Static Analysis of Numerical Algorithms

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Goal of the talk: characterize the loss of precision in programs, due to floating-point arithmetic, at compile time

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  - Join and meet operations, order
- Relational domain for values and errors, main ideas

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- Relational domain for values and errors, main ideas
- Example based on an extract from instrumentation software

Static analysis

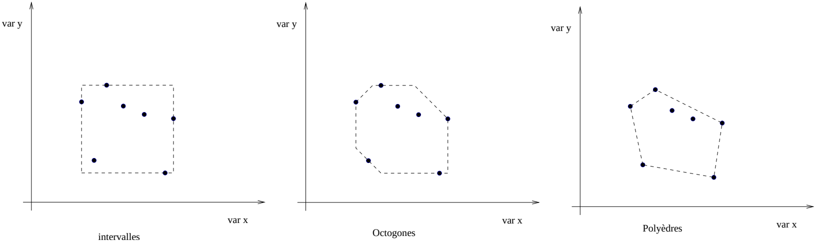
- A program is considered as a dynamical system (discrete in general)
- We can be interested in two main types of properties:
  - *safety*, through invariant true on all trajectories - for all inputs or parameters. Application: give bounds for variables, prove absence of RTEs etc.
  - *liveness* properties which become true at a certain time, on one or all of the trajectories. Application: reachability of a state, termination etc.

Similarity with certain concepts (and methods) of numerical mathematics and control theory.

Theory and tools for *automatic* analysis of such properties, given a program

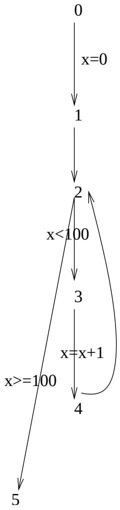
But automatic (or algorithmic) means...

...undecidability (ex. Turing halting problem). So we use abstractions to find over-approximations of these sets of values (sometimes under-approximations too).



$\longrightarrow$ abstract interpretation

```
void main() { [0]
int x=[-100,50]; [1]
while [2] (x<100) {
[3]
x=x+1; [4]
} [5]
}
```



$$
\begin{aligned}
x_0 &= \top \\
x_1 &= [-100, 50] \\
x_2 &= x_1 \cup x_4 \\
x_3 &=]-\infty, 99] \cap x_2 \\
x_4 &= x_3 + [1, 1] \\
x_5 &= [100, +\infty[\cap x_2
\end{aligned}
$$

Resolution of semantic equations

- (Tarsky) $(\wp(\mathbb{Z}), \subseteq)$ (similarly, intervals) is a complete lattice and the functional is monotonic $\Rightarrow$ there is a least fixed point

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- (Tarsky) $(\wp(\mathbb{Z}), \subseteq)$ (similarly, intervals) is a complete lattice and the functional is monotonic $\Rightarrow$ there is a least fixed point
- We compute the Kleene iteration (f is actually order-theoretically continuous here)

$$lfp(f) = \bigsqcup_{n \in \mathbb{N}} f^n(\perp)$$

for the functional:

$$F \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \top \\ [-100, 50] \\ x_1 \cup x_4 \\ ]-\infty, 99] \cap x_2 \\ x_3 + [1, 1] \\ [100, +\infty[\cap x_2 \end{pmatrix}$$

Iteration 1

```
void main() { [0]
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$$
\begin{aligned}
x_0^1 &= \top \\
x_1^1 &= [-100, 50] \\
x_2^1 &= [-100, 50] \\
x_3^1 &=]-\infty, 99] \cap [-100, 50] \\
&= [-100, 50] \\
x_4^1 &= [-100, 50] + [1, 1] \\
&= [-99, 51] \\
x_5^1 &= [100, +\infty[\cap [-100, 50] \\
&= \bot
\end{aligned}
$$

(choatic iteration here/Gauss-Seidel like)

Iteration 50

