Static Analysis
of Numerical Programs and Systems

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During the last ten years...

Some steps

- Starting point (E. Goubault, SAS 2001): bound and propagate rounding errors (intervals) and indicate provenance
  - hired Jan. 01 (PhD Num. Analysis): first implement these ideas (ESOP'02)
- Zonotopic abstract domains
  - first ideas for the error quite early (Dagstuhl Seminar 2002)
  - first focus on the real value (SAS 2006, arxiv 2008, arxiv 2009)
  - back to errors (VMCAI 2011, much sooner in Fluctuat...)
  - extensions (SAS 2007, Computing 2012)
- Meanwhile fixpoint computation (CAV 2005), development of FLUCTUAT

Environment

  - most of them: interactions between program analysis and numerical computation, optimisation, control, probabilities
- Industrial collaborations: many projects and some publications (FMICS 2007, FMICS 2009, DASIA 2009)
Automatic validation of numerical programs and systems

What is correction for numerical computations?

- No run-time error (division by 0, overflow, etc)
- The program computes a result close to what is expected
  - accuracy (and behaviour) of finite precision computations
  - method error

Context: safety-critical programs

- Typically flight control or industrial installation control (signal processing, instrumentation software)

Sound and automatic methods

- Guaranteed methods, that prove good behaviour or else try to give counter-examples
- Automatic methods, given a source code, and sets of (possibly uncertain) inputs and parameters

Abstract interpretation based static analysis
Automatic invariant synthesis

- Program seen as system of equations $X^{n+1} = F(X^n)$
  - Based on a notion of control points in the program
  - Equations describe how values of variables are collected at each control point, for all possible executions (collecting semantics)

Example

```plaintext
int x = [-100, 50]; [1]  
while [2] (x < 100) [3]  
    x = x + 1; [4]  
[5]  
```

\[
\begin{align*}
x_1 &= [-100, 50] \\
x_2 &= x_1 \cup x_4 \\
x_3 &= ] - \infty, 99[ \cap x_2 \\
x_4 &= x_3 + [1, 1] \\
x_5 &= [100, +\infty[ \cap x_2 \\
\end{align*}
\]
Automatic invariant synthesis

- Program seen as a system of equations $X^{n+1} = F(X^n)$
- Want to compute reachable or invariant sets at control points
- Least fixpoint computation on partially ordered structure
  - classically computed as the limit of the Kleene iteration
    $$X^0 = ot, X^1 = F(X^0), \ldots, X^{k+1} = X^k \cup F(X^k)$$
  - or policy iteration (Newton-like method)
- Generally not computable
Automatic invariant synthesis

- Program seen as a system of equations $X^{n+1} = F(X^n)$
- Want to compute reachable or invariant sets at control points
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  - classically computed as the limit of the Kleene iteration
    \[
    X^0 = \bot, X^1 = F(X^0), \ldots, X^{k+1} = X^k \cup F(X^k)
    \]
  - or policy iteration (Newton-like method)
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Sound abstractions heavily relying on set-based methods

- Choose a computable abstraction that defines an over or under-approximation of set of values
- Need a partially ordered structure, with join and meet operators
- Transfer concrete fixpoint computation in the abstract world
Outline

- Affine sets: zonotopic abstraction of real computations
- A word on fixpoint computations
- Extensions of the zonotopic approach
- Analyzing finite precision computations: Fluctuat
Affine Arithmetic (Comba & Stolfi 93)

Affine forms

- Affine form for variable $x$:

$$\hat{x} = x_0 + x_1 \varepsilon_1 + \ldots + x_n \varepsilon_n, \ x_i \in \mathbb{R}$$

where the $\varepsilon_i$ are symbolic variables (noise symbols), with value in $[-1, 1]$.

- Sharing $\varepsilon_i$ between variables expresses implicit dependency

- Interval concretization of affine form $\hat{x}$:

$$\left[ x_0 - \sum_{i=0}^{n} |x_i|, x_0 + \sum_{i=0}^{n} |x_i| \right]$$

Geometric concretization as zonotopes in $\mathbb{R}^p$

$$\hat{x} = 20 - 4\varepsilon_1 + 2\varepsilon_3 + 3\varepsilon_4$$

$$\hat{y} = 10 - 2\varepsilon_1 + \varepsilon_2 - \varepsilon_4$$
**Affine arithmetic**

- **Assignment** $x := [a, b]$ introduces a noise symbol:
  \[
  \hat{x} = \frac{(a + b)}{2} + \frac{(b - a)}{2} \varepsilon_i.
  \]

- **Addition/subtraction** are exact:
  \[
  \hat{x} + \hat{y} = (x_0 + y_0) + (x_1 + y_1)\varepsilon_1 + \ldots + (x_n + y_n)\varepsilon_n
  \]

- **Non linear operations**: approximate linear form, new noise term bounding the approximation error

