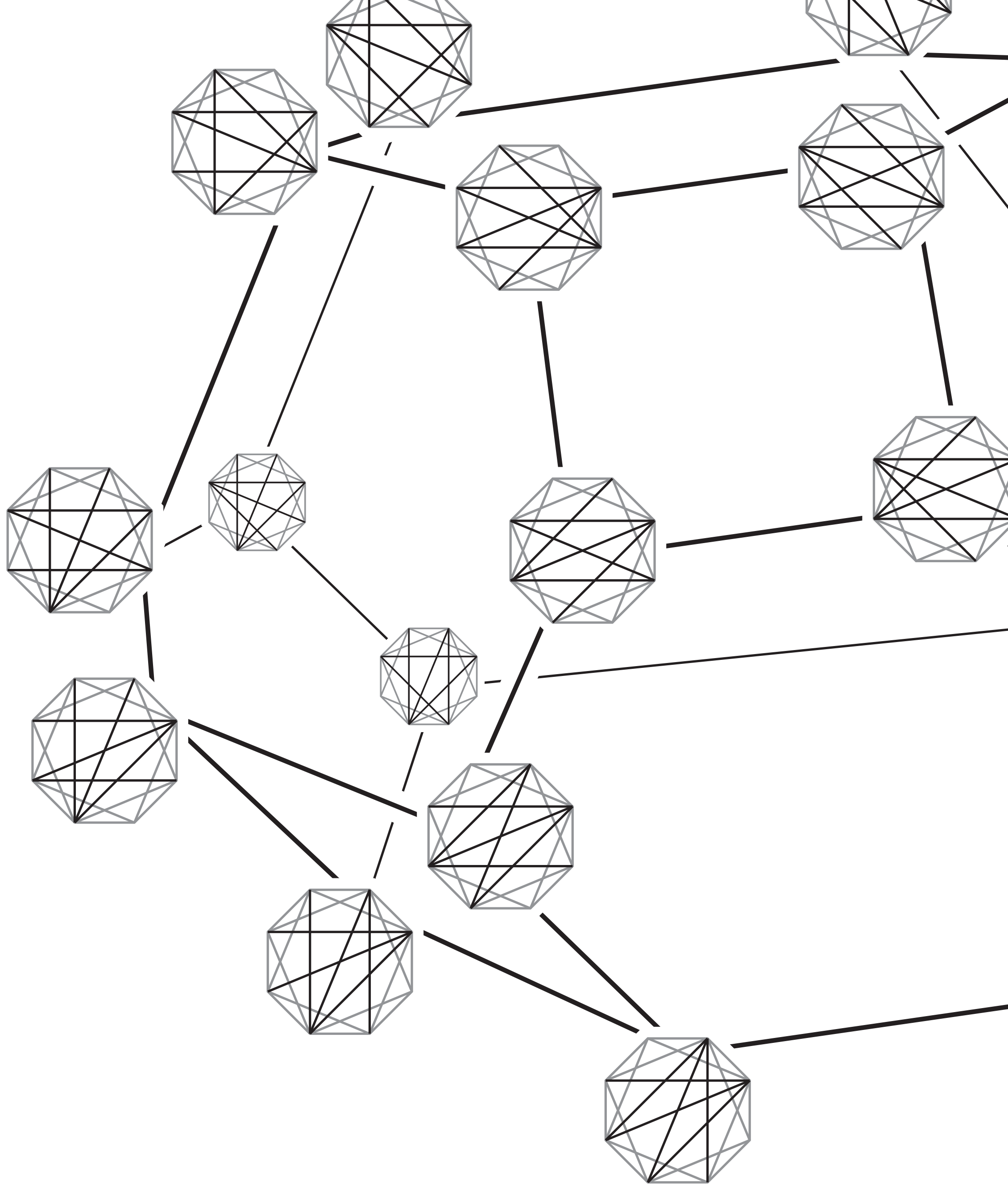
$\Rightarrow$ enumeration algorithm for pseudoline arrangements covering a given support.

CHARACTERIZATION THEOREM

THEOREM. A set Σ of k-stars of the n-gon such that:
(i) any k-relevant edge is contained in zero or two k-stars of Σ, one on each side, and
(ii) any k-boundary edge is contained in exactly one k-star of Σ,
is the set of k-stars of a k-triangulation of the n-gon.

LOWER BOUND THEOREM ... for d-polytopes with $d + 3$ vertices.

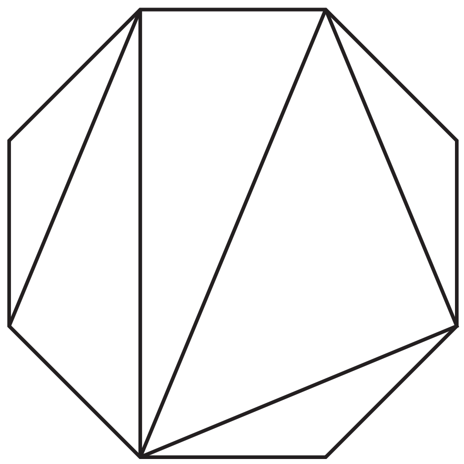
The brick polytope



V. P. & F. Santos, A generalization of Loday's associahedron, 2010.

LODAY'S ASSOCIAHEDRON REVISITED

$$T \text{ triangulation of the } n\text{-gon} \quad \longmapsto \quad \text{vector } !\,(T) \in \mathbb{R}^{n-2}.$$
Loday's associahedron $\Omega(n) = \operatorname{conv}\{\,!\,(T) \mid T \text{ triangulation of the } n\text{-gon}\}$.



J.-L. Loday, Realization of the Stasheff polytope, 2004.

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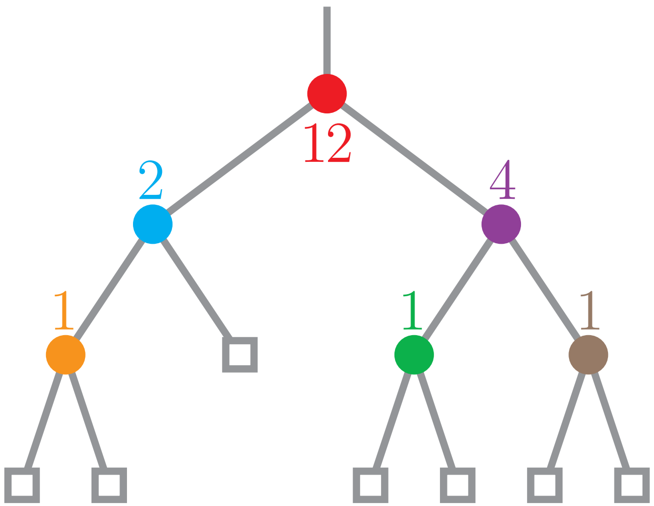

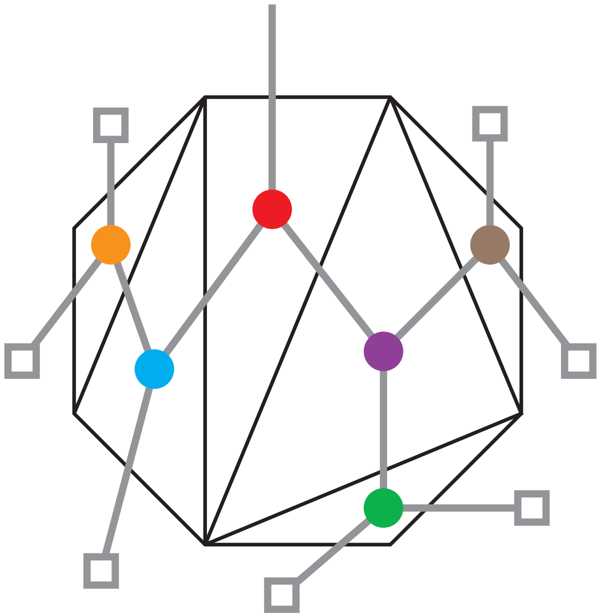

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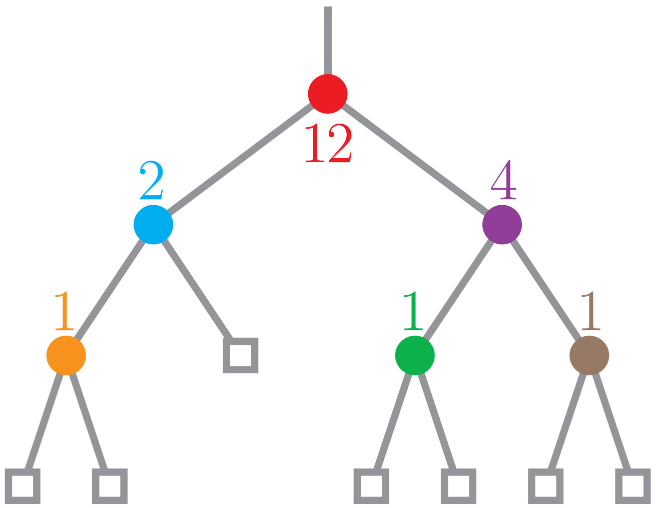

