Acyclicity of Preferences, Nash Equilibria, and Subgame Perfect Equilibria: a Formal and Constructive Equivalence

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Thesis chapters 4 and 5

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Game Theory in Short

Developments I rely on

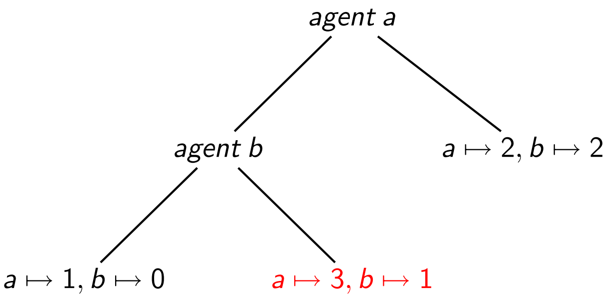
- 1950: Nash equilibrium for strategic games.
- 1953: Kuhn's Nash equilibrium existence for sequential games.
- 1965: Selten's subgame perfect equilibria.
- 2006: using Coq, Vestergaard proved part of Kuhn's result (binary trees instead of trees).

This Presentation

1. Traditional sequential game theory.
2. - Abstraction of sequential games (new).
   - Generalisation of Kuhn's result (new).
   - Fully proved using Coq (new).
3. Applications (new).
4. Another proof (new).
5. Inductive formalism (new for game theory).

Sequential Game

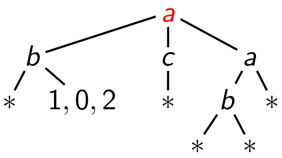
- Finite rooted tree.
- Internal nodes labelled with agents.
- Leaves labelled with payoff functions (from agents to real numbers).



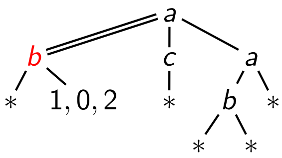
Play in a Sequential Game

*'s are arbitrary payoff functions.

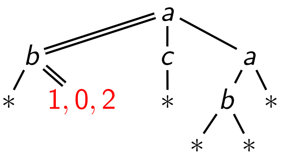
Start



First move

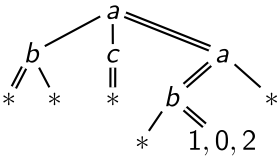


Last move

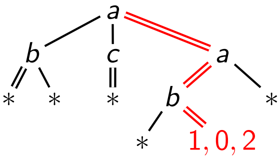


Strategy Profile and Induced Payoff Function

Strategy profile

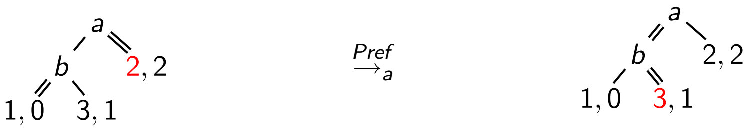


Induced payoff function

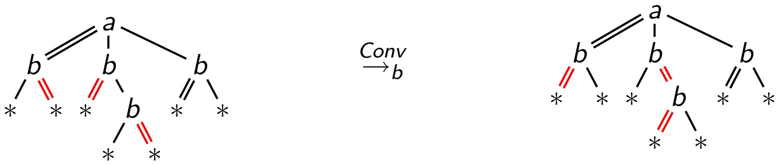


Preference and Convertibility

Preference of agent *a*



Conversion ability of agent *b*

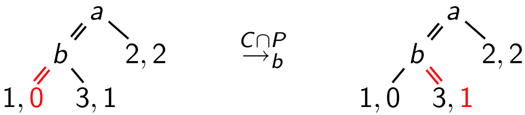


Nash Equilibrium

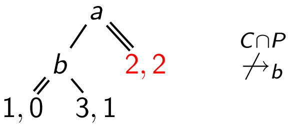
Given a strategy profile,

- Agent happiness: Conversion ability $\cap$ Preference $= \emptyset$.
- Nash equilibrium: Every agent is happy.

A strategy profile that is not a Nash equilibrium



A Nash equilibrium

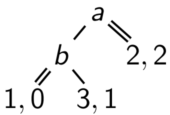


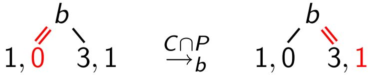
Subgame Perfect Equilibrium (S.P.E.)