```
void main() { [0]
int x=[-100,50]; [1]
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x=x+1; [4]
} [5]
}
```

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x_2^{100} &= [-100, 100] \\
x_2^{100} &=]-\infty, 99] \cap ([-100, 100]) \\
&= [-100, 99] \\
x_3^{100} &= [-100, 99] + [1, 1] \\
&= [-99, 100] \\
x_4^{100} &= [100, +\infty[\cap ([-99, 100] \\
&= [100, 100]
\end{aligned}
$$

Of course this is naive: acceleration of convergence, relational domains etc.

Context of the present work

- Static analysis by abstract interpretation for inaccuracy errors in floating-point computations (FLUCTUAT tool)
  - Follows the floating-point control flow (given an evaluation order!)
  - Guaranteed bounds on errors between real number computation (what is expected) and the implementation in floating-point numbers
  - Identify operations responsible for the accuracy losses
- Applications
  - Safety-critical instrumentation software
  - Towards numerically more intensive programs
- Need for a very accurate real number value analysis

Representation of values (concrete)

The set of floating-point values that a variable x can take is expressed as:

$$\begin{aligned} f^x &= r^x + e_1^x + e_{ho}^x \\ &= r^x + \bigoplus_{i \in I} \alpha_i^x + e_{ho}^x \end{aligned}$$

where:

- r^x is the real-number value that should have been computed if we had exact arithmetic available
- the α_i^x are coefficients expressing the propagation in x of the initial first-order error introduced by the arithmetic operation labelled i in the program
- e_{ho}^x is the higher-order error

Example

