Generalisation and formalisation in game theory

Stéphane Le Roux

École normale supérieure de Lyon

Thesis overview

Generalisation and formalisation in game theory

Game theory: mathematical study of competitive interactions.

Origins: sparse and unrelated works.

Qualitative
- Hobbes's political philosophy (*Leviathan*, 1651).
- Darwin (*The Origin of Species*, 1859).

Quantitative
- Using probabilities to study a card game (Montmort's letter to Nicholas Bernouilly, 1713).
- Cournot (*Recherche sur les Principes Mathématiques de la Théorie des Richesses*, 1838).

Generalisation and formalisation in game theory

Von Neumann and Morgenstern's book,
Theory of Games and Economic Behavior, 1944:

- First clear research directions, game theory community.
- Conscious and justified simplification:
  real-life is measurable with real numbers.

Generalisation and formalisation in game theory

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- First clear research directions, game theory community.
- Conscious and justified simplification:
  real-life is measurable with real numbers.

After 1944:

- Fast development, *e.g.* Probabilistic Nash equilibrium (1950).
- Qualitative and quantitative applications in economics, law, political science, biology, computer science, etc.

My thesis

Prior works:

- Real-life is measurable with real numbers: conscious simplification $\rightsquigarrow$ principle.
- Probabilistic Nash equilibrium: possible solution $\rightsquigarrow$ principle.

My thesis

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My thesis

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- Real-life is measurable with real numbers: conscious simplification $\rightsquigarrow$ principle.
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Motivations: These principles inhibit alternative approaches that may be interesting for themselves or may help understand better the traditional approach.

My work:
- Generalise most basic notions of game theory.
- Generalise most basic results of game theory.
- Formalise a few of my results.

Generalisation and formalisation in game theory

What: formal proof = proof that is computer-verifiable.

How: software (*e.g.* Coq).

Motivations:

- Unambiguous statements.
- Additional guarantee of proof correctness.
- Proof clarification: structure and key points.

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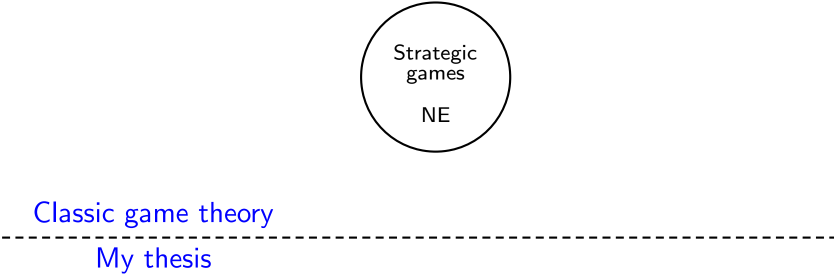

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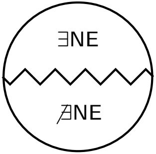

Classic game theory

My thesis

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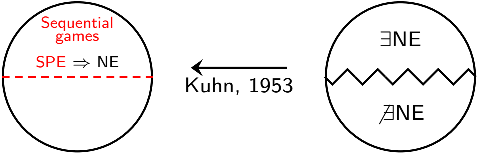

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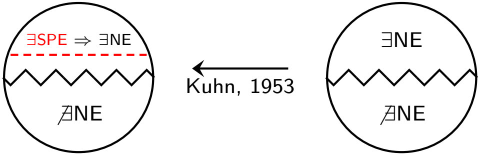

Classic game theory

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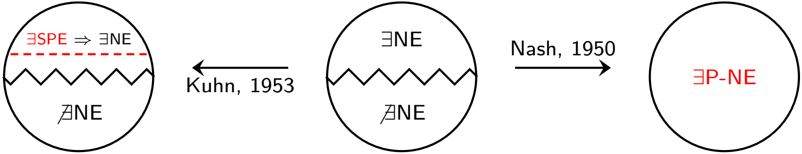

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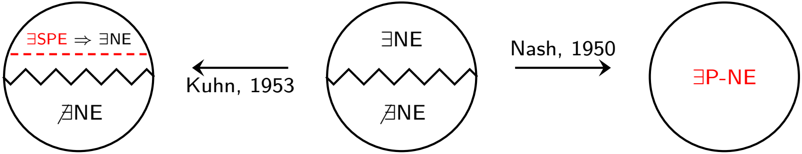

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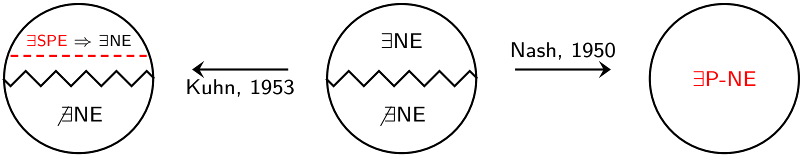

Classic game theory

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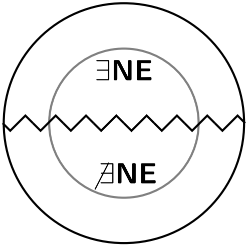

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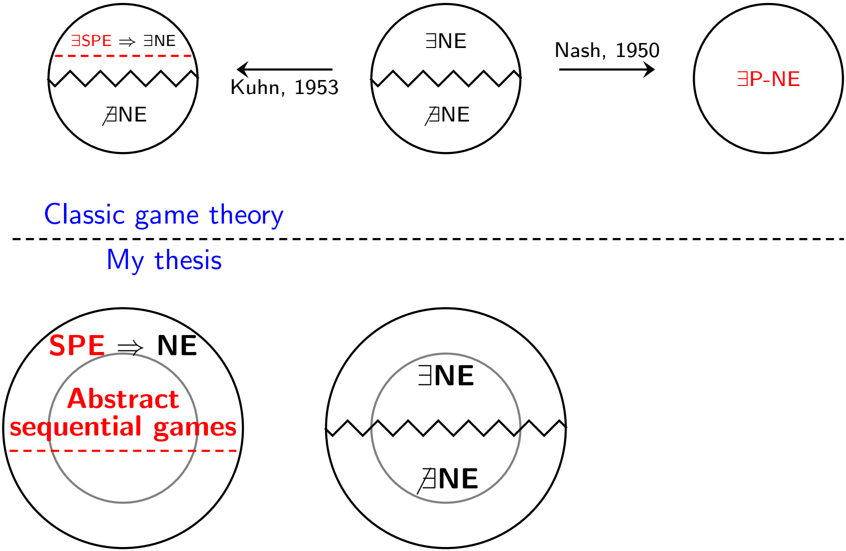
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Kuhn, 1953
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Nash, 1950
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Abstract sequential games
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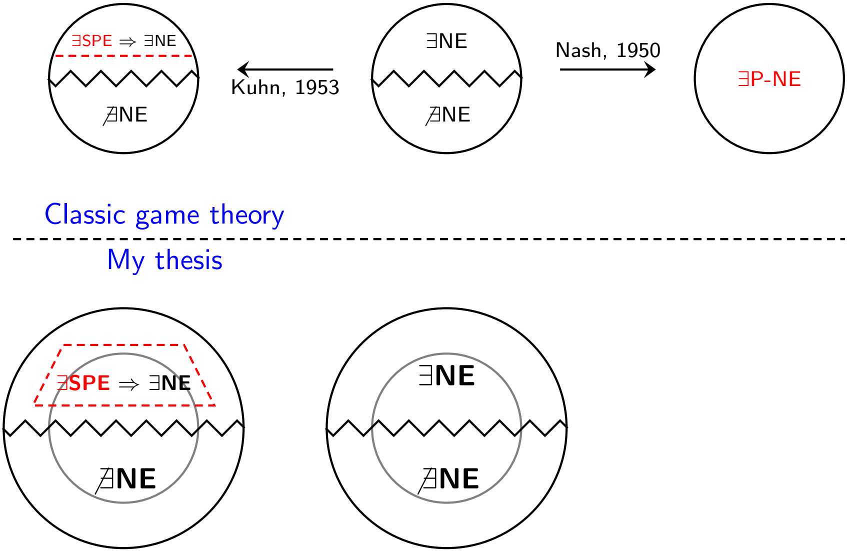
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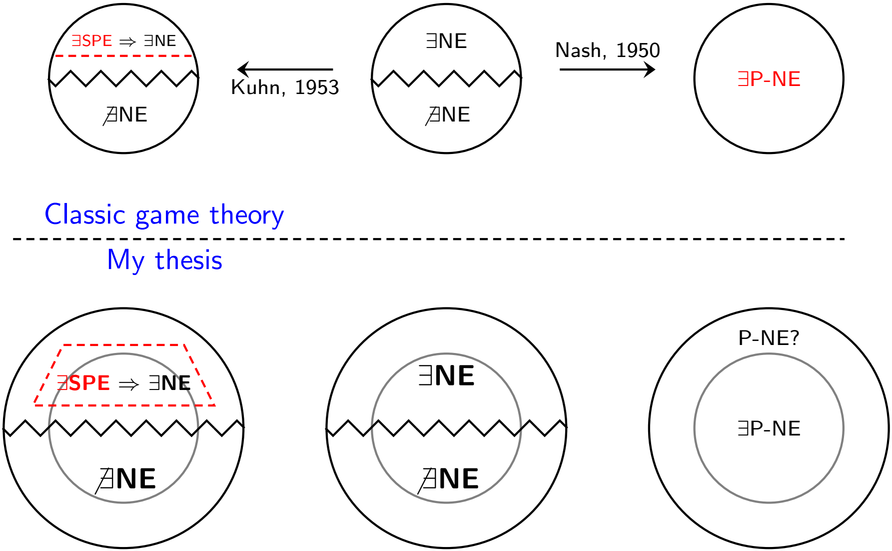
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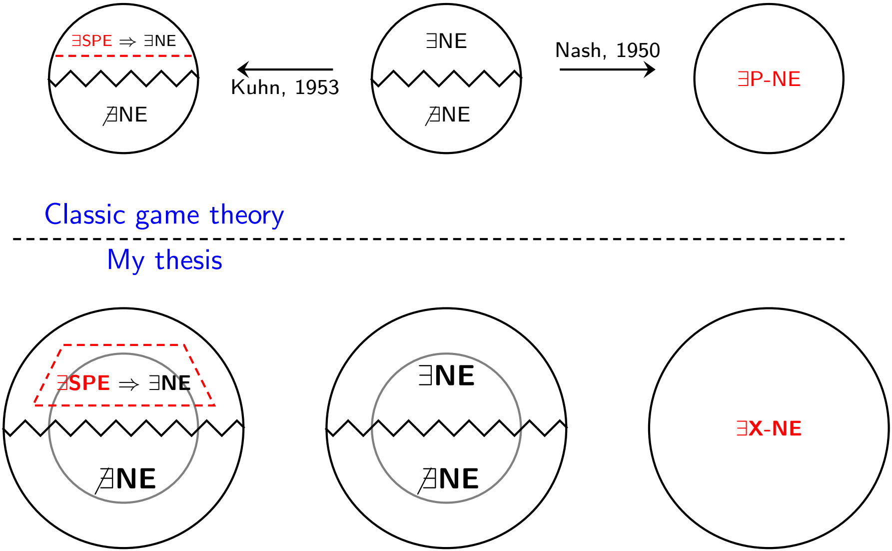
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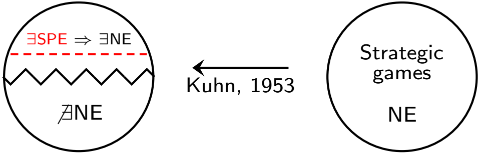
Contents: part 1, preliminaries