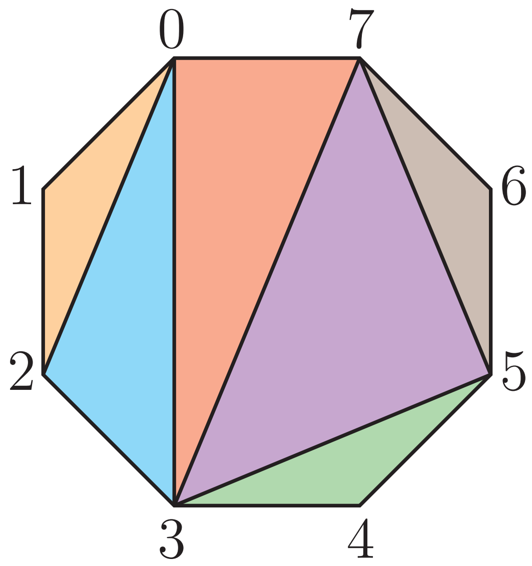

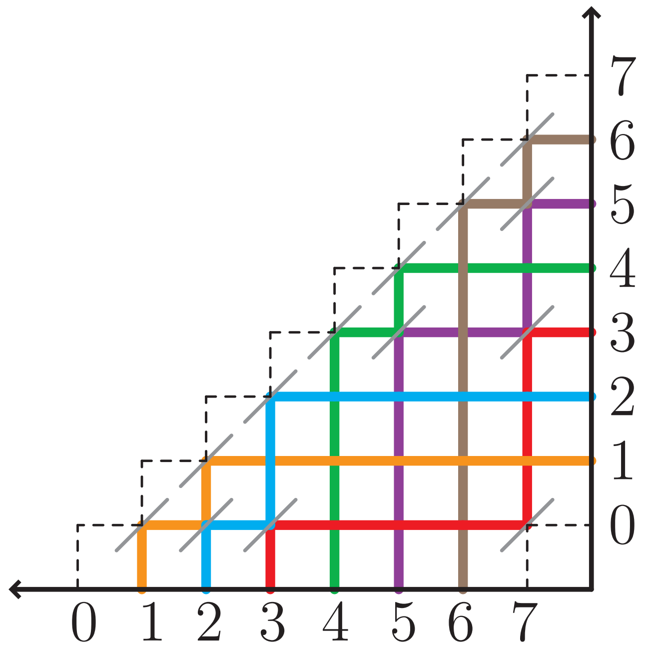

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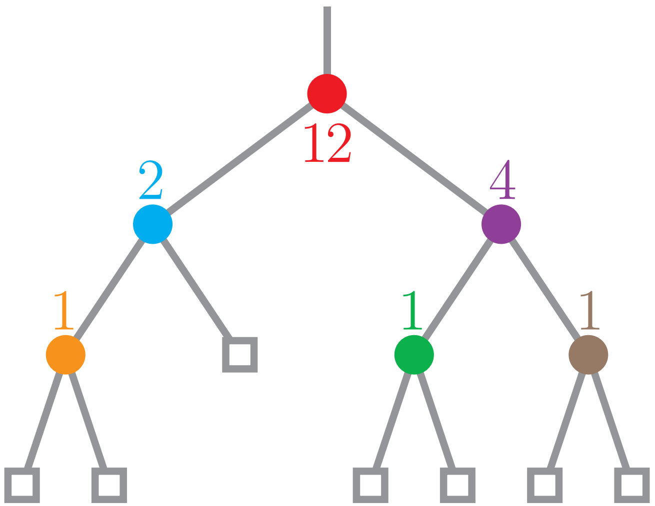
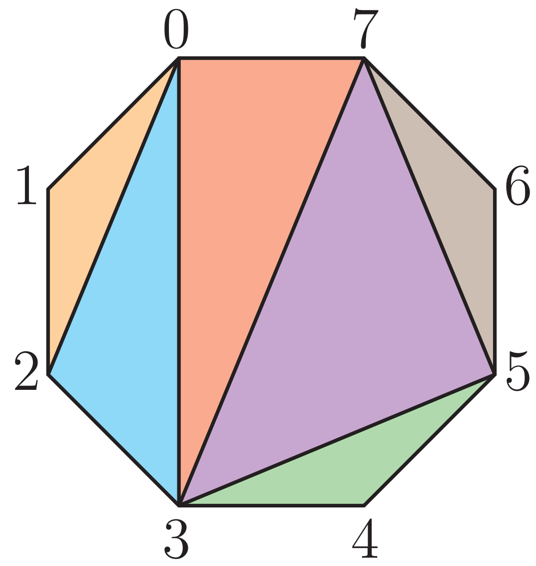
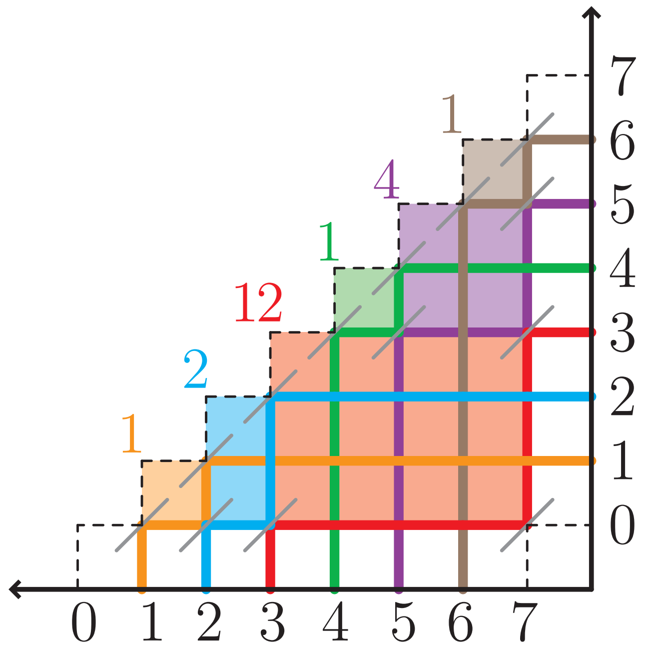


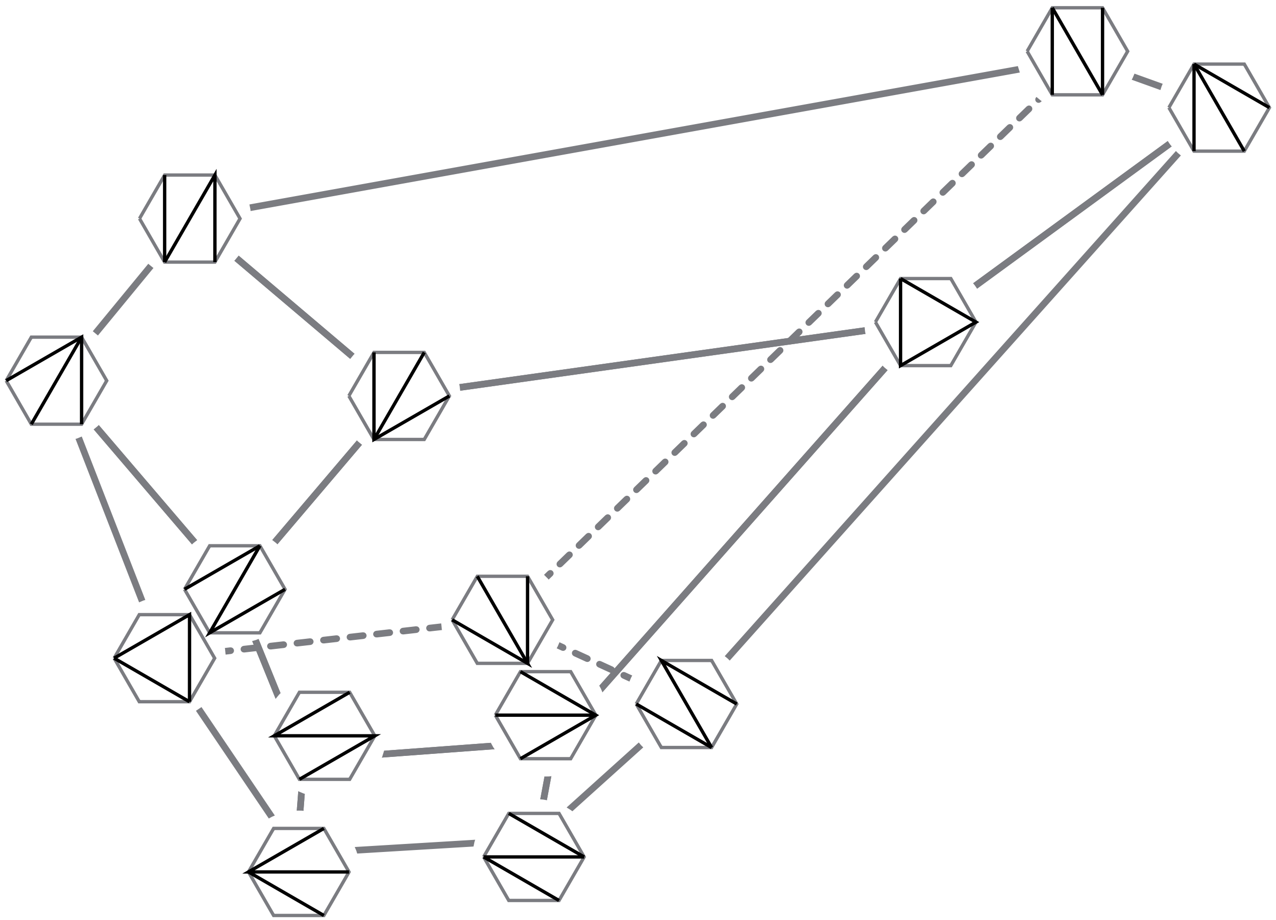




J.-L. Loday, Realization of the Stasheff polytope, 2004.

LODAY'S ASSOCIAHEDRON REVISITED

THEOREM. $\Omega(n)$ is a realization of the $(n-3)$-dimensional associahedron.



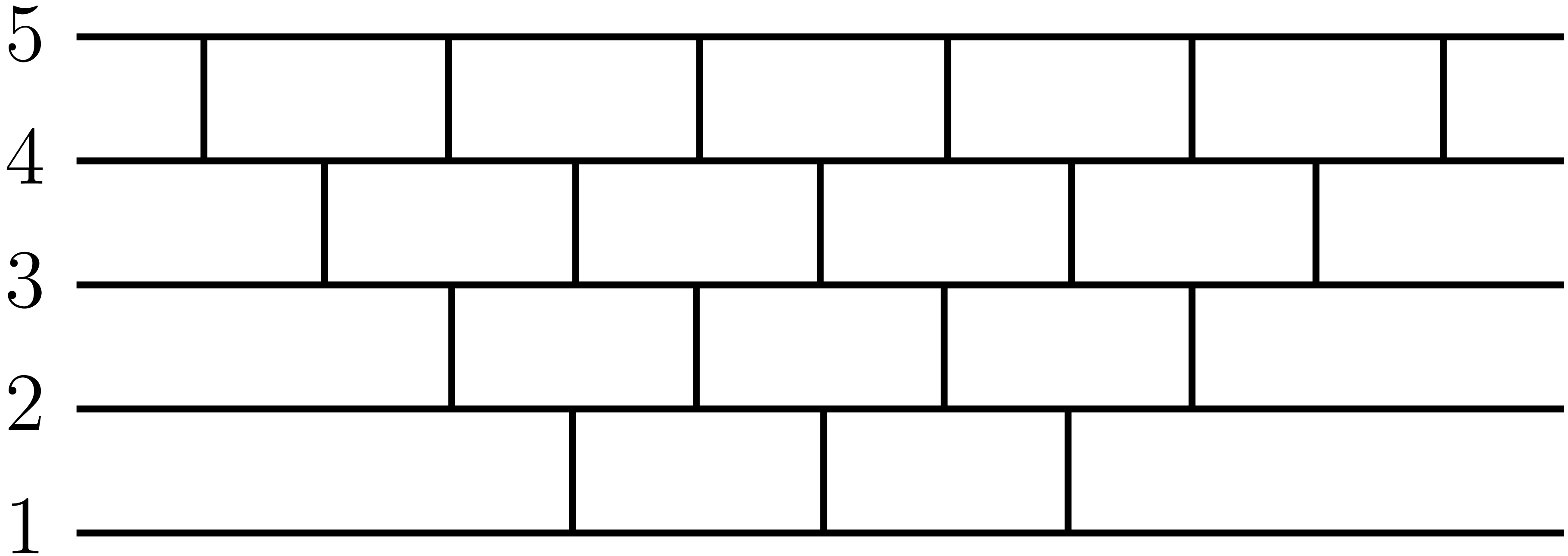
J.-L. Loday, Realization of the Stasheff polytope, 2004.

THE BRICK POLYTOPE

$\mathcal{S}$ = sorting network = support of pseudoline arrangements.

Λ pseudoline arrangement supported by $\mathcal{S}$ $\quad\longmapsto\quad$ vector $!(\Lambda) \in \mathbb{R}^m$.

Brick polytope $\Omega(\mathcal{S}) = \mathrm{conv}\{\,!(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{S}\}$.