- Close to Taylor models of low degree (large ranges for static analysis)
Concretization-based analysis

- Machine-representable abstract values $X$ (affine sets)
- A concretization function $\gamma_f$ defining the set of concrete values represented by an abstract value
- A partial order on these abstract values, induced by $\gamma_f$:
  \[ X \subseteq Y \iff \gamma_f(X) \subseteq \gamma_f(Y) \]

Abstract transfer functions

- Arithmetic operations: $F$ is a sound abstraction of $f$ iff
  \[ \forall x \in \gamma_f(X), \ f(x) \in \gamma_f(F(X)) \]
- Set operations: join ($\cup$), meet ($\cap$), widening
  - no least upper bound / greatest lower bound on affine sets
  - (minimal) upper bounds / over-approximations of the intersection ...

and ... hopefully accurate and effective to compute!!!
Two kinds of noise symbols

- Input noise symbols ($\varepsilon_i$): created by uncertain inputs
- Perturbation noise symbols ($\eta_j$): created by uncertainty in analysis

Perturbed affine sets $X = (C^X, P^X)$

$$
\begin{pmatrix}
\hat{x}_1 \\
\hat{x}_2 \\
\vdots \\
\hat{x}_p
\end{pmatrix} = tC^X \begin{pmatrix}
1 \\
\varepsilon_1 \\
\vdots \\
\varepsilon_n
\end{pmatrix} + tP^X \begin{pmatrix}
\eta_1 \\
\eta_2 \\
\vdots \\
\eta_m
\end{pmatrix}
$$

- Central part links the current values of the program variables to the initial values of the input variables (linear functional)
- Perturbation part encodes the uncertainty in the description of values of program variables due to non-linear computations (multiplication, join etc.)

Zonotopes define input-output relations (parameterization by the $\varepsilon_i$)

- Want an order that preserves these input-output relations
A simple example: functional interpretation

\[
x \in [0, 10];
\]
\[
y = x^2 - x;
\]

Abstraction of function \( x \to y = x^2 - x \) as

\[
y = 32.5 + 45\varepsilon_1 + 12.5\eta_1
\]
A simple example: functional interpretation

\[
\text{real } x = [0, 10]; \\
\text{real } y = x^2 - x;
\]

Abstraction of function \( x \rightarrow y = x^2 - x \) as

\[
y = 32.5 + 45\varepsilon_1 + 12.5\eta_1 \\
= -12.5 + 9x + 12.5\eta_1
\]

(since \( x = 5 + 5\varepsilon_1 \))
Concretization in terms of sets of functions from $\mathbb{R}^n$ to $\mathbb{R}^p$:

$$
\gamma_f(X) = \left\{ f : \mathbb{R}^n \to \mathbb{R}^p \mid \forall \epsilon \in [-1, 1]^n, \exists \eta \in [-1, 1]^m, f(\epsilon) = tC^X \left( \frac{1}{\epsilon} \right) + tP^X \eta \right\}.
$$

- Equivalent to the geometric ordering on augmented space $\mathbb{R}^{p+n}$: zonotopes enclosing current values of variables $+$ their initial values $\epsilon_i$

$$
\gamma(\tilde{X}) \subseteq \gamma(\tilde{Y}), \text{ where } \tilde{X} = \begin{pmatrix}
0_n \\
l_{n \times n} \\
0_{m \times n} \\
P^X
\end{pmatrix} C^X
$$

- Implies the geometric ordering in $\mathbb{R}^p$

- Computable inclusion test:

$$
X \subseteq Y \iff \sup_{u \in \mathbb{R}^p} (\|(C^Y - C^X)u\|_1 + \|P^X u\|_1 - \|P^Y u\|_1) \leq 0
$$

Example

- $x_1 = 2 + \epsilon_1$, $x_2 = 2 - \epsilon_1$
- $x_1$ and $x_2$ are incomparable
Concretization in terms of sets of functions from $\mathbb{R}^n$ to $\mathbb{R}^p$:

$$\gamma_f(X) = \left\{ f : \mathbb{R}^n \to \mathbb{R}^p \mid \forall \epsilon \in [-1, 1]^n, \exists \eta \in [-1, 1]^m, f(\epsilon) = tC^X \left( \frac{1}{\epsilon} \right) + tP^X \eta \right\}.$$

- Equivalent to the geometric ordering on augmented space $\mathbb{R}^{p+n}$: zonotopes enclosing current values of variables $+ \text{their initial values } \epsilon_i$

$$\gamma(\tilde{X}) \subseteq \gamma(\tilde{Y}), \text{ where } \tilde{X} = \begin{pmatrix} 0_n \\ I_{n \times n} \\ 0_{m \times n} \end{pmatrix} \begin{pmatrix} C^X \\ P^X \end{pmatrix}$$