Classic game theory

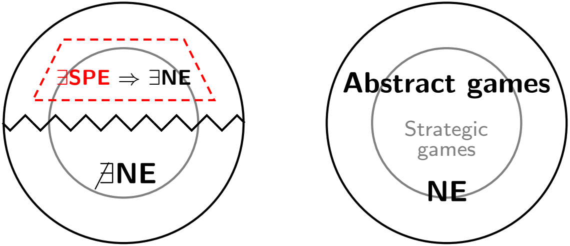
My thesis

Contents: part 2, smaller classes of games

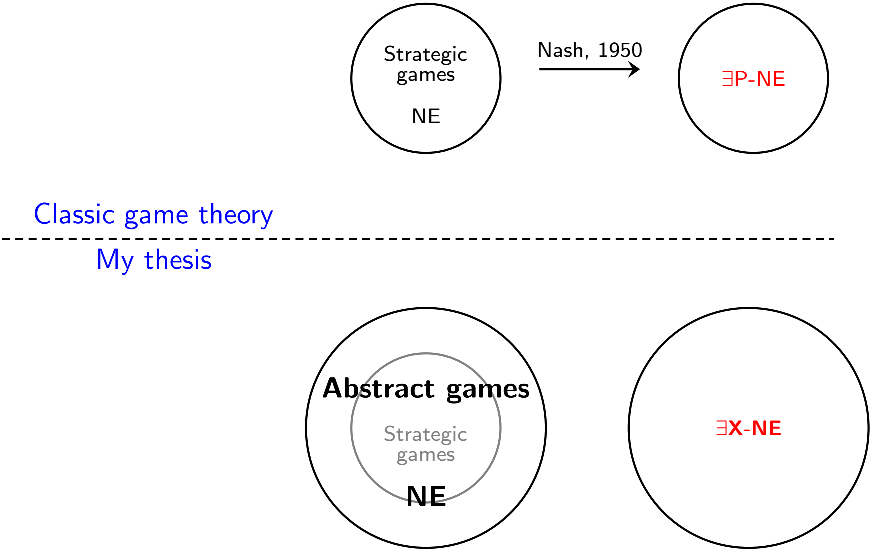


Classic game theory

My thesis



Contents: part 3, weakening Nash equilibrium



Outline

Very abstract game and Nash equilibrium (new)

Definition (Very abstract game)

Very abstract games are 3-tuples $\langle \mathcal{A}, \mathcal{S}, (Happy_a)_{a \in \mathcal{A}} \rangle$ *where:*

- $\mathcal{A}$ *is a non-empty set of agents.*
- $\mathcal{S}$ *is a non-empty set of situations.*
- *For* $a \in \mathcal{A}$, $Happy_a$ *is a predicate over* $\mathcal{S}$.

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Let $s \in \mathcal{S}$. $Eq(s) \quad \stackrel{\Delta}{=} \quad \forall a \in \mathcal{A},\ Happy_a(s)$

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Example (Trivial game: one agent, one situation)

$Happy_a(s)$ $\boxed{s}$ $\qquad$ $\neg Happy_a(s)$ $\boxed{s}$

Convertibility/preference games (new)

Definition (CP Games)

CP games are 4-tuples $\langle \mathcal{A}, \mathcal{S}, (\xrightarrow{+a})_{a\in\mathcal{A}}, (\xrightarrow{\odot a})_{a\in\mathcal{A}} \rangle$ *where:*

- $s \xrightarrow{+a} s'$ *means that agent a can convert s to s'.*
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Definition (Happiness and Nash equilibrium)

$$Eq(s) \quad \triangleq \quad \forall a \in \mathcal{A},\ Happy_a(s)$$

$$\xrightarrow{\oplus a} \quad \triangleq \quad \xrightarrow{+a} \cap \xrightarrow{\odot a}$$

$$Happy_a(s) \quad \triangleq \quad \forall s' \in \mathcal{S}, \quad \neg(s \xrightarrow{\oplus a} s')$$

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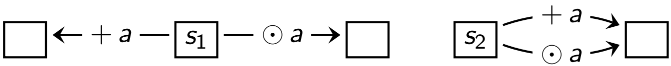
Example



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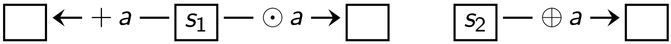
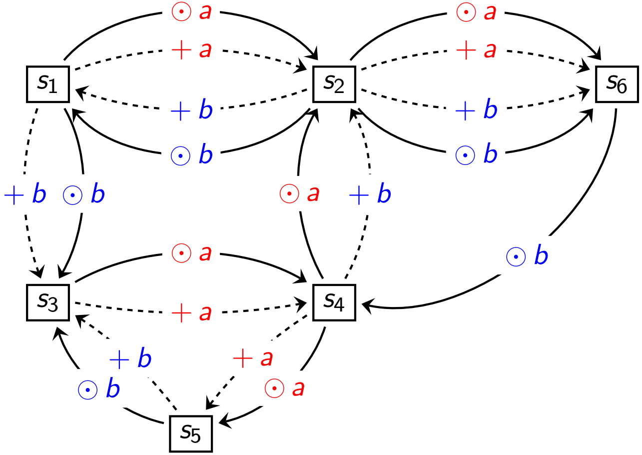


Finding Nash equilibria in CP games



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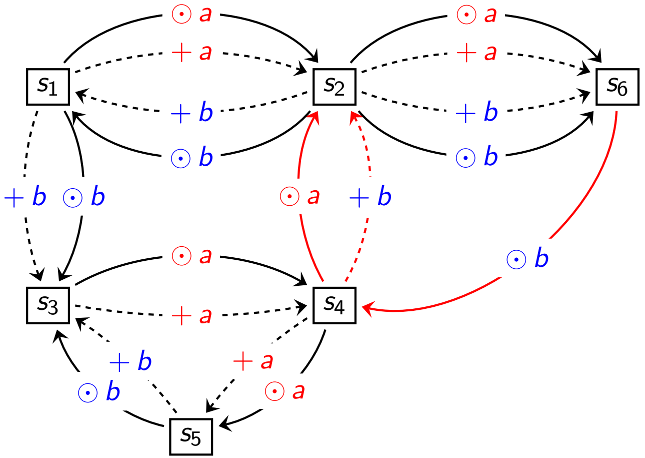

