CAMBRIAN TREES

Grégory CHATEL
(Univ. MIV)

Carsten LANGE
(Univ. Munich)

Vincent PILAUD
(CNRS & LIX)
### MOTIVATION

#### Combinatorics

- **Permutations**
- **Binary Trees**
- **Binary Sequences**

#### Geometry

#### Algebra

**Malvenuto-Reutenauer algebra**

\[
\text{FQSym} = \text{vect} \left\langle \ F_\tau \mid \ \tau \in \mathcal{S} \right\rangle \\
F_\tau \cdot F_\tau' = \sum_{\sigma \in \tau \sqcup \tau'} F_\sigma \\

\Delta F_\sigma = \sum_{\sigma \in \tau \sqcup \tau'} F_\tau \otimes F_\tau'
\]

**Loday-Ronco algebra**

\[
P_{\mathrm{BT}} = \text{vect} \left\langle \ P_T \mid \ T \in \mathcal{BT} \right\rangle \\
P_T \cdot P_{T'} = \sum_{T' \leq T'' \leq T_\uparrow_{T'}} P_{T''} \\

\Delta P_\gamma = \sum_{\gamma \text{ cut}} B(T, \gamma) \otimes A(T, \gamma)
\]

**Solomon algebra**

\[
\text{Rec} = \text{vect} \left\langle \ X_\eta \mid \ \eta \in \pm^* \right\rangle \\
X_\eta \cdot X_\eta' = X_{\eta + \eta'} + X_{\eta - \eta'} \\

\Delta X_\eta = \sum_{\gamma \text{ cut}} B(\eta, \gamma) \otimes A(\eta, \gamma)
\]
COMBINATORICS
Cambrian tree = directed and labeled tree such that

\[
\begin{align*}
\text{Cambrian labeling} & : j < j > j \\
\text{Increasing labeling} & : <j >j
\end{align*}
\]

increasing tree = directed and labeled tree such that labels increase along arcs

leveled Cambrian tree = directed tree with a Cambrian labeling and an increasing labeling
Cambrian trees are dual to triangulations of polygons

signature $\leftrightarrow$ vertices above or below $[0, 9]$

node $j$ $\leftrightarrow$ triangle $i < j < k$

For any signature $\varepsilon$, there are $C_n = \frac{1}{n+1} \binom{2n}{n}$ $\varepsilon$-Cambrian trees
Cambrian correspondence = signed permutation $\mapsto$ leveled Cambrian tree

Exm: signed permutation $\begin{array}{c}2 \begin{array}{c}7 \begin{array}{c}5 \begin{array}{c}1 \begin{array}{c}3 \begin{array}{c}4 \begin{array}{c}6 \end{array} \end{array} \end{array} \end{array} \end{array} \end{array}$

\[
\begin{array}{cccccccccc}
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
\end{array}
\]
Cambrian correspondence = signed permutation $\rightarrow$ leveled Cambrian tree

Exm: signed permutation $\overline{2751346}$

![Cambrian tree diagram](image-url)
Cambrian correspondence = signed permutation $\mapsto$ leveled Cambrian tree

Exm: signed permutation $\begin{array}{c} 2751346 \end{array}$
Cambrian correspondence = signed permutation $\mapsto$ leveled Cambrian tree

Exm: signed permutation $2\overline{751346}$
Cambrian correspondence $= \text{signed permutation} \leftrightarrow \text{leveled Cambrian tree}$

Exm: signed permutation $2\overline{7}5\overline{1}3\overline{4}6$
Cambrian correspondence $=$ signed permutation $\mapsto$ leveled Cambrian tree

Exm: signed permutation $2\overline{751\overline{346}}$
Cambrian correspondence \(=\) signed permutation \(\longrightarrow\) leveled Cambrian tree

Exm: signed permutation \(2 \rightarrow 751 \rightarrow 34 \rightarrow 6 \rightarrow 1\)
Cambrian correspondence = signed permutation $\mapsto$ leveled Cambrian tree

Exm: signed permutation $2751346$
**Cambrian Correspondence**

Cambrian correspondence = signed permutation $\rightarrow$ leveled Cambrian tree

Exm: signed permutation $2751346$

$P(\tau) = P$-symbol of $\tau = \text{Cambrian tree produced by Cambrian correspondence}$