- Implies the geometric ordering in $\mathbb{R}^p$
- Computable inclusion test:

$$X \sqsubseteq Y \iff \sup_{u \in \mathbb{R}^p} \left( \| (C^Y - C^X)u \|_1 + \| P^X u \|_1 - \| P^Y u \|_1 \right) \leq 0$$

**Example**

- $x_1 = 2 + \epsilon_1, \ x_2 = 2 - \epsilon_1$ (geometric concretization $[1, 3]$)
- $x_1$ and $x_2$ are incomparable
Concretization in terms of sets of functions from $\mathbb{R}^n$ to $\mathbb{R}^p$:

$$\gamma_f(X) = \left\{ f : \mathbb{R}^n \to \mathbb{R}^p \mid \forall \epsilon \in [-1, 1]^n, \exists \eta \in [-1, 1]^m, f(\epsilon) = tC^X \begin{pmatrix} 1 \\ \epsilon \end{pmatrix} + tP^X \eta \right\}.$$ 

- Equivalent to the geometric ordering on augmented space $\mathbb{R}^{p+n}$: zonotopes enclosing current values of variables + their initial values $\epsilon_i$


$$\gamma(\tilde{X}) \subseteq \gamma(\tilde{Y}), \text{ where } \tilde{X} = \begin{pmatrix} 0_n \\ I_{n \times n} \\ 0_{m \times n} \end{pmatrix} \begin{pmatrix} C^X \\ P^X \end{pmatrix}$$

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- Computable inclusion test:

$$X \subseteq Y \iff \sup_{u \in \mathbb{R}^p} \left( \| (C^Y - C^X)u \|_1 + \| P^X u \|_1 - \| P^Y u \|_1 \right) \leq 0$$

Example

- $x_1 = 2 + \epsilon_1$, $x_2 = 2 - \epsilon_1$, $x_3 = 2 + \eta_1$ (geometric concretization $[1, 3]$)

- $x_1$ and $x_2$ are incomparable, both are included in $x_3$. 
The choice of “home-made” functional join and meet operations

- Keep the parameterization by the $\varepsilon_i$
- These operations should not be expensive

A lot of literature on zonotopes

- Control theory and hybrid systems analysis: same problem of intersection of zonotopes with guards (Girard, Le Guernic etc)
- But these methods are geometrical
- Still, could be used on the perturbation part

Now: our join operator (2006-present, with E. Goubault)

- Join on coefficients of the forms (interval coefficients): no!
- Central form plus deviation (SAS 2006): $\gamma(\hat{x} \cup \hat{y})$ in general larger than $\gamma(\hat{x}) \cup \gamma(\hat{y})$, bad for fixpoint computation
- Arxiv 2008 and 2009: the componentwise upper bound presented here
Component-wise upper bound

\[
\begin{align*}
\hat{x} &= 3 + \varepsilon_1 + 2\varepsilon_2 \\
\hat{\mu} &= 0 + \varepsilon_1 + \varepsilon_2 \\
\end{align*}
\]

\[
\begin{align*}
\hat{y} &= 1 - 2\varepsilon_1 + \varepsilon_2 \\
\hat{\mu} &= 0 + \varepsilon_1 + \varepsilon_2 \\
\end{align*}
\]

\[
\hat{x} \cup \hat{y} = 2 + \varepsilon_2 + 3\eta_1 \\
\hat{\mu} = 0 + \varepsilon_1 + \varepsilon_2 \\
\]

Construction (cost $\mathcal{O}(n \times p)$)

- Keep “minimal common dependencies”

\[
z_i = \arg\min_{x_i \wedge y_i \leq r \leq x_i \vee y_i} |r|, \; \forall i \geq 1
\]
Component-wise upper bound

\[
\left( \hat{x} = 3 + \varepsilon_1 + 2\varepsilon_2 \right) \cup \left( \hat{y} = 1 - 2\varepsilon_1 + \varepsilon_2 \right) = \left( \hat{x} \cup \hat{y} = 2 + \varepsilon_2 + 3\eta_1 \right)
\]

Construction (cost $O(n \times p)$)

- Keep “minimal common dependencies”

\[
z_i = \arg\min x_i \land y_i \leq r \leq x_i \lor y_i \quad |r|, \forall i \geq 1
\]

- For each dimension, concretization is the interval union of the concretizations: $\gamma(\hat{x} \cup \hat{y}) = \gamma(\hat{x}) \cup \gamma(\hat{y})$
Fixpoint computation with componentwise join (arxiv 2008, with E. Goubault)

General result on recursive linear filters, pervasive in embedded programs:

\[ x_{k+n+1} = \sum_{i=1}^{n} a_i x_{k+i} + \sum_{j=1}^{n+1} b_j e_{k+j}, \quad e[*] = \text{input}(m, M); \]

