Discrete Non Determinism and Nash Equilibria for Strategy-Based Games

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Outline

Real-valued strategic game theory

- Traditional strategic game theory
- Non-traditional real-valued strategic game theory

Abstract strategy-based game theory (new)

- Abstract strategic game theory
- Abstract multi-strategic game theory

Strategic game and Nash equilibrium

Definition by an example

- 2 players V (vertical) and H (horizontal).
- 2 strategies available for V (resp. for H).
- V and H pick 1 strategy each $\rightsquigarrow$ strategy profile.
- Strategy profile $\rightsquigarrow$ 1 real-valued payoff per player.

	h_1		h_2	
v_1	0	2	1	1
v_2	0	1	0	2

	h_1		h_2	
v_1	0	1	1	0
v_2	1	0	0	1

$$Happy_V(v_2, h_1) \triangleq (v_1, h_1) \leq_V (v_2, h_1)$$

$$NashEquilibrium(v_i, h_j) \triangleq Happy_V(v_i, h_j) \wedge Happy_H(v_i, h_j)$$

There are games with **no** Nash equilibrium.

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Probabilistic Nash equilibrium

- Strategy $\rightarrow$ Probability distribution over strategies.

	$h_1 \mapsto y$	$h_2 \mapsto 1-y$
$v_1 \mapsto x$	0, 1	1, 0
$v_2 \mapsto 1-x$	1, 0	0, 1

- Payoff $\rightarrow$ Expected payoff

  $xy \cdot 0 + (1-x)y \cdot 1 + x(1-y) \cdot 0 + (1-x)(1-y) \cdot 1$

- Nash equilibrium $\rightarrow$ Probabilistic Nash equilibrium.

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Theorem (Nash, 1950)

Every finite strategic game has a probabilistic Nash equilibrium.

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Finding probabilistic Nash equilibrium

Difficult computation, but reducible search space.

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Non-traditional real-valued strategic games

Payoff → **vector** of payoffs:

	h_1	h_2
v_1	$[0,0]$, $[2,1]$	$[1,0]$, $[1,2]$
v_2	$[0,0]$, $[1,2]$	$[0,1]$, $[2,1]$

- Expresses criteria's **incommensurability**. (Used by Simon, Blackwell, . . . since the 1950's).
- The notion of expected vector of payoffs makes sense, but there are games with **no probabilistic** Nash equilibrium.

Payoff → **set** of payoffs: (new)

	h_1	h_2
v_1	$\{1,4\}$, $\{0\}$	$\{0\}$, $\{3\}$
v_2	$\{0\}$, $\{2,3\}$	$\{3\}$, $\{1,2\}$

- Expresses payoffs' **discrete non-determinism**.
- $\{1,4\}$ means V gets either 1 or 4 (but V doesn't know how).
- The notion of expected set of payoffs makes **no** sense.

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Abstract strategic game and Nash equilibrium

Definition by an example

Abstraction over payoff function by outcomes o_1, o_2, $o_3 \ldots$

	h_1	h_2
v_1	o_1	o_2
v_2	o_3	o_4

- Preference between outcomes: **arbitrary** binary relation, *e.g.* $o_1 \prec_V o_2$ and $o_3 \prec_H o_1$.
- $NashEquilibrium(v_i, h_j) \triangleq Happy_V(v_i, h_j) \wedge Happy_H(v_i, h_j)$
- $Happy_V(v_1, h_1) \triangleq v_1 \in BestResponse_V(h_1)$, where $v_1 \in BestResponse_V(h_1) \triangleq \mathbf{o_1} \not\prec_V \mathbf{o_1} \wedge \mathbf{o_1} \not\prec_V \mathbf{o_3}$

- There are abstract games with no Nash equilibrium.
- The notion of expected abstract outcome makes no sense.

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Discrete non-detererministic Nash equilibrium

Probability distribution over strategies makes no sense.

	$h_1 \mapsto \frac{1}{\sqrt{2}}$	$h_2 \mapsto 1 - \frac{1}{\sqrt{2}}$	$h_3 \mapsto 0$
$v_1 \mapsto \frac{1}{3}$	$\mathbf{o}_{1,1}$	$\mathbf{o}_{1,2}$	$o_{1,3}$
$v_2 \mapsto \frac{2}{3}$	$\mathbf{o}_{2,1}$	$\mathbf{o}_{2,2}$	$o_{2,3}$
$v_3 \mapsto 0$	$o_{3,1}$	$o_{3,2}$	$o_{3,2}$

Discrete non-detererministic Nash equilibrium

Strategy: pure < **discrete non-deterministic** < probabilistic.