$Q(\tau) = Q$-symbol of $\tau = \text{increasing tree produced by Cambrian correspondence}$

(analogy to Robinson-Schensted algorithm)
Cambrian map $=$ signed permutation $\mapsto$ triangulation

Exm: signed permutation $43816257$
Cambrian map = signed permutation \rightarrow \text{triangulation}

Exm: signed permutation \begin{array}{c}4\overline{3}\overline{8}\overline{1}\overline{6}27\end{array}

Diagram of Cambrian lattice with vertices and edges labeled accordingly.
Cambrian map = signed permutation $\mapsto$ triangulation

Exm: signed permutation $\underline{43816257}$
Cambrian map = signed permutation $\mapsto$ triangulation

Exm: signed permutation $4\underline{3}8\underline{1}6\underline{2}5\underline{7}$
**Cambrian Correspondence and Triangulations**

**Cambrian map** = signed permutation $\mapsto$ triangulation

**Exm:** signed permutation $\overline{43816257}$

![Diagram showing a Cambrian lattice with a signed permutation and triangulation indicated.](image)

Reading: Cambrian lattices 2006
Cambrian map $= \text{signed permutation} \leftrightarrow \text{triangulation}$

Exm: signed permutation $\underline{43816257}$
Cambrian map = signed permutation $\leftrightarrow$ triangulation

Exm: signed permutation $43816257$
Cambrian map = signed permutation $\rightarrow$ triangulation

Exm: signed permutation 43816257
 Cambrian map = signed permutation \rightarrow\rightarrow triangulation

Exm: signed permutation $\begin{pmatrix} 4 & 3 & 8 & 1 & 6 & 2 & 5 & 7 \end{pmatrix}$

Reading. Cambrian lattices 2006
ε-Cambrian congruence = transitive closure of the rewriting rules

\[ UacVbW \equiv_\varepsilon UcaVbW \] if \( a < b < c \) and \( \varepsilon_b = - \)

\[ UbVacW \equiv_\varepsilon UbVcaW \] if \( a < b < c \) and \( \varepsilon_b = + \)

where \( a, b, c \) are elements of \([n]\) while \( U, V, W \) are words on \([n]\)

\[ \text{PROP. } \tau \equiv_\varepsilon \tau' \iff P(\tau) = P(\tau') \]
**CAMBRIAN CONGRUENCE**

$\varepsilon$-Cambrian congruence = transitive closure of the rewriting rules

\[
UacVbW \equiv_{\varepsilon} UcaVbW \quad \text{if } a < b < c \text{ and } \varepsilon_b = -
\]

\[
UbVacW \equiv_{\varepsilon} UbVcaW \quad \text{if } a < b < c \text{ and } \varepsilon_b = +
\]

where \(a, b, c\) are elements of \([n]\) while \(U, V, W\) are words on \([n]\)

**PROP.** \(\tau \equiv_{\varepsilon} \tau' \iff P(\tau) = P(\tau')\)

**PROP.** Cambrian congruence class labeled by Cambrian tree \(T\)

\[
\{ \tau \in \mathcal{G}^{\varepsilon} \mid P(\tau) = T \} = \{ \text{linear extensions of } T \}\]

**PROP.** Cambrian classes are intervals of the weak order

Minimums avoid \(\overline{231}\) and \(312\) while maximums avoid \(\overline{213}\) and \(132\)

Reading. Cambrian lattices. 2006
Rotation operation preserves Cambrian trees:

$$T \xrightarrow{\text{rotation of } i \to j} T'$$

**increasing rotation** = rotation of edge $i \to j$ where $i < j$

**PROP.** The transitive closure of the increasing rotation graph is the Cambrian lattice

$\mathbf{P}$ defines a lattice homomorphism from weak order to Cambrian lattice

(rotation on Cambrian trees correspond to flips in triangulations)
Rotation on Cambrian trees $\leftrightarrow$ flips on triangulations
ROTATIONS AND CAMBRIAN LATTICES
vertices $i$ and $i + 1$ are always comparable in a Cambrian tree

**Canopy** of a Cambrian tree $T = \text{sequence } \text{can}(T) \in \pm^{n-1}$ defined by

\[
\text{can}(T)_i = \begin{cases} 
- & \text{if } i \text{ above } i + 1 \text{ in } T \\
+ & \text{if } i \text{ below } i + 1 \text{ in } T
\end{cases}
\]

