Inner Approximation of the Range of Vector-valued Functions, Episode 2

SWIM 2011

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Aim

Static Analysis

For a given program

- **Input** Sets defining the possible values for the variables of this program;
- **Output** Results guaranteed to be reached with input variables.

Robust Control

For a given model

- **Input** Initial state and control input (with uncertainty)
- **Output** Intermediate and final states

In both cases

Find range of vector-valued function $f : \mathbb{R}^m \to \mathbb{R}^n$

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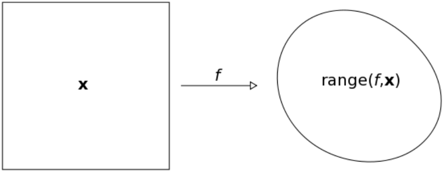
Find range of vector-valued function $f : \mathbb{R}^m \to \mathbb{R}^n$

The range

In general

Find $\mathbb{Y} = \mathrm{range}(f, \mathbf{x})$



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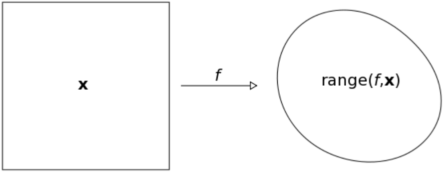


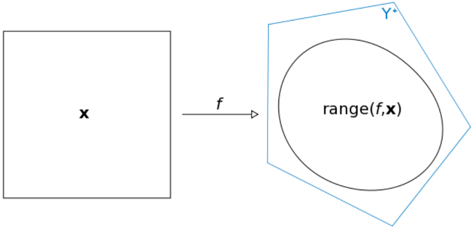
Issue

Computing the range of a function is intractable in general

The range

In general

Find $\mathbb{Y}^+ \supseteq \text{range}(f, \mathbf{x})$



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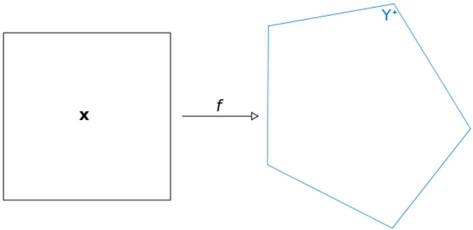


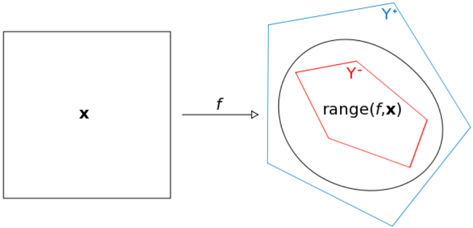
Issue

How pessimistic is the over estimation?

The range

Here

Find $\mathbb{Y}^- \subseteq \text{range}(f, \mathbf{x}) \subseteq \mathbb{Y}^+$



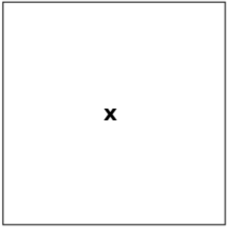
State of the art

Inner-approximation

(Kieffer, Jaulin & Walter 1998) f must be *globally* invertible

(Goldsztejn & Jaulin 2010) f must be *locally* invertible (dimension of domain and codomain must be equal)

Computing an inner approximation via set inversion (Kieffer et al. 1998)



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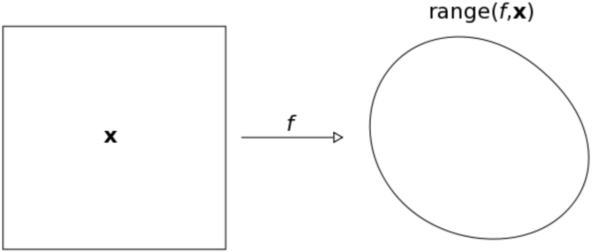

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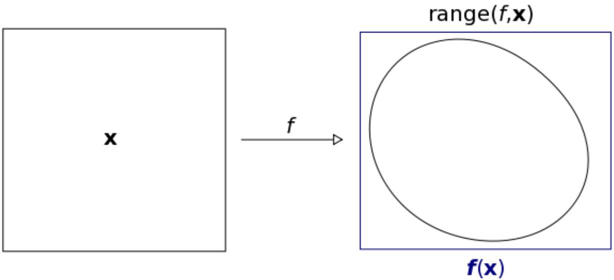

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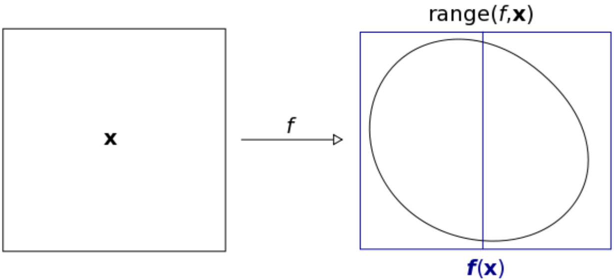

