Robust under-approximations and application to reachability of non-linear control systems with disturbances

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Abstract—**We describe a set-based approach, relying on mean-value extensions, for computing guaranteed under-approximations of ranges (or images) of continuously differentiable functions f from $\mathbb{R}^m$ to $\mathbb{R}^n$, including what we call robust ranges, i.e. ranges of functions under adversarial uncertainties. Our method is capable of computing efficiently, at a low computational cost, full n-dimensional subsets of the image of f. As an application, we show how to compute under-approximations of robust reachable sets of non-linear controlled dynamical systems under time-varying uncertainties, which is central to many verification problems in control theory.**

Index Terms—**Uncertain systems, Computer-aided control design**

I. Introduction

COMPUTING the set of values a function can reach for some input domain is central to many problems in control, such as robust control of dynamical systems, or in global optimization. Computing the exact image of a domain by a function is intractable in general. This is all the more true when this function is the flow of a continuous or hybrid system. Moreover, for general controlled systems, the reachability properties will depend on the initial conditions of the system, but also on the sensitivity of the system to some control inputs and external disturbances, as reflected by the notions of minimal and maximal reachability [1], which we generalize here to robust reachability, when both control inputs and adversarial disturbances are present.

Most existing approximation techniques, often based on extensions of interval methods, compute over-approximations of images or reachable sets. We are interested here in the much less studied problem of computing under-approximations, that is sets of states guaranteed to be reached for some inputs. When the over-approximation is not sufficient to prove a property, computing in addition an under-approximation is helpful to state the quality of the over-approximation. Additionally, when an under-approximation of the reachable set intersects the set of error states, it provides counter-examples to the property, by proving that error states are actually reached.

We start Section II by recalling the general formulation of function image under disturbances. Solving the under and over approximation problem for robust ranges of functions relies mostly on mean-value theorems, revisiting some work of A. Goldsztejn on modal intervals [2], [3]. With respect to our previous work on the under-approximation of reachable sets for control systems [4], [5], we develop a systematic way to under-approximate the image of vector-valued functions f and no longer just projections of f, without resorting to computer intensive set inversion methods [6]. This is necessary to be able to verify more general properties of control systems. In Section III, these results are used, combined with a Taylor-based over-approximating reachability analysis, for computing under-approximations of reachable sets of non-linear continuous systems, with controls and time-varying uncertainties. Section IV discusses a variety of examples and results obtained with our prototype RINO.

Related work: Our approach is related to over-approximations of non-linear continuous and controlled systems, since we rely on such over-approximations to compute under-approximations. Many methods for over-approximating reachable sets have been developed, among which Taylor methods [7] or polytopes [8].

There are far less methods for under-approximating images of functions or sets of reachable states. Interval-based methods, relying on space discretization, have been used for under-approximating the image of functions [9]. They were also used to over and under approximate solutions of differential systems with uncertain initial conditions [10]. Our approach is directly linked to previous work on modal intervals and mean-value theorems [2], [3].

Tight approximations for reachable sets of continuous systems can be found via expensive Eulerian methods: the zero sub-level set of the Lipschitz viscosity solution to a Hamilton-Jacobi (HJB) partial differential equation gives the (backward) reachable set [11]. We hence refer to some of their examples in Section IV as grounds for asserting the precision of our analyses. Some decomposition methods have been designed to reduce the curse of dimensionality for HJB [12], [11], which is solved in general by finite difference or finite elements methods, involving gridding of the state space. HJB has also been used for under-approximations with time-varying uncertainties [14], using LMI relaxations to reduce the computational cost. Other approaches using SoS methods and LMI relaxations have been proposed for inner approximations [15].

In comparison, our method is a fairly inexpensive Lagrangian method. Taylor models are also used on the inverse flow map to derive inner-approximations [16], but

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using topological conditions that are checked with interval constraints solving, which have difficulties to scale up with dimension. Our work includes the calculation of under and over approximations of robust reachable sets as defined in e.g. [5], with time-varying inputs and disturbances. Other methods for under-approximating reachable sets include [17] and [13] for robust under-approximations, based on an analysis of the boundary of the reachable sets. Finally, any method for integrating differential inclusions with error bounds, such as [18] could be used to derive inner-approximations of reachable sets of differential inclusions.

II. Mean-value AE extensions

The results of this section are inspired by work on modal intervals [2], [3]. We avoid for simplicity this formalism.

Notations and Preliminaries: For a vector-valued function $f : \mathbb{R}^m \to \mathbb{R}^n$, we note f_i its i-th component and $\nabla f = (\nabla_j f_i)_{ij} = (\frac{\partial f_i}{\partial x_j})_{1 \le i \le n, 1 \le j \le m}$ its Jacobian matrix. We note $\langle x, y \rangle$ the scalar product of vectors x and y. Set valued quantities, scalar or vector valued will be noted with bold letters, e.g $\boldsymbol{x}$, throughout the paper. An *over-approximating extension*, also called *outer-approximating extension*, of a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is a function $\boldsymbol{f} : \mathcal{P}(\mathbb{R}^m) \to \mathcal{P}(\mathbb{R}^n)$, such that for all $\boldsymbol{x}$ in $\mathcal{P}(\mathbb{R}^m)$, $\text{range}(f, \boldsymbol{x}) = \{f(x), x \in \boldsymbol{x}\} \subseteq \boldsymbol{f}(\boldsymbol{x})$. Dually, under-approximations determine a set of values proved to belong to the range of the function over some input set. An *under-approximating extension*, also called *inner-approximating extension*, of f is a function $\boldsymbol{f} : \mathcal{P}(\mathbb{R}^m) \to \mathcal{P}(\mathbb{R}^n)$, such that for all $\boldsymbol{x}$ in $\mathcal{P}(\mathbb{R}^m)$, $\boldsymbol{f}(\boldsymbol{x}) \subseteq \text{range}(f, \boldsymbol{x})$. Under- and over-approximations can be interpreted as quantified propositions: $\text{range}(f, \boldsymbol{x}) \subseteq \boldsymbol{z}$ can be written $\forall x \in \boldsymbol{x}$, $\exists z \in \boldsymbol{z}$, $f(x) = z$, while $\boldsymbol{z} \subseteq \text{range}(f, \boldsymbol{x})$ can be written $\forall z \in \boldsymbol{z}$, $\exists x \in \boldsymbol{x}$, $f(x) = z$. Both these propositions are what we will call throughout the paper *AE propositions*, for quantified propositions where universal quantifiers (A) precede existential quantifiers (E).