- Suppose this concrete scheme has bounded outputs (zeros of \( x^n - \sum_{i=0}^{n-1} a_{i+1} x^i \) have modules strictly lower than 1).
- Then there exists \( q \) such that the Kleene abstract scheme “unfolded modulo \( q \)” converges towards a finite over-approximation of the outputs

\[ \hat{x}_i = \hat{x}_{i-1} \cup f^q(E_i, \ldots, E_{i-k}, \hat{x}_{i-1}, \ldots, \hat{x}_{i-k}) \]

in finite time, potentially with a widening partly losing dependency information

- The abstract scheme is a perturbation (by the join operation) of the concrete scheme
- Uses the stability property of our join operator: for each dimension

\[ \gamma(\hat{x} \cup \hat{y}) = \gamma(\hat{x}) \cup \gamma(\hat{y}) \]
Some results with the APRON library (K. Ghorbal’s Taylor1+ abstract domain, CAV 2009)

**without widening (for \( p = 5 \) and \( p = 16 \))**

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<thead>
<tr>
<th>filter o2</th>
<th>fixpoint</th>
<th>t(s)</th>
</tr>
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<td>Boxes</td>
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</tr>
<tr>
<td>Octagons</td>
<td>( \top )</td>
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<td>Polyhedra</td>
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<td>T.1+(16)</td>
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<td>abort</td>
<td>( &gt;24h )</td>
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<tr>
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<td>( \top )</td>
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</tr>
<tr>
<td>T.1+(16)</td>
<td>( \top )</td>
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</tr>
</tbody>
</table>

**with widening after 10 Kleene iterations (for \( p = 5 \) and \( p = 20 \))**

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<td>([-8.9,10.6])</td>
<td>0.1</td>
</tr>
<tr>
<td>T.1+(20)</td>
<td>([-5.4,7.07])</td>
<td>0.2</td>
</tr>
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</table>

<table>
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<th>filter o8</th>
<th>fixpoint</th>
<th>t(s)</th>
</tr>
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<tr>
<td>Boxes</td>
<td>( \top )</td>
<td>&lt;0.01</td>
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<td>( \top )</td>
<td>2.56</td>
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<td>Polyhedra</td>
<td>abort</td>
<td>( &gt;24h )</td>
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<tr>
<td>T.1+(5)</td>
<td>([-8.9,10.6])</td>
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</table>
Improved join (NSAD 2012, with T. Le Gall, E. Goubault)

Compute and preserve relations between variables and noise symbols

- Compute $k < p$ independent affine relations common to the 2 abstract values joined (solving a linear system)
- Componentwise join on $p - k$ components
- Global join for the $k$ remaining components using the relations
- mub on the $p - k$ components $\Rightarrow$ mub on all components (else ub)

Example

$\left( \hat{x}_1 = 2 + \varepsilon_1 \atop \hat{x}_2 = 2 + \varepsilon_2 \right) \cup \left( \hat{y}_1 = 4 + \varepsilon_1 \atop \hat{y}_2 = 4 + \varepsilon_2 \right)$
Improved join (NSAD 2012, with T. Le Gall, E. Goubault)

Compute and preserve relations between variables and noise symbols

- Compute $k < p$ independent affine relations common to the 2 abstract values joined (solving a linear system)
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- $\text{mub}$ on the $p - k$ components $\Rightarrow \text{mub}$ on all components (else $\text{ub}$)

Example

$$
\begin{pmatrix}
\hat{x}_1 = 2 + \varepsilon_1 \\
\hat{x}_2 = 2 + \varepsilon_2
\end{pmatrix}
\cup
\begin{pmatrix}
\hat{y}_1 = 4 + \varepsilon_1 \\
\hat{y}_2 = 4 + \varepsilon_2
\end{pmatrix}
$$

Componentwise:

$$
\begin{pmatrix}
\hat{z}_1 = 3 + \varepsilon_1 + \eta_1 \\
\hat{z}_2 = 3 + \varepsilon_2 + \eta_2
\end{pmatrix}
$$
Improved join (NSAD 2012, with T. Le Gall, E. Goubault)

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Example

\[
\left( \begin{array}{c}
\hat{x}_1 = 2 + \varepsilon_1 \\
\hat{x}_2 = 2 + \varepsilon_2
\end{array} \right) \cup \left( \begin{array}{c}
\hat{y}_1 = 4 + \varepsilon_1 \\
\hat{y}_2 = 4 + \varepsilon_2
\end{array} \right)
\]

Relation-preserving: \( x_1 - x_2 = \varepsilon_1 - \varepsilon_2 \)

\[
\left( \begin{array}{c}
\hat{w}_1 = 3 + \varepsilon_1 + \eta_1 \\
\hat{w}_2 = 3 + \varepsilon_2 + \eta_1
\end{array} \right)
\]
Improved join (NSAD 2012, with T. Le Gall, E. Goubault)