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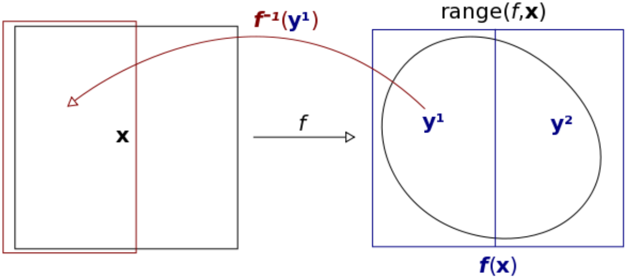

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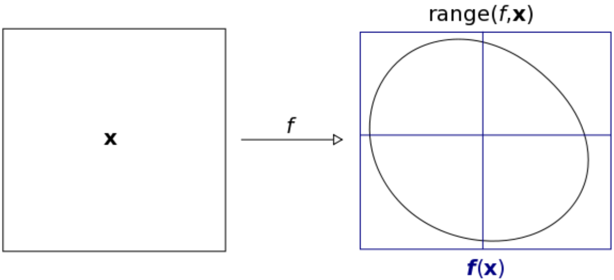

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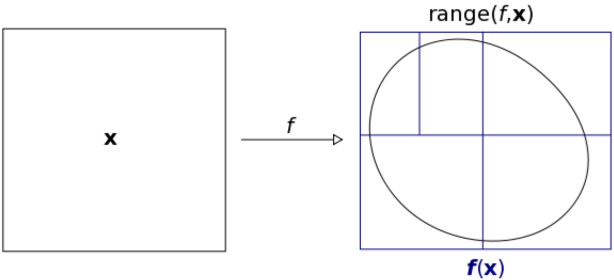

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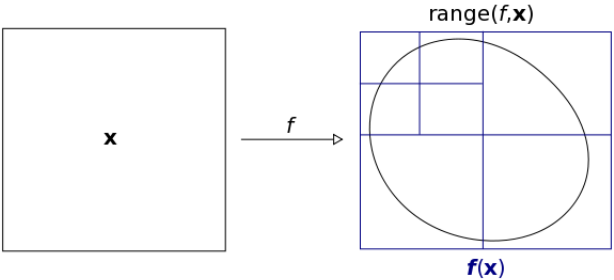

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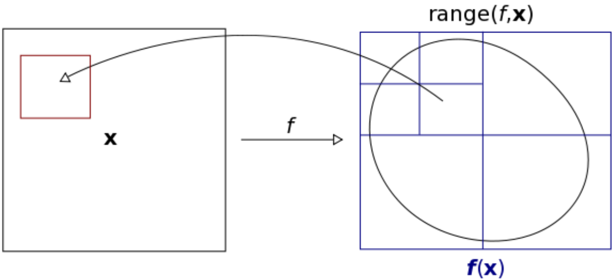

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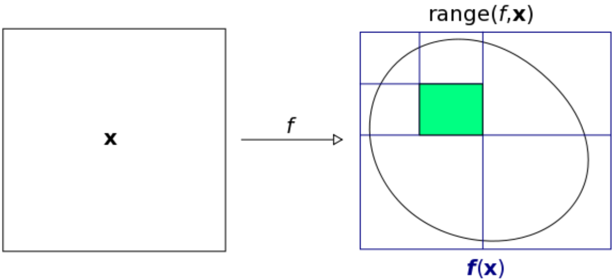

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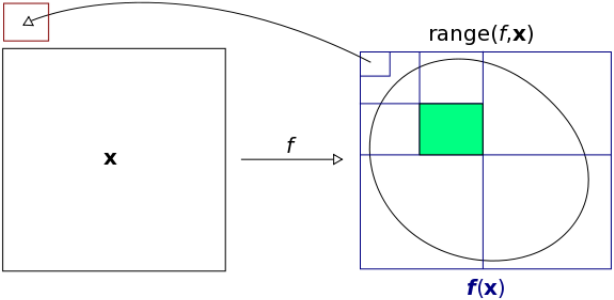

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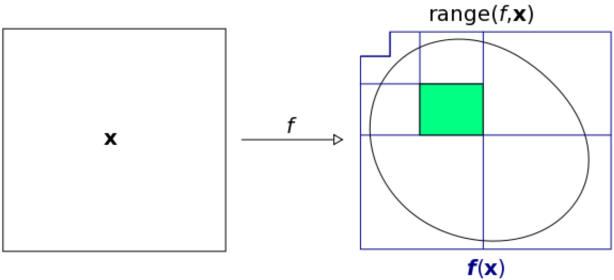