Intervals are used in many situations to rigorously compute with interval domains instead of reals, usually leading to over-approximations of function ranges over boxes. We denote $\mathbb{IR} = \{\boldsymbol{x} = [\underline{x}, \overline{x}]$, $\underline{x} \in \mathbb{R}$, $\overline{x} \in R$, the set of intervals with real-valued bounds. If $\overline{x} < \underline{x}$, then the corresponding interval represents the empty set.

A. Mean-value AE extensions for real-valued functions

We consider in this section a function $f : \mathbb{R}^m \to \mathbb{R}$. The natural interval extension consists in replacing real operations by their interval counterparts in the expression of the function. A generally more accurate extension relies on a linearization by the mean-value theorem.

1) Mean-value AE extensions: Suppose f is differentiable over the box $\boldsymbol{x}$. The mean-value theorem implies that $\forall x^0 \in \boldsymbol{x}$, $\forall x \in \boldsymbol{x}, \exists \xi \in \boldsymbol{x}$, $f(x) = f(x^0) + \langle \nabla f(\xi), x - x^0 \rangle$. If we can bound the range of absolute value of the gradient of f over $\boldsymbol{x}$, by $\boldsymbol{\nabla f}(\boldsymbol{x})$, then we can derive an interval enclosure, called the mean-value extension. Let us choose x^0 to be the center $c(\boldsymbol{x}) = (\overline{x} + \underline{x})/2$ of $\boldsymbol{x}$ and note $r(\boldsymbol{x}) = (\overline{x} - \underline{x})/2$ its radius.

Theorem 1: Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function, $\boldsymbol{x} \in \mathbb{IR}^m$. Let $\boldsymbol{f}^0 = [\underline{f^0}, \overline{f^0}]$ include $f(c(\boldsymbol{x}))$ and $\boldsymbol{\nabla}$ a vector of intervals $\boldsymbol{\nabla_i} = [\underline{\nabla}_i, \overline{\nabla}_i]$ for $i \in \{1, \ldots, m\}$ such that $\{|\nabla_i f(c(\boldsymbol{x}_1), \ldots, c(\boldsymbol{x}_{i-1}), x_i, \ldots, x_m)|, x \in \boldsymbol{x}\} \subseteq \boldsymbol{\nabla}_i$. We have the over- and under-approximating extensions

$$\text{range}(f, \boldsymbol{x}) \subseteq [\underline{f^0}, \overline{f^0}] + \langle \overline{\nabla}, r(\boldsymbol{x}) \rangle [-1, 1] \quad (1)$$

$$[\overline{f^0} - \langle \underline{\nabla}, r(\boldsymbol{x}) \rangle, \underline{f^0} + \langle \underline{\nabla}, r(\boldsymbol{x}) \rangle] \subseteq \text{range}(f, \boldsymbol{x}) \quad (2)$$

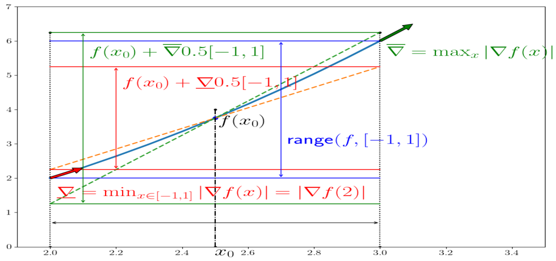
Proof: The mean-value theorem implies that $|f(x) - f(c(\boldsymbol{x}))| = |\langle \nabla f(\xi), x - c(\boldsymbol{x}) \rangle| \le \langle \overline{\nabla}, r(\boldsymbol{x}) \rangle$. As $f(c(\boldsymbol{x})) \in [\underline{f^0}, \overline{f^0}]$, $f(x) \in [\underline{f^0}, \overline{f^0}] + \langle \overline{\nabla}, r(\boldsymbol{x}) \rangle [-1, 1]$.

If $\underline{\nabla} = 0$ then the inner approximation estimate is trivial, so suppose $\underline{\nabla} \neq 0$, meaning f is either strictly increasing or decreasing over $\boldsymbol{x}$. Consider the first case, the other one being symmetric. Function f being continuous, and $\boldsymbol{x}$ being connected, the image of f over $\boldsymbol{x}$ is connected, thus includes the pair of segments L linking $f(\underline{x})$ to $f(c(\boldsymbol{x}))$ and $f(c(\boldsymbol{x}))$ to $f(\overline{x})$ on the real line. By the mean-value theorem applied to $x = \underline{x}$ and $x = \overline{x}$, $f(\underline{x}) - f(c(\boldsymbol{x})) \le -\langle \underline{\nabla}, r(\boldsymbol{x}) \rangle$ implying $[f(c(\boldsymbol{x})) - \langle \underline{\nabla}, r(\boldsymbol{x}) \rangle, f(c(\boldsymbol{x})]$ is included in L, and $\langle \underline{\nabla}, r(\boldsymbol{x}) \rangle \le f(\overline{x}) - f(c(\boldsymbol{x}))$ implying $[f(c(\boldsymbol{x})), f(c(\boldsymbol{x})) + \langle \underline{\nabla}, r(\boldsymbol{x}) \rangle] \subseteq L$. Therefore $[f(c(\boldsymbol{x})) - \langle \underline{\nabla}, r(\boldsymbol{x}) \rangle, f(c(\boldsymbol{x})) + \langle \underline{\nabla}, r(\boldsymbol{x}) \rangle]$ is included in L, thus in $\text{range}(f, \boldsymbol{x})$. Finally, $[\overline{f^0} - \langle \underline{\nabla}, r(\boldsymbol{x}) \rangle, \underline{f^0} + \langle \underline{\nabla}, r(\boldsymbol{x}) \rangle] \subseteq [f(c(\boldsymbol{x})) - \langle \underline{\nabla}, r(\boldsymbol{x}) \rangle, f(c(\boldsymbol{x})) + \langle \underline{\nabla}, r(\boldsymbol{x}) \rangle]$. ■

Note that the wider, lesser quality, are the over-approximations of f and its derivatives, the tighter, lesser quality, are the under-approximations. In particular, this allows us to soundly use floating-point implementations. The under-approximation can even become empty if the width $\overline{f_0} - \underline{f_0}$ of the approximation of $f(c(\boldsymbol{x}))$ exceeds $2\langle \overline{\nabla}, r(\boldsymbol{x}) \rangle$: the lower bound of the resulting interval is larger than the upper bound, which we identify with the empty interval.

