A discrete Nash Theorem with quadratic complexity and dynamic equilibria

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Abstract

Nash's Theorem guarantees the existence of *Nash equilibria* for *strategic-form games*. The typical proof of the result uses Brouwer's Fixed Point Theorem on probabilistic strategies. We show that Tarski's Fixed Point Theorem can be used to establish a similar result for discrete equilibria in a larger class of games that we call *conversion/preference games*. Our result rests on a graph characterisation of Nash equilibria that i) reifies the decision procedure for *pure* Nash equilibria, ii) allows us to compute the equilibria in quadratic time in the number of game situations, and iii) makes the equilibria explicitly dynamic in nature. We conclude by discussing the extended range of technical applications of *non-cooperative game theory* that results from the new theorem, including for gene regulation and cell-level signal transduction.

1 Introduction

In (1), Nash proved that all finite strategic-form games have a *mixed* (i.e., a probabilistic) Nash equilibrium. A detailed proof using Brouwer's Fixed Point Theorem (2) is given in (3). To be precise, Nash observed that the set of finite strategic-form games can be embedded into a set of (continuous) strategic-form games where each agent's set of strategies is comprised of probability distributions over the agent's original strategies and then proved that the latter set of games always have Nash equilibria, in the form of fixed points of a given continuous function.

	h_1	h_2
v_1	0, 1	1, 0
v_2	1, 0	0, 1

Each cell of the array above is an outcome of a possible play of the exemplified game, with the first number in a cell being the resulting *payoff* to player 'v', who chooses the row, and the second number the payoff to player 'h', who chooses the column. The example has no one outcome that satisfies all players in Nash's sense: an agent is happy if he cannot single-handedly improve his payoff, see Definition 2. For example, 'v' would be unhappy with the upper-left outcome because the lower-left is a feasible alternative to him that would increase his payoff; in turn, 'h' would be unhappy with the lower-left outcome because of the lower-right one, and so on counter-clockwise in the array. Instead, a probabilistic Nash equilibrium arises if both agents choose between their two options with equal probability, with *expected* payoffs of a half to each. The usual reading of Nash's probabilistic construction is that it prescribes (weighted) compromises.

In Section 2, we discuss the original Nash The-

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orem; in Section 3, we introduce the new formalism of conversion/preference (C/P) games; in Section 4, we pursue a discrete fixed-point construction on C/P games; in Section 5, we prove our discrete Nash Theorem directly and establish its complexity; in Section 6, we show that the new result admits dynamic equilibria; in Section 7, we connect our earlier fixed-point construction to our new Nash Theorem; in Section 9, we compare the Nash equilibria found by Nash's probabilistic and our discrete Nash Theorems; in Section 10, we discuss the additional modelling capabilities of C/P games compared with strategic-form games; in Section 11, we summarise two applications of our new Nash Theorem in life science.

2 Nash's Theorem

We have informally described strategic-form games as being arrays. Formally, we have the following.

Definition 1 (Strategic-Form Games) G^{sf} *are 3-tuples* $\langle \mathcal{A}, \mathrm{S}_{\mathcal{A}}, \mathrm{P}\rangle$*, where:*

- $\mathcal{A}$ *is a non-empty set of* agents*,*
- $\mathrm{S}_{\mathcal{A}}$ *is the cartesian product,* $\bigotimes_{\mathrm{a}\in\mathcal{A}} \mathrm{S}_{\mathrm{a}}$*, called* strategy profiles*, of non-empty sets of* individual strategies*,* S_{a}*, for each agent,* a*,*
- $\mathrm{P} : \mathrm{S}_{\mathcal{A}} \rightarrow \mathcal{A} \rightarrow \mathbb{R}$ *awards real-valued* payoffs*.*

Let s *range over* $\mathrm{S}_{\mathcal{A}}$*; let* s_{a} *be the* a*-projection of* s*.*

In terms of Nash equilibria, agents in a strategic-form game are free to change the entry in their dimension of the cartesian product of individual strategies but must leave any other entries unchanged. The question of whether a better outcome single-handedly can be obtained by any particular agent is therefore answered (in the negative) by the following predicate.