```
float x = 0.1; // [1]
float y = 0.5; // [2]
float z = x+y; // [3]
float t = x*y; // [4]
```

$$
\begin{aligned}
x &= 0.1 + 1.49011612e^{-9}\ [1] \\
y &= 0.5 \\
z &= 0.6 + 1.49011612e^{-9}\ [1] + \\
&\ 2.23517418e^{-8}\ [3] \\
t &= 0.06 + 1.04308132e^{-9}\ [1] \\
&\ + 2.23517422e^{-9}\ [3] \\
&\ - 8.94069707e^{-10}\ [4] \\
&\ - 3.55271366e^{-17}\ [ho]
\end{aligned}
$$

Abstraction

- First natural idea: use interval arithmetic for coefficients r^x, α_i^x and e_{ho}^x
- Rounding errors (α_i^x) given by the IEEE 754 standard:
  - in general, an interval of width $\texttt{ulp}(x)$ when x is not just a singleton
- But of course, we run into dependency problems, wrapping effect

Specificities

Each variable of a program has values given as a function (at some control point)

$$g(r^{x_1}, \ldots, r^{x_k}, e^{x_1}, \ldots, e^{x_k})$$

where r^{x_i} and e^{x_i} are respectively the enclosure of the real number values, and of the inaccuracy error, of variables x_i

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- non-continuity of g in general (`if` statements) - "unstable" tests
- g can be $>$ 100KLoC, with $>$ 10K variables
- g is constructed on the fly (part of the analysis is actually to find g!)
    - interprocedural calls, depending on context
    - aliases between variables, to be discovered
- we are looking for *invariant sets* of g in a large space of values, if possible, or else the result of an iteration of g over a long period of time
- hence computations in an algebra with union and intersection operations as well

Not only do we have uncertain rounding errors but...

...there are in fact two kinds of uncertainties to propagate:

- Uncertainties on the initial values of the variables (which represent inputs to the program) or uncertainties on the parameters of the program (the implemented model)
    - a priori large intervals [given through user-defined assertions]
- Rounding errors, deterministic but only known in general as belonging to some interval
    - a priori much smaller intervals

Abstraction of the real number computation

Recall that:

$$
\begin{aligned}
f^x &= r^x + e_1^x + e_{ho}^x \\
&= r^x + \bigoplus_{i \in I} \alpha_i^x + e_{ho}^x
\end{aligned}
$$

- We use some form of affine arithmetic for r^x (and for the errors too as we shall see)
- We can refine further the floating-point enclosure, using *error on bounds*

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- Consider:
  x=[0,1]*[0,1];
  - Error is in $[-\frac{\mathtt{ulp}(1)}{2}, \frac{\mathtt{ulp}(1)}{2}]$ for any value of $\mathtt{x}$ (this is accounted for by terms e_1^x and e_{ho}^x)
  - But the error is null on $\mathtt{x}$=0 and $\mathtt{x}$=1

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  - Error is in [-$\frac{\mathtt{ulp}(1)}{2}$,$\frac{\mathtt{ulp}(1)}{2}$] for any value of `x` (this is accounted for by terms e_1^x and e_{ho}^x)
  - But the error is null on `x`=0 and `x`=1
- Hence we maintain a correction on bounds (δ_-^x, δ_+^x) which controls a potential drift of the bounds
  - we compute r^x, then the real number enclosure of $r^x + e_1^x + e_{ho}^x$
  - then we round these bounds and deduce (δ_-^x, δ_+^x) and the new first-order error
- The enclosure is then of the form is $[\mathit{inf}\ \ r^x + \delta_-^x, \mathit{sup}\ \ r^x + \delta_+^x]$

Affine Arithmetic for real number computation (r^x)

Proposed in 93 by Comba, De Figueiredo and Stolfi as a more accurate extension of Interval Arithmetic

- *Assignment* of a of a variable x whose value is given in a range $[a, b]$ at label i, introduces a noise symbol ε_i :

$$\hat{x} = \frac{(a+b)}{2} + \frac{(b-a)}{2}\varepsilon_i .$$

- *Addition* of affine forms is computed componentwise:

$$\hat{x} + \hat{y} = (\alpha_0^x + \alpha_0^y) + (\alpha_1^x + \alpha_1^y)\varepsilon_1 + \ldots + (\alpha_n^x + \alpha_n^y)\varepsilon_n$$

- *Multiplication* : we select an approximate linear form, the approximation error creates a new noise term :

$$\hat{x} \times \hat{y} = \alpha_0^x\alpha_0^y + \sum_{i=1}^{n}(\alpha_i^x\alpha_0^y + \alpha_i^y\alpha_0^x)\varepsilon_i + (\sum_{i=1}^{n}|\alpha_i^x|.|\sum_{i=1}^{n}|\alpha_i^y|)\varepsilon_{n+1}.$$

(can be improved, in particular with SDP)

Interval affine arithmetic

- The analyzer represents the real coefficients α_i^x by small intervals with MPFR bounds
- When the width of such intervals gets larger, we use new noise symbols
- Extended abstract domain $\mathbb{AI}$ $\hat{x} = \boldsymbol{\alpha_0^x} + \boldsymbol{\alpha_1^x}\varepsilon_1 + \ldots + \boldsymbol{\alpha_n^x}\varepsilon_n$ with $\boldsymbol{\alpha_0^x} \in \mathbb{IR}$ and $\boldsymbol{\alpha_i^x} \in \mathbb{IR}$ $(i > 0)$

Join (and meet) operations on affine forms

- A natural join between $\hat{r^x}$ and $\hat{r^y}$ is

$$\hat{r}^{x \cup y} = \boldsymbol{\alpha_0^x} \cup \boldsymbol{\alpha_0^y} + \sum_{i \in L} (\boldsymbol{\alpha_i^x} \cup \boldsymbol{\alpha_i^y})\, \varepsilon_i \qquad (1)$$

  Result might be greater than the union of enclosing intervals (partly corrected by the (δ_-^x, δ_+^x)).
- But with interval coefficients $\hat{r}^{x \cup y} - \hat{r}^{x \cup y} \neq 0$!

Join (and meet) operations on affine forms

For an interval $\boldsymbol{i}$, we note

$$\text{mid}(\boldsymbol{i}) = \frac{\underline{i} + \overline{i}}{2}, \quad \text{dev}(\boldsymbol{i}) = \overline{i} - \text{mid}(\boldsymbol{i})$$

the center and deviation of the interval.

- A better join is

$$\hat{r}^{x \cup y} = \text{mid}([\alpha_0^x, \alpha_0^y]) + \sum_{i \in L} \text{mid}([\alpha_i^x, \alpha_i^y])\, \varepsilon_i + \sum_{i \geq 0} \text{dev}([\alpha_i^x, \alpha_i^y])\, \varepsilon_k^u \tag{2}$$

- Then we have affine forms with real coefficients again
- Order on affine forms considers noise symbols due to join operations differently than noise symbols due to arithmetic operations

Example (join)

$$\text{Let } \hat{r^x} = 1 + 2\varepsilon_1 + \varepsilon_2 \text{ and } \hat{r^y} = 2 - \varepsilon_1.$$

- Join on intervals $r^x \cup r^y \in [-2, 4]$
- First join on affine forms

$$\hat{r}^{x \cup y} = [1, 2] + [-1, 2]\varepsilon_1 + [0, 1]\varepsilon_2 \subset [-2, 5]$$

  (larger enclosure than on intervals but still interesting for further computations to keep relations, over-approximation compensated by (δ_-^x, δ_+^x))
- Second join on affine forms

$$\hat{r}^{x \cup y} = 1.5 + 0.5\varepsilon_1 + 0.5\varepsilon_2 + 2.5\varepsilon_3^u \subset [-2, 5]$$

  Same enclosure in this case, but above all $\hat{r}^{x \cup y} - \hat{r}^{x \cup y} = 0$

(Ongoing work on good join and meet operators, order on affine forms, widening and fixpoint computations)

First-order errors

Also represented in affine arithmetic (with other noise symbols):

$$e_1^x = \bigoplus_{l \in L_2} t'^x_l \, \eta_l$$

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(Notice: values [large intervals] are considered to be of order 0)

Higher-order error terms

- The multiplication of errors introduce higher-order error terms, which are modelled in the following manner:

$$e^x_{ho} = (t^x_h + \bigoplus_{l \in L_2} {t'}^x_{h,l}\, \eta_l + \bigoplus_{i \in I} {t''}^x_{h,i}\, \varepsilon_i + \bigoplus_{p \in P} \beta^x_{h,p}\, \vartheta_p).$$

Newton method (non-linear) for the "inverse"