Compute and preserve relations between variables and noise symbols

- Compute $k < p$ independent affine relations common to the 2 abstract values joined (solving a linear system)
- Componentwise join on $p - k$ components
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Example

$$(\hat{x}_1 = 2 + \varepsilon_1, \hat{x}_2 = 2 + \varepsilon_2) \cup (\hat{y}_1 = 4 + \varepsilon_1, \hat{y}_2 = 4 + \varepsilon_2)$$

Relation-preserving: $x_1 - x_2 = \varepsilon_1 - \varepsilon_2$

$$(\hat{w}_1 = 3 + \varepsilon_1 + \eta_1, \hat{w}_2 = 3 + \varepsilon_2 + \eta_1)$$

Future work: imprecise relations
Policy iteration: Newton-like method to replace the widening process

- Kleene iteration with widening can be inefficient and/or inaccurate
  - especially for bounds on rounding errors (no narrowing)
- Policy iteration introduced for stochastic control problems (Howard 60)
- Pointed out to our attention by S. Gaubert and proposed together for static analysis on intervals (CAV 2005)
  - selection principle: fixpoint problem can be reduced to computing the inf of fixpoint of simpler problems
  - decreasing iteration from post-fixpoints: can stop at any time
  - often faster and more accurate than Kleene iteration, but complexity not well known (exponential worst case?)
- For relational domains (Adje-Gaubert-Goubault-Taly-Zennou, 2007-2012)
- Dual method by Gawlitza and Seidl (2007-2012)

Policy iteration and zonotopes?

- Mainly future work, 2 approaches investigated (with intern M. Buchet and T. Le Gall): policies on the argmin operator (join) or on the constraints on noise symbols (meet)
Main idea to interpret test, informally

- Translate the condition on noise symbols: constrained affine sets
- Abstract domain for the noise symbols: intervals, octagons, etc.
- Equality tests are interpreted by the substitution of one noise symbol of the constraint (cf summary instantiation for modular analysis)
- More general constraints in the future?

Example

```c
real x = [0, 10];
real y = 2*x;
if (y >= 10)
    y = x;
```

- Affine forms before tests: $x = 5 + 5\varepsilon_1$, $y = 10 + 10\varepsilon_1$
- In the if branch $\varepsilon_1 \geq 0$: condition acts on both $x$ and $y$

(Minimal) upper bound computation on constrained affine sets is difficult
Remember: example of functional abstraction

\[
\begin{align*}
\text{real } & x = [0, 10]; \\
\text{real } & y = x \times x - x;
\end{align*}
\]

Abstraction of function \( x \to y = x^2 - x \) as

\[
\begin{align*}
y & = 32.5 + 45\varepsilon_1 + 12.5\eta_1 \\
& = -12.5 + 9x + 12.5\eta_1
\end{align*}
\]

- Almost modular by construction!
- But valid only for inputs in \([0,10]\) ⇒ partition of contexts though a summary-based algorithm.
Summary-based modular analysis

A summary of a program function is a pair of zonotopes \((I, O)\)

- \(I\) abstracts the calling context, \(O\) the output
- Both zonotopes abstract functions of the inputs of the program (linear form of the \(\varepsilon_i\))
- Output \(O\) can be instantiated for a particular calling context \(C \subseteq I\) (constraints on the noise symbols that define a restriction of the function)

Summary instantiation: \([I == C] O\)

- Write the constraint \(I == C\) in the space of noise symbols:

\[
(c_{0i}^I - c_{0i}^C) + \sum_{r=1}^{n}(c_{ri}^I - c_{ri}^C)\varepsilon_r + \sum_{r=1}^{m}(p_{ri}^I - p_{ri}^C)\eta_r = 0 \quad (i = 1, \ldots, p)
\]

- We derive relations of the form (by Gauss elimination):

\[
\eta_{k+1} = R_i(\eta_i, \ldots, \eta_1, \varepsilon_n, \ldots, \varepsilon_1) \quad (i = 0, \ldots, r - 1)
\]

- We eliminate \(\eta_1, \ldots, \eta_r\) in \(O\) using the relations above
Example: summary-based analysis

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
</table>
| 0 | ```
function mult(real a, real b) {
    return a*(b-2);
}
``` |
| 1 | ```
x := [-1,1]; // x = eps_1
``` |
| 2 | ```
real y1 = mult(x+1, x);
``` |
| 3 | ```
real y2 = mult(x, 2*x);
``` |

Summary construction
Example: summary-based analysis

\[
\text{(0)} \quad \text{mult}(\text{real } a, \text{ real } b) \{ \text{return } a \cdot (b - 2); \}
\]

\[
\text{(1)} \quad x := [-1, 1]; \quad \text{// } x = \text{eps}_1
\]

\[
\text{(2)} \quad \text{real } y_1 = \text{mult}(x+1, x);
\]

\[
\text{(3)} \quad \text{real } y_2 = \text{mult}(x, 2 \cdot x);
\]