Definition 2 *A strategy,* s*, is a* Nash equilibrium*,* $\mathrm{Eq}^{\mathrm{N}}_{\mathrm{G}^{\mathrm{sf}}}(\mathrm{s})$*, for a given strategic-form game,* G^{sf}*, if*[1]

$$\begin{array}{lll} \forall \mathrm{a}\in\mathcal{A}, \mathrm{s}'\in\mathrm{S}_{\mathcal{A}} & . & (\forall \mathrm{a}'\in\mathcal{A}\ .\ \mathrm{a}\neq\mathrm{a}' \Rightarrow \mathrm{s}_{\mathrm{a}'} = \mathrm{s}'_{\mathrm{a}'}) \\ & & \Downarrow \\ & & \neg(\mathrm{P}(\mathrm{s})(\mathrm{a}) < \mathrm{P}(\mathrm{s}')(\mathrm{a})) \end{array}$$

[1] Nash's original notation for our s′ is "$\mathrm{s}_{\text{-a}};\sigma$", i.e., s with something else (from S_{a}) in the a-position.

As noted, Nash equilibria do not always exist directly in a strategic-form game and we now define *individual probabilities*, *probability profiles*, and, for a given probability profile, the overall *probability* that the agents collectively assign to a strategy profile.

Definition 3 (Strategic-Form Probabilities) *Given finite* $\langle \mathcal{A}, \mathrm{S}_{\mathcal{A}}, \mathrm{P}\rangle$*.*

$$\begin{array}{rcl} \mathrm{W}^{\mathrm{S}_{\mathrm{a}}} & \triangleq & \{\mathrm{w}_{\mathrm{a}} : \mathrm{S}_{\mathrm{a}} \rightarrow [0,1] \mid (\sum_{\sigma\in\mathrm{S}_{\mathrm{a}}} \mathrm{w}_{\mathrm{a}}(\sigma)) = 1\} \\ \mathrm{W}^{\mathrm{S}_{\mathcal{A}}} & \triangleq & \bigotimes_{\mathrm{a}\in\mathcal{A}} \mathrm{W}^{\mathrm{S}_{\mathrm{a}}} \\ \mu^{\mathrm{S}_{\mathcal{A}}}(\mathrm{w},\mathrm{s}) & \triangleq & \prod_{\mathrm{a}\in\mathcal{A}} \mathrm{w}_{\mathrm{a}}(\mathrm{s}_{\mathrm{a}}) \end{array}$$

The *expected-payoff* function associated with a probability profile is as follows.

Definition 4 (Expected Payoff Function) *Given finite* $\langle \mathcal{A}, \mathrm{S}_{\mathcal{A}}, \mathrm{P}\rangle$ *with associated probabilities.*

$$\mathrm{EP}^{\mathrm{S}_{\mathcal{A}}}_{\mathrm{P}}(\mathrm{w})(\mathrm{a}) \quad \triangleq \quad \sum_{\mathrm{s}\in\mathrm{S}_{\mathcal{A}}} \mu^{\mathrm{S}_{\mathcal{A}}}(\mathrm{w},\mathrm{s}) \cdot \mathrm{P}(\mathrm{s})(\mathrm{a})$$

Nash's insight, result, existence argument, and the employed construction can now be articulated thus.

Theorem 5 (Nash (1, 3)) *Consider a finite strategic-form game,* $\langle \mathcal{A}, \mathrm{S}_{\mathcal{A}}, \mathrm{P}\rangle$*, given with probabilities. The strategic-form game* $\langle \mathcal{A}, \mathrm{W}^{\mathrm{S}_{\mathcal{A}}}, \mathrm{EP}^{\mathrm{S}_{\mathcal{A}}}_{\mathrm{P}}\rangle$ *has a Nash equilibrium. Informally, we say that* $\langle \mathcal{A}, \mathrm{S}_{\mathcal{A}}, \mathrm{P}\rangle$ *has a* probabilistic Nash equilibrium*.*

Proof We follow (3). $\mathrm{W}^{\mathrm{S}_{\mathcal{A}}}$ is the cartesian product of each agent's $\mathrm{W}^{\mathrm{S}_{\mathrm{a}}}$. Because they involve a summation to 1, each $\mathrm{W}^{\mathrm{S}_{\mathrm{a}}}$ is the standard simplex of the vector space spanned by the elements of S_{a} taken as unit vectors. As a result, $\mathrm{W}^{\mathrm{S}_{\mathcal{A}}}$ is a convex polytope in the vector space spanned by $\mathrm{S}_{\mathcal{A}}$, which in particular means that it is compact. More, it is possible to define a continuous function on probability profiles that, for each agent, speculatively puts more weight where that agent can benefit from it *relative to the other agents' unchanged weights* and then makes a combined change. This function has a fixed point by the generalised Brouwer's Fixed Point Theorem[2] (2) and any such fixed point is a Nash equilibrium (3). □

[2] "A continuous function on a compact, convex set has a fixed point".