Definition (Subgame Perfect Equilibrium)

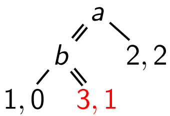
An S.P.E. is a Nash equilibrium for which every child is an S.P.E.

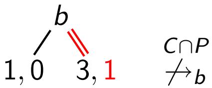
A Nash equilibrium
that is not an S.P.E.





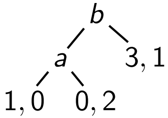
An S.P.E.



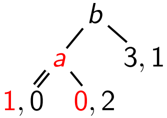


“Backward Induction”

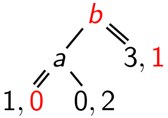
Input: game below



First step



Last step: ouput



Theorem (Kuhn)

Every sequential game has a Nash equilibrium.

Proof.

"Backward induction" yields S.P.E. □

Outline

Introduction

Traditional Sequential Game Theory

Generalisation of Kuhn's Result (new)

Applications (new)

Another Proof (new)

Inductive Formalism (new for game theory)

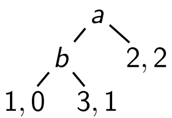
Towards Abstraction

- Why only real-valued payoff functions?
- Why payoff functions?

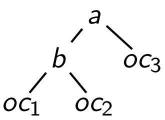
Abstract Sequential Game (new)

- For leaves: abstract objects named outcomes instead of concrete payoff functions.
- For preferences: arbitrary binary relations instead of the usual total order over the reals.

Traditional



Abstract

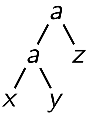


Similar notions of strategy profile, induced outcome, convertibility, Nash equilibrium with respect to given preferences.

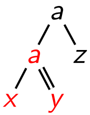
"Backward Induction" $\not\rightsquigarrow$ Nash Equilibrium

The preference for agent a is defined by $z \xrightarrow{Pref}_a x$ only.

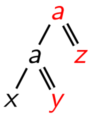
Game input



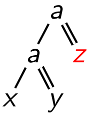
First step



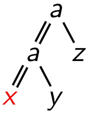
Last step



Not Nash equilibrium



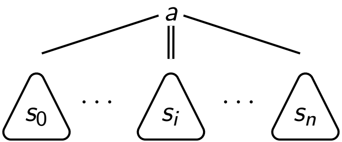
$$\xrightarrow{C\cap P}_a$$



Subequilibrium Lemma

Lemma

Agent b ($\neq a$) is happy with the strategy profile below iff he is happy with s_i.



Lemma (Subequilibrium)

A non-leaf strategy profile is a Nash equilibrium iff the agent owning the root is happy and choses a Nash equilibrium.

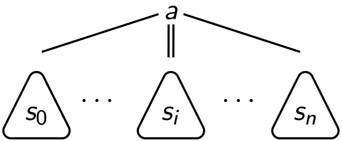
Total Ordering of Preferences Guarantees S.P.E.

Lemma

If preferences are total orders, then all games have S.P.E.

Proof.

By structural induction on games and the subequilibrium lemma (akin to Kuhn's proof).



□

Acyclicity of Preferences Guarantees S.P.E. (new)

Lemma (new)

If preferences are acyclic, then all games have S.P.E.

Proof.

- acyclicity $\Rightarrow$ linear extensions (total order bigger preferences).
- There is S.P.E. with respect to the linear extensions (previous slide).
- It is also S.P.E. with respect to the original preferences, since smaller preferences imply more equilibria.

□

Nash Equilibria Requires Acyclicity of Preferences

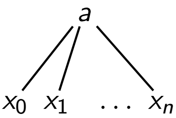
Lemma

If there exists a cycle in the preferences, then there exists a game without Nash equilibrium.

Proof.

Assume a cycle $x_0 \overset{Pref}{\longrightarrow}_a x_1 \ldots x_n \overset{Pref}{\longrightarrow}_a x_0$.

The following game has no Nash equilibrium.



□

Triple Equivalence (new)

Theorem (new)

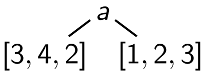
The following three propositions are equivalent:

- *The preferences are acyclic.*
- *Every game has a Nash equilibrium.*
- *Every game has a subgame perfect equilibrium.*

Outline

Multi-Criteria Sequential Games

- 1950's: Simon and Blackwell's vector payoff.
- For each agent, payoffs are vectors of fixed length.