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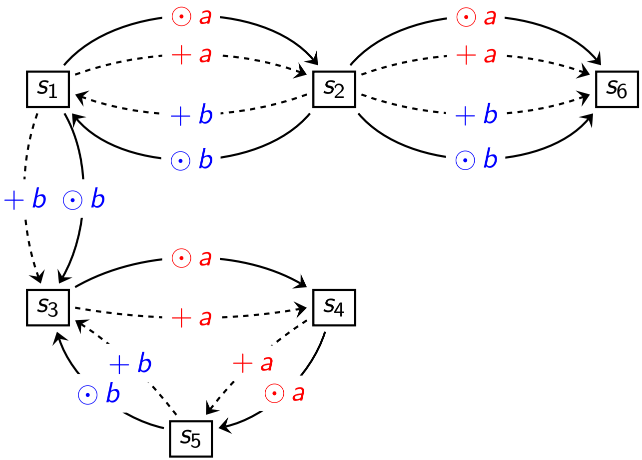

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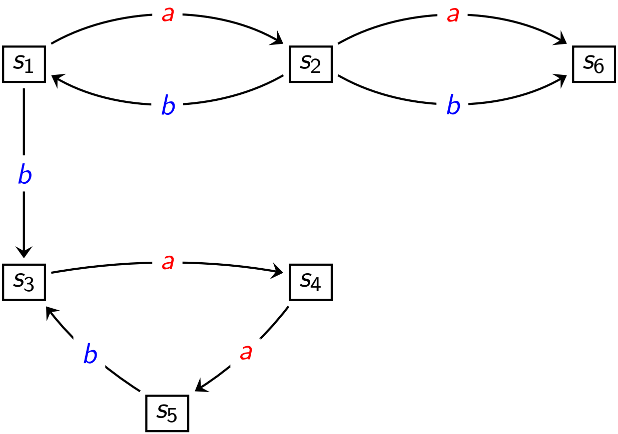

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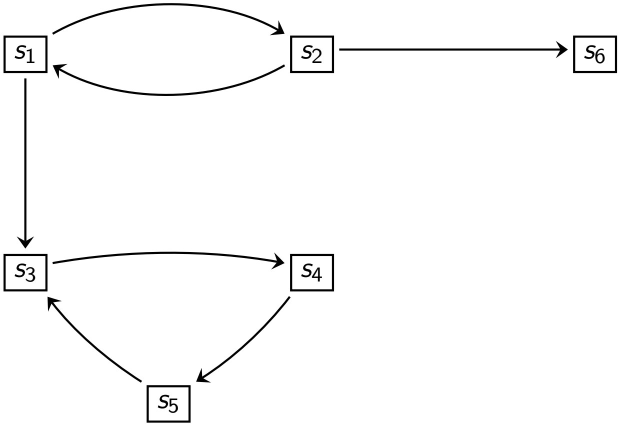

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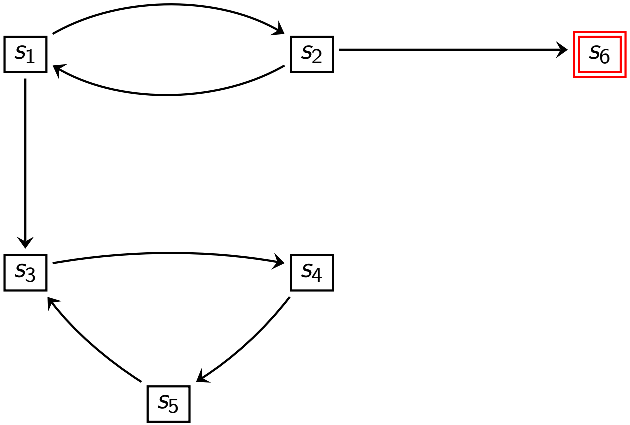


Strategic games (classic)

Informal definition using the new CP approach

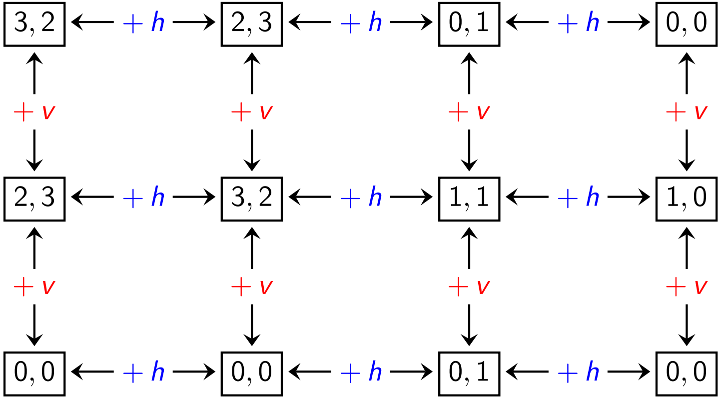
Definition (Strategic games)

A strategic game is a CP game such that:

- *Situations form a Cartesian product, and they are called strategy profiles.*
- *Convertibilities are equivalence relations along a Cartesian product and orthogonal to each other.*
- *Each strategy profile encloses a payoff function:* $\mathcal{A} \to \mathbb{R}$.
- *Preferences follow the payoff functions.*

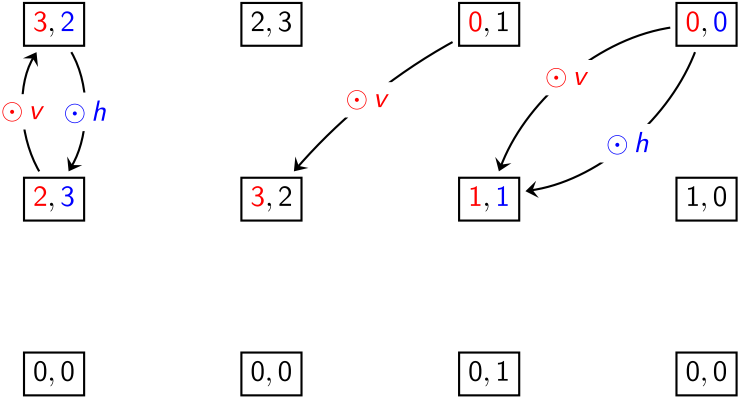
Convertibilities in strategic games

Reflexive and transitive convertibility arrows are not displayed.

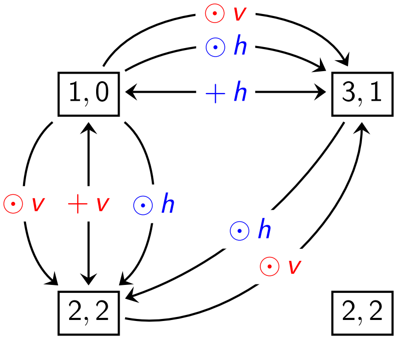


Preferences in strategic games

Only a few preference arrows are displayed.



Array representation of strategic games



	h_1	h_2
v_1	1, 0	3, 1
v_2	2, 2	2, 2

Example of a play in a strategic game

1. 2-agent game:

	h_1	h_2
v_1	1, 0	3, 1
v_2	2, 2	2, 2

2. v's choice:

	h_1	h_2
v_1	1, 0	3, 1
v_2	2, 2	2, 2

h's choice:

	h_1	h_2
v_1	1, 0	3, 1
v_2	2, 2	2, 2

3. Strategy profile:

	h_1	h_2
v_1	1, 0	3, 1
v_2	2, 2	2, 2

4. Payoff function:

	h_1	h_2
v_1	1, 0	3, 1
v_2	2, 2	2, 2

Nash equilibrium in strategic games

Two Nash equilibria

	h_1	h_2
v_1	1, 0	3, 1
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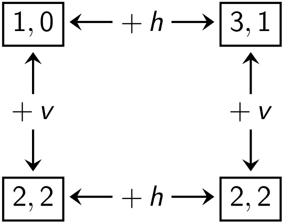

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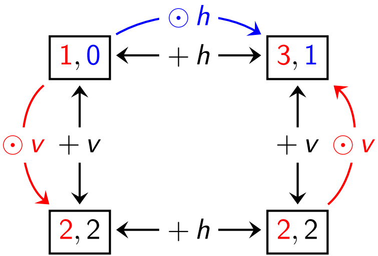