The problem of finding probabilistic Nash equilibria for finite strategic-form games with at least two agents is in PPAD in the size of $\mathrm{S}_{\mathcal{A}}$ (4). In fact, it is PPAD-complete (5, 6). Informally, this means that it is likely that we will not know whether the problem is polynomial or exponential for some time.

3 C/P Games

To facilitate our discrete development, we now introduce a new game formalism called conversion/preference (C/P) games. It is based on strategic-form games, with conversion and preference accounting for the views and options available to the partaking players. The formalism distinguishes itself by seemingly being the most general structure that allows for the definition of Nash equilibria; in a sense, C/P games capture the essence of Nash equilibria.

Definition 6 (C/P Games) G^{cp} *are 4-tuples* $\langle \mathcal{A}, \mathcal{S}, (\underset{\mathrm{a}}{\succ})_{\mathrm{a}\in\mathcal{A}}, (\lhd_{\mathrm{a}})_{\mathrm{a}\in\mathcal{A}} \rangle$*, where:*

- $\mathcal{A}$ *is a non-empty set of* agents.
- $\mathcal{S}$ *is a non-empty set of* synopses.[3]
- *For* $\mathrm{a} \in \mathcal{A}$*,* $\underset{\mathrm{a}}{\succ}$ *is a binary relation over* $\mathcal{S}$*, associating two synopses if agent* a *can* convert *the first to the second.*
- *For* $\mathrm{a} \in \mathcal{A}$*,* $\lhd_{\mathrm{a}}$ *is a binary relation over* $\mathcal{S}$*, associating two synopsis if agent* a prefers *the second to the first.*

Synopses are abstractions over *strategy profiles* but, in combination with conversion and preference, they can also be seen more generally as denoting the state of (a play of) a game, e.g., in terms of the players' possessions or even in a purely abstract sense, see Sections 10 and 11.

In strategic-form games, the cartesian-product nature of the set of strategy profiles, $\mathrm{S}_{\mathcal{A}}$, determines what alternatives are available to a particular agent as far as Nash equilibria are concerned: agent a can change the a-projection of an s to something else from S_{a}. The dynamic view of an agent "changing" one strategy profile to another by contributing a different individual strategy is what we capture directly in the definition of C/P games, by $\underset{\mathrm{a}}{\succ}$, without relying on an underlying structure of, in this case, $\mathcal{S}$.

The preference relations, $\lhd_{\mathrm{a}}$, account relatively for the penalties and rewards the agents receive in the different outcomes, without explicit payoffs (7).

Definition 7 *A synopsis,* s*, is an* abstract Nash equilibrium*,* $\mathrm{Eq}^{\mathrm{aN}}_{\mathrm{G^{cp}}}(\mathrm{s})$*, for a given* G^{cp}*, if*

$$\forall \mathrm{a} \in \mathcal{A}, \mathrm{s}' \in \mathcal{S} \quad . \quad \mathrm{s} \underset{\mathrm{a}}{\succ} \mathrm{s}' \ \Rightarrow\ \neg(\mathrm{s} \lhd_{\mathrm{a}} \mathrm{s}')$$

The canonical embedding of strategic-form games into C/P games preserves and reflects (abstract) Nash equilibria and we therefore suppress "abstract".

Proposition 8 *For a strategic-form game,* G^{sf}*, let*

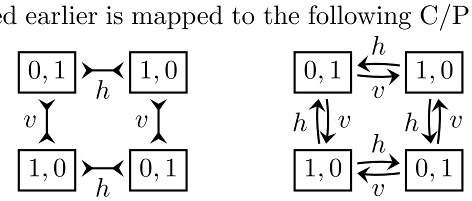
$$\mathrm{s} \underset{\mathrm{a}}{\succ\!\prec} \mathrm{s}' \quad \triangleq \quad \forall \mathrm{a}' . (\mathrm{a} \neq \mathrm{a}' \Rightarrow \mathrm{s}_{\mathrm{a}'} = \mathrm{s}'_{\mathrm{a}'})$$

$$\mathrm{s} \lhd_{\mathrm{a}} \mathrm{s}' \quad \triangleq \quad \mathrm{P}(\mathrm{s})(\mathrm{a}) < \mathrm{P}(\mathrm{s}')(\mathrm{a})$$