	$h_1 \mapsto\ > 0$	$h_2 \mapsto\ > 0$	$h_3 \mapsto\ = 0$
$v_1 \mapsto\ > 0$	$\mathbf{o_{1,1}}$	$\mathbf{o_{1,2}}$	$o_{1,3}$
$v_2 \mapsto\ > 0$	$\mathbf{o_{2,1}}$	$\mathbf{o_{2,2}}$	$o_{2,3}$
$v_3 \mapsto\ = 0$	$o_{3,1}$	$o_{3,2}$	$o_{3,2}$

- Let $S_V = \{v_1, v_2\}$ and $S_H = \{h_1, h_2\}$.

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- Let $S_V = \{v_1, v_2\}$ and $S_H = \{h_1, h_2\}$.
- $NashEq(S_V, S_H) \triangleq Happy_V(S_V, S_H) \wedge Happy_H(S_V, S_H)$
- $Happy_V(S_V, S_H) \triangleq S_V \subseteq BestResp_V(S_H)$
- Non-det *BestResp*: 5 relevant extensions of pure *BestResp*. (Example coming soon.)

Existence of discrete non-det Nash equilibrium

Theorem (new)

If preferences ($\prec_V$, $\prec_H$, etc) are transitive and irreflexive, every abstract strategic game has a non-det Nash equilibrium.
(For any of the 5 relevant extensions of pure BestResp)

Proof.

Prove and invoke a dedicated fixed-point lemma.
(constructive proof with polynomial complexity.) □

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Corollary (new)

Every vector-payoff (resp. set-payoff) strategic game has a discrete non-deterministic Nash equilibrium (for the 5 extensions).

Proof.

vector-payoff (resp. set-payoff) preferences are transitive and irreflexive + theorem above. □

Computing discrete non-det Nash equilibrium

With 1 specific non-det *BestResp* among the 5.

	$h_1 \mapsto Maybe$		$h_2 \mapsto Maybe$		$h_3 \mapsto Maybe$	
$v_1 \mapsto Maybe$	3,	2	2,	3	3,	1
$v_2 \mapsto Maybe$	2,	3	3,	2	3,	1
$v_3 \mapsto Maybe$	0,	0	0,	0	0,	3

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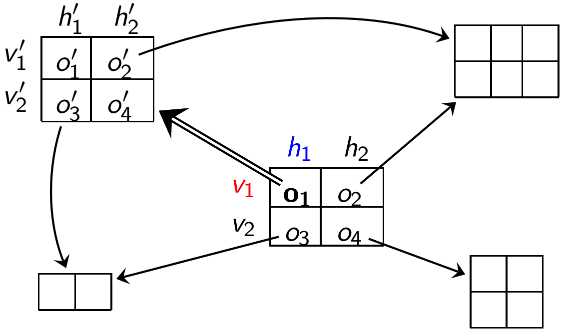
Multi-strategic game

- Digraph whose nodes are abstract strategic games.
- At a node: local profile yields outcome and next node.



- **Global** profile $\triangleq$ 1 **local** profile per node.
- Nash equilibrium $\triangleq$ **global** profile inducing (starting from each node) a **maximal** infinite sequence of outcomes.
- Games without NashEq, but discrete non-det is a solution.

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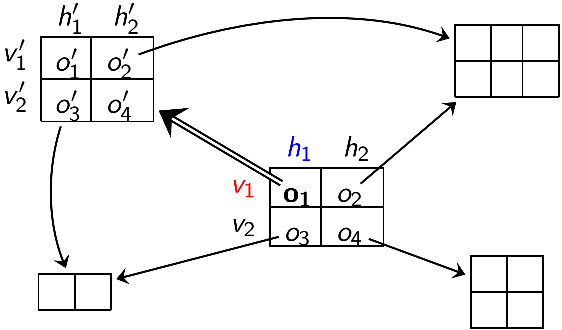

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- Non-det strategy: continuous $\rightarrow$ discrete.
- Existence of discrete Nash equilibrium.
  (for 5 definitions.)
- polynomial complexity, similarity discrete/continuous.
  (+ algebraic structure, recommendation.)
- Other definitions of discrete non-det Nash equilibrium?
  (polynomial complexity?)
- Multi-strategic games. (simultaneous and sequential.)

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