Note also that when $0 \in \boldsymbol{\nabla_i f}$, then $\underline{\nabla}_i = 0$ and if this is the case for all i, the under-approximation is empty or reduced to a point. The extensions (1) and (2) are a simplified formulation of the results presented with modal intervals and generalized interval arithmetic in [2], [3] and Theorem 3.3 in [5]. We will refer to them as *AE extensions*, as they can be interpreted as *AE propositions*.

Example 1: Let us consider the range of f defined by $f(x) = x^2 - x$ over $\boldsymbol{x} = [2, 3]$. We can compute $f(2.5) = 3.75$ and $\boldsymbol{\nabla f}([2, 3]) \subseteq [3, 5]$. Then (1) and (2) yield $3.75 + 1.5[-1, 1] \subseteq \text{range}(f, [2, 3]) \subseteq 3.75 + 2.5[-1, 1]$, from which we deduce $[2.25, 5.25] \subseteq \text{range}(f, [2, 3]) \subseteq [1.25, 6.25]$. The result is illustrated on the figure below:



2) Robust mean-value AE extensions: We now introduce a generalization of the mean-value AE extensions to compute ranges that are robust to disturbances, identified as some input components. Let us partition the indices of the input space in two subsets $I_{\mathcal{A}}$ and $I_{\mathcal{E}}$, where $I_{\mathcal{A}}$ defines the indices of the inputs that correspond to disturbances, and $I_{\mathcal{E}}$ the remaining dimensions. We decompose the input box $\boldsymbol{x}$ accordingly by $\boldsymbol{x} = \boldsymbol{x}_{\mathcal{A}} \times \boldsymbol{x}_{\mathcal{E}}$. We define the robust range of function f on $\boldsymbol{x}$, robust on $\boldsymbol{x}_{\mathcal{E}}$ with respect to disturbances $\boldsymbol{x}_{\mathcal{A}}$, as $\text{range}(f, \boldsymbol{x}, I_{\mathcal{A}}, I_{\mathcal{E}}) = \{z \,|\, \forall w \in \boldsymbol{x}_{\mathcal{A}},\ \exists u \in \boldsymbol{x}_{\mathcal{E}},\ z = f(w, u)\}$. Intuitively, u will be control components, w disturbances to which the output range should be robust. When $I_{\mathcal{A}}$ is empty, the robust range equals the classical range.

Theorem 2: Let $f : \mathbb{R}^m \to \mathbb{R}$ be continuously differentiable, $\boldsymbol{x} = \boldsymbol{x}_{\mathcal{A}} \times \boldsymbol{x}_{\mathcal{E}} \in \mathbb{IR}^m$. Let $\boldsymbol{f}^0$, $\boldsymbol{\nabla}_w$ and $\boldsymbol{\nabla}_u$ be vectors of intervals such that $f(c(\boldsymbol{x})) \subseteq \boldsymbol{f}^0$, $\{|\nabla_w f(w, c(\boldsymbol{x}_{\mathcal{E}}))|\,, w \in \boldsymbol{x}_{\mathcal{A}}\} \subseteq \boldsymbol{\nabla}_w$ and $\{|\nabla_u f(w, u)|\,, w \in \boldsymbol{x}_{\mathcal{A}},\ u \in \boldsymbol{x}_{\mathcal{E}}\} \subseteq \boldsymbol{\nabla}_u$.

The mean-value AE extensions for the robust range are

$$\text{range}(f, \boldsymbol{x}, I_{\mathcal{A}}, I_{\mathcal{E}}) \subseteq [\underline{f^0} - \langle \overline{\nabla}_u, r(\boldsymbol{x}_{\mathcal{E}})\rangle + \langle \underline{\nabla}_w, r(\boldsymbol{x}_{\mathcal{A}})\rangle, \\ \overline{f^0} + \langle \overline{\nabla}_u, r(\boldsymbol{x}_{\mathcal{E}})\rangle - \langle \underline{\nabla}_w, r(\boldsymbol{x}_{\mathcal{A}})\rangle] \quad (3)$$

$$[\overline{f^0} - \langle \underline{\nabla}_u, r(\boldsymbol{x}_{\mathcal{E}})\rangle + \langle \overline{\nabla}_w, r(\boldsymbol{x}_{\mathcal{A}})\rangle, \underline{f^0} + \\ \langle \underline{\nabla}_u, r(\boldsymbol{x}_{\mathcal{E}})\rangle - \langle \overline{\nabla}_w, r(\boldsymbol{x}_{\mathcal{A}})\rangle] \subseteq \text{range}(f, \boldsymbol{x}, I_{\mathcal{A}}, I_{\mathcal{E}}) \quad (4)$$

Proof: Consider functions $g^w : u \to f(w, u) - f(w, c(\boldsymbol{x}_{\mathcal{E}}))$ (for any $w \in \boldsymbol{x}_{\mathcal{A}}$) and $h : w \to f(w, c(\boldsymbol{x}_{\mathcal{E}}))$. We see that $\text{range}(f, \boldsymbol{x}, I_{\mathcal{A}}, I_{\mathcal{E}})$ can be expressed as $C = \{\gamma \in \mathbb{R} \mid \forall \alpha \in \text{range}(h, \boldsymbol{x}_{\mathcal{A}}), \exists \beta \in \text{range}(g^w, \boldsymbol{x}_{\mathcal{E}}), \gamma = \alpha + \beta\}$. Under and over-approximating $C = \text{range}(f, \boldsymbol{x}, I_{\mathcal{A}}, I_{\mathcal{E}})$ C thus means under and over-approximating $C(A, B) = \{\gamma \in \mathbb{R} \mid \forall \alpha \in A, \exists \beta \in B, \gamma = \alpha + \beta\}$. First note that $C(A, B) = AB$ where $AB = [\overline{A} + \underline{B}, \underline{A} + \overline{B}]$. Let us prove that $AB \subseteq C(A, B)$. Take $\gamma \in AB$, now solving for $\gamma = \alpha + \beta$ means that $\beta = \gamma - \alpha \in [\overline{A} + \underline{B} - \overline{A}, \underline{A} + \overline{B} - \underline{A}]$ which is B, therefore we have a solution β in B and $AB \subseteq C(A, B)$. Conversely, take $\gamma \in C(A, B)$, and consider first $\alpha = \overline{A}$, then $\beta = \gamma - \overline{A}$ must be in B so $\gamma - \overline{A} \geq \underline{B}$ and $\gamma \geq \underline{B} + \overline{A}$. Similarly, consider $\alpha = \underline{A}$, then $\beta = \gamma - \underline{A}$ must be in B so we must have $\gamma - \underline{A} \leq \overline{B}$. This implies $\gamma \leq \underline{A} + \overline{B}$. This means that $C(A, B) \subseteq AB$, thus $C = AB$.