$\mathrm{G}^{\mathrm{cp}} = \langle \mathcal{A}, \mathrm{S}_{\mathcal{A}}, (\underset{\mathrm{a}}{\succ\!\prec})_{\mathrm{a}\in\mathcal{A}}, (\lhd_{\mathrm{a}})_{\mathrm{a}\in\mathcal{A}} \rangle$ *is a C/P game and*

$$\mathrm{Eq}^{\mathrm{N}}_{\mathrm{G^{sf}}}(\mathrm{s}) \ \Leftrightarrow\ \mathrm{Eq}^{\mathrm{aN}}_{\mathrm{G^{cp}}}(\mathrm{s})$$

We note that the result covers both the original and the derived probabilistic games considered in Theorem 5. To illustrate, we note that the example considered earlier is mapped to the following C/P game.



The C/P-game formalism accommodates more than just strategic-form games. We shall address what this means in Section 10 and take more substantial advantage of it in Section 11.

4 Discrete Fixed Points

The starting point of our discrete Nash Theorem is the "unhappiness" relation used in Section 1, capturing when agents *can* and *want* to change their minds.

[3] The name synopsis is inspired by the thespian meaning of 'abstract of a play'.

Definition 9 *The* (free) change-of-mind *relation for agent* a *is* $\rightarrow_{\mathrm{a}} \triangleq \succ_{\mathrm{a}} \cap \lhd_{\mathrm{a}}$. *Let* $\rightarrow \triangleq \bigcup_{\mathrm{a} \in \mathcal{A}} \rightarrow_{\mathrm{a}}$ *and let* $\rightarrow^*$ *be the reflexive, transitive closure of* $\rightarrow$.

Our initial interest in the change-of-mind relation is that it allows us to define a discrete equivalent of the update function (for probabilities) sketched in the proof of Theorem 5. We shall objectively compare the notions of compromise induced by the two resulting flavours of Nash equilibria in Section 9.[4]

Definition 10 $\mathscr{U}(\mathrm{S}) \triangleq \bigcup_{\mathrm{s} \in \mathrm{S}} \{\mathrm{s}' \mid \mathrm{s} \rightarrow^* \mathrm{s}'\}$

The empty set, $\emptyset$, and the full set, $\mathcal{S}$, are straightforwardly seen to be fixed points of this function. More generally, we have the following result.

Lemma 11 *The set of fixed points of* $\mathscr{U}$ *forms a non-empty, complete lattice.*

Proof $\mathscr{U}$ is monotonic on the complete lattice $\mathcal{P}(\mathcal{S})$ ordered by inclusion because $\rightarrow^*$ is reflexive; we are done by Tarski's Fixed Point Theorem (8). □

Interestingly, not all discrete fixed points will be Nash equilibria and we therefore pursue an alternative presentation of our technology next. We shall return to the fixed-point construction in Section 7.

5 A Discrete Nash Theorem

In this section, we take inspiration from the fact that the change-of-mind relation can be used to characterise Nash equilibria. Specifically, change-of-mind reifies the already-considered "happiness" decision procedure for (pure) Nash equilibria — note that a *terminal* in a graph is a node with no out-edges.

Proposition 12 $\mathrm{Eq}^{\mathrm{aN}}_{\mathrm{G}^{\mathrm{cp}}}(\mathrm{s}) \Leftrightarrow \mathrm{Terminal}_{\rightarrow}(\mathrm{s})$

Proof $\rightarrow_{\mathrm{a}}$ is the intersection of $\succ_{\mathrm{a}}$ and $\lhd_{\mathrm{a}}$. □

[4] Additionally, Section 11.1 will detail a compelling analogue of the difference between the two styles of update functions that is well-established in the area of gene-regulation analysis.

The result implies that only cycles can prevent the existence of Nash equilibria for a finite C/P game. Thinking graph theoretically, we note that generalised cycles in graphs are called *strongly connected components* (SCCs): $\lfloor v \rfloor \triangleq \{v' \mid v \rightarrow^* v' \wedge v' \rightarrow^* v\}$, see Appendix A.