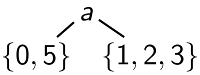
$$\frac{\forall i \leq n, x_i \leq y_i \qquad \exists k \leq n, x_k < y_k}{(x_0, \ldots, x_n) <_{vect} (y_0, \ldots, y_n)}$$

Corollary (new)

Every multi-criteria sequential game has a Nash equilibrium/subgame perfect equilibrium.

Non-Deterministic Payoffs (new)

Payoffs: real number $\rightsquigarrow$ set of real numbers.



$$\frac{\forall x \in X, y \in Y,\ x \leq y \qquad \exists x \in X, y \in Y,\ x < y}{X <_{set} Y}$$

Corollary (new)

Every sequential game with non-deterministic payoffs has a Nash equilibrium/subgame perfect equilibrium.

Benevolent Selfishness (new)

Traditional sequential games, but different preferences.
Let P_1 and P_2 be two payoff functions.

$$\frac{P_1(a) < P_2(a)}{P_1 <_a P_2} \qquad \frac{\forall b, P_1(b) \leq P_2(b) \qquad \exists b', P_1(b') < P_2(b')}{P_1 <_a P_2}$$

Corollary (new)

Every sequential game with selfish-benevolent agents has a Nash equilibrium/subgame perfect equilibrium.

Outline

Second Proof

Cut and use I.H.

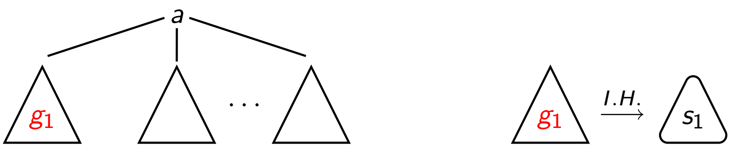
Lemma

If preferences are strict partial orders, then all games have Nash equilibria.

New proof (Five slides)

Prove by induction on the natural number n that any sequential game with at most n nodes has a Nash equilibrium.

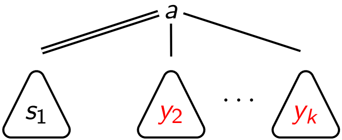
Case $n + 1$, let g be as follows.



Second Proof

Case split

Consider all the possible $y_2, \ldots, y_k$ such that g is the underlying game of the following strategy profile.

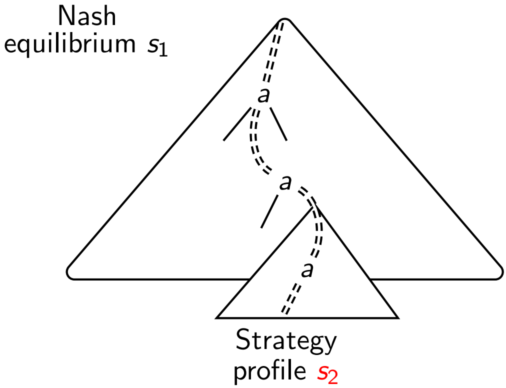


- First case, for some $y_2, \ldots, y_k$ agent a is happy with the profile, which is a Nash equilibrium by the subequilibrium lemma.
- Second case, for all $y_2, \ldots, y_k$ agent a is not happy with the profile.

Second Proof

Cut

On the path induced by s_1, consider nodes that both are owned by agent a and have two or more children.

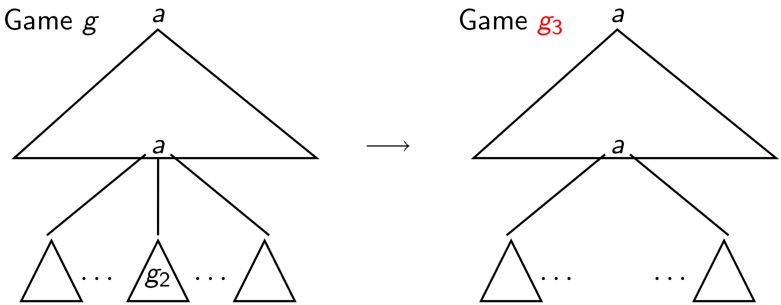


By the subequilibrium lemma, s_2 is a Nash equilibrium.

Second Proof

Cut and use I.H.