Moreover, suppose we have an under approximation I_A (resp. over approximation O_A) of A and an under approximation I_B (resp. over approximation O_B) of B. We have $C(O_A, I_B) \subseteq C(A, B) \subseteq C(I_A, O_B)$.

We can now use Theorem 1 on function g^w. The absolute value of its gradient is bounded by $\underline{\nabla}_u$ and $\overline{\nabla}_u$, thus, for a given $w \in \mathbf{x}_{\mathcal{A}}$, $\text{range}(g^w, \boldsymbol{x}_{\mathcal{E}}) \subseteq \langle \overline{\nabla}_u, r(\boldsymbol{x}_{\mathcal{E}})\rangle[-1, 1]$ and $[-\langle \underline{\nabla}_u, r(\boldsymbol{x}_{\mathcal{E}})\rangle, \langle \underline{\nabla}_u, r(\boldsymbol{x}_{\mathcal{E}})\rangle] \subseteq \text{range}(g^w, \boldsymbol{x}_{\mathcal{E}})$.

We now apply Theorem 1 on function h to get $\text{range}(h, \boldsymbol{x}_{\mathcal{A}}) \subseteq [\underline{h^0}, \overline{h^0}] + \langle \overline{\nabla}_w, r(\boldsymbol{x}_{\mathcal{A}})\rangle[-1, 1]$ and $[\overline{h^0} - \langle \underline{\nabla}_w, r(\boldsymbol{x}_{\mathcal{A}})\rangle, \underline{h^0} + \langle \underline{\nabla}_w, r(\boldsymbol{x}_{\mathcal{A}})\rangle] \subseteq \text{range}(h, \boldsymbol{x}_{\mathcal{A}})$. Note that we can take $\underline{h^0} = \underline{f^0}$ and $\overline{h^0} = \overline{f^0}$.

We can now use $O_A = [\underline{f^0}, \overline{f^0}] + \langle \overline{\nabla}_w, r(\boldsymbol{x}_{\mathcal{A}})\rangle[-1, 1]$, $I_B = [-\langle \underline{\nabla}_u, r(\boldsymbol{x}_{\mathcal{E}})\rangle, \langle \underline{\nabla}_u, r(\boldsymbol{x}_{\mathcal{E}})\rangle]$ as an over approximation of $\text{range}(h, \boldsymbol{x}_{\mathcal{A}})$ and an under approximation of $\text{range}(g^w, \boldsymbol{x}_{\mathcal{E}})$. We obtain $C(O_A, I_B) = [\overline{f^0} + \langle \overline{\nabla}_w, r(\boldsymbol{x}_{\mathcal{A}})\rangle - \langle \underline{\nabla}_u, r(\boldsymbol{x}_{\mathcal{E}})\rangle, \underline{f^0} - \langle \overline{\nabla}_w, r(\boldsymbol{x}_{\mathcal{A}})\rangle + \langle \underline{\nabla}_u, r(\boldsymbol{x}_{\mathcal{E}})\rangle] \subseteq C = \text{range}(f, \boldsymbol{x}, I_{\mathcal{A}}, I_{\mathcal{E}})$.

Finally, we use $I_A = [\overline{f^0} - \langle \underline{\nabla}_w, r(\boldsymbol{x}_{\mathcal{A}})\rangle, \underline{f^0} + \langle \underline{\nabla}_w, r(\boldsymbol{x}_{\mathcal{A}})\rangle]$ and $O_B = \langle \overline{\nabla}_u, r(\boldsymbol{x}_{\mathcal{E}})\rangle[-1, 1]$ as an inner approximation of $\text{range}(h, \boldsymbol{x}_{\mathcal{A}})$ and an outer approximation of $\text{range}(g^w, \boldsymbol{x}_{\mathcal{E}})$. We obtain $C(I_A, O_B) = [\underline{f^0} + \langle \underline{\nabla}_w, r(\boldsymbol{x}_{\mathcal{A}})\rangle - \langle \overline{\nabla}_u, r(\boldsymbol{x}_{\mathcal{E}})\rangle, \overline{f^0} - \langle \underline{\nabla}_w, r(\boldsymbol{x}_{\mathcal{A}})\rangle + \langle \overline{\nabla}_u, r(\boldsymbol{x}_{\mathcal{E}})\rangle$ as an over approximation of $C = \text{range}(f, \boldsymbol{x}, I_{\mathcal{A}}, I_{\mathcal{E}})$. ■

Note that changing the order of the components on which the mean-value theorem is applied sequentially, either here or in Theorem 1, leads to different variations of (3) and (4).