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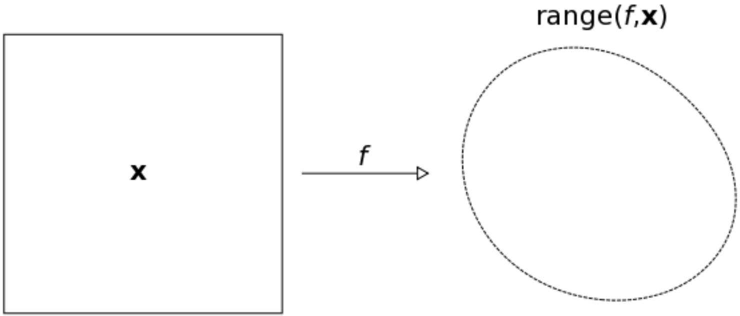

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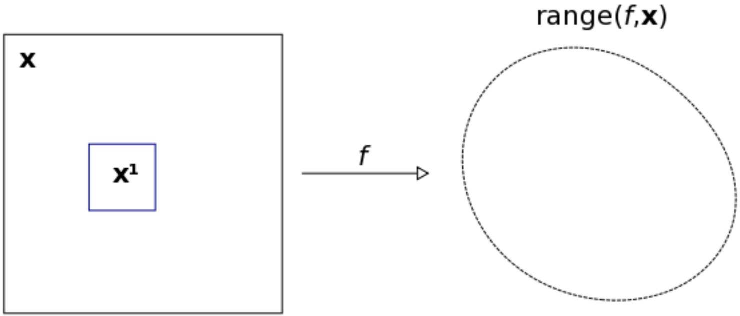

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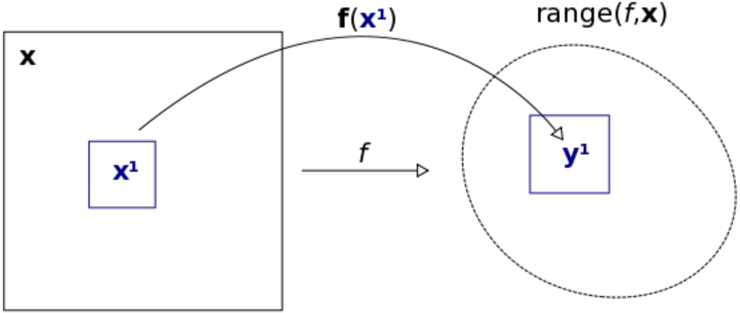

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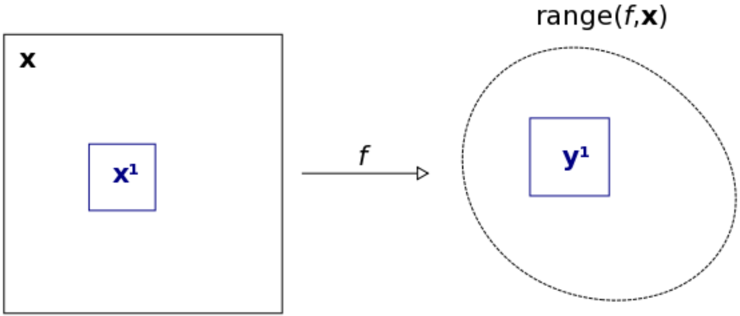

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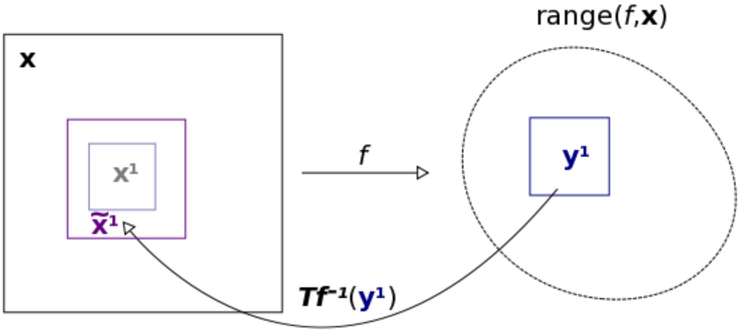

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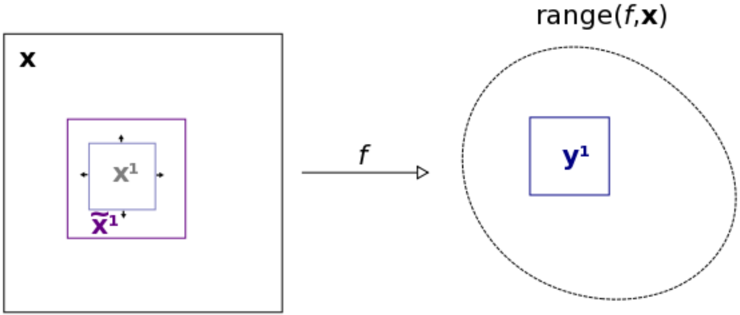