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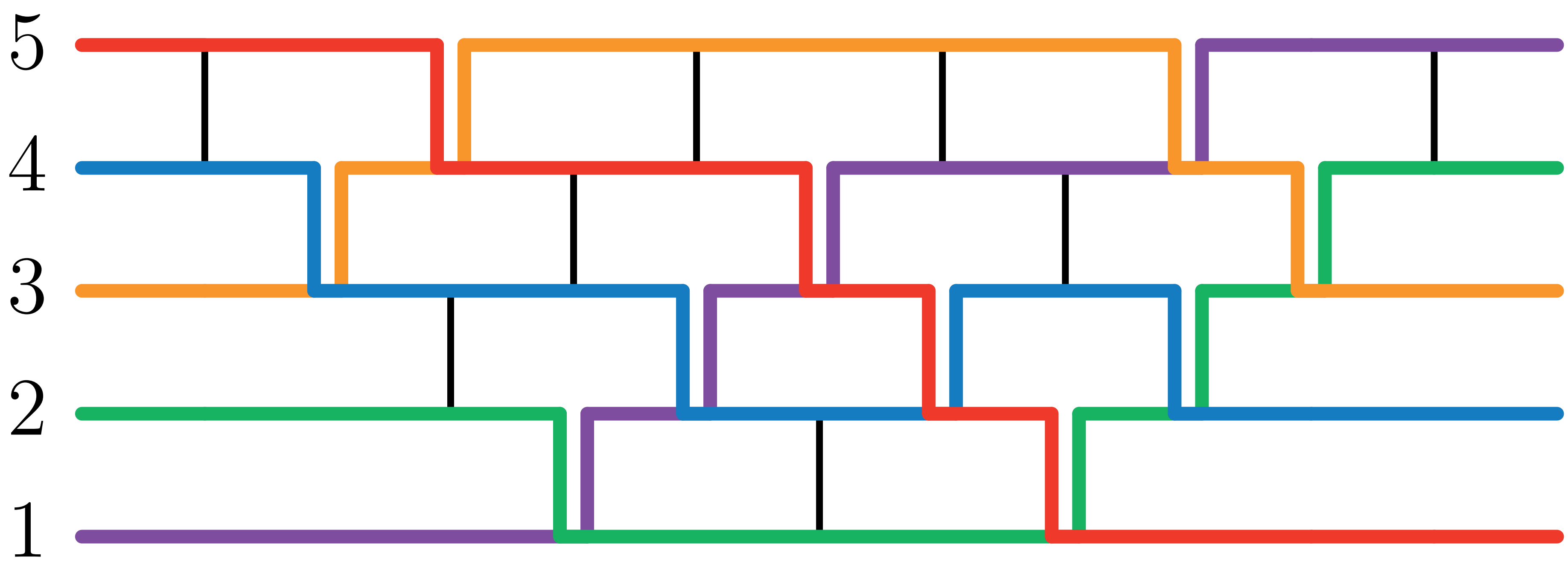

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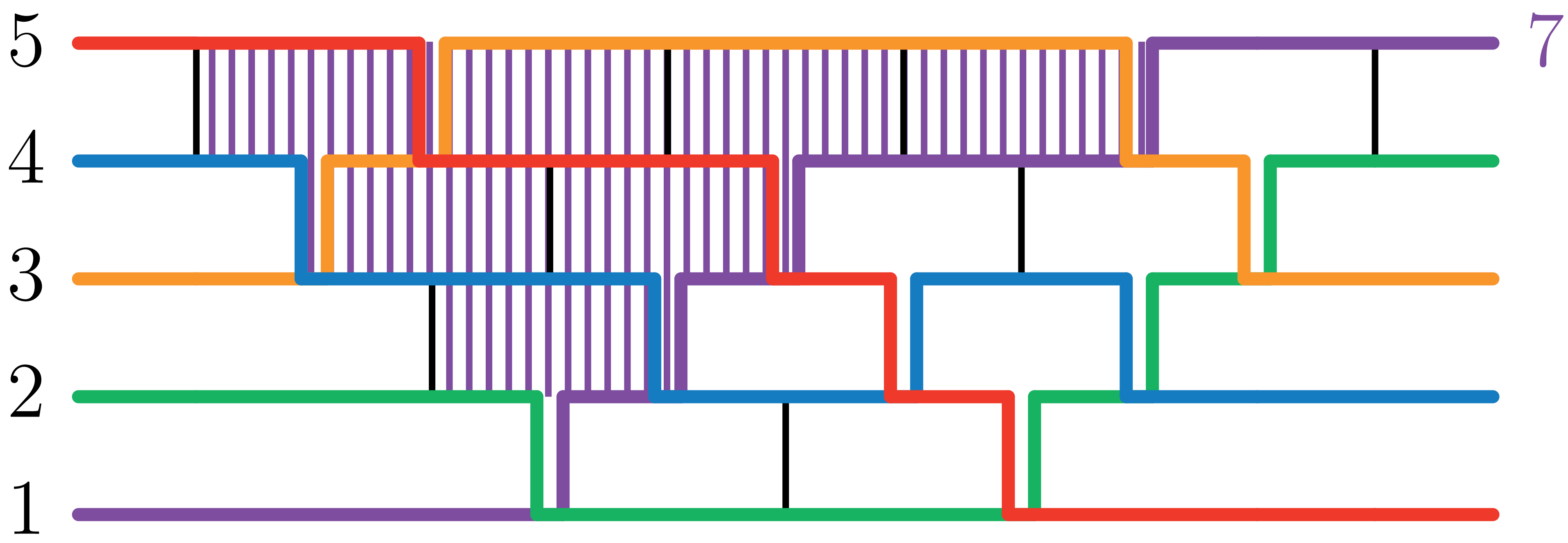

$!(\Lambda)_j$ = number of bricks above the jth pseudoline of Λ.

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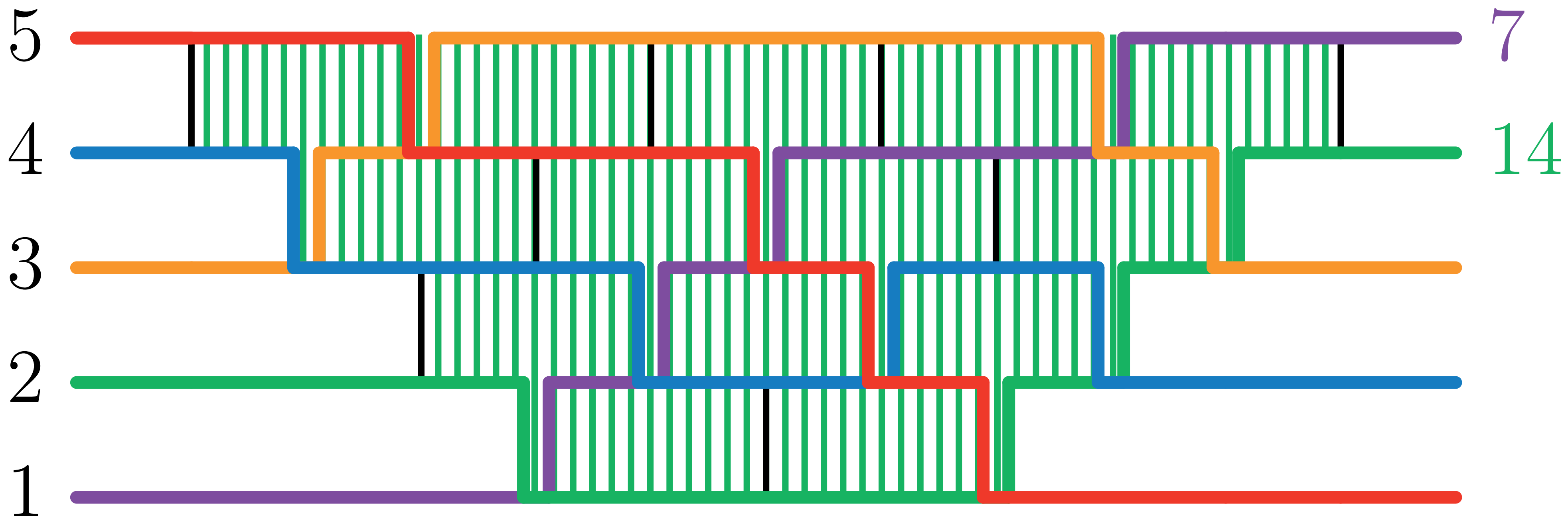

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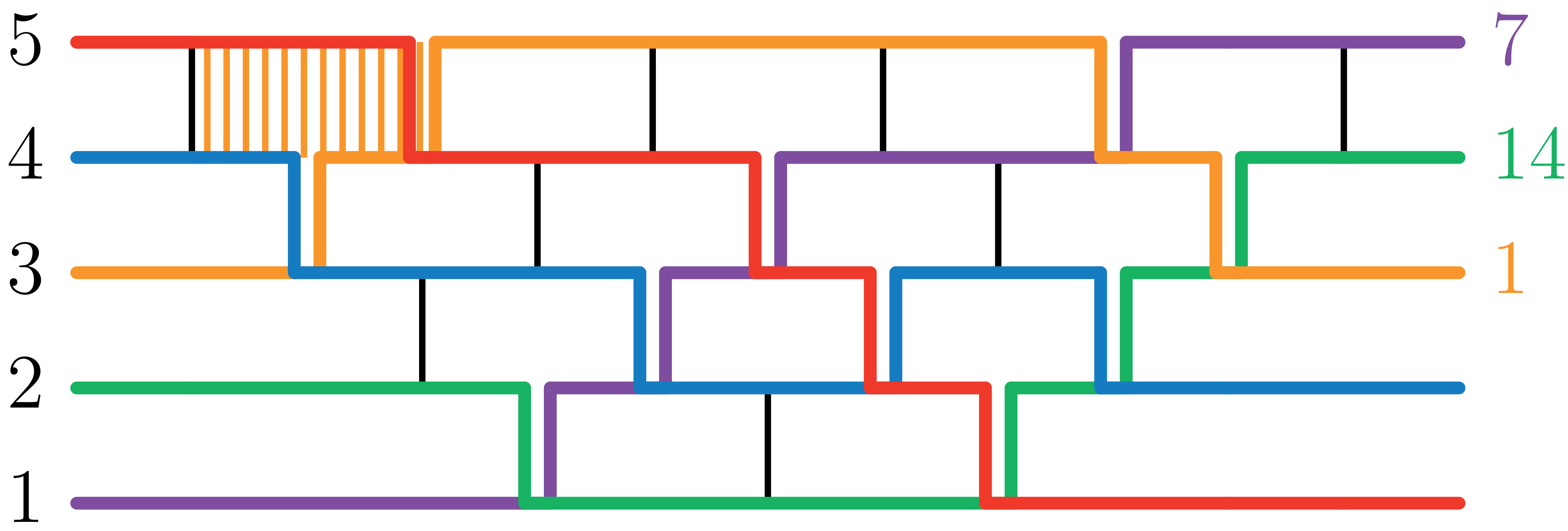

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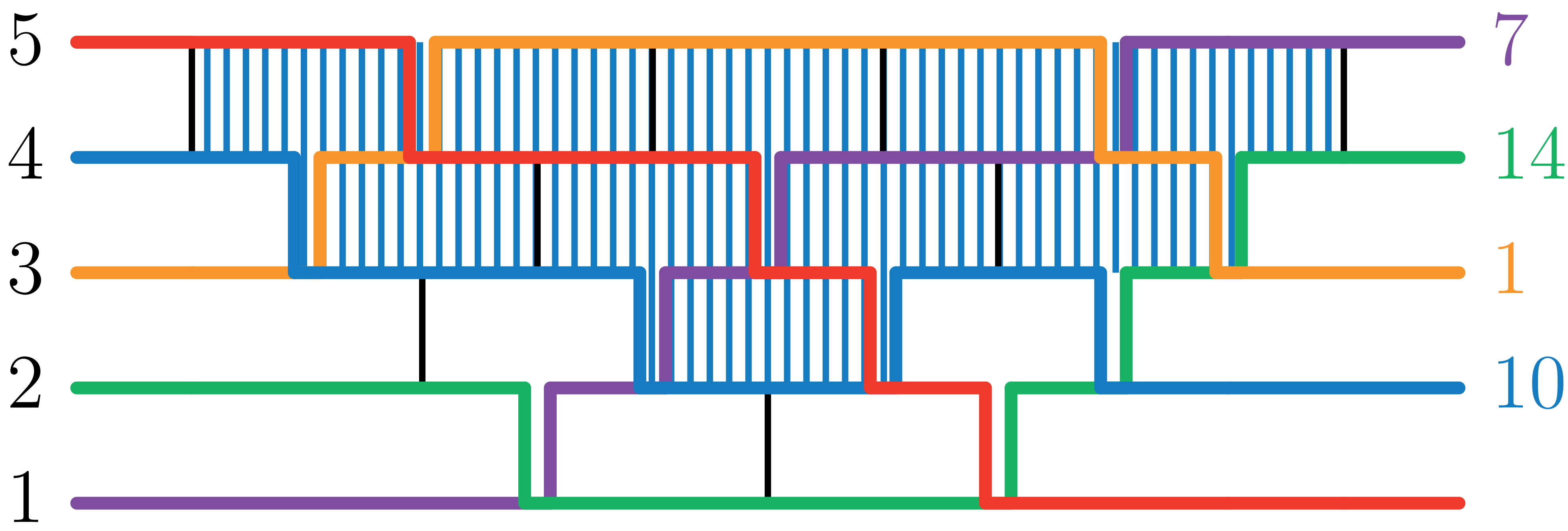