Example 2: Let us consider the range of f defined by $f(x_1, x_2) = x_2^2 - 2x_1$ for $x \in [2, 3] \times [2, 3]$. Let us first approximate $\text{range}(f, \boldsymbol{x})$. We have $f(2.5, 2.5) = 1.25$, $\nabla f(\boldsymbol{x}) \subseteq ([-2, -2], [4, 6])$, and Theorem 1 yields $1.25 + 4 * 0.5[-1, 1] + 2 * 0.5[-1, 1] \subseteq \text{range}(f, \boldsymbol{x}) \subseteq 1.25 + 6 * 0.5[-1, 1] + 2 * 0.5[-1, 1]$ which simplifies to $[-1.75, 4.25] \subseteq \text{range}(f, \boldsymbol{x}) \subseteq [-2.75, 5.25]$. Let us now consider $\text{range}(f, \boldsymbol{x}, 1, 2)$, which means we take $f(w, u) = u^2 - 2w$. Theorem 2 yields $[1.25 - 2 + 1, 1.25 + 2 - 1] \subseteq \text{range}(f, \boldsymbol{x}, 1, 2) \subseteq [1.25 - 3 + 1, 1.25 + 3 - 1]$, which simplifies to $[0.25, 2.25] \subseteq \text{range}(f, \boldsymbol{x}, 1, 2) \subseteq [-0.75, 3.25]$.

B. Mean-value AE extensions for vector-valued functions

The mean-value extensions of Theorem 1 give us interval under and over-approximations of projections of the image of the function. The Cartesian product of the over-approximations of each component provides an over-approximation of a vector-valued function $f : \mathbb{R}^m \to \mathbb{R}^n$. This is not the case for under-approximation. Suppose for example that we have $\forall z_1 \in \boldsymbol{z}_1, \exists x_1 \in \boldsymbol{x}_1, \exists x_2 \in \boldsymbol{x}_2,\ z_1 = f_1(x)$ and $\forall z_2 \in \boldsymbol{z}_2, \exists x_1 \in \boldsymbol{x}_1, \exists x_2 \in \boldsymbol{x}_2,\ z_2 = f_2(x)$. We obviously cannot deduce directly that for all $\forall z_1 \in \boldsymbol{z}_1$ and $\forall z_2 \in \boldsymbol{z}_2$ there exist x_1 and x_2 such that $z = f(x)$.

Suppose now that we have the following properties: $\forall z_1 \in \boldsymbol{z}_1, \forall x_1 \in \boldsymbol{x}_1, \exists x_2 \in \boldsymbol{x}_2,\ z_1 = f_1(x)$ and $\forall z_2 \in \boldsymbol{z}_2, \forall x_2 \in \boldsymbol{x}_2, \exists x_1 \in \boldsymbol{x}_1,\ z_2 = f_2(x)$ with continuous selections x_2 and x_1 (it is the case, by a result of [2] when f is an elementary function). This means that there exist functions $g_2(z_1, x_1) : \boldsymbol{z}_1 \times \boldsymbol{x}_1 \to \boldsymbol{x}_2$ and $g_1(z_2, x_2) : \boldsymbol{z}_2 \times \boldsymbol{x}_2 \to \boldsymbol{x}_1$ that are continuous in x_1 (resp. x_2), and such that $\forall (z_1, z_2) \in \boldsymbol{z},\ \forall (x_1, x_2) \in \boldsymbol{x},\ z_1 = f_1(x_1, g_2(z_1, x_1))$ and $z_2 = f_2(g_1(z_2, x_2), x_2)$. Using the Brouwer fixed point theorem, for each $z_1 \in \boldsymbol{z}_1$ and $z_2 \in \boldsymbol{z}_2$, on the continuous map $g : (x_1, x_2) \to (g_1(z_2, x_2), g_2(z_1, x_1))$ from the compact set $\boldsymbol{x}_1 \times \boldsymbol{x}_2$ into itself, we know that g has a fixed point $(x_1^z, x_2^z) \in \boldsymbol{x}_1 \times \boldsymbol{x}_2$. This means that for all $(z_1, z_2) \in \boldsymbol{z}$ there exist $(x_1^z, x_2^z) \in \boldsymbol{x}$ such that $(z_1, z_2) = f(x_1^z, x_2^z)$.

This result can be generalized to functions $f : \mathbb{R}^m \to \mathbb{R}^n$ for any n, as shown in Theorem 3. However, in some cases, only the projection of a subset of the n output components will be non-empty: for example, we cannot include a full n dimensional box within the image of f if $n > m$.

Theorem 3: Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be an elementary function and $\pi : \{1, \ldots, m\} \to \{1, \ldots, n\}$. Let us note, for all $i \in \{1, \ldots n\}$, $J_E^{(z_i)} = \{j \in \{1, \ldots, m\},\ \pi(j) = i\}$ and $J_A^{(z_i)} =$

$\{j \in \{1,\dots,m\}\} \setminus J_E^{(z_i)}$. Consider the n AE-extensions $i \in \{1,\dots,n\}$, built from Theorem 2,

$$\forall z_i \in \boldsymbol{z}_i,\ (\forall x_j \in \boldsymbol{x}_j)_{j \in J_A^{(z_i)}},\ (\exists x_j \in \boldsymbol{x}_j)_{j \in J_E^{(z_i)}},\ z_i = f_i(x) \tag{5}$$

Then $\boldsymbol{z} = \boldsymbol{z}_1 \times \boldsymbol{z}_2 \times \ldots \times \boldsymbol{z}_n$, if non-empty, is an under-approximation of the image of f: $\forall z \in \boldsymbol{z}$, $\exists x \in \boldsymbol{x}$, $z = f(x)$.

Proof: The principle is the same as in the case of 2 components. Function π associates to each x_j for $j \in \{1,\dots,m\}$ the index $i \in \{1,\dots,n\}$ of the unique output component of the function in which it will be existentially quantified. First suppose π surjective. For each AE-extension (5) for z_i, for all $k_i \in J_E^{(z_i)}$ we can associate the continuous selection $g_{k_i}(z_i,(x_j)_{j\in J_A^{(z_i)}})$ by [3], since f is elementary. For a given $(z_1,\dots,z_n) \in \boldsymbol{z}$, let us define the continuous map g that associates to each $(x_1,\dots,x_m) \in \boldsymbol{x}$, $((g_{k_1}(z_1,(x_j)_{j\in J_A^{(z_1)}}))_{k_1\in J_E^{(z_1)}}, \dots, (g_{k_n}(z_n,(x_j)_{j\in J_A^{(z_n)}}))_{k_n\in J_E^{(z_n)}})$ in $\boldsymbol{x} \subseteq \mathbb{R}^m$, since π being surjective, $\{J_E^{(z_i)} | i = 1,\dots,n\}$ forms a partition of $\{1,\dots,m\}$. By Brouwer fixed point theorem, $\forall z \in \boldsymbol{z}$, there is a fixed point $x^z \in \boldsymbol{x}$ of g, which thus satisfies $z = f(x^z)$.