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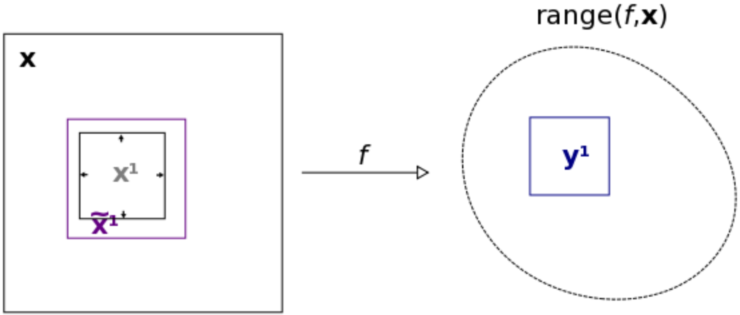

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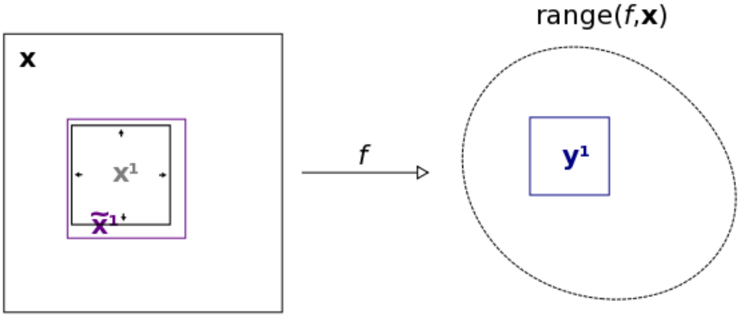

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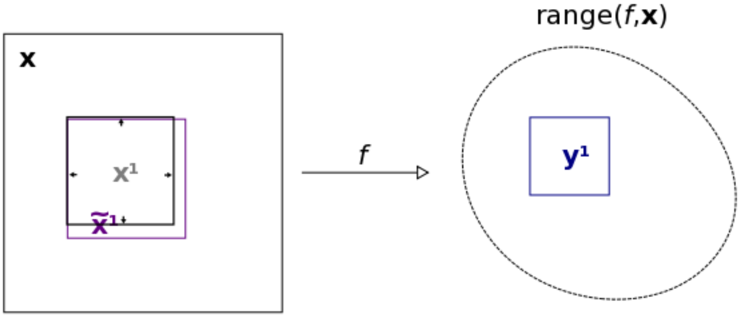

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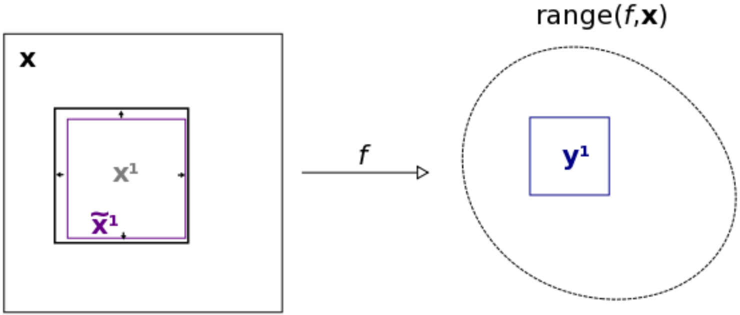

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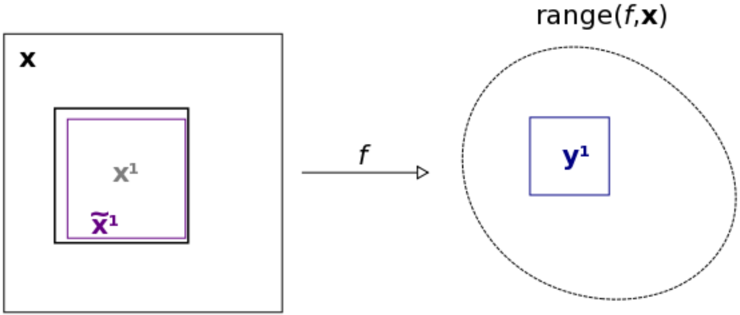

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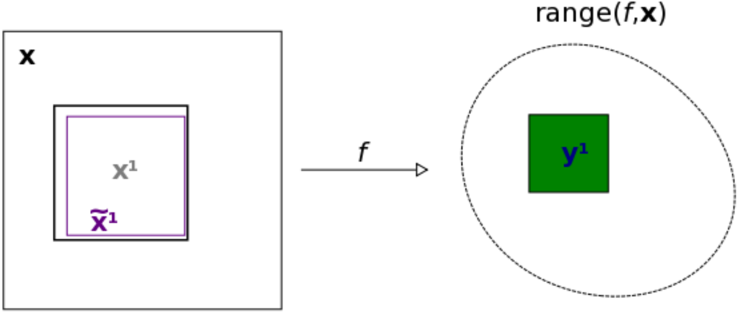