Summary construction
Example: summary-based analysis

\begin{verbatim}
(0) mult(real a, real b) { return a*(b-2); }
(1) x := [-1,1]; // x = eps_1
(2) real y1 = mult(x+1, x);
(3) real y2 = mult(x, 2*x);
\end{verbatim}

Summary instantiation to context $C_2$
Affine sets: some current and future variations

Keep same parameterization $x = \sum_i x_i \varepsilon_i$ but with

- Interval coefficients $x_i$: generalized affine sets for under-approximation
  - under-approximation: sets of values of the outputs, that are sure to be reached for some inputs in the specified ranges
  - interval coefficients $x_i$, noise symbols in generalized intervals ($\varepsilon_i = [-1, 1]$ or $\varepsilon_i^* = [1, -1]$), Kaucher arithmetic extends classical interval arithmetic (SAS 2007, with E. Goubault)
  - extensions (higher order forms, under-approximations of discrete values, interval methods of Goldsztejn/Jaulin, application to robust control): O. Mullier’s PhD thesis (with E. Goubault, M. Kieffer)

- Noise symbols $\varepsilon_i$ being no longer defined in intervals:
  - probabilistic affine forms: $\varepsilon_i$ take values in p-boxes (Computing 2012, with O. Bouissou, E. Goubault, J. Goubault-Larrecq)
  - ellipsoids (clear potential for program invariants): $\|\varepsilon\|_2 \leq 1$ (instead of $\|\varepsilon\|_\infty \leq 1$)?

But also, polynomial/rational invariants (extensions of changes of bases by Sankaranarayanan for instance)
Mean-value theorem (à la Goldsztejn 2005)
Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) differentiable, \( (t_1, \ldots, t_n) \) a point in \([-1, 1]^n\) and \( \Delta_i \) such that
\[
\left\{ \frac{\partial f}{\partial \varepsilon_i}(\varepsilon_1, \ldots, \varepsilon_i, t_{i+1}, \ldots, t_n), \; \varepsilon_i \in [-1, 1] \right\} \subseteq \Delta_i.
\]
Then
\[
\tilde{f}(\varepsilon_1, \ldots, \varepsilon_n) = f(t_1, \ldots, t_n) + \sum_{i=1}^{n} \Delta_i(\varepsilon_i - t_i),
\]
is interpretable in the following way:
- if \( \tilde{f}(\varepsilon_1^*, \ldots, \varepsilon_n^*) \), computed with Kaucher arithmetic, is an improper interval, then \( \text{pro} \tilde{f}(\varepsilon_1^*, \ldots, \varepsilon_n^*) \) is an under-approx of \( f(\varepsilon_1, \ldots, \varepsilon_n) \).
- \( \tilde{f}(\varepsilon_1, \ldots, \varepsilon_n) \) is an over-approx of \( f(\varepsilon_1, \ldots, \varepsilon_n) \).

Generalized affine forms
- Affine forms with interval coefficients, defined on the \( \varepsilon_i \) (no \( \eta_j \) symbols)
- Under-approximation by over-approximation of dependencies
- Joint use of under-/over-approximation: quality of analysis results
- Extract scenarios giving extreme values
Example

\[ f(x) = x^2 - x \text{ when } x \in [2, 3] \text{ (real result } [2, 6]) \]

- **Affine form**
  
  \[ x = 2.5 + 0.5\varepsilon_1, \quad f^\varepsilon(\varepsilon_1) = (2.5 + 0.5\varepsilon_1)^2 - (2.5 + 0.5\varepsilon_1) \]

- **Bounds on partial derivative**
  
  \[
  \frac{\partial f^\varepsilon}{\partial \varepsilon_1}(\varepsilon_1) = 2 \times 0.5 \times (2.5 + 0.5\varepsilon_1) - 0.5 \subseteq [1.5, 2.5]
  \]

- **Mean value theorem with } t_1 = 0**
  
  \[
  \tilde{f}^\varepsilon(\varepsilon_1) = 3.75 + [1.5, 2.5]\varepsilon_1
  \]

  Under-approximating concretization
  
  \[ 3.75 + [1.5, 2.5][1, -1] = 3.75 + [1.5, -1.5] = [5.25, 2.25] \]

  Over-approximating concretization
  
  \[ 3.75 + [1.5, 2.5][-1, 1] = 3.75 + [-2.5, 2.5] = [1.25, 6.25] \]

- **Affine arithmetic (over-approximation)**
  
  \[ x^2 - x = [3.75, 4] + 2\varepsilon_1 \text{ (concretization } [1.75, 6]) \]
Square-root algorithm (Householder method)

double Input, x, xp1, residue, shouldbezero;
double EPS = 0.00002;