Finally, if π is not surjective, there exist z_i in which no input variable is existentially quantified. The corresponding under-approximation will be empty or reduced to a point and the previous proof still holds on the other components. ■

Remark 1: Theorem 3 gives an under-approximation of range$(f,\boldsymbol{x})$ for $f: \mathbb{R}^m \to \mathbb{R}^n$. It can also be used to compute an under-approximation of the robust range$(f,\boldsymbol{x},I_{\mathcal{A}},I_{\mathcal{E}})$. For this, we need to choose $\pi : \{1,\dots,m\} \to (\{1,\dots,n\} \setminus I_{\mathcal{A}})$, which means that the disturbance part of the input components will always be quantified universally.

We now illustrate these computations and the choice of π.

Example 3: We consider $f(x) = (5x_1^2 + x_2^2 - 2x_1x_2 - 4, x_1^2 + 5x_2^2 - 2x_1x_2 - 4)^{\intercal}$ with $\boldsymbol{x} = [0.9, 1.1]^2$. We have

$$\nabla f(x) = (\frac{\partial f_i}{\partial x_j}) = \begin{pmatrix} 10x_1 - 2x_2 & 2x_2 - 2x_1 \\ 2x_1 - 2x_2 & 10x_2 - 2x_1 \end{pmatrix}$$

Evaluation on $\boldsymbol{x}$ yields $\nabla f(\boldsymbol{x}) \subseteq (([6.8, 9.2], [-0.4, 0.4])^{\intercal}, ([-0.4, 0.4], [6.8, 9.2])^{\intercal})$. Using Theorem 2 yields range$(f,\boldsymbol{x}) \subseteq [-0.96, 0.96]^2$. The projection of the under-approximation on each component is $[-0.68, 0.68]$. For example, $f_1(1,1) + 0.1 * 6.8 * [-1,1] + 0 = [-0.68, 0.68] \subseteq$ range$(f_1,\boldsymbol{x})$. The under-approximation for $\pi : (1 \to 2, 2 \to 1)$ is empty: the contributions of x_1 on the under-approximation of f_2 and of x_2 on f_1 are 0, as the corresponding coefficients in the Jacobian contain 0. Choosing $\pi : (1 \to 1, 2 \to 2)$ yields under-approximation $[-0.64, 0.64]^2$: for AE extension $\forall z_1 \in \boldsymbol{z}_1, \forall x_2 \in \boldsymbol{x}_2, \exists x_1 \in \boldsymbol{x}_1, z_1 = f(\boldsymbol{x})$, we have $[-0.68 + 0.4 * 0.1, 0.68 - 0.4 * 0.1] \subseteq$ range$(f_1,\boldsymbol{x},2)$.

Preconditioning for computing inner skewed boxes: The n-dimensional inner boxes that we compute can sometimes be small or empty, in particular when the image cannot be precisely approximated by a centered box. This problem can be partly solved by computing a skewed box as under-approximation, that is the image of a box by a linear map, instead of a box. Let $C \in \mathbb{R}^{n\times n}$ be a non-singular matrix. If $\boldsymbol{z}$ is an interval vector such that $\boldsymbol{z} \subseteq$ range$(Cf,\boldsymbol{x})$, we can deduce the skewed box $\{C^{-1}z | z \in \boldsymbol{z}\}$ to be in range$(f,\boldsymbol{x})$. A natural choice for C is the inverse of the center of the interval Jacobian matrix $C = (c(\boldsymbol{\nabla}))^{-1}$.

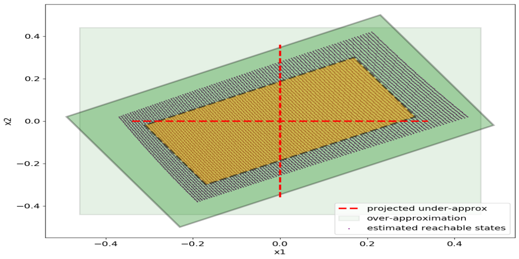


Fig. 1: Example 4: samples, under and over-approximation.

Example 4: We consider $f(x) = (2x_1^2 - x_1x_2 - 1, x_1^2 + x_2^2 - 2)^{\intercal}$ with $\boldsymbol{x} = [0.9, 1.1]^2$. We only find empty inner boxes with the mean-value extension. Using the preconditioning and $\pi : (1 \to 1, 2 \to 2)$, we obtain for (f_1, f_2) the yellow under-approximating parallelotope of Figure 1.

We also estimate range$(f,\boldsymbol{x})$ by sampling points in the input domain. This sampling-based estimation is represented as the dark dots-filled region. The green parallelotope and box are the over-approximations with and without preconditioning.

Implementation: Successfully applying the results of this section implies bounding the partial derivatives over $\boldsymbol{x}$. Our implementation relies on a combination of automatic differentiation and evaluation with affine arithmetic. When the partial derivatives vary a lot over $\boldsymbol{x}$, it may be interesting to reduce the input set or use quadrature formulae to evaluate the mean-value theorem, but this is out of the scope here, and the results described in this work do not use such techniques. Finally, for these examples, we could easily enumerate the possible π, but future work includes defining heuristics.

III. Application to reachable sets

We consider general non-linear systems of ODEs

$$\begin{cases} \dot{z}(t) = f(z(t), u(t)) & \text{if } t \geq 0 \\ z(t) = z_0 & \text{if } t = 0 \end{cases} \tag{6}$$

where $z(t) \in \mathbb{R}^n$, the initial value is defined by $z(0) = z_0$, and the input signal $u(t)$ belongs to $\mathbb{U} = \{\phi : \mathbb{R}^+ \to \mathcal{U}$ piecewise constant with finitely many discontinuities$\}$ with $\mathcal{U} \subseteq \mathbb{R}^p$. Function $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ is assumed sufficiently smooth on $\mathbb{R}^n$ (at least $\mathcal{C}^1$, and more when we will use higher order Taylor models). We suppose that given an initial state $z(0) = z_0$ and an input signal u, there exists a unique solution or trajectory $\varphi^f(t; z_0, u)$ to system (6) for all time $t \in \mathbb{T} = [0, T_{\max}]$. We are interested in *reachable sets*, the sets of states reachable by trajectories of the system, starting with z_0 in a set $\boldsymbol{Z}_0$. Following [1], we define maximal and minimal reachability.

