Quillen Model Categories
Model
Martin-Löf Type Theory with Identity Types

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Disclaimer

Ideas are not from me (Awodey & Warren, Voevodsky, . . .),
errors are mine.

λ-calculus

- Introduction rule:

$$\frac{\Gamma, x : A \vdash f : B}{\Gamma \vdash \lambda x.f : A \to B}$$

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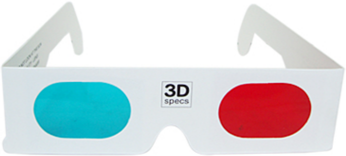

Now with dependent types.

Dependent types

```
Array.make :  int -> array
```

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```
Array.make :  n:int -> n array
```

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$$\frac{}{\Gamma \vdash [] : \texttt{array}(0)} \qquad \frac{\Gamma \vdash k : \texttt{int} \qquad \Gamma, n : \texttt{int} \vdash a : \texttt{array}(n)}{\Gamma, n : \texttt{int} \vdash (k :: a) : \texttt{array}(n+1)}$$

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$$\frac{x : A \vdash f(x) : B(x) \qquad \vdash a : A}{\vdash (\lambda_{x:A}.f(x))a = f(a) : B(a)}$$

Remark

The usual arrow type $A \to B$ is recovered as

$$\Pi_{x:A}.B$$

where x does not occur in B.

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Categories

A category $\mathcal{C}$ consists of

- objects: $\mathrm{Ob}(\mathcal{C})$
- morphisms: $\forall A, B \in \mathrm{Ob}(\mathcal{C}), \quad \mathrm{Hom}(A, B)$
- compositions:

$$\frac{f : A \to B \qquad g : B \to C}{g \circ f : A \to C}$$

- identities:

$$\forall A \in \mathrm{Ob}(\mathcal{C}), \quad \mathrm{id}_A : A \to A$$

such that

- composition is associative:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- admits identities as neutral elements

$$\mathrm{id} \circ f = f = f \circ \mathrm{id}$$

The category **Set**

The category **Set** has

- objects: sets
- morphisms: functions $f : A \to B$
- with usual composition and identities

Modeling programming languages

From a programming language, we can build a category Π whose

- objects: types
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- composition: usual composition of programs

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Definition

A **model** of the programming language is a functor

$$F : \Pi \to \mathcal{C}$$

Models of simply typed λ-calculus

Take the category with

- objects: types

$$A \quad ::= \quad X \mid A \Rightarrow B \mid A \times B$$

- morphisms $A \to B$: λ-terms $f : A \Rightarrow B$

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Example:

$$\lambda x.\lambda y.x : A \to (B \Rightarrow A)$$

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Exercise: give a model of this language into **Set**.

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More generally, it can be modeled in any cartesian closed category.

Cartesian closed categories

Definition

A **cartesian closed category** is a category which has

- *products*:

$$\forall f : A \to B, g : A \to C, \qquad \begin{tikzcd} & A \arrow[d, dotted] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\ & B \times C \arrow[dl, "\pi_B"] \arrow[dr, "\pi_C"'] & \\ B & & C \end{tikzcd}$$

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- which is *closed*:

$$\frac{A \times B \to C}{A \to (B \Rightarrow C)}$$

A model of Martin-Löf type theory

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Definition

A **locally cartesian closed category** is a category $\mathcal{C}$ in which for every object A the slice category $\mathcal{C}/A$ is cartesian closed.

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Theorem

An LCCC is a category with pullbacks in which for every $f : A \to B$, the base change functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ has a right adjoint $\Pi_f : \mathcal{C}/A \to \mathcal{C}/B$.

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Example

$$\frac{\Gamma, x : A \vdash B(x) : \mathrm{type}}{\Gamma \vdash \Pi_{x:A}.B(x) : \mathrm{type}}$$

Problem

Every LCCC is also a model of MLTT with the rule of *extensionality*:

$$\frac{\vdash p : \mathrm{Id}_A(a, b)}{\vdash a = b : A}$$

...and type checking is indecidable in extensional MLTT!

Half of the title

We explain here that Quillen model categories model identity types in Martin-Löf type theory:

$$F : \mathcal{M} \to \mathcal{Q}$$

Which provides non-extensional models.

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The idea here is that identity types behave like homotopies between topological spaces.

Homotopy

A homotopy between two continuous functions $f, g : A \to B$ between topological spaces A and B is a continuous function

$$h : I \times A \to B$$