Definition 13 Progressive change-of-mind *for agent* a *is* $\lfloor \mathrm{s} \rfloor \curvearrowright_{\mathrm{a}} \lfloor \mathrm{s}' \rfloor \triangleq \mathrm{s} \rightarrow_{\mathrm{a}} \mathrm{s}' \wedge \lfloor \mathrm{s} \rfloor \neq \lfloor \mathrm{s}' \rfloor$. *Let* $\curvearrowright \triangleq \bigcup_{\mathrm{a} \in \mathcal{A}} \curvearrowright_{\mathrm{a}}$.

The progressive change-of-mind relation is the *shrunken graph* of the change-of-mind relation, i.e., the cycle-free super-structure over SCCs. This means that we can prove a first version of our alternative Nash Theorem, establishing the guaranteed existence of a discrete notion of Nash equilibria.

Theorem 14 (Light Version) *Consider a finite C/P game,* G^{cp}. *The "shrunken" C/P game* $\lfloor \mathrm{G}^{\mathrm{cp}} \rfloor$, *i.e.,* $\langle \mathcal{A}, \lfloor \mathcal{S} \rfloor, (\curvearrowright_{\mathrm{a}})_{\mathrm{a} \in \mathcal{A}}, (\curvearrowright_{\mathrm{a}})_{\mathrm{a} \in \mathcal{A}} \rangle$ *has Nash equilibria, and all of them can be found in quadratic time in the number of synopses.*

Proof The progressive change-of-mind relation used in a "shrunken" C/P game is cycle-free, see Proposition 21, Appendix A. This means that the length of any progressive change-of-mind path is bounded by the size of $\lfloor \mathcal{S} \rfloor$, guaranteeing the existence of terminal nodes and thus, by Proposition 12, compromises that are "shrunken" Nash equilibria. The complexity measure is due to Tarjan (9), see Theorem 20, Appendix A. □

The theorem exploits what could be called the *Nash Construction*, i.e., the guarantee first used in Theorem 5 that some derived game has Nash equilibria that are meaningful as constituting a compromise when retracted back to the original game. We will spend the latter part of this article, from Section 9 onwards, clarifying the notion of compromise that is invoked in Theorem 14 and on comparing it to Nash's payoff-driven probabilistic notion.

6 Change-of-Mind Equilibria

Unlike Theorem 5, the compromises fingered as Nash equilibria in Theorem 14 have a formal, direct characterisation in the originating C/P game. Specifically, our equilibria are clusters of synopses that, while potentially improvable in the view of some agents, can only be immaterially improved upon.

Definition 15 *Write* $\xrightarrow{\mathrm{S}}$ *for* $\rightarrow \cap (\mathrm{S} \times \mathrm{S})$*, i.e., the graph of a set of synopses. For non-empty* S*,* $\xrightarrow{\mathrm{S}}$ *is a* change-of-mind equilibrium*,* $\mathrm{Eq}^{\mathrm{com}}_{\mathrm{G}^{\mathrm{cp}}}(\xrightarrow{\mathrm{S}})$*, for* G^{cp} *if*

$$\forall \mathrm{s} \in \mathrm{S}, \mathrm{s}' \in \mathcal{S} \quad . \quad \mathrm{s} \rightarrow^* \mathrm{s}' \;\Leftrightarrow\; \mathrm{s}' \in \mathrm{S}$$

Lemma 16 $\mathrm{Eq}^{\mathrm{com}}_{\mathrm{G}^{\mathrm{cp}}}(\xrightarrow{\mathrm{S}}) \quad \Leftrightarrow \quad \mathrm{Eq}^{\mathrm{aN}}_{\lfloor \mathrm{G}^{\mathrm{cp}} \rfloor}(\mathrm{S})$

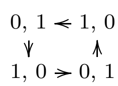
Proof By two direct arguments, using Proposition 12. The only interesting step is from left to right and showing that S necessarily is an $\lfloor - \rfloor$-equivalence class. As S is non-empty, we have an $\mathrm{s}_1 \in \mathrm{S}$. To prove $\mathrm{S} \subseteq \lfloor \mathrm{s}_1 \rfloor$, consider some $\mathrm{s}_2 \in \mathrm{S}$. By the $\Leftarrow$-direction of the assumed $\mathrm{Eq}^{\mathrm{com}}_{\mathrm{G}^{\mathrm{cp}}}(\xrightarrow{\mathrm{S}})$ applied twice, we have $\mathrm{s}_1 \rightarrow^* \mathrm{s}_2$ and $\mathrm{s}_2 \rightarrow^* \mathrm{s}_1$, i.e., $\mathrm{s}_2 \in \lfloor \mathrm{s}_1 \rfloor$, as required. Conversely, if $\mathrm{s}_2 \in \lfloor \mathrm{s}_1 \rfloor$, we have $\mathrm{s}_1 \rightarrow^* \mathrm{s}_2$, and by the $\Rightarrow$-direction of the assumed $\mathrm{Eq}^{\mathrm{com}}_{\mathrm{G}^{\mathrm{cp}}}(\xrightarrow{\mathrm{S}})$, $\mathrm{s}_2 \in \mathrm{S}$. □