Outline

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Idea

Starting Point : Order 1 Taylor Expansion

$$f(x) = f(\tilde{x}) + J(x - \tilde{x})$$

Idea

Let consider $y = f(\tilde{x}) + J(x - \tilde{x})$

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$$y - f(\tilde{x}) = J(x - \tilde{x})$$
$$y - f(\tilde{x}) = (\text{Diag } J + \text{OffDiag } J)(x - \tilde{x})$$

Split

$$(\text{Diag } J)_{ij} = \begin{cases} (J)_{ij} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad (\text{OffDiag } J)_{ij} = \begin{cases} (J)_{ij} & \text{if } i \neq j \\ 0 & \text{otherwise} \end{cases}$$

Limitation

This split implies that dimensions of domain and codomain have to be equal.

Idea

Let consider $y = f(\tilde{x}) + J(x - \tilde{x})$

$$
\begin{aligned}
y - f(\tilde{x}) &= J(x - \tilde{x}) \\
y - f(\tilde{x}) &= (\text{Diag } J + \text{OffDiag } J)(x - \tilde{x}) \\
x &= \tilde{x} + \text{Diag}^{-1}\ J(y - f(\tilde{x}) - \text{OffDiag } J(x - \tilde{x}))
\end{aligned}
$$

Constraint

J_{ii} must not be equal to 0

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$$
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Criteria

For a box $\mathbf{y} \in \mathbb{R}^n$:

$$\tilde{x} + \text{Diag}^{-1}\ \mathbf{J}(\mathbf{y} - f(\tilde{x}) - \text{OffDiag } \mathbf{J}(\mathbf{x} - \tilde{x})) \subseteq \text{int } \mathbf{x}$$

$$\Downarrow$$

$$\mathbf{y} \subseteq \text{range}(f, \mathbf{x})$$

Extension: $m \neq n$

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Limitation

This criteria is only valid when dimensions of domain m and codomain n of f are equal

Idea

Transformation of $f : \mathbb{R}^m \to \mathbb{R}^n$ to a function $g : \mathbb{R}^\beta \to \mathbb{R}^\beta$, with $\beta = \max(m, n)$ and with similar range properties.

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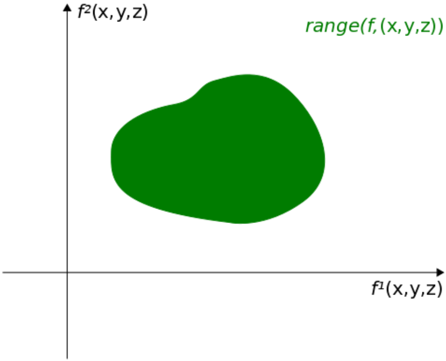
Outline

Examples: $m > n$

Example 1

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} f^1(x,y,z) \\ f^2(x,y,z) \end{pmatrix}$$



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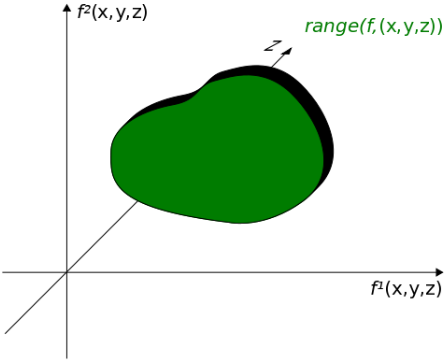

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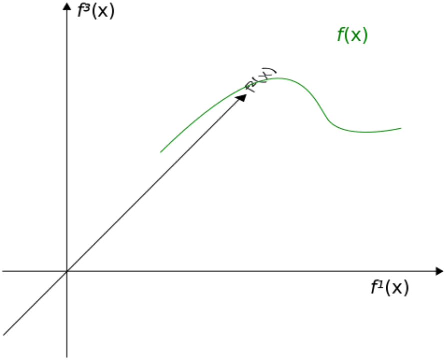

Construction

f is of constant rank 2 in a given box **x**

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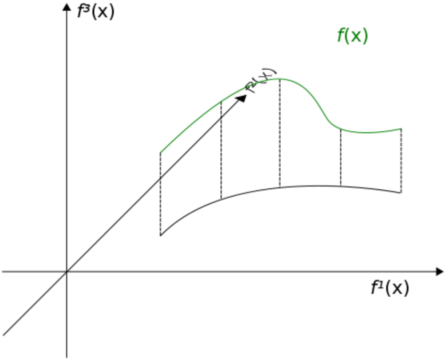