Input = __BUILTIN_DAED_DBETWEEN(16.0, 20.0);
x = 1.0/Input; xp1 = x; residue = 2.0*EPS;
while (fabs(residue) > EPS) {
    xp1 = x*(1.875+Input*x*x*x*(-1.25+0.375*Input*x*x*x));
    residue = 2.0*(xp1-x)/(x+xp1);
    x = xp1;
}
shouldbezero = x*x-1.0/Input;

---

With 32 subdivisions of the input
- Stopping criterion of the Householder algorithm is satisfied after 5 iterations:
  \[ [0, 0] \subseteq \text{residue}(x_4, x_5) \subseteq [-1.44e^{-5}, 1.44e^{-5}] \]
- Tight enclosure of the iterate:
  \[ [0.22395, 0.24951] \subseteq x_5 \subseteq [0.22360, 0.25000] \]
- Functional proof:
  \[ [0, 0] \subseteq \text{shouldbezero} \subseteq [-1.49e^{-6}, 1.49e^{-6}] \]
Combining deterministic and probabilistic methods

- Discrete p-boxes or Dempster-Shafer structures
  - Generalize probability distributions and interval computations: less pessimistic than intervals but still guaranteed
  - Represent sets of probability distributions: between an upper and a lower Cumulative Distribution Function $P(X \leq x)$

- Encode as much deterministic dependencies as possible by affine arithmetic
  - because arithmetic on p-boxes/DS not very efficient
  - associate a DS structure to each noise symbol
  - both more accurate and faster than direct DS arithmetic:
Example: recursive filter with independent inputs in \([-1,1]\)

Prove that dangerous worst case occur with very low probability

- Deterministic analysis (left): outputs in \([-3.25,3.25]\) (exact)
- Mixed probabilistic/deterministic analysis (right): outputs in \([-3.25,3.25]\), and in \([-1,1]\) with very strong probability

Some current work

- Set operations (extending the deterministic ones)
- Handling dependency information between noise symbols
- Efficient rounding error computation
Validate finite precision implementations
Prove that the program computes something close to what is expected (in real numbers)
  ■ Accuracy of results
  ■ Behaviour of the program (control flow, number of iterations)

Validate algorithms
Bound when possible the method/approximation error
  ■ Check functional properties in real-number semantics

Ideally, find good match between method and implementation errors
Floating-point numbers (defined by the IEEE 754 norm)

- Normalized floating-point numbers
  \[ (-1)^s 1.x_1 x_2 \ldots x_n \times 2^e \] (radix 2 in general)
  - implicit 1 convention \((x_0 = 1)\)
  - \(n = 23\) for simple precision, \(n = 52\) for double precision
  - exponent \(e\) is an integer represented on \(k\) bits \((k = 8\) for simple precision, \(k = 11\) for double precision\)

- Denormalized numbers (gradual underflow),
  \[ (-1)^s 0.x_1 x_2 \ldots x_n \times 2^{e_{\text{min}}} \]

- Consequences and difficulties:
  - limited range and precision: potentially inaccurate results, run-time errors
  - no associativity, representation error for harmless-looking reals such as 0.1
  - re-ordering by the compiler, use of registers with different precision, etc
Concrete semantics

- **Aim:** compute rounding errors and their propagation
  - we need the floating-point values
  - relational (thus accurate) analysis more natural on real values
  - for each variable, we compute \((f^x, r^x, e^x)\)
  - then we will abstract each term (real value and errors)

```plaintext
float x, y, z;
x = 0.1; // [1]
y = 0.5; // [2]
z = x+y; // [3]
t = x*z; // [4]

f^x = 0.1 + 1.49e^{-9} [1]
f^y = 0.5
f^z = 0.6 + 1.49e^{-9} [1] + 2.23e^{-8} [3]
```
```c
#include "daed_builtins.h"

int main() {
    int i;
    double y=0.7;
    double x=y;
    for (i=1; i<=20; i++) {
        x=11*x-7;
    }
    return 0;
}
```
IEEE 754 norm on f.p. numbers specifies the rounding error:

- Elementary rounding error when rounding $r^x$ to $\uparrow_\circ r^x$:

  $$\exists (\delta_r > 0, \delta_a > 0), \ |r^x - \uparrow_\circ r^x| \leq \max(\delta_r|\uparrow_\circ r^x|, \delta_a)$$

- The f.p. result of arithmetic elementary operations $+,-,\times,/,\sqrt{}$ is the rounded value of the real result

  - unit in the Last Place $ulp(x) =$ distance between two consecutive floating-point numbers around $x =$ maximal rounding error around $x$

    $$ulp(1) = 2^{-23} \approx 1.19200928955 \times 10^{-7}$$

  - rounding error of elementary operation always less than the ulp of the result
  - also more refined properties, such as Sterbenz lemma (if $x$ and $y$ are two float such that $\frac{y}{2} \leq x \leq y$, then f.p. operation $x - y$ is exact)