To illustrate the sense in which this makes our equilibria dynamic, we first take note of the static case.

Proposition 17 $\mathrm{Eq}^{\mathrm{com}}_{\mathrm{G}^{\mathrm{cp}}}(\xrightarrow{\{\mathrm{s}\}}) \wedge (\mathrm{s} \not\rightarrow \mathrm{s}) \;\Leftrightarrow\; \mathrm{Eq}^{\mathrm{aN}}_{\mathrm{G}^{\mathrm{cp}}}(\mathrm{s})$

Next, we revisit our motivating example in Section 1.

	h_1	h_2
v_1	**0, 1**	**1, 0**
v_2	**1, 0**	**0, 1**



As mentioned, the only probabilistic Nash equilibrium arises when both agents choose between their two options with equal probability, for expected payoffs of a half to each (and we indicate the outcomes involved in the compromise with boldface). The only change-of-mind equilibrium is shown on the right. It, too, involves all four outcomes of the game in the prescribed compromise. The main virtue of Nash's probabilistic compromise is that it dictates an exact expected payoff to each agent. A change-of-mind equilibrium, on the other hand, makes it clear why the considered outcomes are included in the compromise. The upper-left outcome in the example, say, is included because it is a better alternative for 'h' than the upper-right one, and so on clockwise in the array.

7 Fixed Points Revisited

Returning to our discrete fixed-point construction in Section 4, we can now finally identify the interesting $\mathscr{U}$-fixed points as those that are least non-empty.

Lemma 18 *Given* G^{cp} *with change-of-mind* $\rightarrow$*.*

$$\mathrm{Eq}^{\mathrm{com}}_{\mathrm{G}^{\mathrm{cp}}}(\xrightarrow{\mathrm{S}})$$
$$\Updownarrow$$
$$\mathscr{U}(\mathrm{S}) = \mathrm{S} \wedge (\forall \mathrm{S}' \,.\, \emptyset \subsetneq \mathrm{S}' \subsetneq \mathrm{S} \Rightarrow \mathscr{U}(\mathrm{S}') \not\subseteq \mathrm{S}') \wedge \mathrm{S} \neq \emptyset$$

Proof By two reasonably direct arguments. The only interesting step is from bottom to top and showing that, for any two $\mathrm{s}_1, \mathrm{s}_2 \in \mathrm{S}$, we have $\mathrm{s}_1 \rightarrow^* \mathrm{s}_2$. We first note that $\mathscr{U}$ is post-fixpointed: $\mathrm{S} \subseteq \mathscr{U}(\mathrm{S})$, idempotent: $\mathscr{U}(\mathscr{U}(\mathrm{S})) = \mathscr{U}(\mathrm{S})$, and monotone: $\mathrm{S}_1 \subseteq \mathrm{S}_2 \Rightarrow \mathscr{U}(\mathrm{S}_1) \subseteq \mathscr{U}(\mathrm{S}_2)$. By monotonicity and the assumed $\mathscr{U}(\mathrm{S}) = \mathrm{S}$, we have $\mathscr{U}(\{\mathrm{s}_1\}) \subseteq \mathrm{S}$. If $\neg(\mathrm{s}_1 \rightarrow^* \mathrm{s}_2)$, then $\mathrm{s}_2 \in \mathrm{S} \setminus \mathscr{U}(\{\mathrm{s}_1\})$, i.e., $\mathscr{U}(\{\mathrm{s}_1\}) \subsetneq \mathrm{S}$. By post-fixpointed-ness, $\mathscr{U}(\{\mathrm{s}_1\})$ is non-empty and, by assumption of least-ness, we may therefore conclude $\mathscr{U}(\mathscr{U}(\{\mathrm{s}_1\})) \not\subseteq \mathscr{U}(\{\mathrm{s}_1\})$. This contradicts idempotency, which means that, indeed, $\mathrm{s}_1 \rightarrow^* \mathrm{s}_2$. □

8 Technical Summary

Summarising the preceding sections, we have.