```
double xi, xsi, A, temp;
signed int *PtrA, *Ptrxi, cond, exp, i;
A = __BUILTIN_DAED_DBETWEEN(20.0,30.0);
/* inverse power of 2 closest to A */
PtrA = (signed int *) (&A);
Ptrxi = (signed int *) (&xi);
exp = (signed int) ((PtrA[0] & 0x7FF00000) >> 20) - 1023;
xi = 1; Ptrxi[0] = ((1023-exp) << 20);
cond = 1; i = 0;
while (abs(temp)>e-8) {
  xsi = 2*xi-A*xi*xi;
  temp = xsi-xi;
  xi = xsi;
  i++;  }
```

Computation of the inverse

- Symbolic execution:
  - Input = 20.0 : i = 5, xi = 5.000000e-2 + [-2.81893e-18,-2.76471e-18]
  - Output = 30.0 : i = 9, xi = 3.333333e-2 + [-5.28429e-18,6.21309e-18]
- With intervals
  - does not converge, even when subdividing
- With the relational model, finds $i \in [5, 9]$ for input $A \in [20, 30]$ (with subdivisions)

A closest look at results (relational)

Input plus initial error [20,20.001] + [-1e-05,1e-05]:

- (0.03 sec, 4.1M) :
  - `xi` in [4.999750e-2,5.000000e-2] + [-2.68644e-08,2.68644e-08]
  - `temp=xsi-xi` in [-5.06890974e-9,5.06891107e-9] + [-1.89053e-09,1.89053e-09] (the precise estimate of the error allows for a precise computation of the floating-point value)

For larger value domains: subdivision.

Example : second-order filter

A new independent input `E` at each iteration of the filter:

```
double S,S0,S1,E,E0,E1;
int i;

S=0.0; S0=0.0;
E=__BUILTIN_DAED_DBETWEEN(0,1.0);
E0=__BUILTIN_DAED_DBETWEEN(0,1.0);

for (i=1;i<=170;i++) {
  E1 = E0;
  E0 = E;
  E = __BUILTIN_DAED_DBETWEEN(0,1.0);
  S1 = S0;
  S0 = S;
  S = 0.7 * E - E0 * 1.3 + E1 * 1.1 + S0 * 1.4 - S1 * 0.7 ;
}
```

Second-order filter

- Relational analysis on values and errors
  - with the default precision of the analysis (60 bits) :
    `S` in [-4.e26,4.e26], error [-5.e+11,5.e+11] in 5.1 sec, 25M
  - with 200 bits:
    `S` in [-1.09,2.76], error [-1.1e-14,1.1e-14] in 5.2 sec, 27M

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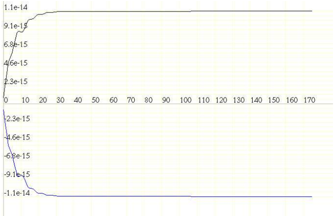
(Notice the importance of using MPFR for representing the coefficients in the relational model)

Values and errors stabilized with MPFRbits=200

Values in [-1.09,2.76]



Error in [-1.1e-14,1.1e-14]

Propagation of an error on the input:

- Each input has now an error in [0,0.001]
- Relational on errors : `S` in [-1.09,2.76], with a stabilized error in [-0.00109,0.00276]

Current research

- For embedded systems:
  - the integrators (and everything built on that, i.e. PID controllers): probabilistic methods, CVFs?
  - More generally, analysis of hybrid systems, i.e. systems combining the discrete semantics of the program with a system of PDEs/ODEs for the continuous physical environment (see O. Bouissou's talk) - see ERTS'06, SCAN'06
  - Analysis of code/specification in MatLab/Simulink [fragment]
- Scientific codes: analysis of the methods to solve the linear equations (i.e. conjugate gradient etc.) used for instance when solving PDEs by a finite element method
- General improvements:
  - Computation of under-approximations as well $\longrightarrow$ show the quality of the results
  - Improvement of the resolution of the semantic equations by policy iteration; faster and better precision, incremental analysis etc. See CAV'05, ESOP'07