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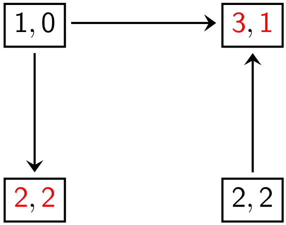

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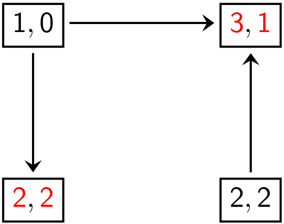
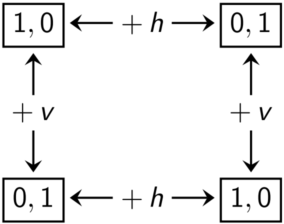


No Nash equilibrium

	h_1	h_2
v_1	1, 0	0, 1
v_2	0, 1	1, 0



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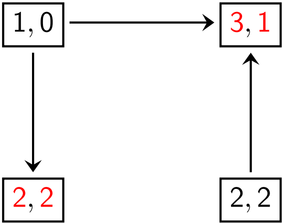

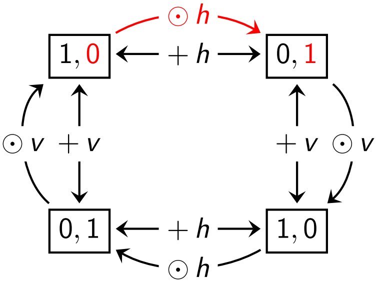

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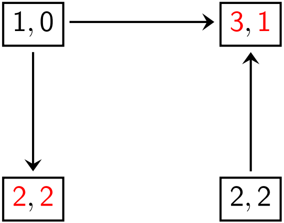

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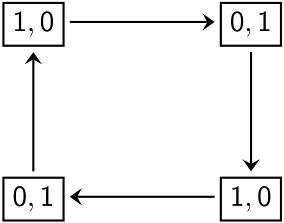
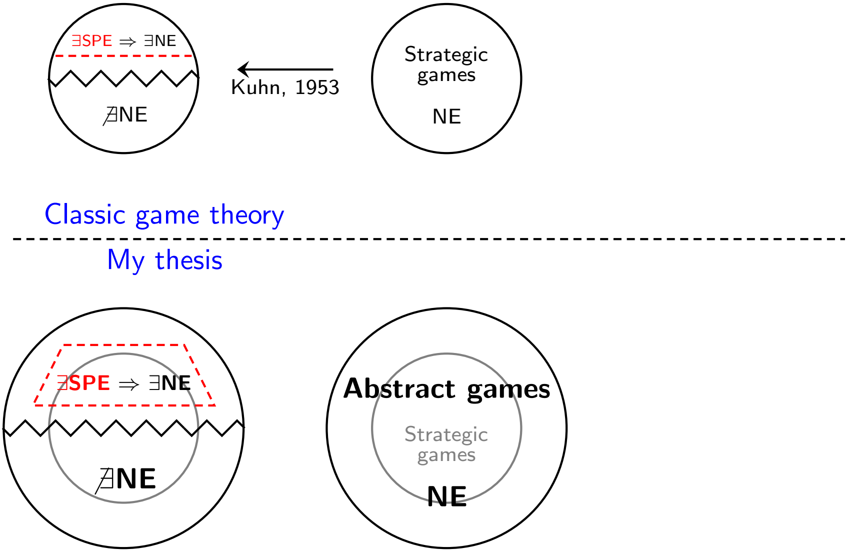


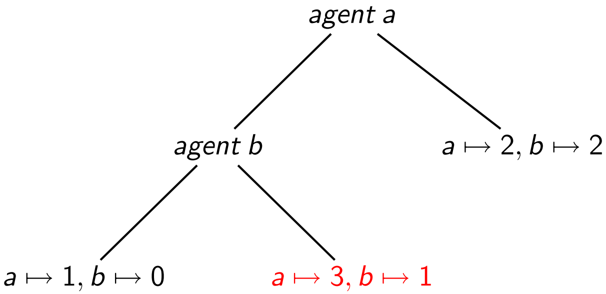
Part 2, smaller classes of games



Outline

Sequential Game (classic)

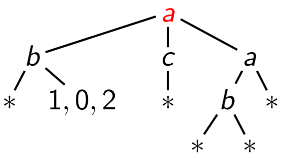
- Finite rooted tree.
- Internal nodes labelled with agents.
- Leaves labelled with payoff functions (from agents to real numbers).



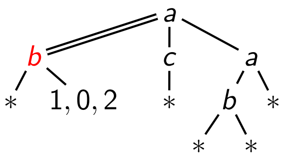
Play in a Sequential Game

*'s are arbitrary payoff functions.

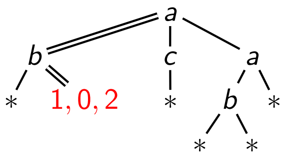
Start



First move

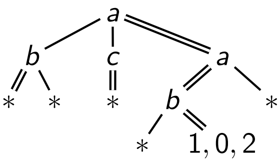


Last move

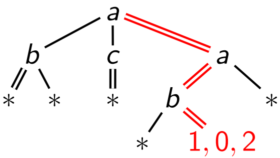


Strategy Profile and Induced Payoff Function

Strategy profile



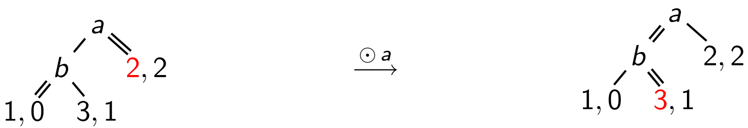
Induced payoff function



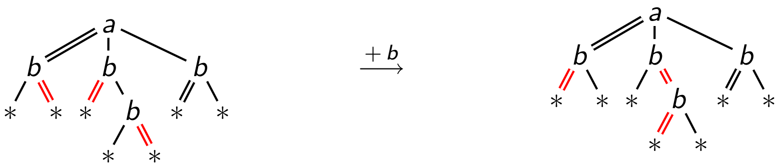
Nash Equilibrium of sequential games

By embedding into CP games

Preference of agent a



Convertibility of agent b

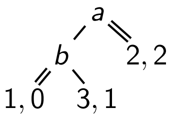


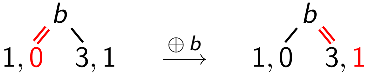
Subgame Perfect Equilibrium (S.P.E.)

Definition (Subgame Perfect Equilibrium)

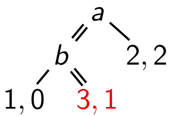
An S.P.E. is a Nash equilibrium for which every child is an S.P.E.

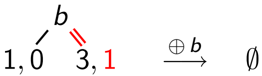
A Nash equilibrium
that is not a S.P.E.





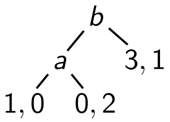
An S.P.E.



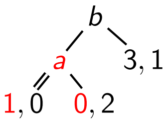


"Backward Induction"

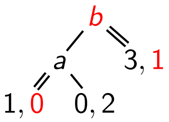
Input: game below



First step



Last step: ouput



Theorem (Kuhn, 1953)

Every sequential game has a Nash equilibrium.

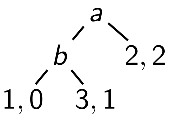
Proof.

"Backward induction" yields S.P.E. □

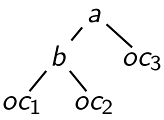
Abstract sequential games (new)

- Abstract outcomes instead of concrete payoff functions.
- Arbitrary preferences over outcomes.

Traditional



Abstract



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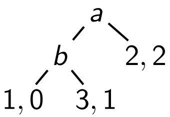
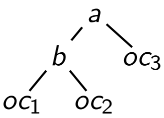
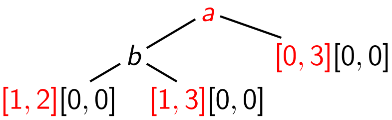


Abstract



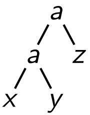
Multi-criteria payoffs (1950's: Simon and Blackwell)



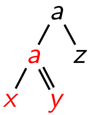
"Backward Induction" $\not\rightsquigarrow$ Nash Equilibrium

The preference for agent a is defined by $z \overset{\odot a}{\longrightarrow} x$ only.

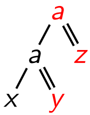
Game input



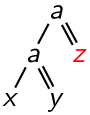
First step



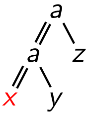
Last step



Not Nash equilibrium



$\xrightarrow{\oplus a}$



Equilibria in abstract sequential games

Theorem (new and proved using Coq)

The following three propositions are equivalent:

- *The preference of each agent is acyclic.*
- *Every game has a Nash equilibrium.*
- *Every game has a subgame perfect equilibrium.*

Proof.

Topological sorting + classic "backward induction". □

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Proof.