**PROP.** $P$, $\text{can}$, and $\text{rec}$ define lattice homomorphisms:

\[
\varepsilon \xrightarrow{P} \text{Camb}(\varepsilon) \xrightarrow{\text{can}} \pm^{n-1}
\]
vertices $i$ and $i + 1$ are always comparable in a Cambrian tree

**Canopy** of a Cambrian tree $T = \text{sequence } \text{can}(T) \in \pm^{n-1}$ defined by

$$\text{can}(T)_i = \begin{cases} - & \text{if } i \text{ above } i + 1 \text{ in } T \\ + & \text{if } i \text{ below } i + 1 \text{ in } T \end{cases}$$

**PROP.** $P$, $\text{can}$, and $\text{rec}$ define lattice homomorphisms:

\[ \mathcal{G}_\varepsilon \xrightarrow{\text{rec}} \varepsilon \xrightarrow{\text{can}} \pm^{n-1} \]

\[ P \xrightarrow{\text{Camb}(\varepsilon)} \]

\[ \text{P} \xrightarrow{\text{Camb}(\varepsilon)} \]
GEOMETRY
**POLYTOPES & COMBINATORICICS**

polytope = convex hull of a finite set of $\mathbb{R}^d$

= bounded intersection of finitely many half-spaces

face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations

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Given a set of points, determine the face lattice of its convex hull.

Given a lattice, is there a polytope which realizes it?
Permutohedron $\text{Perm}(n)$

$= \text{conv} \{ (\sigma(1), \ldots, \sigma(n + 1)) \mid \sigma \in \Sigma_{n+1} \}$

$= \mathbb{H} \cap \bigcap_{\emptyset \neq J \subseteq [n\!+\!1]} \mathbb{H}^{\geq}(J)$
Permutohedron \( \text{Perm}(n) \)

\[ = \text{conv} \left\{ (\sigma(1), \ldots, \sigma(n + 1)) \mid \sigma \in \Sigma_{n+1} \right\} \]

\[ = \mathbb{H} \cap \bigcap_{\emptyset \neq J \subseteq [n+1]} \mathbf{H}^{\geq}(J) \]

\( k \)-faces of \( \text{Perm}(n) \)

\[ \equiv \text{surjections from} \ [n + 1] \text{ to} \ [n + 1 - k] \]
Permutohedron $\text{Perm}(n)$

$= \text{conv} \{(\sigma(1), \ldots, \sigma(n + 1)) \mid \sigma \in \Sigma_{n+1}\}$

$= \mathbb{H} \cap \bigcap_{\emptyset \neq J \subseteq [n+1]} \mathbb{H}^\geq(J)$

$k$-faces of $\text{Perm}(n)$

$\equiv$ surjections from $[n + 1]$ to $[n + 1 - k]$

$\equiv$ ordered partitions of $[n + 1]$ into $n + 1 - k$ parts
Permutohedron \( \text{Perm}(n) \)

\[
\begin{align*}
\text{Permutohedron } \text{Perm}(n) &= \text{conv } \{(\sigma(1), \ldots, \sigma(n + 1)) \mid \sigma \in \Sigma_{n+1}\} \\
&= \mathbb{H} \cap \bigcap_{\emptyset \neq J \subseteq [n+1]} H^{\geq}(J)
\end{align*}
\]

\(k\)-faces of \( \text{Perm}(n) \)

\[
\equiv \text{surjections from } [n + 1] \text{ to } [n + 1 - k] \\
\equiv \text{ordered partitions of } [n + 1] \text{ into } n + 1 - k \text{ parts}
\]

connections to

- inversion sets
- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group
**ASSOCIAHEDRON**

**Associahedron** = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex \((n + 3)\)-gon, ordered by reverse inclusion

- vertices ↔ triangulations
- edges ↔ flips
- faces ↔ dissections

vertices ↔ binary trees
edges ↔ rotations
faces ↔ Schröder trees
**VARIOUS ASSOCIAHEDRA**

**Associahedron** = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex \((n + 3)\)-gon, ordered by reverse inclusion

(Pictures by Ceballos-Santos-Ziegler)