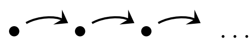
Theorem 19 (Main Result) *For arbitrary C/P game,* $\mathrm{G}^{\mathrm{cp}} = \langle \mathcal{A}, \mathcal{S}, (\underset{\mathrm{a}}{\succ})_{\mathrm{a} \in \mathcal{A}}, (\lhd_{\mathrm{a}})_{\mathrm{a} \in \mathcal{A}} \rangle$*, and non-empty* $\mathrm{S} \subseteq \mathcal{S}$*, the following are equivalent.*

- S *underpins a* change-of-mind equilibrium*,* $\mathrm{Eq}^{\mathrm{com}}_{\mathrm{G}^{\mathrm{cp}}}(\xrightarrow{\mathrm{S}})$*.*
- S *is a* "shrunken" Nash equilibrium*,* $\mathrm{Eq}^{\mathrm{aN}}_{\lfloor \mathrm{G}^{\mathrm{cp}} \rfloor}(\mathrm{S})$*.*
- S *is a least non-empty fixed point of* $\mathscr{U}$*.*

For finite C/P games, such S *exist and all can be found in quadratic time in the size of* $\mathcal{S}$*.*

The first equilibrium characterisation above is defined entirely inside the considered game and is what gives our equilibria their explicitly dynamic nature. The second characterisation mimics Theorem 5, while the last aligns itself with the proof of Theorem 5. More, the first characterisation makes any of our equilibria *sustainable* in the sense that, once there, noone can defect; the second makes some equilibrium *inevitable* in the sense that if not in an equilibrium, someone wants to go towards one; finally, the third makes them *atomic* in the sense that nothing smaller will have similar properties.

Interestingly, and in contrast to the construction used in Theorem 5, nothing in our definitions requires finiteness of the games, which means that we, in principle, can prove that some infinite C/P game has change-of-mind equilibria. That said, any C/P game with, e.g., the following infinitely-ascending progressive change-of-mind relation will not because it has no terminal node/all tails are $\mathscr{U}$-fixed points.



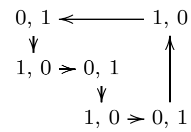
We leave the infinite case for future work.

9 Compromises

Our running example strategic-form game betrays the substantial differences that may exist between the compromises prescribed by Nash's probabilistic Theorem 5 and our discrete Theorem 19. In particular, the considered compromises coincide, i.e., the only probabilistic Nash equilibrium assigns non-zero probabilities[5] to the same outcomes as are involved in the only change-of-mind equilibrium. We will now show that also all other possible configurations can be observed when comparing compromises.

Change-of-Mind are Probabilistic Generalising our running example to a three-by-three game highlights perhaps the most interesting feature of change-of-mind equilibria, namely the ability to carve out a part of a game as constituting a Nash equilibrium, below right with six involved outcomes.

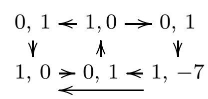
	h_1	h_2	h_3
v_1	**0, 1**	**0, 0**	**1, 0**
v_2	**1, 0**	**0, 1**	**0, 0**
v_3	**0, 0**	**1, 0**	**0, 1**



The only probabilistic Nash equilibrium arises again if both agents choose between their options with equal probability, for expected payoffs of a third to each and involving all nine possible outcomes.

Probabilistic are Change-of-Mind A different generalisation of our two-by-two example arises by adding an extra, 'h'-undesirable column.

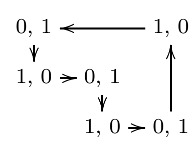
	h_1	h_2	h_3
v_1	**0, 1**	**1, 0**	0, 1
v_2	**1, 0**	**0, 1**	1, −7



If 'v' puts all weight on one row, 'h' will want to put all weight on the columns where he gets a payoff of 1, which will make 'v' reassign weights toward the other row. If 'v' puts weight on both rows, 'h' will always prefer the first column to the third, i.e., 'h3' will be assigned weight 0. In other words, the only probabilistic Nash equilibrium involves v_1, v_2, h_1, and h_2 with equal probabilities. The only change-of-mind equilibrium involves all six outcomes.