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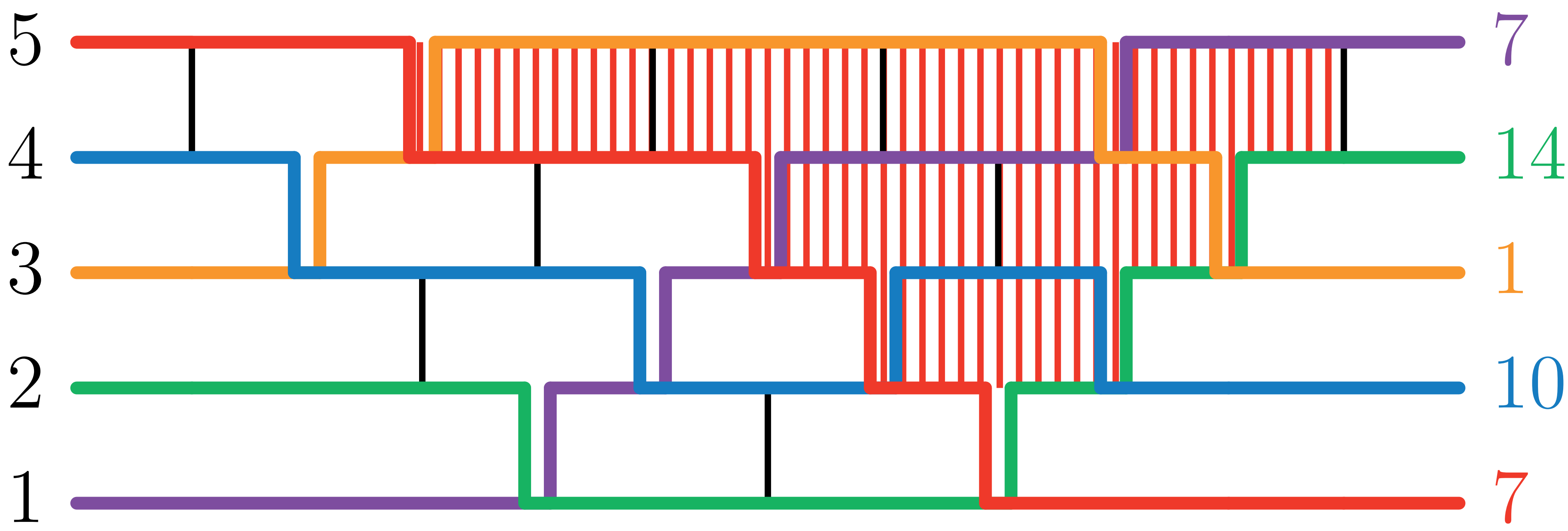

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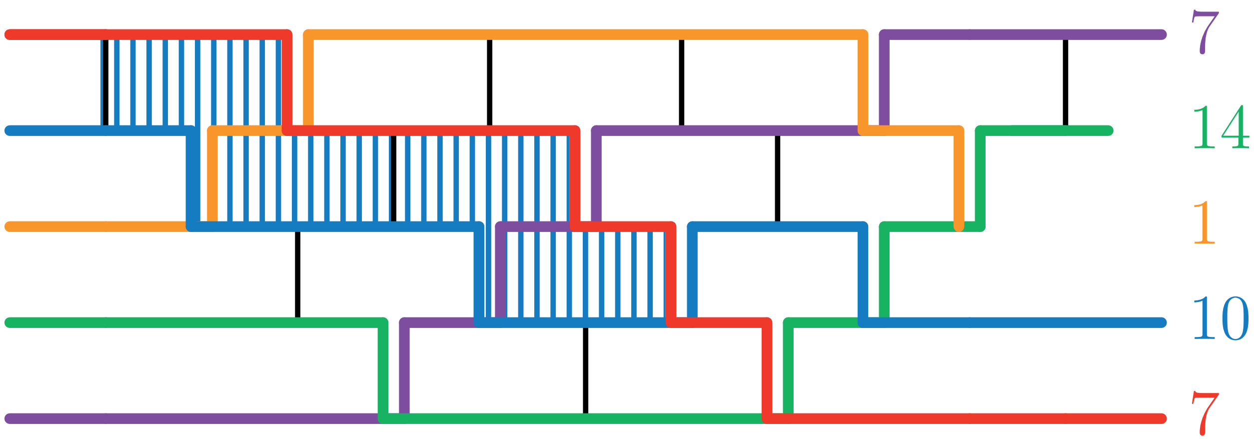
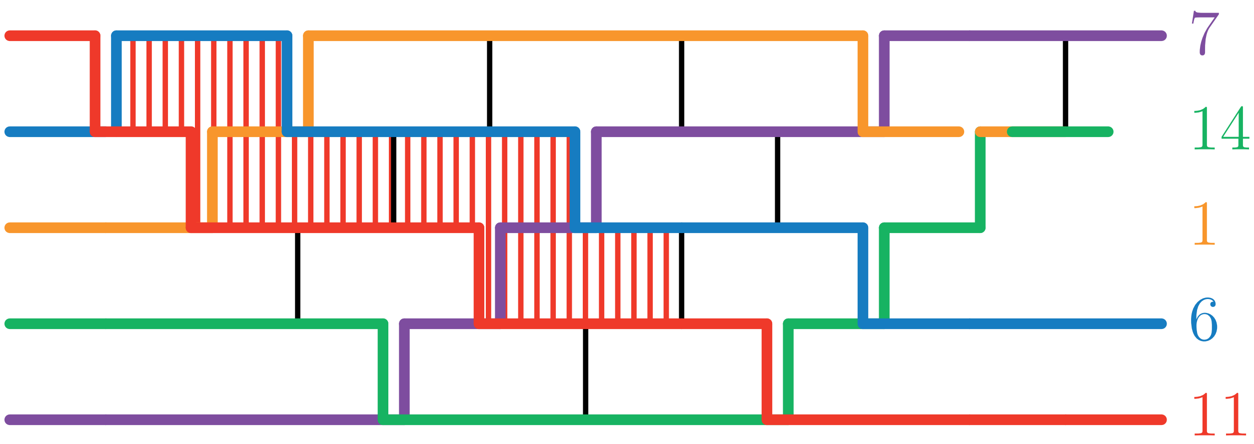
BRICK VECTORS AND FLIPS



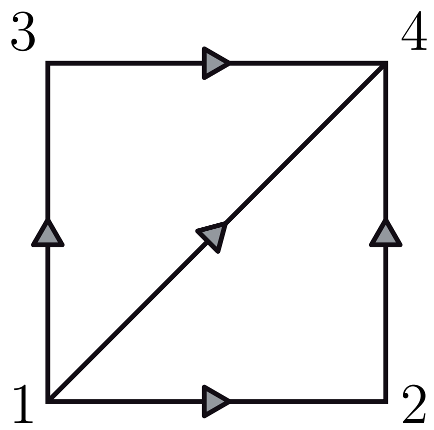


LEMMA. Λ and Λ' related by a flip between their ith and jth pseudolines

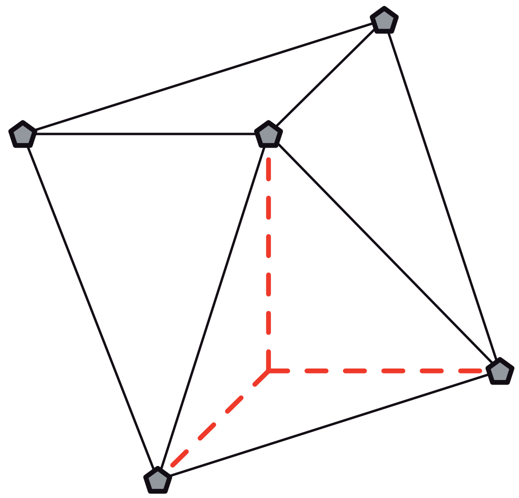
$$\Rightarrow \; !(\Lambda) - !(\Lambda') = \; (e_i - e_j).$$

THE INCIDENCE CONE OF A MULTIGRAPH

G oriented (multi)graph $\longmapsto$ Incidence configuration $I(G) = \{e_i - e_j \mid (i;j) \in G\}$,
$\longmapsto$ Incidence cone $C(G)$ = cone generated by $I(G)$.



REMARK. circuits in $I(G) \longleftrightarrow$ simple cycles in G,
cocircuits in $I(G) \longleftrightarrow$ minimal cuts in G,
(and signs correspond to the orientations of the edges).



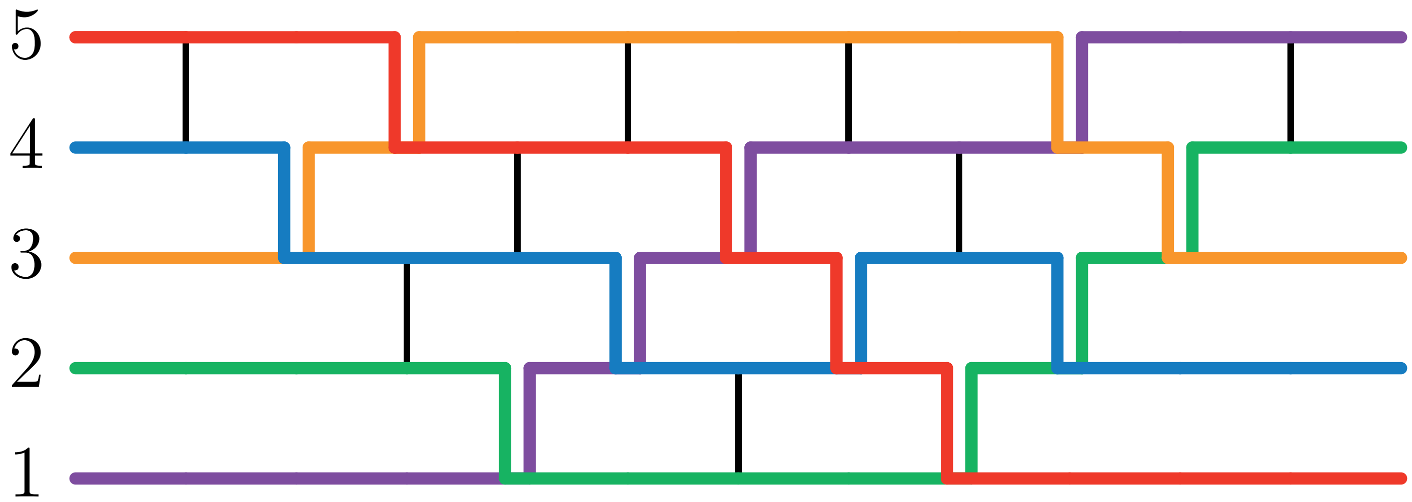
REMARK. $C(G)$ is pointed $\longleftrightarrow$ G is acyclic.