Lee ('89), Gel’fand-Kapranov-Zelevinski ('94), Billera-Filliman-Sturmfels ('90), . . . , Ceballos-Santos-Ziegler ('11) Loday ('04), Hohlweg-Lange ('07), Hohlweg-Lange-Thomas ('12), P.-Santos ('12), P.-Stump ('12'), Lange-P. ('13')
Loday’s associahedron $= \text{conv} \{ L(T) \mid T \text{ triangulation of the } (n+3)\text{-gon} \} = \bigcap_{\delta \text{ diagonal of the } (n+3)\text{-gon}} H^{\geq}(\delta)$

$L(T) = (\ell(T,j) \cdot r(T,j))_{j \in [n+1]}$

$H^{\geq}(\delta) = \left\{ x \in \mathbb{R}^{n+1} \mid \sum_{j \in B(\delta)} x_j \geq \left( |B(\delta)| + 1 \right) \right\}$

Loday, Realization of the Stasheff polytope ('04)
The associahedron is obtained from the permutahedron by removing facets.
Can also replace Loday’s \((n + 3)\)-gon by others...

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 0 & + \\
+ & + & +
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 0 & - \\
+ & + & -
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 0 & + \\
- & + & +
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 0 & - \\
- & + & -
\end{array}
\]

...to obtain different realizations of the associahedron

Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)
\[ \text{Asso}(P) = \text{conv} \left\{ \text{HL}(T) \mid T \text{ triangulation of } P \right\} = H \cap \bigcap_{\delta \text{ diagonal of } P} H^{\geq}(\delta) \]

\[ \text{HL}(T)_j = \begin{cases} \ell(T, j) \cdot r(T, j) & \text{if } j \text{ down} \\ n + 2 - \ell(T, j) \cdot r(T, j) & \text{if } j \text{ up} \end{cases} \]

\[ H^{\geq}(\delta) = \left\{ x \mid \sum_{j \in B(\delta)} x_j \geq \binom{|B(\delta)| + 1}{2} \right\} \]

Hohlweg-Lange, Realizations of the associahedron and cyclohedron (’07)
Cambrian trees = labeled and oriented dual binary trees

Alternative vertex description of Hohlweg-Lange’s associahedra:

\[
HL(T)_j = \begin{cases} 
|\{\pi \text{ maximal path in } T \text{ with 2 incoming arcs at } j\}| & \text{if } j \text{ down} \\
 n + 2 - |\{\pi \text{ maximal path in } T \text{ with 2 outgoing arcs at } j\}| & \text{if } j \text{ up}
\end{cases}
\]
CAMBRIAN TREES AND NORMAL CONES

Incidence cone $C(T) = \text{cone } \{ e_i - e_j \mid \text{for all } i \rightarrow j \text{ in } T \}$

Braid cone $C^\diamond(T) = \{ x \in \mathbb{R}^n \mid x_i \leq x_j \text{ for all } i \rightarrow j \text{ in } T \}$

THEO. The cones form complete simplicial fans:

(i) $\{ C^\diamond(\tau) \mid \tau \in \mathfrak{S}_n \}$ = braid fan = normal fan of the permutahedron

(ii) $\{ C^\diamond(T) \mid T \in \text{Camb}(\varepsilon) \}$ = $\varepsilon$-Cambrian fan = normal fan of the $\varepsilon$-associahedron

(iii) $\{ C^\diamond(\chi) \mid \chi \in \pm^{n-1} \}$ = boolean fan = normal fan of the parallelepiped
Incidence cone $C(T) = \text{cone}\{e_i - e_j | \text{for all } i \to j \text{ in } T\}$
Braid cone $C_\diamond(T) = \{x \in \mathbb{H} | x_i \leq x_j \text{ for all } i \to j \text{ in } T\}$

**THEO.** The cones form complete simplicial fans:

(i) $\{C_\diamond(\tau) | \tau \in \mathcal{S}_n\} = \text{braid fan} = \text{normal fan of the permutahedron}$
(ii) $\{C_\diamond(T) | T \in \text{Camb}(\varepsilon)\} = \varepsilon\text{-Cambrian fan} = \text{normal fan of the } \varepsilon\text{-associahedron}$
(iii) $\{C_\diamond(\chi) | \chi \in \pm^{n-1}\} = \text{boolean fan} = \text{normal fan of the parallelepiped}$
CAMBRIAN TREES AND NORMAL CONES