Topological sorting + classic "backward induction". □

Corollary (new)

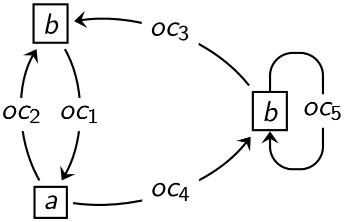
Every multi-criteria sequential game has an equilibrium.

Proof.

Multi-criteria preferences are acyclic. □

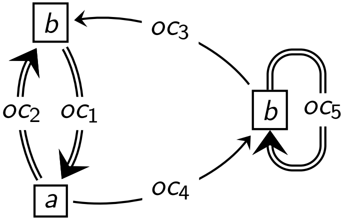
Sequential graph games (new)

- Finite directed graph,
- Non-zero outdegree,
- Nodes labelled with agents,
- Arcs labelled with outcomes,
- Outcome $\rightsquigarrow$ sequence of outcomes.



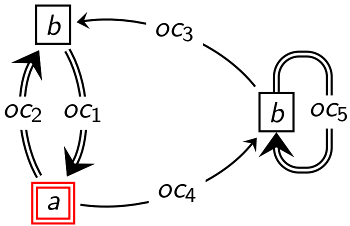
Strategy profile

- Built on sequential graph games,
- Each node chooses one and only one next node.



Induced sequences of outcomes

- Starting from any node,
- following the choices,
- yields an infinite sequence of outcomes.



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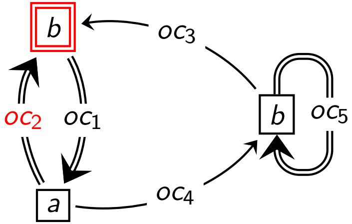

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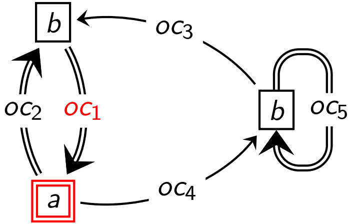

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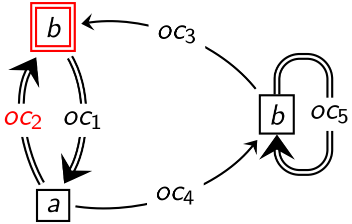
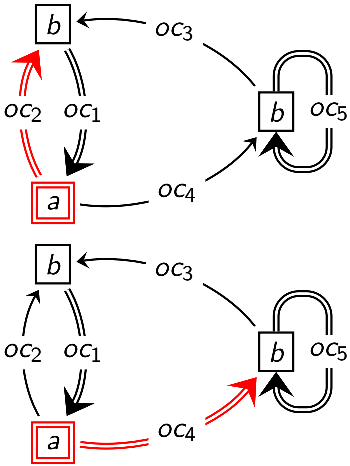


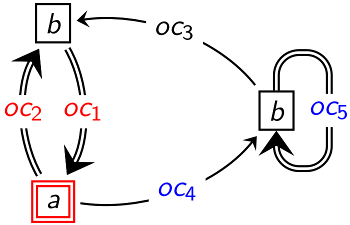
Convertibility



Local and global equilibria

Definition (Local equilibrium, explained by an example)

Preference over infinite sequences of outcomes.



Local hapiness of agent a at the node $\boxed{a}$

$$\Updownarrow$$

$$\neg((oc_2\, oc_1)^\omega \xrightarrow{\odot\, a} oc_4\, oc_5{}^\omega)$$

Local and global equilibria

Definition (Local equilibrium, explained by an example)

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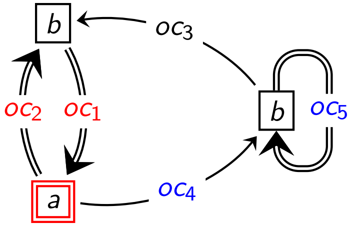

Local hapiness of agent a at the node $\boxed{a}$

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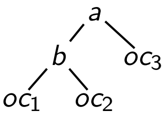
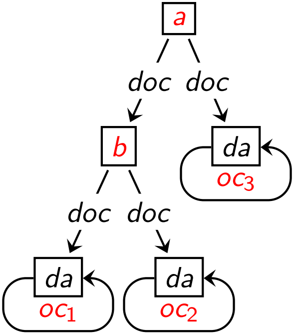
Definition

Global equilibrium = Local equilibrium at all nodes.

A sequential tree game is a sequential graph game





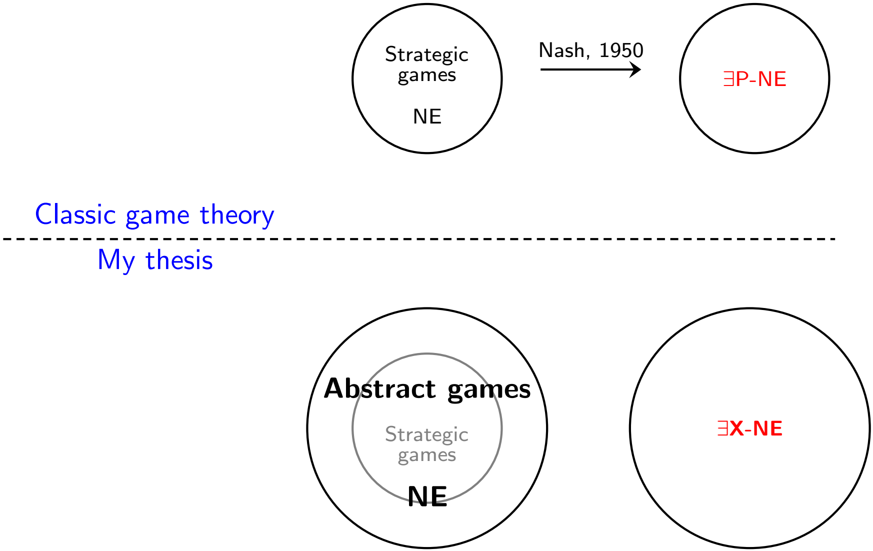
Lemma

Local equilibrium $\leftrightarrow$ *Nash equilibrium*
Global equilibrium $\leftrightarrow$ *S.P.E.*

Global equilibrium in sequential graph games (new)

- Sufficient condition for existence.
- Necessary condition for existence.
- Coinciding when preferences are total orders.

Part 3, weakening Nash equilibrium



Outline

Probabilistic Nash equilibrium (classic)

No Nash equilibrium

	h_1	h_2
v_1	1, 0	0, 1
v_2	0, 1	1, 0

Probabilistic strategies

	$h_1 \mapsto x$	$h_2 \mapsto 1 - x$
$v_1 \mapsto y$	1, 0	0, 1
$v_2 \mapsto 1 - y$	0, 1	1, 0

Probabilistic NE

	$h_1 \mapsto \frac{1}{2}$	$h_2 \mapsto \frac{1}{2}$
$v_1 \mapsto \frac{1}{2}$	1, 0	0, 1
$v_2 \mapsto \frac{1}{2}$	0, 1	1, 0

Theorem (Nash, 1950)

Every finite strategic game has a probabilistic Nash equilibrium.

Abstract strategic games (new)

- Abstract outcomes instead of concrete payoff functions.
- Arbitrary preferences over outcomes.

Traditional

	h_1	h_2
v_1	1, 0	3, 1
v_2	2, 2	2, 2

Abstract

	h_1	h_2
v_1	oc_1	oc_2
v_2	oc_3	oc_4

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Abstract

	h_1	h_2
v_1	oc_1	oc_2
v_2	oc_3	oc_4

Multi-criteria strategic game without probabilistic NE

	h_1	h_2
v_1	$[2, 0]$, $[2, 0]$	$[1, 1]$, $[1, 1]$
v_2	$[1, 1]$, $[1, 1]$	$[0, 2]$, $[0, 2]$

Discrete non-deterministic Strategies

Expected abstract outcome? $\frac{1}{3}oc_2 + \frac{2}{3}oc_3 \xrightarrow{\odot\, v} \frac{1}{3}oc_5 + \frac{2}{3}oc_6$?

	$h_1 \mapsto 0$	$h_2 \mapsto \frac{1}{3}$	$h_3 \mapsto \frac{2}{3}$
$v_1 \mapsto 1$	oc_1	oc_2	oc_3
$v_2 \mapsto 0$	oc_4	oc_5	oc_6

Discrete non-determinism:

	h_1	h_2	h_3
v_1	oc_1	oc_2	oc_3
v_2	oc_4	oc_5	oc_6

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Discrete non-determinism:

	h_1	h_2	h_3
v_1	oc_1	oc_2	oc_3
v_2	oc_4	oc_5	oc_6

Pure strategy

$\downarrow$

Discrete non-deterministic strategy

$\downarrow$

Probabilistic strategy

Discrete non-deterministic Nash equilibrium

- Five different notions of such equilibrium.
- Two notions are related to the classic strategy dominance.