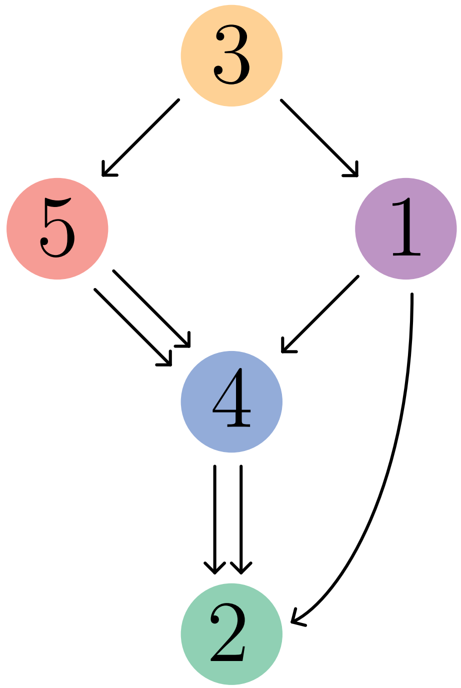
facets of $C(G) \longleftrightarrow$ complements of the minimal directed cuts of G.

CONTACT GRAPH OF A PSEUDOLINE ARRANGEMENT

Contact graph $\Lambda^\#$ of a pseudoline arrangement $\Lambda =$

- a node for each pseudoline of Λ, and
- an arc for each contact point of Λ oriented from top to bottom.





VERTEX CHARACTERIZATION AND FACET DESCRIPTION

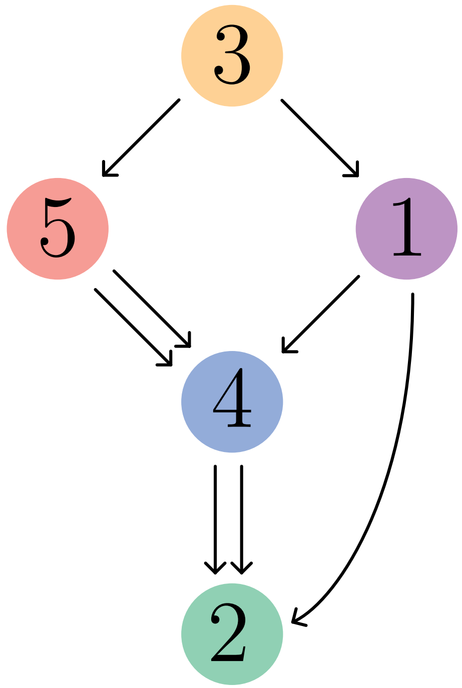
Contact graph $\Lambda^{\#}$ of a pseudoline arrangement $\Lambda =$

- a node for each pseudoline of Λ, and
- an arc for each contact point of Λ oriented from top to bottom.

THEOREM. Cone of $\Omega(\mathcal{S})$ at $!(\Lambda)$ $=$ incidence cone $C(\Lambda^{\#})$.

COROLLARY. $!(\Lambda)$ vertex of $\Omega(\mathcal{S})$ $\longleftrightarrow$ $\Lambda^{\#}$ acyclic.

COROLLARY. Normal vectors of $\Omega(\mathcal{S})$ = characteristic vectors of sinks of directed cuts of acyclic contact graphs of pseudoline arrangements supported by $\mathcal{S}$.



TRIANGULATIONS AND MULTITRIANGULATIONS

> T k-triangulation of the n-gon. Then:
> $$\begin{aligned}(T^*)^\# &= \text{contact graph of the dual pseudoline arrangement of } T\\ &= \text{dual graph of } T \text{ as complex of } k\text{-stars.}\end{aligned}$$

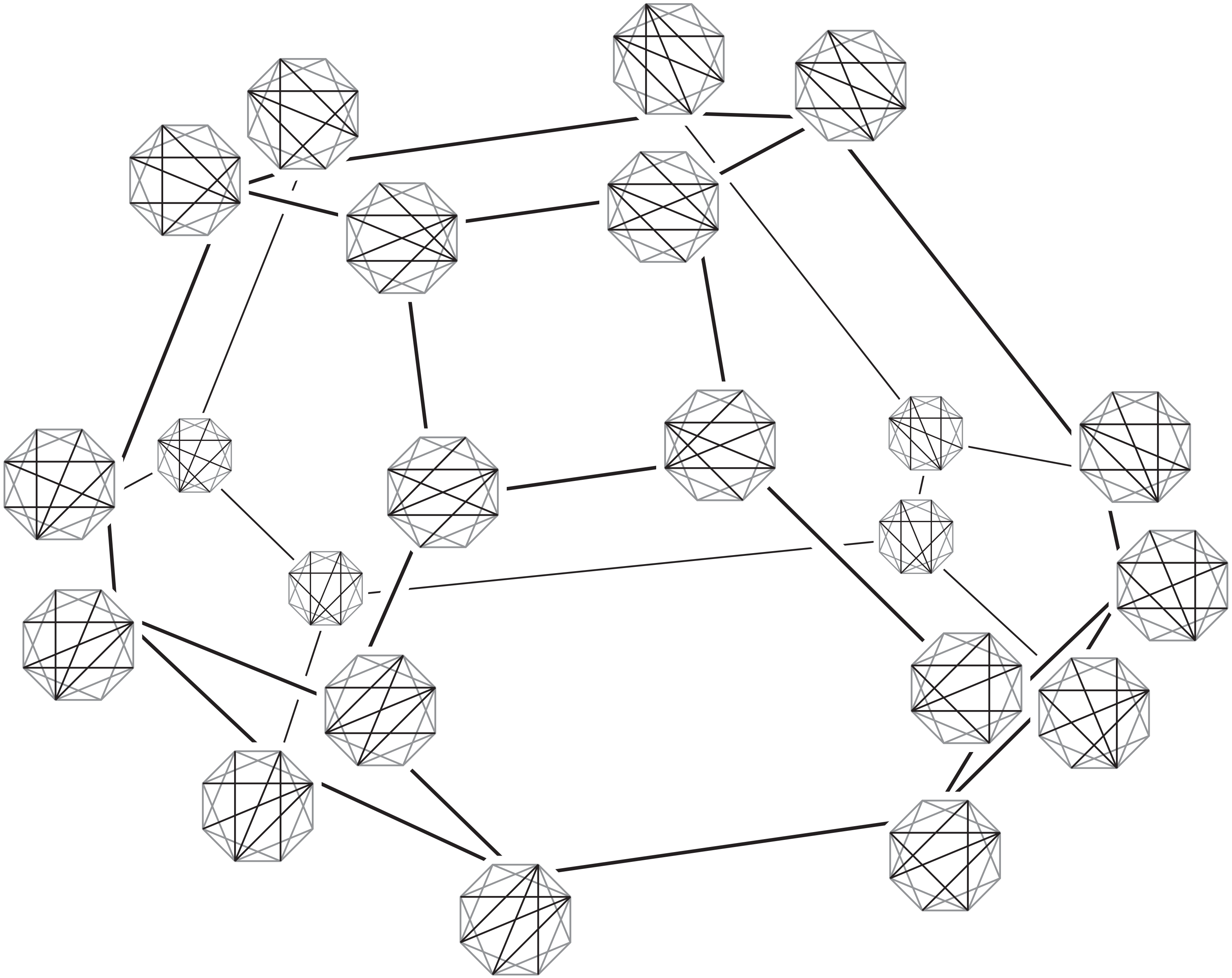
TRIANGULATIONS

1. Up to translation, Loday's associahedron = Brick polytope of V_n^* minus its first level.
2. Contact graph = dual tree $\Rightarrow$ each triangulation appears as a simple vertex.
3. Normal vectors of facets of $\Omega(n) = \ 0^{i-1}1^{j-i-1}0^{n-j} \mid [i;j]$ internal diagonal .

MULTITRIANGULATIONS

1. Not all k-triangulations appear as vertices of $\Omega(V_n^{*k})$, and not all vertices are simple.
2. Normal vectors of facets of $\Omega(V_n^{*k}) =$ 0=1-sequences of length $n-2k$,
   distinct from 0^{n-2k} and 1^{n-2k}
   and not containing 10^r1, for $r \geq k$.

TRIANGULATIONS AND MULTITRIANGULATIONS



Open problems and perspectives

OPEN PROBLEMS AND PERSPECTIVES

1. Multi Dyck paths

THEOREM. The number of k-triangulations of the n-gon is

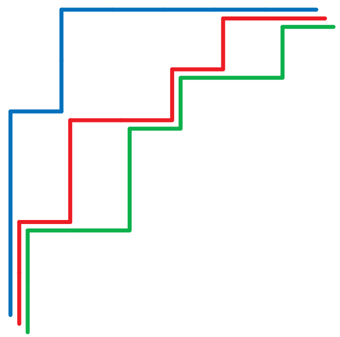
$$\det(C_{n-i-j})_{1\le i,j\le k} = \begin{pmatrix} C_{n-2} & C_{n-3} & \cdots & C_{n-k} & C_{n-k-1} \\ C_{n-3} & C_{n-4} & \cdots & C_{n-k-1} & C_{n-k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-k-1} & C_{n-k-2} & \cdots & C_{n-2k+1} & C_{n-2k} \end{pmatrix},$$

where $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the mth Catalan number.

J. Jonsson, Generalized triangulations and diagonal-free subsets of stack polyominoes, 2005.

PROBLEM. Find an explicit bijection

k-triangulations $\longleftrightarrow$ k-Dyck paths.



S. Elizalde, A bijection between 2-triangulations and pairs of non-crossing Dyck paths, 2006.

C. Nicolas, Another bijection between 2-triangulations and pairs of non-crossing Dyck paths, 2009.

OPEN PROBLEMS AND PERSPECTIVES

1. Multi Dyck paths
2. Pseudotriangulations and multipseudotriangulations in higher dimension

multipseudotriangulations of 2-dimensional point sets

$\longrightarrow$ Positivity of the j-depth for all j

$\longrightarrow$ Lower Bound Theorem for d-polytopes with $d+3$ vertices

E. Welzl, Entering and leaving j-facets, 2001.

PROBLEM. Define (multi)pseudotriangulations in higher dimension.

OPEN PROBLEMS AND PERSPECTIVES

1. Multi Dyck paths
2. Pseudotriangulations and multipseudotriangulations in higher dimension
3. Polytopality of flip graphs

$\mathcal{S}$ support of pseudoline arrangements.
$\Delta(\mathcal{S}) =$ simplicial complex whose maximal simplices are the sets of contact points of pseudoline arrangements supported by $\mathcal{S}$.

PROBLEM. Is $\Delta(\mathcal{S})$ the boundary complex of a polytope?

Remark:

- Multitriangulations are universal.
- First open case: pseudotriangulations of non-realizable pseudoline arrangements.

Thank you

Questions

You will ask about that, right?