**Incidence cone** \( C(T) = \text{cone} \{ e_i - e_j \mid \text{for all } i \to j \text{ in } T \} \)

**Braid cone** \( \mathcal{C}(\tau) = \{ \mathbf{x} \in \mathbb{H} \mid x_i \leq x_j \text{ for all } i \to j \text{ in } T \} \)

**THEO.** The cones form complete simplicial fans:

(i) \( \{ \mathcal{C}(\tau) \mid \tau \in \mathcal{S}_n \} = \text{braid fan} = \text{normal fan of the permutahedron} \)

(ii) \( \{ \mathcal{C}(T) \mid T \in \text{Camb} (\varepsilon) \} = \varepsilon\text{-Cambrian fan} = \text{normal fan of the } \varepsilon\text{-associahedron} \)

(iii) \( \{ \mathcal{C}(\chi) \mid \chi \in \pm^{n-1} \} = \text{boolean fan} = \text{normal fan of the parallelepiped} \)

Characterization of fibers in terms of cones:

\[ T = P(\tau) \iff C(T) \subseteq C(\tau) \iff \mathcal{C}(T) \supseteq \mathcal{C}(\tau), \]

\[ \chi = \text{can}(T) \iff C(\chi) \subseteq C(T) \iff \mathcal{C}(\chi) \supseteq \mathcal{C}(T), \]

\[ \chi = \text{rec}(\tau) \iff C(\chi) \subseteq C(\tau) \iff \mathcal{C}(\chi) \supseteq \mathcal{C}(\tau). \]
ALGEBRA
For \( n, n' \in \mathbb{N} \), consider the set of perms of \( S_{n+n'} \) with at most one descent, at position \( n \):
\[
S^{(n,n')} := \{ \tau \in S_{n+n'} \mid \tau(1) < \cdots < \tau(n) \text{ and } \tau(n+1) < \cdots < \tau(n+n') \}
\]

For \( \tau \in S_n \) and \( \tau' \in S_{n'} \), define
- shifted concatenation \( \tau \bar{\|} \tau' = [\tau(1), \ldots, \tau(n), \tau'(1)+n, \ldots, \tau'(n')+n] \in S_{n+n'} \)
- shifted shuffle product \( \tau \bar{\ll} \tau' = \{ \tau \circ \pi^{-1} \mid \pi \in S^{(n,n')} \} \subset S_{n+n'} \)
- convolution product \( \tau \star \tau' = \{ \pi \circ (\tau \bar{\|} \tau') \mid \pi \in S^{(n,n')} \} \subset S_{n+n'} \)

Exm: \(12 \bar{\|} 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\}\)
\(12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}\)
**MALVENUTO-REUTENAUER ALGEBRA**

**DEF.** Combinatorial Hopf Algebra = combinatorial vector space $\mathcal{B}$ endowed with

- **product** $\cdot : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$
- **coproduct** $\triangle : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$

which are “compatible”, ie.

$$
\begin{array}{ccc}
\mathcal{B} \otimes \mathcal{B} & \cdot & \mathcal{B} \\
\downarrow \triangle \otimes \triangle & & \uparrow \cdot \otimes \cdot \\
\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} & \rightarrow & \mathcal{B} \otimes \mathcal{B} \\
& I \otimes \text{swap} \otimes I &
\end{array}
$$

**Malvenuto-Reteunauer algebra** = Hopf algebra $FQSym$ with basis $(\mathcal{F}_\tau)_{\tau \in S}$ and where

$$
\mathcal{F}_\tau \cdot \mathcal{F}_{\tau'} = \sum_{\sigma \in \tau \sqcup \tau'} \mathcal{F}_\sigma \quad \text{and} \quad \triangle \mathcal{F}_\sigma = \sum_{\sigma \in \tau \ast \tau'} \mathcal{F}_\tau \otimes \mathcal{F}_{\tau'}
$$
For signed permutations:

- signs are attached to values in the shuffle product
- signs are attached to positions in the convolution product