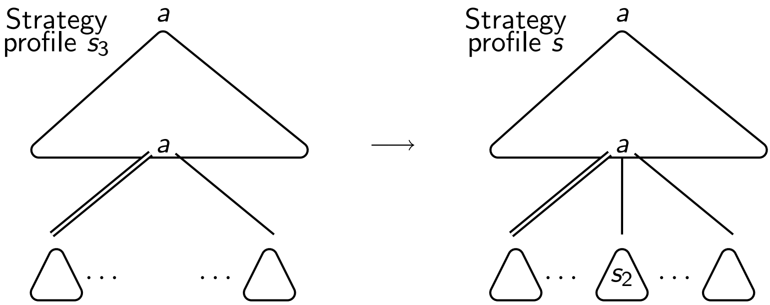
Let g_2 be the underlying game of s_2.



$$g_3 \xrightarrow{I.H.} s_3$$

Second Proof

Paste



- Agent $b \neq a$ is happy since s_3 is a Nash equilibrium.
- Agent a is happy by the first case splitting of the proof.

Acyclicity of Preferences Guarantees Nash Equilibrium

Lemma

If preferences are acyclic, then all games have a Nash equilibrium.

Proof.

- acyclicity $\Rightarrow$ partial orders by transitive closure (bigger preferences).
- There is a Nash equilibrium with respect to the transitive closures (previous slide).
- It is also a Nash equilibrium with respect to the original preferences, since smaller preferences imply more equilibria.

□

Double Equivalence (new)

Theorem (new)

The following two propositions are equivalent:

- *The preferences are acyclic.*
- *Every game has a Nash equilibrium.*

Comparing the Two Proofs

- Triple *vs* double equivalence.
- Through perfect equilibrium *vs* direct proof.
- Minimal hypotheses *vs* decidable equality between agents.
- Induction on games *vs* natural numbers.
- Uniform use of I.H. *vs* I.H. used twice in different contexts.
- Informative *vs* magic.
- Fully formalised using Coq *vs* not fully yet.

Outline

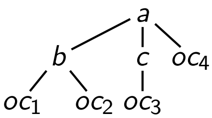
Games, by Induction

- If *oc* is an outcome then the object below is a game.

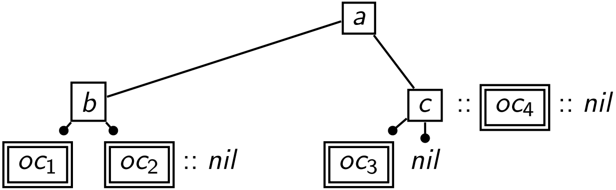
- If *a* is an agent, *g* is a game, and *l* is a list of games, then the rightmost object is also a game.

Example

- Traditional:



- New:



Games in Coq

Variable $Outcome$: Set.
Variable $Agent$: Set.

Inductive $Game$: Set :=
| gL : $Outcome \rightarrow Game$
| gN : $Agent \rightarrow Game \rightarrow list\ Game \rightarrow Game$.

Coq Induction Principle for Games

$\forall\ (P : Game \rightarrow Prop)\ (Q : Game \rightarrow list\ Game \rightarrow Prop),$

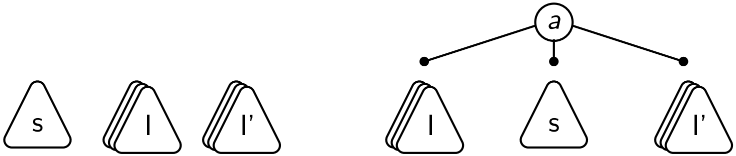
$(\forall\ oc,\ P\ (gL\ oc)) \rightarrow$
$(\forall\ g,\ P\ g \rightarrow Q\ g\ nil) \rightarrow$
$(\forall\ g,\ P\ g \rightarrow \forall\ g'\ l,\ Q\ g'\ l \rightarrow Q\ g'\ (g :: l)) \rightarrow$
$(\forall\ g\ l,\ Q\ g\ l \rightarrow \forall\ a,\ P\ (gN\ a\ g\ l)) \rightarrow$

$\forall\ g,\ Pg$

Strategy profiles, by Induction

- If *oc* is an outcome then the object below is a strategy profile.

- If a is an agent, s is a strategy profile, and l and l' are lists of profiles, then the rightmost object is also a strategy profile.



Summary

- Abstraction of sequential tree games.
- Generalisation of Kuhn's result, by induction on games.
- Converse property.
- Fully proved using Coq.
- Inductive formalism for games and related notions.
- Applications to various natural instances.
- Second proof, by induction on the size of games.