DIAMETER OF $G_{n;k}$

PROPOSITION. The diameter $_{n;k}$ of the graph of flips on k-triangulations of the n-gon is bounded by

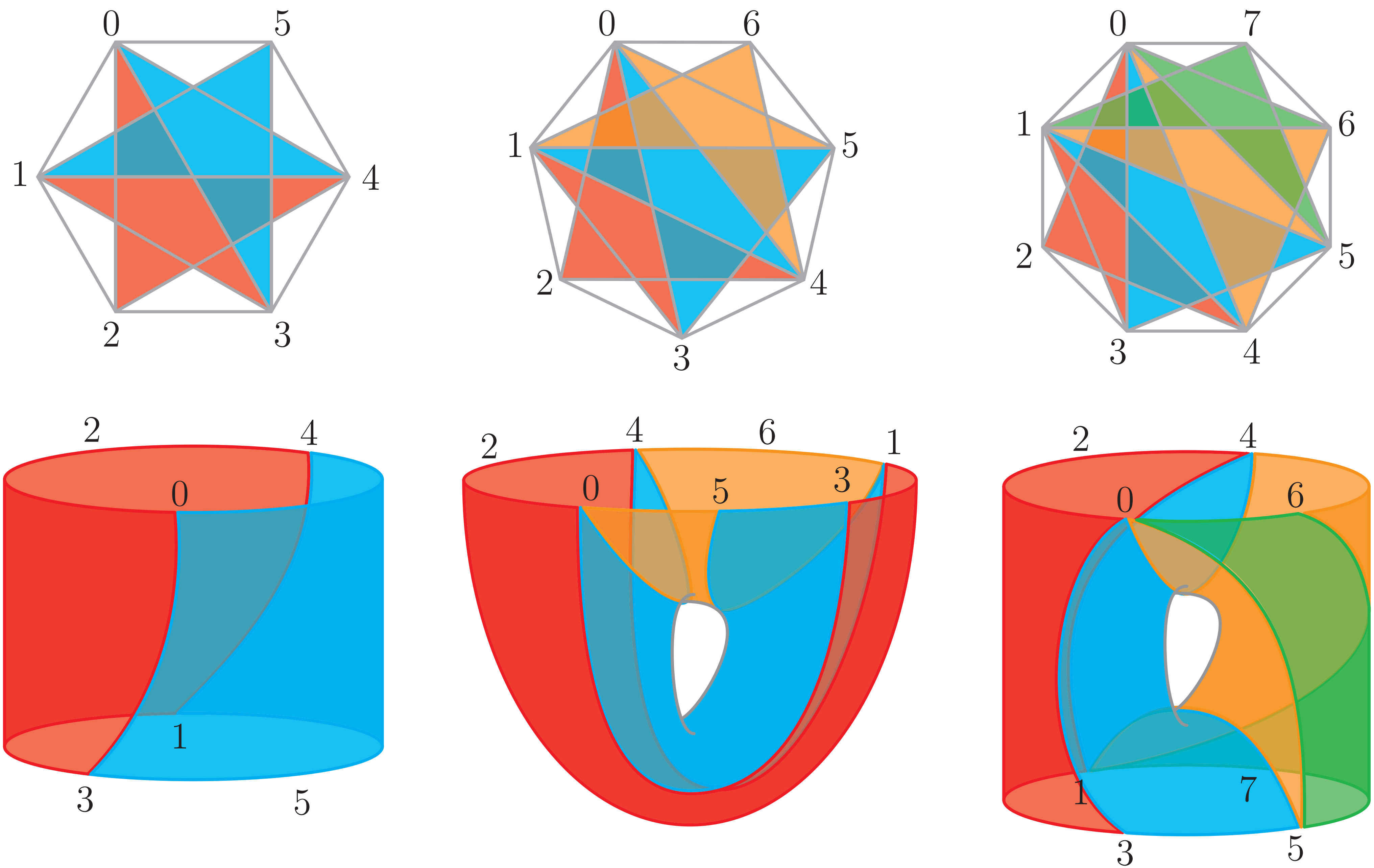
$$2 \left\lfloor \frac{n}{2} \right\rfloor \quad k + \frac{1}{2} \quad - k(2k+3) \le \;_{n;k} \le 2k(n - 4k - 1);$$

when $n > 4k^2(2k+1)$.

Diameter for little values of n and k:

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$_{n;1}$	0	1	2	4	5	7	9	11	12	15	16	18	20	22	24	26
$_{n;2}$	0	1	3	6	8	11	14									
$_{n;3}$	0	1	3	6	10											

MAPS ON SURFACES



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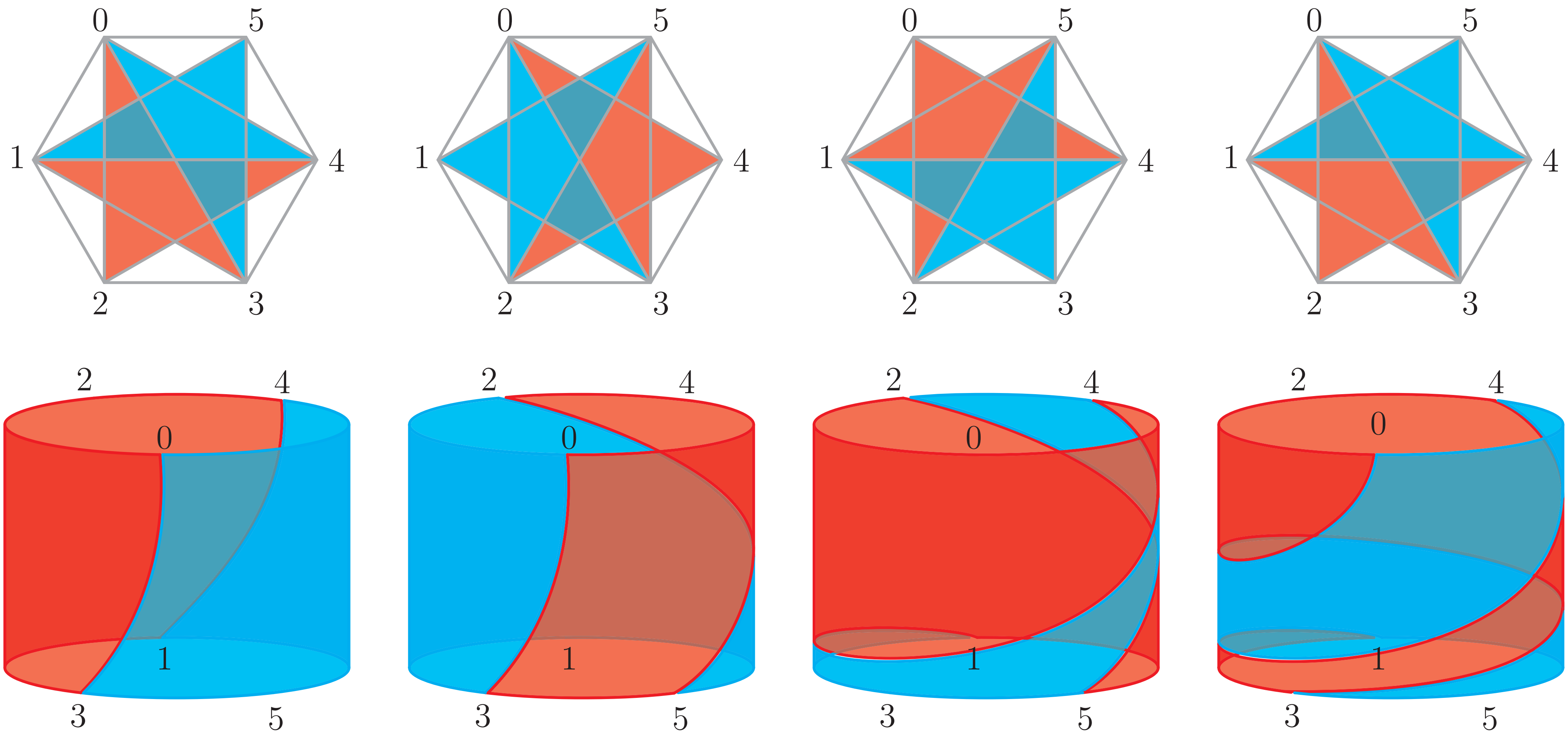
\end{aligned}
$$

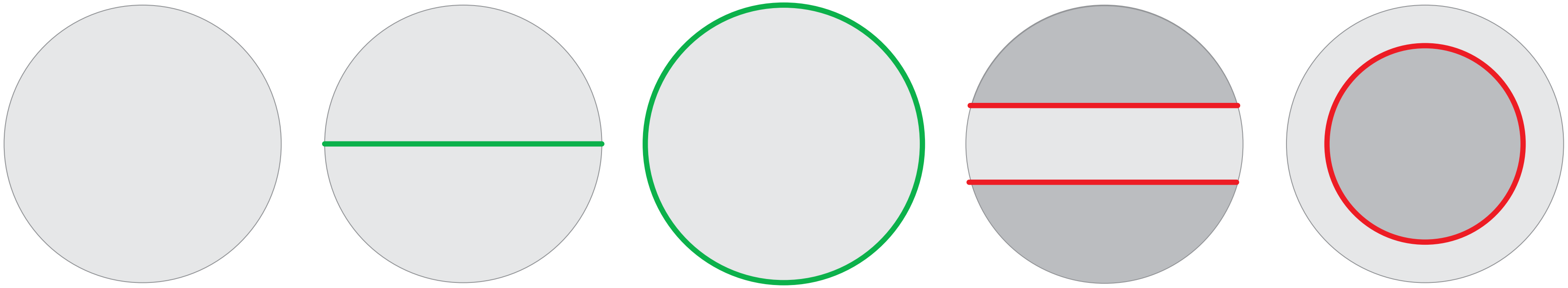
MAPS ON SURFACES



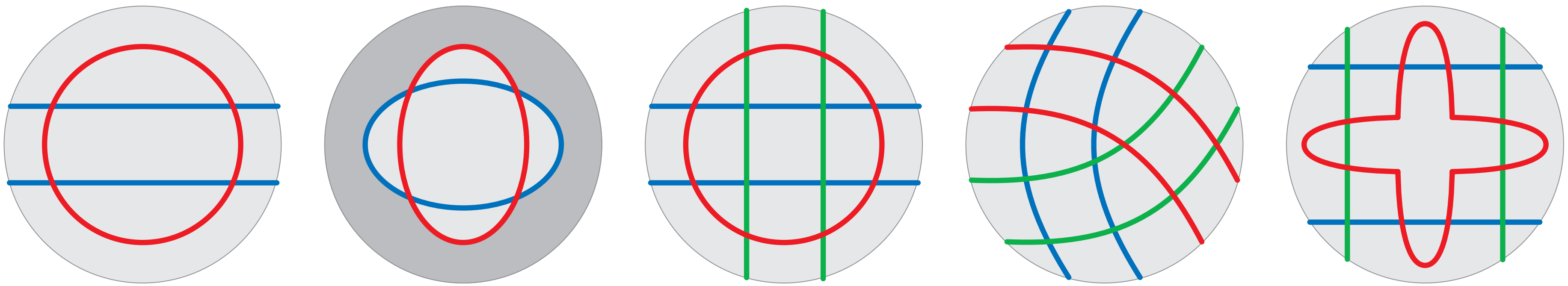
fundamental group of the flip graph $G_{n;k} \longmapsto$ mapping class group of the surface $\mathcal{S}_{n;k}$