Exm:  
\[
\begin{align*}
\overline{12} \shuffle \overline{231} &= \{ \overline{12453}, \overline{14253}, \overline{14523}, \overline{14532}, \overline{41253}, \overline{41523}, \overline{41532}, \overline{45123}, \overline{45132}, \overline{45312} \}, \\
\overline{12} \ast \overline{231} &= \{ \overline{12453}, \overline{13452}, \overline{14352}, \overline{15342}, \overline{23451}, \overline{24351}, \overline{25341}, \overline{34251}, \overline{35241}, \overline{45231} \}.
\end{align*}
\]

\[
\text{concatenation} \quad \text{shuffle} \quad \text{convolution}
\]

\[
F_{\text{QSym}} = \text{Hopf algebra with basis } (F_\tau)_{\tau \in S_\pm} \text{ and where }
\]

\[
F_\tau \cdot F_\tau' = \sum_{\sigma \in \tau \shuffle \tau'} F_\sigma \quad \text{and} \quad \Delta F_\sigma = \sum_{\sigma \in \tau \ast \tau'} F_\tau \otimes F_\tau'
\]
Cambrian algebra = vector subspace Camb of $\text{FQSym}_\pm$ generated by

\[
P_T := \sum_{\tau \in \mathcal{G}_\pm \atop P(\tau) = T} F_{\tau} = \sum_{\tau \in \mathcal{L}(T)} F_{\tau},
\]

for all Cambrian trees $T$.

Exm:

\[
P = \begin{array}{c}
F_{2137546} + F_{2173546} + F_{2175346} + F_{2713546} + F_{2715346} \\
+ F_{2751346} + F_{7213546} + F_{7215346} + F_{7251346} + F_{7521346}
\end{array}
\]

THEO. Camb is a subalgebra of $\text{FQSym}_\pm$

(ie. the Cambrian congruence is “compatible” with the product and coproduct in $\text{FQSym}_\pm$

GAME: Explain the product and coproduct directly on the Cambrian trees...
PROP. For any Cambrian trees $T$ and $T'$,

$$\mathcal{P}_T \cdot \mathcal{P}_{T'} = \sum_{S} \mathcal{P}_S$$

where $S$ runs over the interval $[T ↘ \bar{T}', T ↖ \bar{T}']$ in the $\varepsilon(T)\varepsilon(T')$-Cambrian lattice.
\[ \Delta P = \Delta (F_{213} + F_{231}) \]

\[ = 1 \otimes (F_{213} + F_{231}) + F_1 \otimes F_{12} + F_1 \otimes F_{21} + F_{21} \otimes F_1 + F_{12} \otimes F_1 + (F_{213} + F_{231}) \otimes 1 \]

\[ = 1 \otimes P + P \otimes P + P \otimes P + P \otimes P + P \otimes P + \text{other terms} \]

\[ = 1 \otimes P + P \otimes (P \cdot P) + P \otimes P + P \otimes P + P \otimes P + P \otimes 1. \]

**PROP.** For any Cambrian tree \( S \),

\[ \Delta P_S = \sum_{\gamma} \left( \prod_{T \in B(S, \gamma)} P_T \right) \otimes \left( \prod_{T' \in A(S, \gamma)} P_{T'} \right) \]

where \( \gamma \) runs over all cuts of \( S \), and \( A(S, \gamma) \) and \( B(S, \gamma) \) denote the Cambrian forests above and below \( \gamma \) respectively.
\[ \Delta \mathcal{P} = \Delta (F_{213} + F_{321}) \]

\[ = 1 \otimes (F_{213} + F_{321}) + F_{1} \otimes F_{12} + F_{1} \otimes F_{21} + F_{21} \otimes F_{1} + F_{12} \otimes F_{1} + (F_{213} + F_{321}) \otimes 1 \]

\[ = 1 \otimes \mathcal{P} + \mathcal{P} \otimes \mathcal{P} + \mathcal{P} \otimes \mathcal{P} + \mathcal{P} \otimes \mathcal{P} + \mathcal{P} \otimes \mathcal{P} + \mathcal{P} \otimes \mathcal{P} \]

\[ = 1 \otimes \mathcal{P} + \mathcal{P} \otimes (\mathcal{P} \cdot \mathcal{P}) + \mathcal{P} \otimes \mathcal{P} + \mathcal{P} \otimes \mathcal{P} + \mathcal{P} \otimes \mathcal{P} + \mathcal{P} \otimes \mathcal{P} \]