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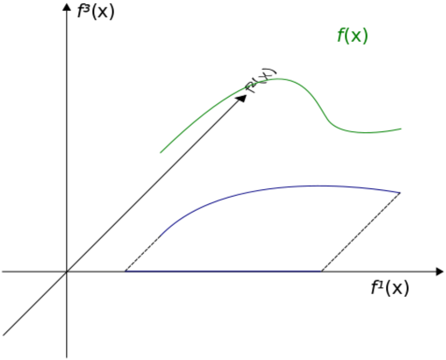


Construction

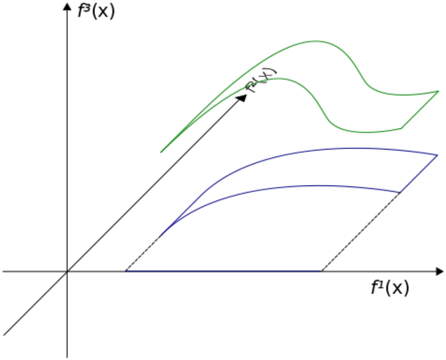
Projection of range(f, $\mathbf{x}$) is made

Examples: $m < n$

Example 1

$$g : \mathbb{R}^2 \to \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} f^1(x) \\ f^2(x) + y \\ f^3(x) \end{pmatrix}$$



Construction

We add y to the domain of f

Examples: $m < n$

Example 1

$$g : \mathbb{R}^3 \to \mathbb{R}^3$$

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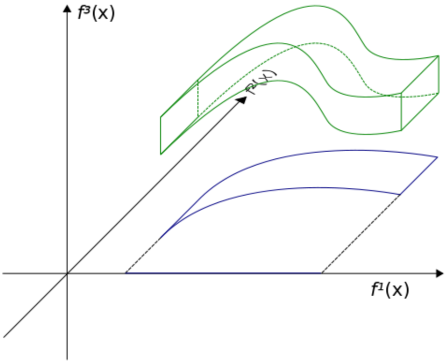

Construction

We add z to the domain of f

Extension

Constant Rank Theorem

This method is supported by functions verifying the *constant rank theorem*.
For a function $f : \mathbb{R}^m \to \mathbb{R}^n$ of constant rank k:

- $k = n$ we obtain the inner-approximation of the range of f
- Otherwise we obtain the inner-approximation of the projection of k components of f

Function g

For $\alpha = \min(m, n)$ and $\beta = \max(m, n)$:

$$\text{Let } g : \mathbb{R}^k \times \mathbb{R}^{\alpha-k} \times \mathbb{R}^{\beta-\alpha} \to \mathbb{R}^k \times \mathbb{R}^{\alpha-k} \times \mathbb{R}^{\beta-\alpha}$$

$$(u, v_1, v_2) \mapsto \begin{pmatrix} f_{[1..k]}(u, v_1, v_2) \\ f_{[k+1..\alpha]}(u, v_1, v_2) + v_1 \\ f_{[\alpha+1..\beta]}(u, v_1, v_2) + v_2 \end{pmatrix}$$

with $f_{[\alpha+1..\beta]}(u, v_1, v_2) = 0_{\beta-\alpha}$ if $\alpha = n$

Jacobian of g : J^g

$$J^g = \begin{pmatrix} J^{f,[1..k],u} & J^{f,[1..k],v_1} & J^{f,[1..k],v_2} \\ J^{f,[k+1..\alpha],u} & \mathrm{I} & 0 \\ J^{f,[\alpha+1..\beta],u} & 0 & \mathrm{I} \end{pmatrix}$$

with $J^{f,[a..b],t} = \left(\frac{\partial f_i}{\partial t}\right)_{i \in [a..b]}$

Extension: criteria

Criteria (Goldsztejn & Jaulin 2010)

For a box $\mathbf{y} \in \mathbb{R}^n$:

$$\tilde{x} + \mathrm{Diag}^{-1}\ \mathbf{J}(\mathbf{y} - f(\tilde{x}) - \mathrm{OffDiag}\ \mathbf{J}(\mathbf{x} - \tilde{x})) \subseteq \mathrm{int}\ \mathbf{x}$$
$$\Downarrow$$
$$\mathbf{y} \subseteq \mathrm{range}(f, \mathbf{x})$$

Criteria: (Extension)

For a box $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) \in \mathbb{R}^k \times \mathbb{R}^{\alpha-k} \times \mathbb{R}^{\beta-\alpha}$:

$$\tilde{u} + \mathrm{Diag}^{-1}_{J_{f,[1..k],u}}(\mathbf{y}_1 - f_{[1..k]}(\tilde{u}, \tilde{v}_1, \tilde{v}_2) - \mathrm{OffDiag}_{J_{f,[1..k],u}}(\mathbf{u} - \tilde{u}) - J_{f,[1..k],v_1}(\mathbf{v}_1 - \tilde{v}_1) - J_{f,[1..k],v_2}(\mathbf{v}_2 - \tilde{v}_2)) \subseteq \mathrm{int}\ \mathbf{u}$$
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$$\mathbf{y}_1 \subseteq \mathrm{Im}(f_1, \ldots, f_k)$$