Maximal reachability: Given a vector of uncertain input signal u defined in the set $\mathbb{U}$, we note $R^f_{\mathcal{E}}(t; \boldsymbol{Z}_0, \mathbb{U}) = \{z \in \mathbb{R}^n | \exists u \in \mathbb{U}, \exists z_0 \in \boldsymbol{Z}_0, z = \varphi^f(t; z_0, u)\}$ the *maximal* *reachable set*, where we seek the input signal that maximizes the size of the reachable set. In this case, u will correspond to a controllable input signal, which is existentially quantified, hence the $\mathcal{E}$ subscript notation.

Minimal reachability: We note $R^f_{\mathcal{A}}(t;\boldsymbol{Z}_0,\mathbb{U}) = \{z \in \mathbb{R}^n | \forall u \in \mathbb{U}, \exists z_0 \in \boldsymbol{Z}_0, z = \varphi^f(t;z_0,u)\}$ the *minimal reachable set*, that contains only states that trajectories will reach whatever the input signal. Here, u will correspond to an uncontrollable disturbance, with respect to which the behavior of the system must be robust, and it is universally quantified, hence the subscript $\mathcal{A}$.

Robust reachability: We generalize the above definitions by using the subscript $\mathcal{AE}$ to define the reachable set which is maximal with respect to some dimensions $u_{\mathcal{E}}$ of the input (vector) u that represent the control, and minimal or robust with respect to the remaining dimensions $u_{\mathcal{A}}$, that represent the disturbance part of the input signal. Let $u = (u_{\mathcal{A}}, u_{\mathcal{E}}) \in \mathbb{U} = (\mathbb{U}_{\mathcal{A}}, \mathbb{U}_{\mathcal{E}})$. We define the *robustly reachable set* by

$$R^f_{\mathcal{AE}}(t;\boldsymbol{Z}_0,\mathbb{U}) = \{z \in \mathbb{R}^n \,|\, \forall u_{\mathcal{A}} \in \mathbb{U}_{\mathcal{A}}, \exists u_{\mathcal{E}} \in \mathbb{U}_{\mathcal{E}}, \exists z_0 \in \boldsymbol{Z}_0,\, z = \varphi^f(t;z_0,u)\}$$

Let $\mathcal{I}_{\mathcal{AE}}$ and $\mathcal{O}_{\mathcal{AE}}$ be two sets such that $\mathcal{I}_{\mathcal{AE}} \subseteq R^f_{\mathcal{AE}}(t;\boldsymbol{Z}_0,\mathbb{U}) \subseteq \mathcal{O}_{\mathcal{AE}}$. We call $\mathcal{I}_{\mathcal{AE}}$ a robust under-approximation and $\mathcal{O}_{\mathcal{AE}}$ a robust over-approximation.

We now use the results of Section II to compute robust under-approximations and over-approximations from maximal over-approximations, following [4], [5]. The main idea is to instantiate in the generalized mean-value theorem, the function f as the solution of system (6), and parameter x as the uncertain initial condition z_0 together with inputs and perturbations u. We need u to be finitely representable for computing under-approximations. Here we consider piecewise constant u, but could also handle more general piecewise continuous u, see [5] in the case of projected under-approximations. We first need to compute:

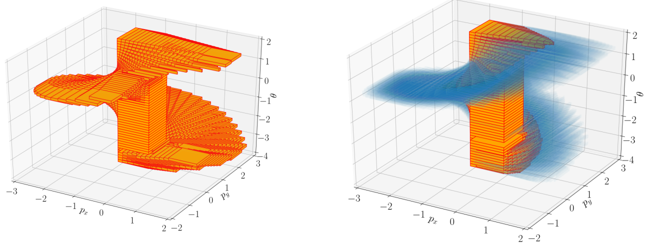
1) a maximal over-approximation $\tilde{\mathcal{O}}^f_{\mathcal{E}}(t)$ of the trajectory $\varphi^f(t;\tilde{z_0},\tilde{u})$ for a given $(\tilde{z_0},\tilde{u}) \in \boldsymbol{Z}_0 \times \mathcal{U}$.
2) a maximal over-approximation $\mathcal{O}^F_{\mathcal{E}}(t)$ of the sensitivity matrix with respect to uncertain initial condition z_0 and input u, over the range $\boldsymbol{Z}_0 \times \mathcal{U}$.

Computing these over-approximations is classical. We can use any approach, for instance Taylor model methods, which are well suited for non-linear systems. In terms of complexity, the overhead of our approach for under-approximation compared to over-approximation corresponds to the computation of the reachable sets for the sensitivity matrix (a $(n+p)$ multiplicative coefficient, n being the dimension of the state-space, p the dimension of the parameter space).

IV. Implementation and Examples

The approach is implemented using Taylor models in the RINO C++ prototype, available from https://github.com/cosynus-lix/RINO. The examples are available in the repository. The timings are given on a Macbook Pro 2.6GHz Intel Core i7 (6 cores) and 32Gb of RAM.

We choose $\pi(i) = i$ for each state space-component i.



(a) With no disturbances

(b) With disturbances

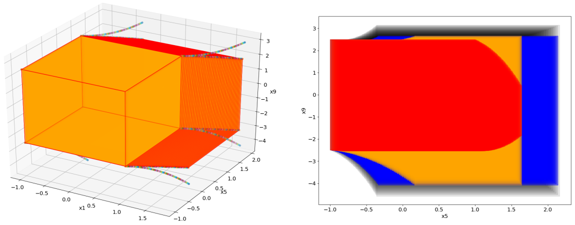
Fig. 2: Joint p_x, p_y and θ for Dubbins, constant controls

1) Dubbins vehicle: We consider the classical Dubbins vehicle studied in [12] with constant speed v and angular control a, giving the dynamics of its position (p_x, p_y) and its heading θ. We set here $v = 5$, $a \in [-1,1]$, and disturbances b_1, b_2, b_3 to be added to each component of the dynamical system: $-1 \leq b_1 \leq 1$, $-1 \leq b_2 \leq 1$ and $-5 \leq b_3 \leq 5$.