**PROP.** For any Cambrian tree \( S \),

\[ \Delta \mathcal{P}_S = \sum_{\gamma} \left( \prod_{T \in B(S, \gamma)} \mathcal{P}_T \right) \otimes \left( \prod_{T' \in A(S, \gamma)} \mathcal{P}_{T'} \right) \]

where \( \gamma \) runs over all cuts of \( S \), and \( A(S, \gamma) \) and \( B(S, \gamma) \) denote the Cambrian forests above and below \( \gamma \) respectively.
DUAL CAMBRIAN ALGEBRA AS QUOTIENT OF FQSym* 

\[ \text{FQSym}_\pm^* = \text{dual Hopf algebra with basis } (G_\tau)_{\tau \in \mathcal{G}_\pm} \quad \text{and where} \]

\[ G_\tau \cdot G_{\tau'} = \sum_{\sigma \in \tau \ast \tau'} G_\sigma \quad \text{and} \quad \Delta G_\sigma = \sum_{\sigma \in \tau \sqcup \tau'} G_\tau \otimes G_{\tau'} \]

**PROP.** The graded dual Camb* of the Cambrian algebra is isomorphic to the image of FQSym* under the canonical projection

\[ \pi : \mathbb{C}\langle A \rangle \longrightarrow \mathbb{C}\langle A \rangle / \equiv, \]

where \( \equiv \) denotes the Cambrian congruence. The dual basis \( Q_T \) of \( P_T \) is expressed as \( Q_T = \pi(G_\tau) \), where \( \tau \) is any linear extension of \( T \).
product in dual cambrian algebra

\[ Q \cdot Q = G_{12} \cdot G_{213} \]
\[ = G_{12}135 + G_{13}125 + G_{14}152 + G_{15}124 + G_{23}415 + G_{24}135 + G_{25}314 + G_{34}215 + G_{35}214 + G_{45}123 \]
\[ = Q + Q + Q + Q + Q + Q + Q + Q + Q + Q \]

**PROP.** For any Cambrian trees \( T \) and \( T' \),

\[ QT \cdot QT' = \sum_s QT_s T' \]

where \( s \) runs over all shuffles of \( \varepsilon(T) \) and \( \varepsilon(T') \)
PROP. For any Cambrian trees $T$ and $T'$,

$$Q_T \cdot Q_{T'} = \sum_s Q_{T s T'}$$

where $s$ runs over all shuffles of $\varepsilon(T)$ and $\varepsilon(T')$
\[
\mathbb{Q} \cdot \mathbb{Q} = G_{12} \cdot G_{213} = G_{12} + G_{13} + G_{1325} + G_{1324} + G_{2315} + G_{234} + G_{25314} + G_{34215} + G_{35214} + G_{45213}
\]

\[
= \mathbb{Q} + \mathbb{Q} + \mathbb{Q} + \mathbb{Q} + \mathbb{Q} + \mathbb{Q} + \mathbb{Q} + \mathbb{Q} + \mathbb{Q} + \mathbb{Q}
\]

**PROP.** For any Cambrian trees \( T \) and \( T' \),

\[
\mathbb{Q}_T \cdot \mathbb{Q}_{T'} = \sum_s \mathbb{Q}_{T s T'}
\]

where \( s \) runs over all shuffles of \( \varepsilon(T) \) and \( \varepsilon(T') \)
\[ Q \cdot Q = G_{12} \cdot G_{213} \]
\[ = G_{12435} + G_{13425} + G_{14325} + G_{15324} + G_{23415} + G_{24315} + G_{25314} + G_{34215} + G_{35214} + G_{45213} \]

\[ = Q + Q + Q + Q + Q + Q + Q + Q + Q + Q \]

**PROP.** For any Cambrian trees \( T \) and \( T' \),
\[ Q_T \cdot Q_{T'} = \sum_s Q_{T_s T'} \]
where \( s \) runs over all shuffles of \( \varepsilon(T) \) and \( \varepsilon(T') \)
**PRODUCT IN DUAL CAMBRIAN ALGEBRA**

\[
\begin{align*}
Q_1 \cdot Q_2 &= G_{12} \cdot G_{213} \\
&= G_{12} + G_{13} + G_{1325} + G_{14325} + G_{15324} + G_{23415} + G_{24315} + G_{25314} + G_{34215} + G_{35214} + G_{45213} \\
&= Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 + Q_7 + Q_8 + Q_9 + Q_{10}
\end{align*}
\]