Theorem (new)

If preferences are transitive and irreflexive, every abstract strategic game has a discrete non-deterministic Nash equilibrium.
(For any of the five notions).

Proof.

By a dedicated fixed-point theorem. □

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Corollary (new)

Every multi-criteria strategic game has a discrete non-deterministic Nash equilibrium.

Proof.

Multi-criteria preferences are transitive and irreflexive. □

Finding an equilibrium in a traditional strategic game

Polynomial in the number of strategy profiles

	h_1	h_2	h_3	h_4
v_1	3,2	2,3	0,1	0,0
v_2	2,3	3,2	1,1	1,0
v_3	0,0	0,0	0,1	0,0

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v_2	2,3	3,2	1,1	1,0
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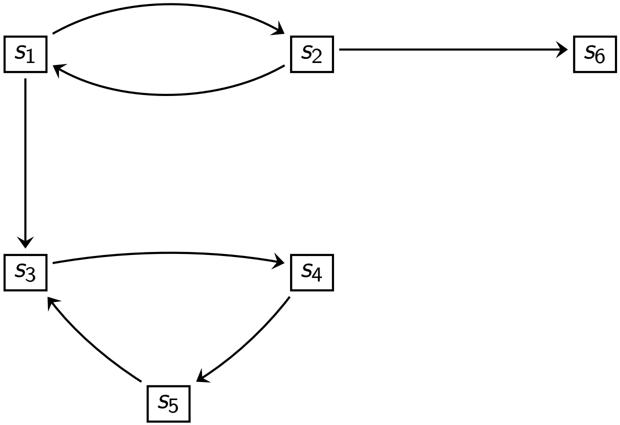
Weakening Nash equilibrium for CP game (new)



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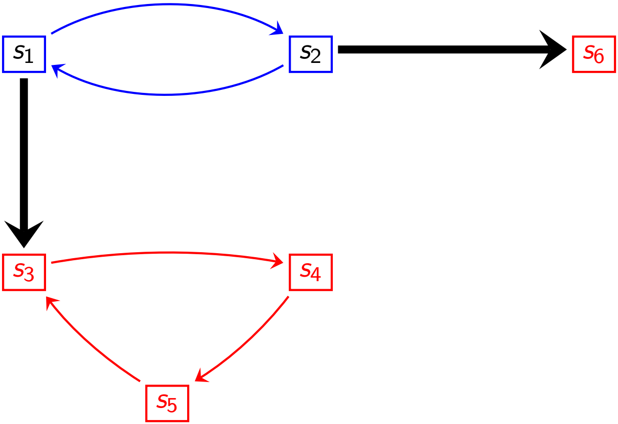
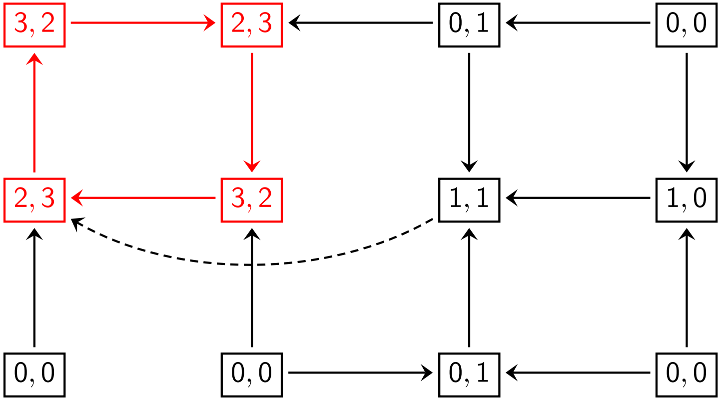


Same game, three kinds of equilibrium



Same game, three kinds of equilibrium

	$\mathbf{h_1}$	$\mathbf{h_2}$	h_3	h_4
$\mathbf{v_1}$	**3,2**	**2,3**	0,1	0,0
$\mathbf{v_2}$	**2,3**	**3,2**	1,1	1,0
v_3	0,0	0,0	0,1	0,0

	$h_1 \mapsto \frac{1}{2}$	$h_2 \mapsto \frac{1}{2}$	$h_3 \mapsto 0$	$h_4 \mapsto 0$
$v_1 \mapsto \frac{1}{2}$	3, 2	2, 3	0, 1	0, 0
$v_2 \mapsto \frac{1}{2}$	2, 3	3, 2	1, 1	1, 0
$v_3 \mapsto 0$	0, 0	0, 0	0, 1	0, 0

Same game, three kinds of equilibrium

	$\mathbf{h_1}$	$\mathbf{h_2}$	h_3	h_4
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	$h_1 \mapsto \frac{1}{2}$	$h_2 \mapsto \frac{1}{2}$	$h_3 \mapsto 0$	$h_4 \mapsto 0$
$v_1 \mapsto \frac{1}{2}$	3, 2	2, 3	0, 1	0, 0
$v_2 \mapsto \frac{1}{2}$	2, 3	3, 2	1, 1	1, 0
$v_3 \mapsto 0$	0, 0	0, 0	0, 1	0, 0

- Probabilistic Nash equilibrium is related to one notion of discrete non-deterministic Nash equilibrium.
- CP equilibrium is related to another notion of discrete non-deterministic Nash equilibrium.

Summary

Classic game theory

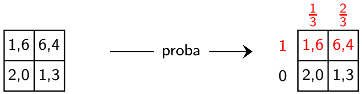
My definitions
and results

1,6	6,4
2,0	1,3

Summary

Classic game theory

My definitions and results



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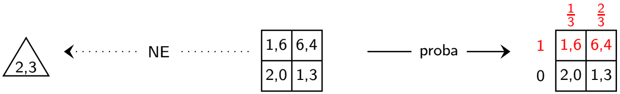
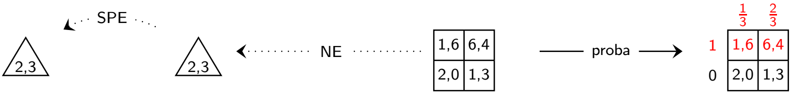


Summary

Classic game theory

My definitions

and results



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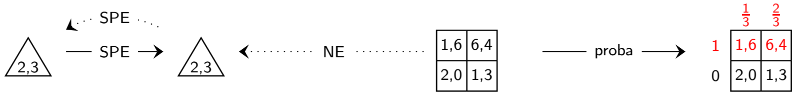

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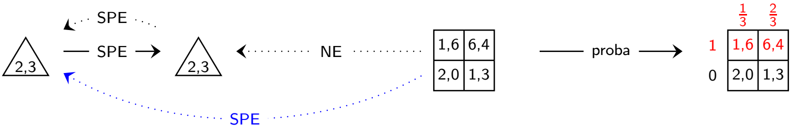

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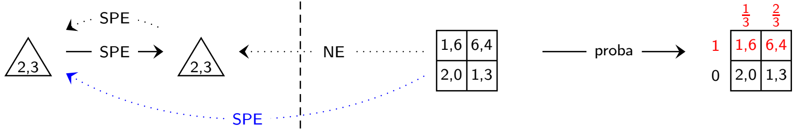