where $I = [0, 1]$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$.

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Two spaces A and B are *homotopy equivalent* when there exists maps $f : A \to B$ and $g : B \to A$ such that

$$g \circ f \sim \mathrm{id}_A \qquad f \circ g \sim \mathrm{id}_B$$

Ex: square $\approx$ circle, coffee mug $\approx$ donut, etc.

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- etc.

Modeling MLTT

We interpret

- a type $\vdash A$: type as a topological space
- a term $\vdash x : A$ as a point in A
- a term $p : \mathrm{Id}_A(a, b)$ as a path $a \to b$
- a term $s : \mathrm{Id}_{\mathrm{Id}(a,b)}(p, q)$ as an homotopy $\begin{tikzcd} a \arrow[r, bend left, "p"{name=U}] \arrow[r, bend right, "q"'{name=D}] \arrow[from=U, to=D, phantom, "s\Downarrow"] & b \end{tikzcd}$
- etc.

Dependent types

As in the case of LCCC we interpret a dependent type

$$x : A \vdash B(x) : \mathrm{type}$$

as a continuous map

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and a term $x : A \vdash f : B(x)$ as a section of this map.

Dependent types and equality

The maps interpreting types should have the *homotopy lifting property*:

$$
\begin{tikzcd}
\{0\} \arrow[r, "\beta"] \arrow[d] & B \arrow[d] \\
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Maps like this are often called *fibrations*.

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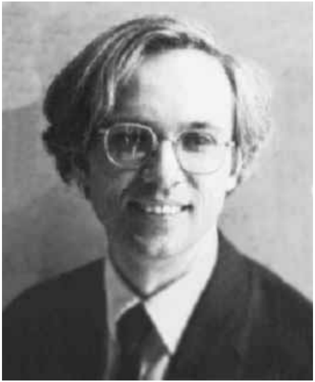
Ex: the interpretation of $x, y : A \vdash \mathrm{Id}_A(x, y)$ is a map

$$\begin{tikzcd} A^I \arrow[d] \\ A \times A \end{tikzcd}$$

Homotopy is more generally carried on in Quillen model categories.



Lifting properties

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Definition

Given maps $f : A \to B$ and $g : C \to D$, f has the *left lifting property* wrt g when every commutative square

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Given a class $\mathfrak{L}$ of maps, we write ${}^{\perp}\mathfrak{L}$ for the class of maps which have LLP wrt every map in $\mathfrak{L}$ (and similarly $\mathfrak{L}^{\perp}$ for RLP).

Weak factorization systems

Definition

A *weak factorization system* $(\mathfrak{L}, \mathfrak{R})$ consists of two classes of maps such that

1. every map $f : A \to B$ factors as

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "f"'] & & C \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

   with $i \in \mathfrak{L}$ and $p \in \mathfrak{R}$
2. $\mathfrak{L}^{\perp} = \mathfrak{R}$ and $\mathfrak{L} = {}^{\perp}\mathfrak{R}$

Definition

A *model category* consists of $\mathcal{C}$ together with subcategories

- $\mathfrak{F}$: *fibrations*
- $\mathfrak{C}$: *cofibrations*
- $\mathfrak{W}$: *weak equivalences*

such that

1. three for two
2. both $(\mathfrak{C}, \mathfrak{W} \cap \mathfrak{F})$ and $(\mathfrak{C} \cap \mathfrak{W}, \mathfrak{F})$ are weak factorization systems.

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Example

On **Top**:

- generating cofibrations are inclusions $i : \Delta^n \to \Delta^n \times I$,
- fibrations are RLP of generating cofibrations (Serre fibrations),
- weak equivalences are weak homotopy equivalences.

Path objects

Definition

A *(very good) path object* A^I for an object A consists of a factorization

$$\begin{tikzcd} A \arrow[rr, "r"] \arrow[dr, "\Delta"'] & & A^I \arrow[dl, "p"] \\ & A \times A & \end{tikzcd}$$

with r acyclic cofibration and p fibration.

Interpretation of MLTT

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The current state of things

Theorem (Awodey & Warren)

MLTT can be interpreted in any model category.

Theorem (Gambino & Garner)

The interpretation is complete.

The Homotopy Hypothesis

$$
\begin{tikzcd}
\text{Homotopy Types} \arrow[rr, equal] \arrow[dr, equal] & & \text{Weak } \omega\text{-groupoids} \arrow[dl, equal] \\
& \text{MLTT} \arrow[u, phantom, "???"] &
\end{tikzcd}
$$

Towards directed algebraic topology?

We could think of a directed variant:

- replace equality by a reduction relation:

$$f \rightsquigarrow g \qquad \Rightarrow \qquad \text{there is a directed path from } f \text{ to } g$$

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- the reduction should be compatible with identity:

$$r : \mathsf{Id}(f, f') \qquad \Rightarrow \qquad \exists g', \quad \exists s : \mathsf{Id}(g, g') \quad \text{and} \quad g \rightsquigarrow g'$$

$$\begin{tikzcd} f \arrow[r, equal] \arrow[d, squiggly] & f' \arrow[d, squiggly] \\ g \arrow[r, equal, dashed] & {\color{red} g'} \end{tikzcd}$$

We can "translate continuously" the directed path $f \rightsquigarrow g$ into the directed path $f' \rightsquigarrow g$