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Outline

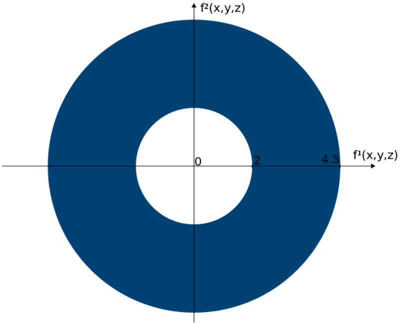
Implementation

- Computation in C++ using Profil/bias library for interval computations;
- Slight changes in algorithms provided in (Goldsztejn & Jaulin 2010)

Experiments

- Experiments have been runned on a MacBook Pro 13" Core i5 2,3 GHz

$m > n$



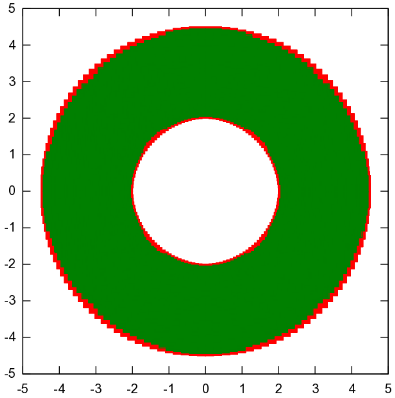
Function definition

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$

$$(x, y, z) \mapsto \begin{cases} (x + r\cos(z))\cos(y) \\ (x + r\sin(z))\sin(y) \end{cases}$$

$$\mathbf{x} = [3, 4]\,;\;\; \mathbf{y} = [0, 2\pi]\,;\;\; \mathbf{z} = [0, 2\pi]$$

$m > n$



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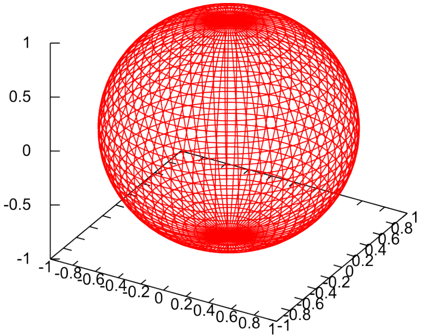


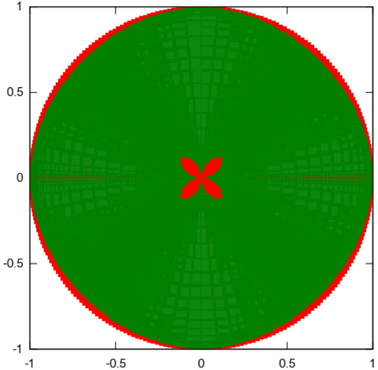
Function definition

$$f : \mathbb{R}^2 \to \mathbb{R}^3$$

$$(u, v) \mapsto \begin{cases} f_1(u, v) = \cos(u)\cos(v) \\ f_2(u, v) = \sin(u)\cos(v) \\ f_3(u, v) = \sin(v) \end{cases}$$

$$\mathbf{u} = [0, 2\pi]; \ \mathbf{v} = [-\pi, \pi]$$

$m < n$



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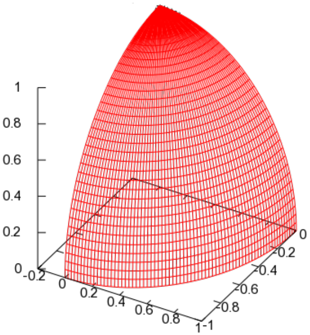
$\varepsilon = 0.04$

$$\mathbf{u} = \left[\frac{3\pi}{2} + \varepsilon, 2\pi - \varepsilon\right]; \ \mathbf{v} = \left[\varepsilon, \frac{\pi}{2} - \varepsilon\right], \ \varepsilon > 0$$

$m < n$



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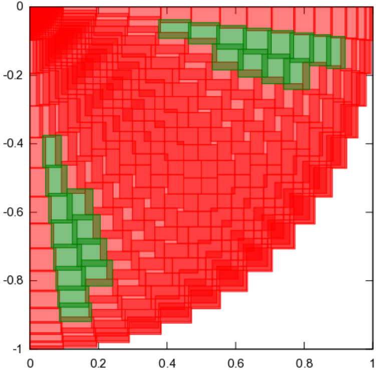