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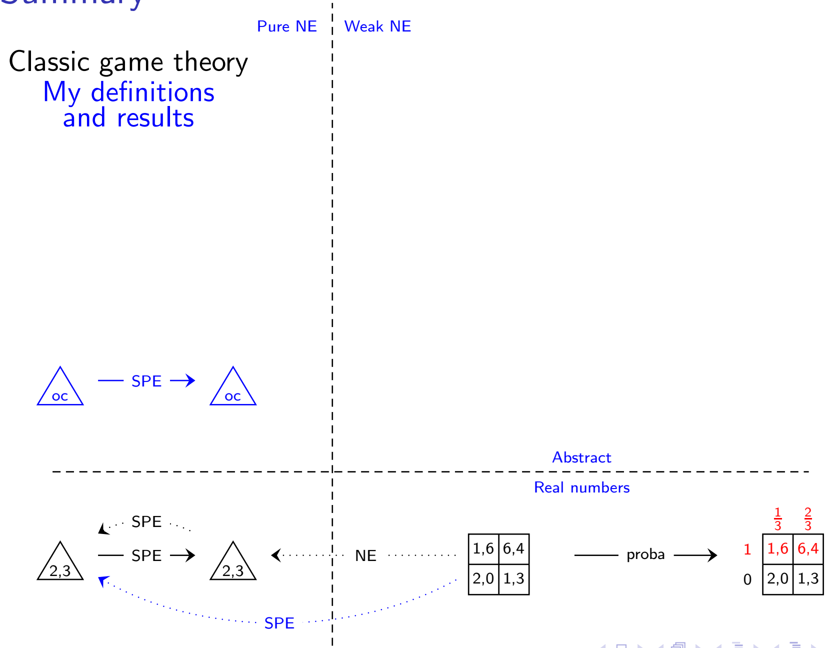
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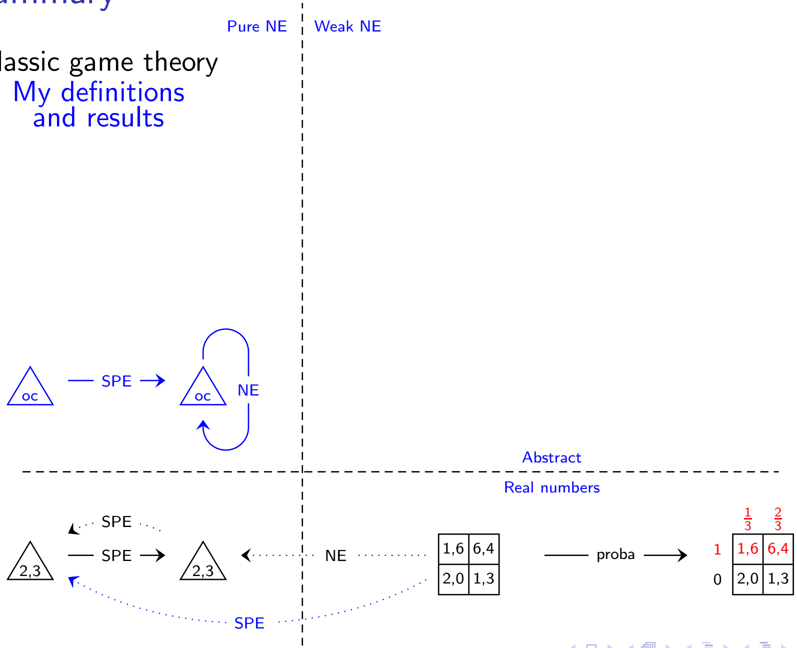
Summary

Classic game theory

My definitions and results



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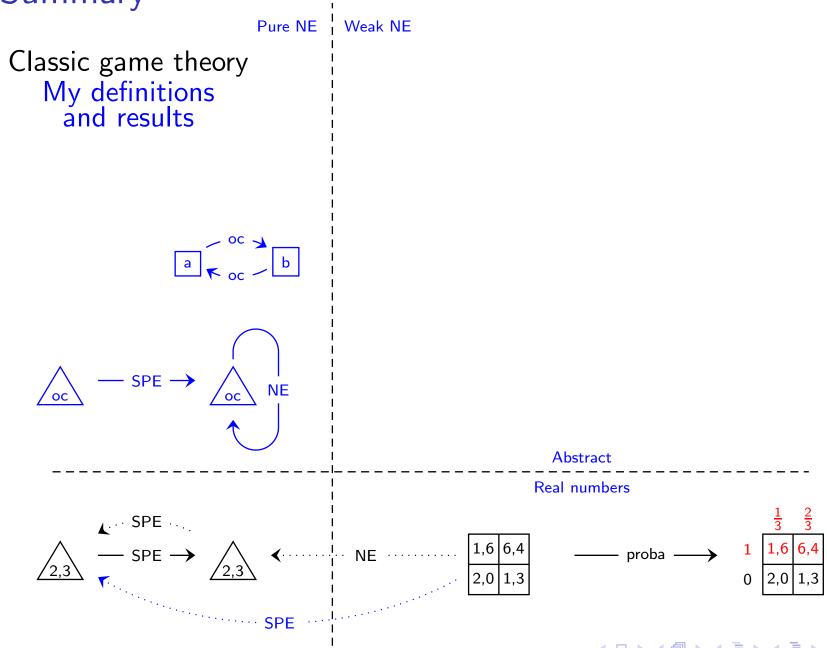
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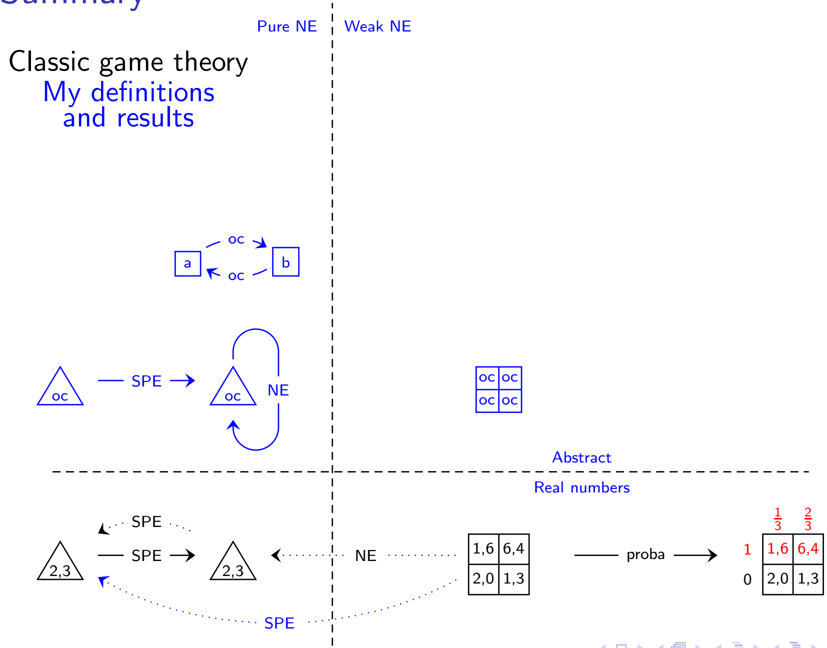
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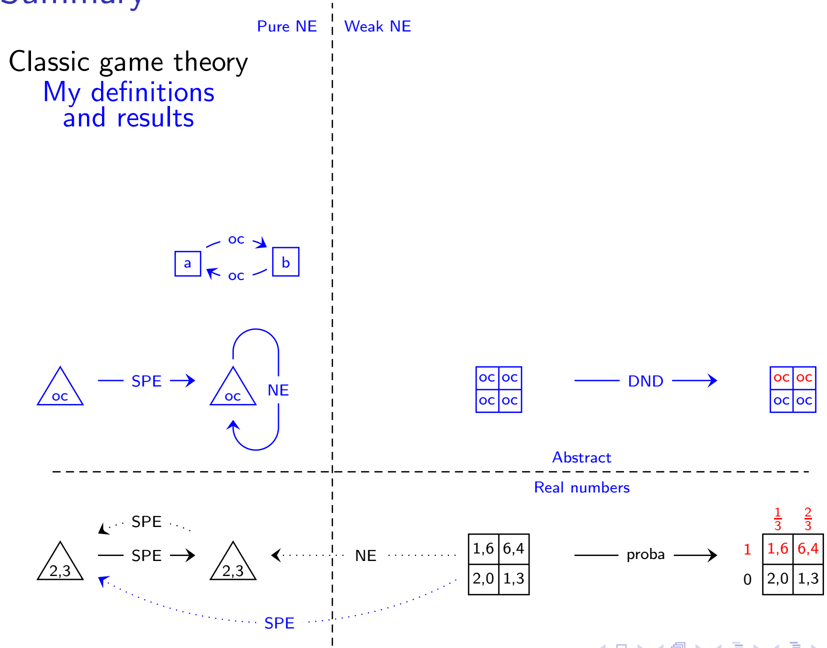
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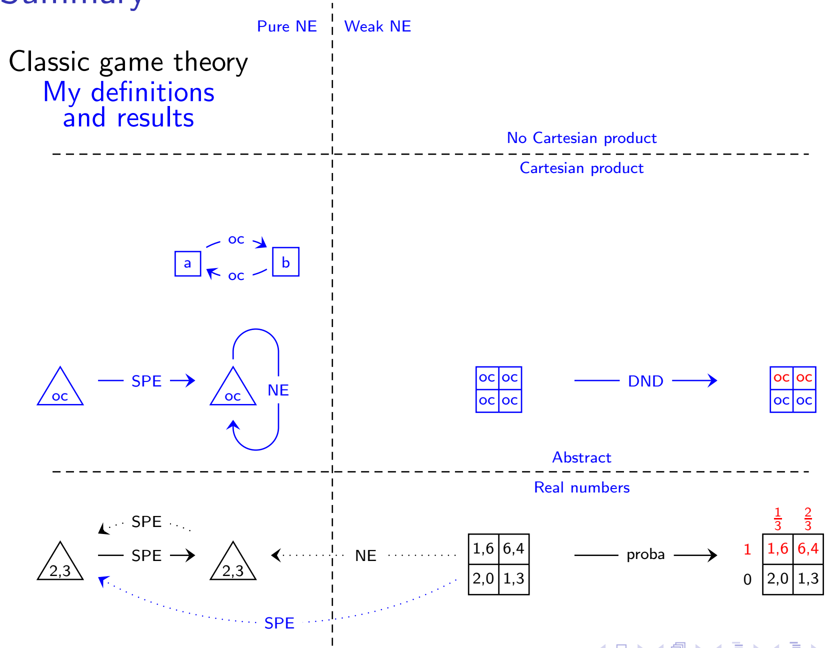
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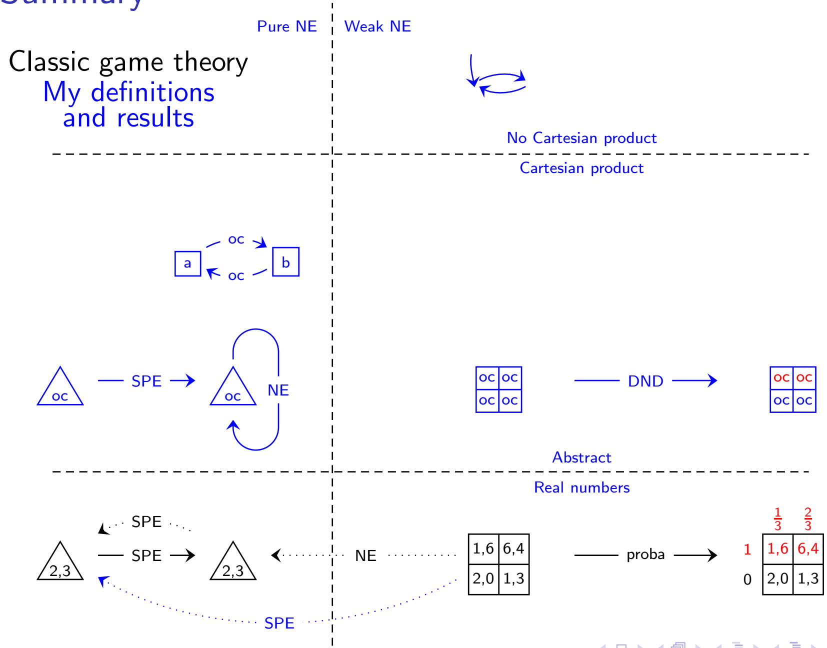
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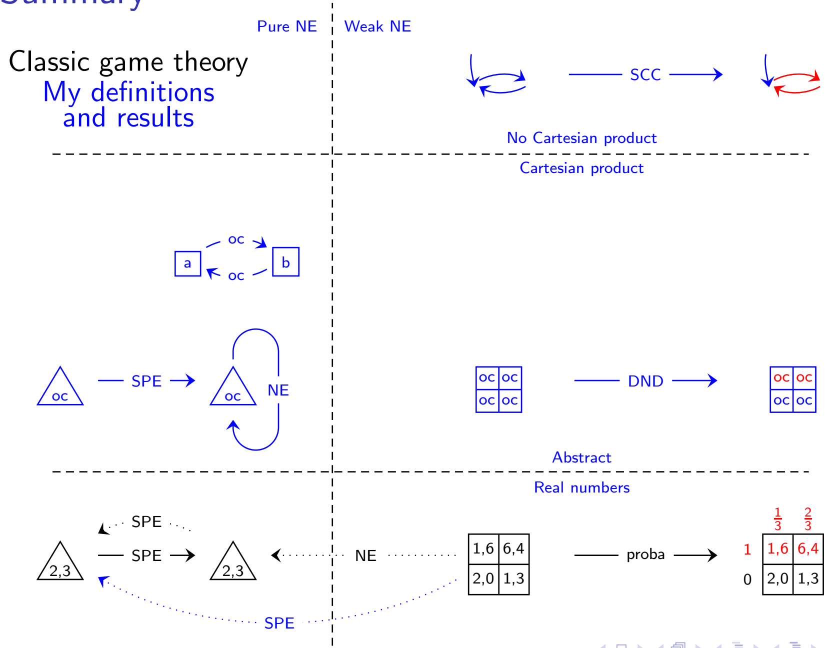
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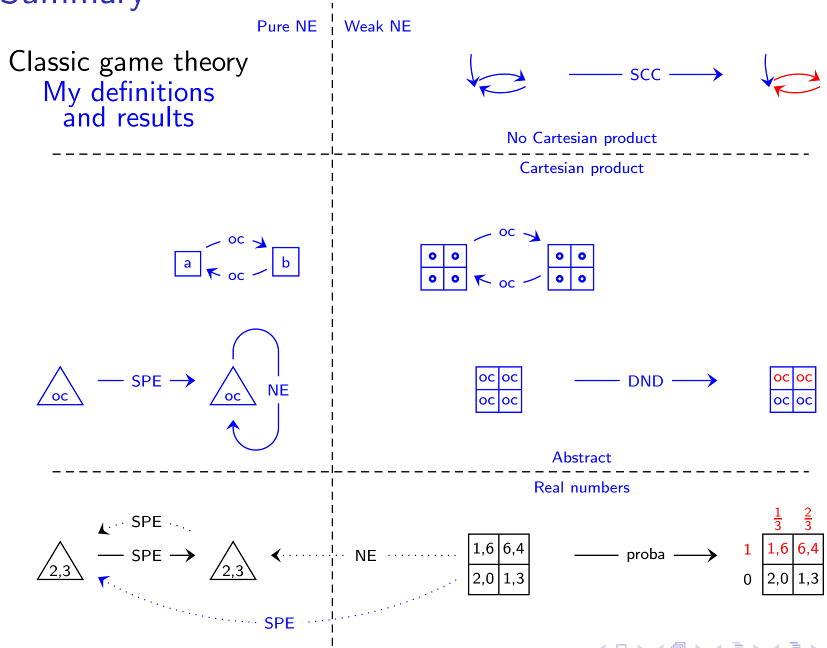