Abstraction: for each variable $x$, a triplet $(f^x, r^x, e^x)$
Abstract value

- For each variable:
  - Interval $f^x = [f^x, \bar{f}^x]$ bounds the finite prec value, $(f^x, \bar{f}^x) \in F \times F$,
  - Affine forms for real value and error; for simplicity no $\eta$ symbols

$$f^x = (\alpha_0^x + \bigoplus_i \alpha_i^x \varepsilon_i^r) + (e_0^x + \bigoplus_l e_l^x \varepsilon_l^e)$$

real value

center of the error

uncertainty on error due to point $l$

$$+ \bigoplus_i m_i^x \varepsilon_i^r$$

propag of uncertainty on value at pt $i$

- Constraints on noise symbols (interval + equality constraints)
  - for finite precision control flow
  - for real control flow
Stable test assumption (old version)

- Control flow of the program is same for the finite precision and real values of the program
- If found not to be the case: unstable test warning
- Joining branches:
  - join values and errors coming from the two branches
  - in case of unstable test, possibly unsound error bounds

Sound unstable test analysis (new version ... November 2012!)

- Tests interpreted over real and float values: two sets of constraints on noise symbols
- Joining branches
  - join fp and real values from the branches
  - errors: join error computed in the two branches with, when it exists (unstable test), the difference between real value in one branch and float value in the other branch for the same execution (ie for same values of the $\varepsilon_i$ = intersections of the constraints for these two branches)

In the line of robustness/continuity analysis of Chaudhuri, Gulwani and al.
Test interpretation: example of unstable test

```c
ajoute 1, sinon

if ((REG_Locale_i3 % 2) != 0)
{
    REG_Locale_i3++;
}
else
{
    REG_Locale_f1 = REG_Locale_f1 *

REG_Locale_f2;

/*****************************/
*******/
/* On remet l'exposant en place, on
multiplie par 2^Exposant/2 */

/*****************************/
*******/

Registro_PtrZoneDoubleVar[0] =
(REG_Locale_i3 / 2 + 1023) <= 20);
(_S1) = _Locale_DoubleVar *

sqrt.c

(Const SQRT C5 (double)
_E1 (double)
_Locale_DoubleConst (double)
_Locale_DoubleVar (double)
_S1 (double)
signgam (integer)

Variables / Files

Variable Interval

Float :
1.37910938 5.82669338

Real :
1.37910938 5.82669338

Global error :
-5.47324865 5.47324587

Relative error :
-3.96868349 3.96868148

Higher Order error :
0 0

At current point (93) :
-1.27995 1.27995

Last analysis : 0.01 sec / 16384 Kilo Bytes
```

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The static analyzer FLUCTUAT

The Fluctuat team

- Continuous support by Airbus and IRSN, more occasional by other users
  - Is/has been used for a wide variety of codes (automotive, nuclear industry, aeronautics, aerospace) of size up to about 50000 LOCs

Assertions in the program analyzed

- Hypotheses on the environment of the program
- Local partitioning and collecting strategies

Exploitation of results

- Warnings: unstable tests, run-time errors
- Identifying problems and refining results: worst case scenarios, symbolic execution, subdivisions
- Library and new interactive version (F. Védrine)
Second order filters
Back to the Householder scheme
First extension to the analysis of hybrid systems by interaction with the ODE guaranteed solver GrkLib (HybridFluctuat, CAV 2009, with O. Bouissou, E. Goubault, K. Tekkal, F. Védrine)

First interface with Esterel’s SCADE (with library version of Fluctuat)

(Present/Future) Interaction with constraint solvers
  - first demonstration of refinement of Fluctuat’s result (Ponsini, Michel, Rueher 2011-2012)
  - natural for us because increasing use of constraints on noise symbols (and we could provide more for non linear operations)

(Present/Future) Floating-point to fixed point automatic conversion (ANR DEFIS project 2012-2015, first steps in 2005 with intern J. Mascunan)

(Future) Interaction with provers, Frama-C platform
  - ACSL language to exchange information on real, float and errors
  - provide provers with loop invariants
  - use locally refined results and properties
  - towards formally proved abstract domain implementations ? (cf D. Pichardie)
Conclusion and Perspectives

Some highlights in the near future

- Under-approximations (O. Mullier’s PhD thesis)
- Fixpoint computations and zonotopes (if possible in Fluctuat): policy iteration, mix with ellipsoids?

Keep the line of numerically-oriented static analysis

- Natural application is industrial safety-critical control-command software
  - needs driven by the users (modularity, interaction with other tools etc)
- Extension to safety-critical control systems (medical instrumentation?)
- Long term goal: programs from scientific computing
  - as a complement to existing sensitivity and uncertainty propagation analyses (Chaos polynomial, Cestac method)
  - specific problems (large arrays)
  - parallelism and reordering of arithmetic operations
  - link with mathematical studies of schemes in finite precision (Wilkinson, Meurant, Demmel etc), view as perturbed schemes