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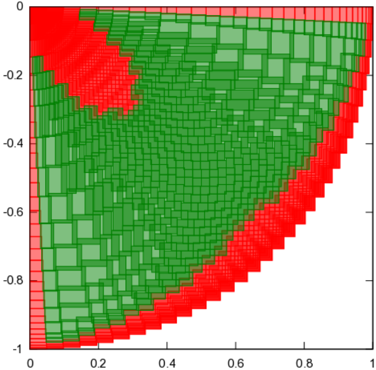
$$\varepsilon = 0.1$$

$$\mathbf{u} = \left[\frac{3\pi}{2} + \varepsilon, 2\pi - \varepsilon\right];\ \mathbf{v} = \left[\varepsilon, \frac{\pi}{2} - \varepsilon\right],\ \varepsilon > 0$$

$m < n$



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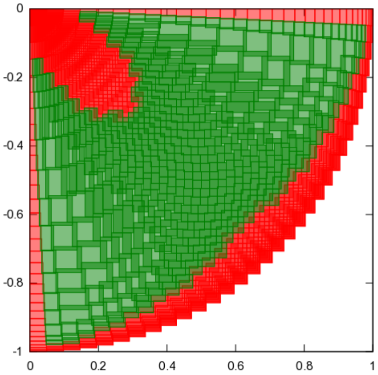
$$\varepsilon = 0.08$$

$$\mathbf{u} = \left[\frac{3\pi}{2} + \varepsilon, 2\pi - \varepsilon\right]; \ \mathbf{v} = \left[\varepsilon, \frac{\pi}{2} - \varepsilon\right], \ \varepsilon > 0$$

$m < n$



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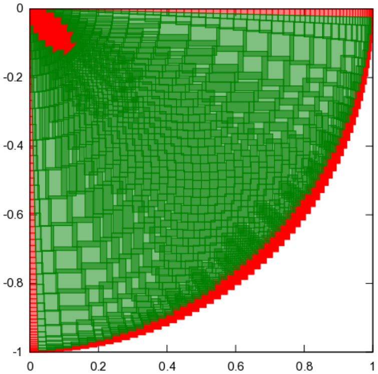
$$\varepsilon = 0.06$$

$$\mathbf{u} = \left[\frac{3\pi}{2} + \varepsilon, 2\pi - \varepsilon\right];\ \mathbf{v} = \left[\varepsilon, \frac{\pi}{2} - \varepsilon\right],\ \varepsilon > 0$$

$m < n$



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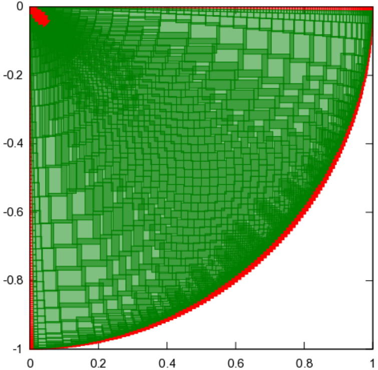
$$\varepsilon = 0.04$$

$$\mathbf{u} = \left[\frac{3\pi}{2} + \varepsilon, 2\pi - \varepsilon\right];\ \mathbf{v} = \left[\varepsilon, \frac{\pi}{2} - \varepsilon\right],\ \varepsilon > 0$$

$m < n$



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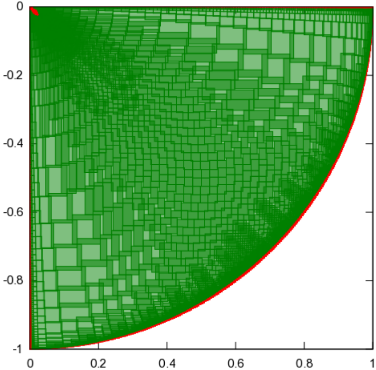
$$\varepsilon = 0.02$$

$$\mathbf{u} = \left[\frac{3\pi}{2} + \varepsilon, 2\pi - \varepsilon\right] ; \ \mathbf{v} = \left[\varepsilon, \frac{\pi}{2} - \varepsilon\right], \ \varepsilon > 0$$

$m < n$



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$$\varepsilon = 0.01$$

$$\mathbf{u} = \left[\frac{3\pi}{2} + \varepsilon, 2\pi - \varepsilon\right]; \ \mathbf{v} = \left[\varepsilon, \frac{\pi}{2} - \varepsilon\right], \ \varepsilon > 0$$

Outline

Conclusion

Extension

This extension is able to

- Compute inner-approximation when $k = n$;
- Compute k components of f otherwise.

Future Works

- Deal with redondant computations when $m > n$;
- Find submatrix of maximal constant rank (hints);
- Extension to other abstract domains (e.g. affine forms).

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References

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Kieffer, M., Jaulin, L. & Walter, E. (1998), Guaranteed recursive nonlinear state estimation using interval analysis, *in* 'Decision and Control, 1998. Proceedings of the 37th IEEE Conference on', Vol. 4, IEEE, pp. 3966–3971.