ENUMERATION OF DOUBLE PSEUDOLINE ARRANGEMENTS

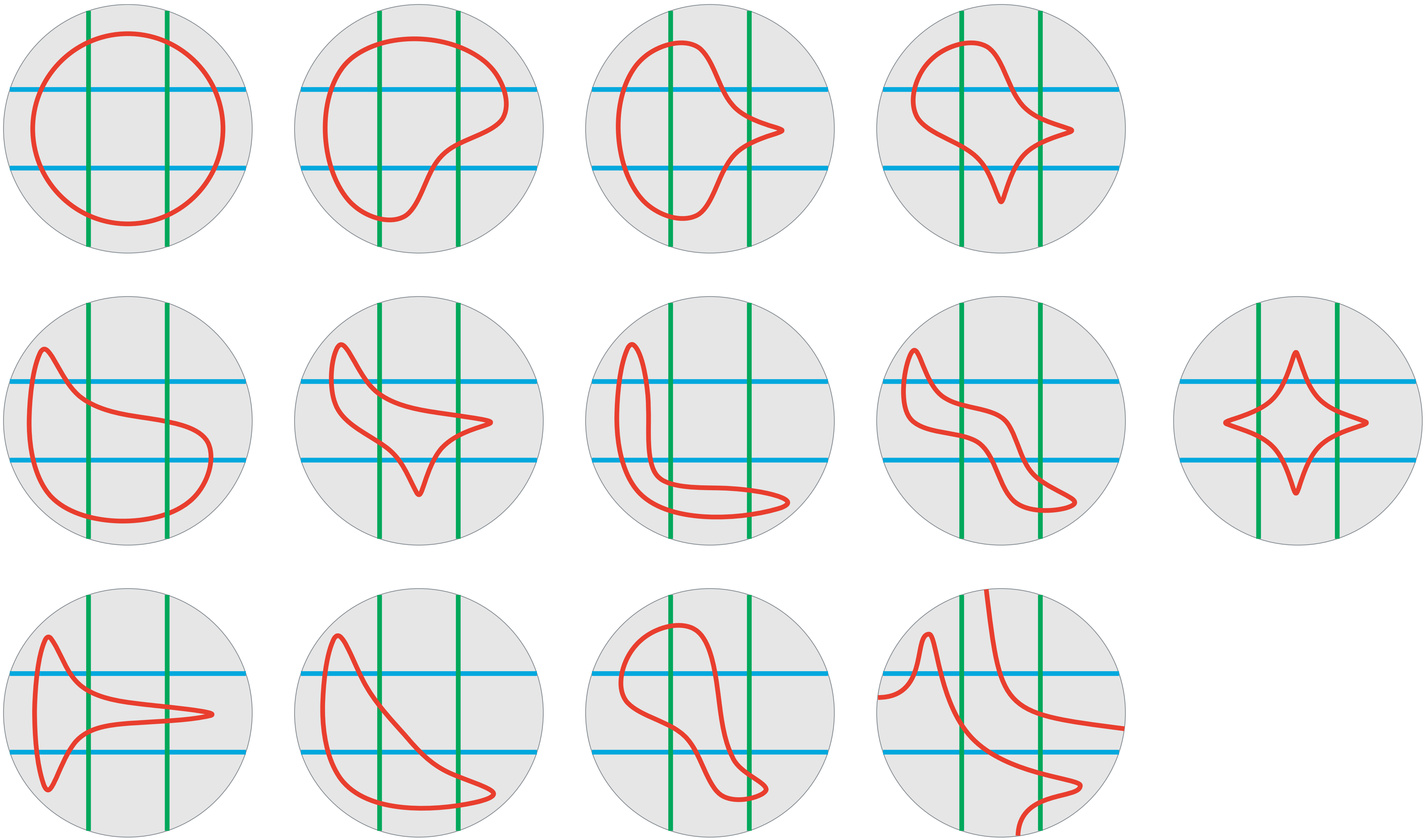
$\mathcal{P}$ = projective plane = disk with antipodal boundary points identified.
pseudoline = non-separating simple closed curve
double pseudoline = separating simple closed curve



double pseudoline arrangement = finite set of double pseudolines such that any two of them cross in exactly four points, transversally at these points and induce a cell decomposition of $\mathcal{P}$.



ENUMERATION OF DOUBLE PSEUDOLINE ARRANGEMENTS



Number of arrangements with n pseudolines and m double pseudolines:

	0	1	2	3	4	5
0			1	13	6 570	181 403 533
1		1	4	626	4 822 394	
2	1	2	48	86 715		
3	1	5	1 329			
4	1	25	80 253			
5	1	302				
6	4	9 194				
7	11	556 298				
8	135					
9	4 382					
10	312 356					

ENUMERATION OF SYMMETRIC REALIZATIONS OF $\triangle_{8;2}$

USE SYMMETRY

$\mathbb{D}_n$ = dihedral group = isometries of the regular n-gon

Natural action of $\mathbb{D}_n$ on $\triangle_{n;k}$:
$$\begin{array}{rcl} \mathbb{D}_n \times \triangle_{n;k} & \longrightarrow & \triangle_{n;k} \\ (\;\;; E) & \longmapsto & E = \{\;\; e \mid e \in E\} \end{array}$$

DECOMPOSE INTO TWO STEPS

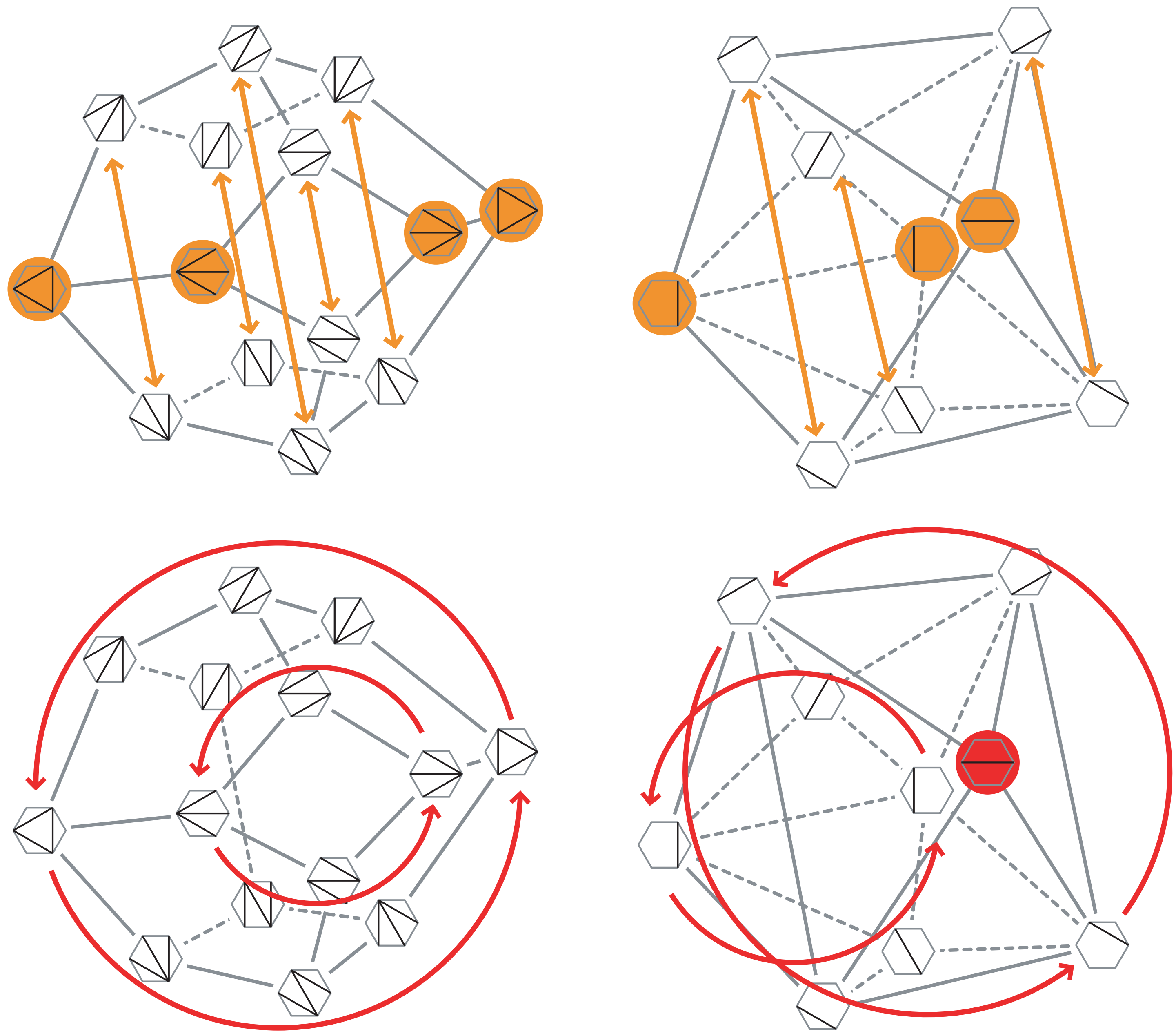
1. From face lattice to oriented matroids

   Find all possible symmetric oriented matroids realizing $\triangle_{n;k}$

2. From oriented matroids to polytopes

   Deduce the space of symmetric realizations of $\triangle_{n;k}$

SYMMETRY



Δ a simplicial complex with an action of a group G

$P \subset \mathbb{R}^d$ a realization of Δ symmetric under G, and V its vertex set

$$
: \left\{ \begin{array}{ccc}
V^{d+1} & \longrightarrow & \{-1, 0, +1\} \\
(v_0, v_1, \dots, v_d) & \longmapsto & \begin{array}{c}\text{orientation of the simplex} \\ \text{spanned by } v_0, v_1, \dots, v_d\end{array} = \operatorname{sign}\det \begin{pmatrix} v_0 & v_1 & \dots & v_d \\ 1 & 1 & 1 & 1 \end{pmatrix}
\end{array} \right.
$$

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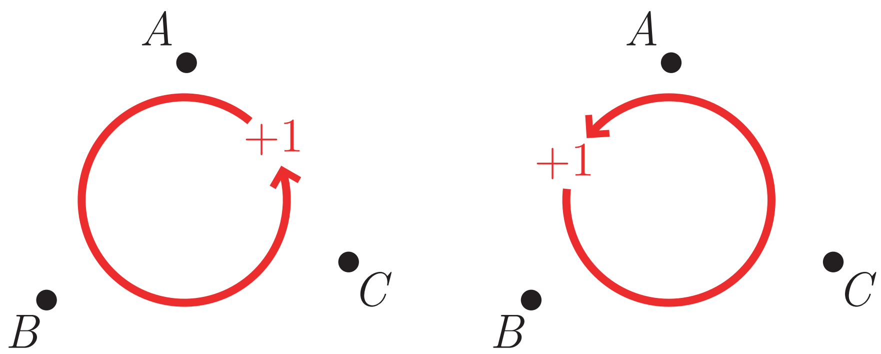
satisfies the relations:

(i) Alternating relations



FROM FACE LATTICE TO ORIENTED MATROIDS

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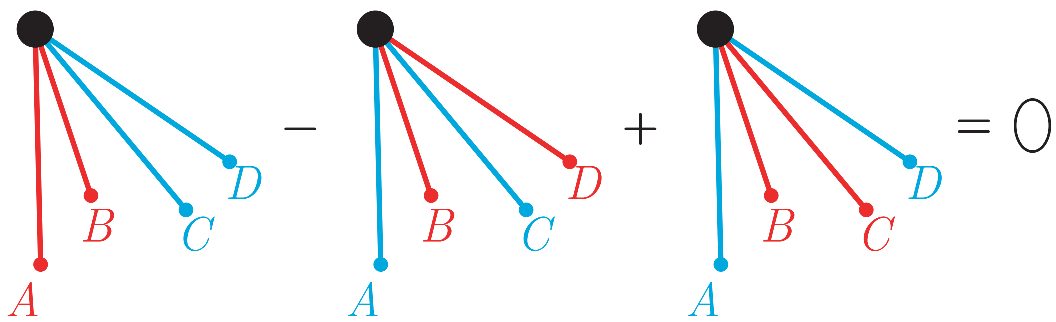
satisfies the relations:

(i) Alternating relations

(ii) Grassmann-Plucker relations



FROM FACE LATTICE TO ORIENTED MATROIDS

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$P \subset \mathbb{R}^d$ a realization of Δ symmetric under G, and V its vertex set

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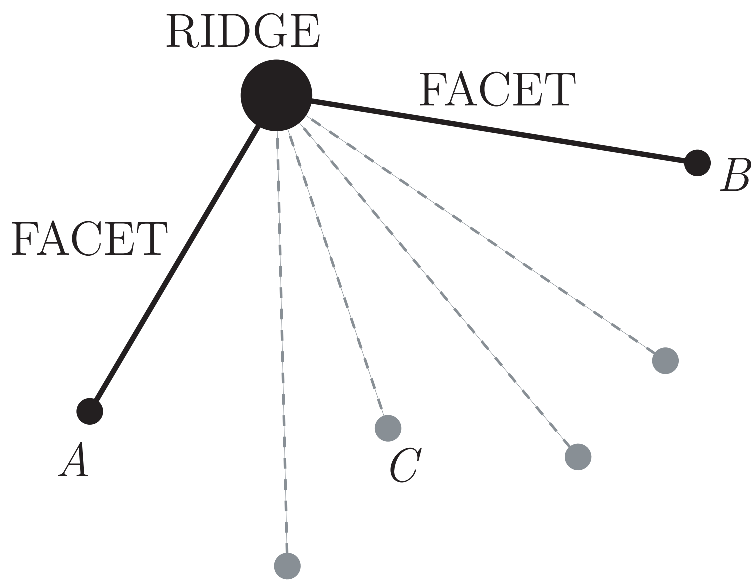
satisfies the relations:

(i) Alternating relations

(ii) Grassmann-Plucker relations

(iii) Necessary simplex orientations



Δ a simplicial complex with an action of a group G
$P \subset \mathbb{R}^d$ a realization of Δ symmetric under G, and V its vertex set

$$:\ \left[\begin{array}{lcl} V^{d+1} & \longrightarrow & \{-1, 0, +1\} \\ (v_0, v_1, \dots, v_d) & \longmapsto & \begin{array}{c}\text{orientation of the simplex}\\ \text{spanned by } v_0, v_1, \dots, v_d\end{array} = \operatorname{sign}\det \begin{pmatrix} v_0 & v_1 & \dots & v_d \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}\right.$$

satisfies the relations:

(i) Alternating relations

(ii) Grassmann-Plucker relations

(iii) Necessary simplex orientations

(iv) Symmetry

FROM ORIENTED MATROIDS TO POLYTOPES

Problem. For a given oriented matroid, find a matrix representing it or a proof that such a matrix is impossible to find.