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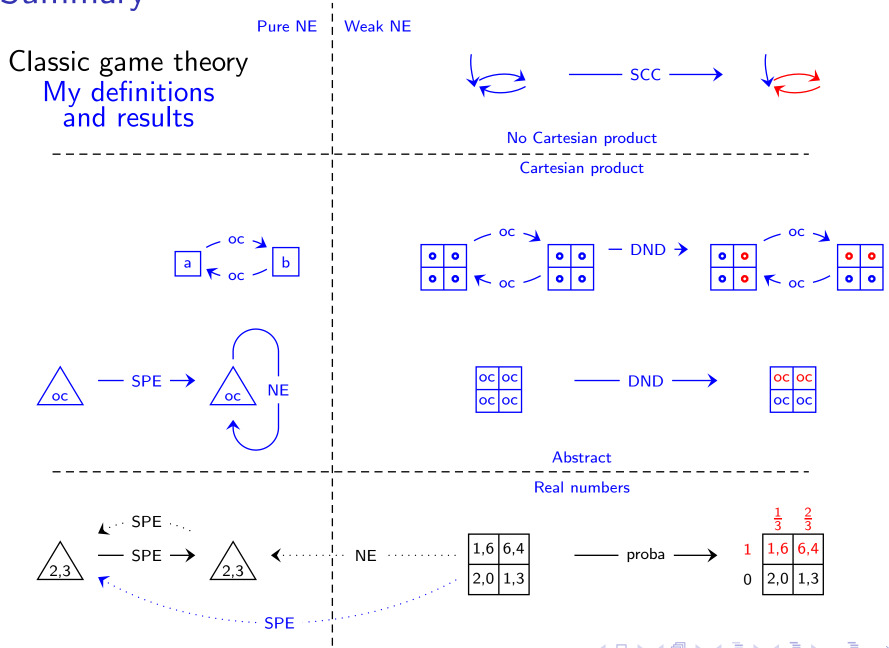
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Contribution w.r.t. formalisation of game theory

Concrete purpose: formalising my generalisation of Kuhn's result.

Prior works: Vestergaard 2006, using Coq.
Weak version of Kuhn's result for binary trees.
Inspiring but not suitable for the general case.

My work:

- About 2000 lines of code, from scratch.
- 60% of topological sorting.
  $\rightsquigarrow$ CoLoR library on rewriting.
- 40% of game theory.
  $\rightsquigarrow$ General formalism for sequential game theory (game tree, strategy profile, induction principles, conversion, preference, induced outcome, Nash equilibrium, SPE, "backward induction", etc).

Possible continuations

- Non-coinciding necessary and sufficient conditions.
- Sequential graph games for network routing.
- Notions of discrete non-deterministic Nash equilibrium.
- Time complexity.
- Equilibrium different from Nash: tacit cooperation.
- More formalisation.