**PROP.** For any Cambrian trees \( T \) and \( T' \),

\[
Q_T \cdot Q_{T'} = \sum_s Q_{T_s T'}
\]

where \( s \) runs over all shuffles of \( \varepsilon(T) \) and \( \varepsilon(T') \)
\[ \Delta Q = \Delta G_{213} = 1 \otimes G_{213} + G_1 \otimes G_{T2} + G_{T1} \otimes G_1 + G_{213} \otimes 1 = 1 \otimes Q + Q \otimes Q + Q \otimes Q + Q \otimes 1. \]

**PROP.** For any Cambrian tree \( S \),

\[ \Delta Q_S = \sum_{\gamma} Q_{L(S, \gamma)} \otimes Q_{R(S, \gamma)} \]

where \( \gamma \) runs over all gaps between vertices of \( S \), and \( L(S, \gamma) \) and \( R(S, \gamma) \) denote the Cambrian trees left and right to \( \gamma \) respectively.
COPRODUCT IN DUAL CAMBRIAN ALGEBRA

\[ \triangle Q = \triangle G_{213} \]

\[ = 1 \otimes G_{213} + G_1 \otimes G_{214} + G_{214} \otimes G_1 + G_{213} \otimes 1 \]

\[ = 1 \otimes Q + Q \otimes Q + Q \otimes Q + Q \otimes Q \otimes 1. \]

**PROP.** For any Cambrian tree $S$,

\[ \triangle Q_S = \sum_{\gamma} Q_{L(S,\gamma)} \otimes Q_{R(S,\gamma)} \]

where $\gamma$ runs over all gaps between vertices of $S$, and $L(S,\gamma)$ and $R(S,\gamma)$ denote the Cambrian trees left and right to $\gamma$ respectively.
COPRODUCT IN DUAL CAMBRIAN ALGEBRA

\[ \Delta Q = \Delta G_{213} = 1 \otimes G_{213} + G_1 \otimes G_{T2} + G_{21} \otimes G_T + G_{213} \otimes 1 \]

\[ = 1 \otimes Q + Q \otimes Q + Q \otimes Q + Q \otimes 1. \]

**PROP.** For any Cambrian tree \( S \),

\[ \Delta Q_S = \sum_{\gamma} Q_{L(S, \gamma)} \otimes Q_{R(S, \gamma)} \]

where \( \gamma \) runs over all gaps between vertices of \( S \), and \( L(S, \gamma) \) and \( R(S, \gamma) \) denote the Cambrian trees left and right to \( \gamma \) respectively.
MULTIPLICATIVE BASES

Define

\[ E_T := \sum_{T \leq T'} P_{T'} \quad \text{and} \quad H_T := \sum_{T' \leq T} P_{T'} . \]

**PROP.** \((E^T)_{T \in \text{Camb}} \quad \text{and} \quad (H^T)_{T \in \text{Camb}} \) are multiplicative bases of Camb, i.e.

\[ E^T \cdot E^{T'} = E^{T \uparrow T'} \quad \text{and} \quad H^T \cdot H^{T'} = H^{T \setminus T'} . \]
INDECOMPOSABLE ELEMENTS

PROP. The following properties are equivalent for a Cambrian tree $S$:

- $E^S$ can be decomposed into a product $E^S = E^T \cdot E^{T'}$ for non-empty $T, T'$
- $([k] \parallel [n] \setminus [k])$ is an edge cut of $S$ for some $k \in [n]$
- at least one linear extension $\tau$ of $S$ is decomposable, i.e. $\tau([k]) = [k]$ for some $k \in [n]$

The tree $S$ is then called $E$-decomposable
PROP. For any signature $\varepsilon \in \pm^n$, the set of $\mathbb{E}$-indecomposable $\varepsilon$-Cambrian trees forms a principal upper ideal of the $\varepsilon$-Cambrian lattice.

PROP. For any signature $\varepsilon \in \pm^n$, there are $C_{n-1}$ $\mathbb{E}$-indecomposable $\varepsilon$-Cambrian trees. Therefore, there are $2^n C_{n-1}$ $\mathbb{E}$-indecomposable Cambrian trees on $n$ vertices.
Extend combinatorial, geometric and algebraic properties of binary trees to further families of trees...
THANK YOU