"On the one hand, there is a general algorithm to solve this problem. On the other hand, it is known that this algorithm from real algebraic geometry is far from applicable for our cases in the theory of oriented matroids."

J. Bokowski, Computational Oriented Matroids, 2006

$\Longrightarrow$ USE HEURISTICAL METHODS

Our heuristic is symmetry

THE SPACE OF SYMMETRIC REALIZATIONS OF $\Delta_{8;2}$

Proposition. The space of symmetric realizations of $\Delta_{8;2}$ has dimension 4.

Example. With some arbitrary values of the 4 parameters, we obtain a particular symmetric realization of $\Delta_{8;2}$:

$$M \simeq \begin{pmatrix} 0.21 & 0 & 0 & -0.21 & 0.52 & -0.74 & 0.74 & -0.52 & 0 & 0 & 0 & 0 \\ -0.95 & 0.66 & 0 & -0.63 & 0.8 & -0.4 & -0.3 & 0.68 & 0 & 0 & 0 & 0 \\ -0.17 & 0.75 & -1 & 0.77 & -0.21 & -0.4 & 0.60 & -0.34 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0.55 & 0.55 & -0.55 & -0.55 & 0.5 & 0.7 & -0.7 & -0.4 & 1 & 0 & -1 & 0 \\ -0.55 & 0.55 & 0.55 & -0.4 & -0.7 & 0.7 & 0.5 & -0.55 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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> polymake multiassociahedron82 F_VECTOR
F_VECTOR
12 66 192 306 252 84
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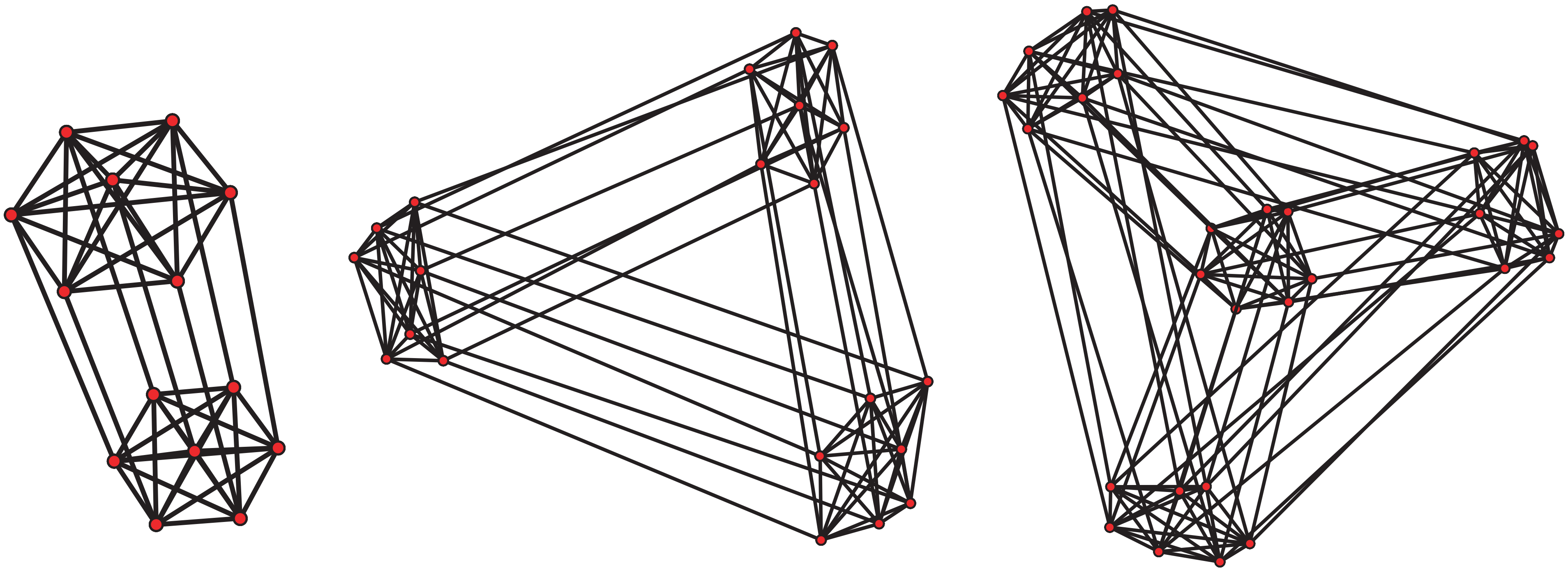
POLYTOPALITY OF PRODUCTS OF NON-POLYTOPAL GRAPHS

Cartesian product of polytopes: $P \times Q := \{(p;q) \mid p \in P; q \in Q\}$.

Cartesian product of graphs: $$V(G \times H) := V(G) \times V(H);$$ $$E(G \times H) := (V(G) \times E(H)) \cup (E(G) \times V(H)):$$



REMARK. graph of $P \times Q$ = (graph of P) $\times$ (graph of Q).

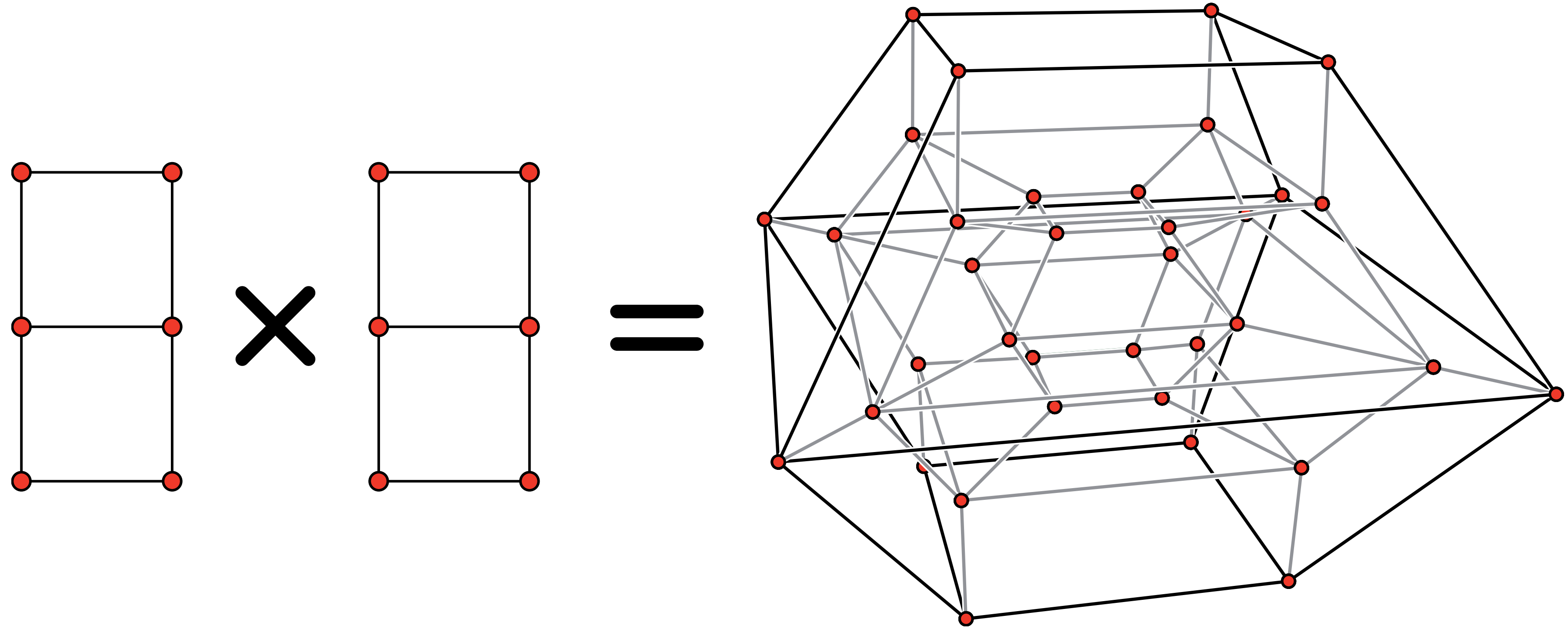
PROBLEM. Does the polytopality of $P \times Q$ imply that of P and Q?

POLYTOPALITY OF PRODUCTS OF NON-POLYTOPAL GRAPHS

PROBLEM. Does the polytopality of $P \times Q$ imply that of P and Q?

THEOREM. $G \times H$ simple polytopal $\iff$ G and H simply polytopal.

THEOREM. The product of a d-polytopal graph by the graph of a regular subdivision of an e-polytope is $(d+e)$-polytopal.



J. Pfeifle, V. P. & F. Santos, On polytopality of Cartesian products of graphs, 2010.

PRODSIMPLICIAL NEIGHBORLY POLYTOPES

$k \geq 0$ and $\underline{n} := (n_1, \ldots, n_r)$.

A polytope is $(k, \underline{n})$-prodsimplicial-neighborly if its k-skeleton is combinatorially equivalent to that of the product of simplices $\triangle_{\underline{n}} := \triangle_{n_1} \times \cdots \times \triangle_{n_r}$.

EXAMPLE.

(i) neighborly polytopes arise when $r = 1$.
For example, the cyclic polytope $C_{2k+2}(n+1)$ is (k, n)-PSN.

(ii) neighborly cubical polytopes arise when $\underline{n} = (1, 1, \ldots, 1)$.

M. Joswig and G. Ziegler, Neighborly cubical polytopes, 2000.

PROBLEM. What is the minimal dimension of a (k, n)-PSN polytope?

PRODSIMPLICIAL NEIGHBORLY POLYTOPES

CONSTRUCTIONS

(i) products of cyclic polytopes.

(ii) reflections of cyclic polytopes.

(iii) Minkowski sums of cyclic polytopes.

(iv) projections of deformed products of polytopes.

OBSTRUCTIONS

A $(k;\underline{n})$-PSN polytope is $(k;\underline{n})$-projected-prodsimplicial-neighborly if it is a projection of a polytope combinatorially equivalent to $\triangle_{\underline{n}}$.

Sanyal's topological obstruction method:

Projection preserving the k-skeleton of $\triangle_{\underline{n}}$

$\longmapsto$ simplicial complex embeddable in a certain dimension (Gale duality)

$\longmapsto$ topological obstruction (Sarkaria's criterion).

B. Matschke, J. Pfeifle, and V. P., Prodsimplicial neighborly polytopes, 2010.