To compare with the HJB methods of [12], we compute the backward reachable set at time t $\mathcal{G}(t) = \{x_0 | \forall u_{\mathcal{A}}, \exists u_{\mathcal{E}}, \exists x \in \mathcal{G}_0, x = \varphi^f(t;x_0,u))\}$ from the target set $\mathcal{G}_0 = \{(p_x,p_y,\theta)||p_x| \leq 0.5, |p_y| \leq 0.5,\ 0 \leq \theta \leq 2\pi\}$. This is equivalent to computing $\{x_0 | \forall u_{\mathcal{A}}, \exists u_{\mathcal{E}}, \exists x \in \mathcal{G}_0, x_0 = \varphi^{-f}(t;x,u))\}$, the forward reachable set with the opposite vector flow. Figure 2a shows the robust under-approximation, computed in 2 seconds, with a Taylor method of order 3, a time horizon of 0.5 seconds and step size of 0.025 seconds, 50 subdivisions on the heading angle θ, with constant controls and no disturbances. The results are very similar to those shown in [12]. Without disturbances, the maximal and robust reachable sets are equal. We then analyze the same system, but with disturbances, with the same parameters. The results, also obtained in 2 seconds, are shown in Figure 2b. The robust approximation, in orange, is now smaller, as expected. Indeed, reaching the small centered unit square from any direction, under some disturbances, limits the potential initial positions, given the (controlled) angular speed's bounds. The maximal under-approximation, in blue, is comparable to that of the unperturbed case.

2) 10D hover quadrotor: We now consider the model of a hovering quadrotor described in [12] and consisting of a 10-dimensional system

defining the 3D position (p_x, p_y, p_z), the linear velocities (v_x, v_y, v_z), the pitch and roll (θ_x, θ_y), the pitch and roll rates (ω_x, ω_y). The controls are S_x, S_y in $[-\frac{\pi}{180}, \frac{\pi}{180}]$ representing the desired pitch and roll angle, and $T_z \in [0, 19.62]$, the vertical thrust. Disturbances (d_x, d_y, d_z) represent for instance the wind in the three axes. The target set is given as $-1 \leq p_x, p_y \leq 1$, $-2.5 \leq p_z \leq 2.5$, $v_x = -1.5$, $\theta_x = 0$, $\omega_x = 0$, $v_y = -1.8$, $\theta_y = 0$, $\omega_y = 0$, $v_z = 1.2$. The results shown in Figure 3a are for the model without disturbances: they are computed in 1.28 seconds for an order 4 Taylor method, a time horizon of 0.5 seconds with a stepsize of 0.01. The robust inner image in orange (the maximal range is equal here to the robust range and thus hidden as there is no disturbance) is very similar to the corresponding one in [12], without the need to consider a decomposition into smaller



(a) Joint p_x, p_y and p_z, no disturbance, constant controls

(b) Joint p_y and p_z, disturbances, time-varying controls

Fig. 3: 10D hovering example

$Ex.$	d	p	T	δ	k	a	v	sd	$time$
Bru	2	2	4	0.02	4				1.26
$B24$	2	1	1	0.1	3	✓	✓		0.02
Dub	3	4	1	0.01	3				0.14
−	−	−	−	−	−			100	11.58
−	−	−	−	−	−	✓	✓	100	428.1
$6D$	6	2	1	0.01	4				0.87
−	−	−	−	−	−		✓		15.56
−	−	−	−	−	−	✓	✓		30.52
$L-L$	7	0	20	0.1	3				24.04
$10D$	10	6	1	0.01	5				1.26
−	−	−	−	−	−		✓		9.98

TABLE I: Timings on various examples

dimensional subsystems. Similar results are obtained for time-varying controls, but in 6.49 seconds. In the disturbed model, where d_x, d_y and d_z are taken to be between -0.5 and 0.5 (analysis time of 1.22 seconds), we see in Figure 3b that the minimal under-approximation of the image in red is close to the robust and maximal under-approximations in orange and blue and maximal over-approximations in grey. It demonstrates both that the analysis is very accurate and the hovering is very stable under the disturbances.

3) Other benchmarks: We show in Table I the results with the RINO prototype on various examples: d is the dimension of the system, p the number of parameters, T the time horizon under which the analysis is done, δ is the step-size of the Taylor-model method, k is the order of the Taylor-model method, a is checked if the analysis is done with adversarial disturbances, sd is the number of subdivisions used, v is checked when (some, at least) uncertainties are time-varying (in all examples below, they are then piecewise-constant on stepsize seconds), time is the total analysis time in seconds. Dub and 10D are respectively the Dubbins and 10D hovering examples we just described in full detail. 6D is a 6-dimensional simplified model of a quadrotor, taken from [12]. L-L is the classical Laub-Loomis example for over-approximations of reachable sets, taken here from [19]. We use them to demonstrate that our under-approximating analysis is not much more costly than classical over-approximation; as an example, Laub-Loomis takes between 1 and 20 seconds on different reachability tools (Flow*, Dynibex, CORA etc.) whereas our analysis took 24 seconds with similar parameters as those reported in [19]. Bru is the classical Brusselator benchmark for reachability tools, that has been considered for under-approximations in [16]. Our analysis takes 1.26 seconds whereas it is reported to take 89 seconds in [16], with a lower precision (γ the minimal relative width of the under-approximation with respect to the over-approximation in the axes directions is 27.7% for us instead of 55% in [16]) with the same parameters (stepsize, time, order etc.). B24 is the first example of [14], which takes us only 0.02 seconds to solve (on a slightly different target set since our analyzer cannot represent exactly Euclidean balls) for a good final precision instead of the 334.20 seconds (plus, from 0.69 to 89.23 seconds for the LMI part) reported in [14]. Still, the full comparison is difficult to make since we do not have the exact results obtained, just pictures, as is also the case when trying to compare Dubbins and 10D with [12]. Generally speaking, the analyzer provides fast results, and only needs rather low orders for the Taylor methods. Adversarial disturbances or controlled inputs yield similar computation time. Time-varying inputs however increase the computational cost, as each degree of liberty adds a column in the Jacobian matrix involved in the mean-value extension.

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