Disjoint Compromises By a similar token, we can make several rows 'v'-undesirable.

	h_1	h_2	h_3
v_1	0, 1	−7, 0	1, 0
v_2	1, 0	0, 1	−7, 0
v_3	−7, 0	1, 0	0, 1
v_4	**0, 0**	**0, 0**	**0, 0**

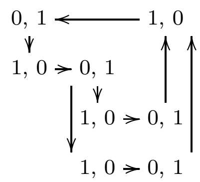


In any Nash probabilistic compromise, player 'v' chooses strategy v_4 with full weight. By contrast, the only change-of-mind equilibrium is disjoint from there, involving the previously-observed cycle around the cells with 1, 0 and 0, 1.

[5] Nash (3) says 'uses' when discussing induced compromises.

Non-Trivial Overlaps The strategic-form game where only the last row is 'v'-undesirable exhibits complementary features.

	h_1	h_2	h_3
v_1	**0, 1**	**0, 0**	**1, 0**
v_2	**1, 0**	**0, 1**	**0, 0**
v_3	**0, 0**	**1, 0**	**0, 1**
v_4	−7, 0	1, 0	0, 1



As before, 'v' will avoid the row with the negative payoff and the only probabilistic Nash equilibrium involves the upper nine cells with equal probability. The only change-of-mind equilibrium is as shown.

Focusing narrowly on their prescribed compromises for strategic-form games, we have not been able to separate change-of-mind and probabilistic Nash equilibria quantitatively, i.e., in terms of a measure. In particular, the examples above show that neither notion consistently results in smaller compromises than the other nor higher average/expected payoffs. Indeed, the two notions appear to be of independent interest and to have distinct (qualitative) meaning.

10 C/P vs Strategic-Form

The prisoner's dilemma is a classic example in game theory that more than being a dilemma highlights the non-cooperative aspect of Nash equilibria. The game is made up of two agents who are accused of a crime. If both confess, they share the mandated 3-year prison term for the crime, with right of parole. If one confesses and the other denies any involvement, the former serves the full sentence. If both deny their involvement (and they are found guilty), they are deemed to have attempted to pervert the course of justice and are punished for that and the crime.

	h_{confess}	h_{deny}
v_{confess}	−1, −1	−3, 0
v_{deny}	0, −3	**−2, −2**

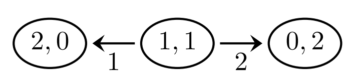
The (perceived) issue at hand is that the lower right cell, with both denying their involvement, is the only Nash equilibrium although they would be better off by jointly confessing. The mechanics of the prisoner's dilemma also show up if we consider two players that share two tokens that they play for: a player with a token can take the token from the other player at will with the aim of acquiring both tokens. We call the game "blink-and-you-lose".

	h_{leave}	h_{take}
v_{leave}	1, 1	0, 2
v_{take}	2, 0	**1, 1**

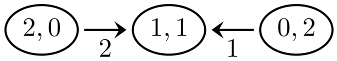
Here, the lower-right cell is again the only Nash equilibrium. The main failure of non-cooperative game theory in this case is not that it steers the players towards the worse of two evils but that it fails to allow for the winning configurations of one player possessing both tokens as equilibria, the same way that avoiding jail is not an equilibrium in the prisoner's dilemma.

By contrast, and because C/P games need not be array-shaped and because no particular set of permissible moves are mandated, we can model aspects of blink-and-you-lose that classic game theory cannot. One way is to form a game consisting not of four strategy profiles but of three "game situations". A: player 1 has both tokens; B: the players have a token each; C: player 2 has both tokens. According to the rules of the game, agent 1 prefers his winning situation, A, to B and C, and the neutral B to C, his losing situation. Similarly, for agent 2 and C over A and B, and B over A. When it comes to specifying the conversion relations there are at least three distinct principles that can be employed, i.e., agents can be assigned different capabilities/intentions.

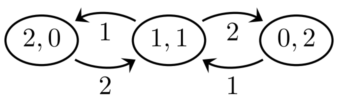
Foresight: A player realises that he can win by taking the opponent's token faster than the opponent can react, i.e., player 1 can convert B to A by outpacing player 2. Player 2, in turn, can convert B to C. This version of the game has two singleton change-of-mind equilibria (that, thus, are also Nash equilibria): A and C.



Hindsight: A player, say 1, analyses what would happen if he does not act. In case 2 acts, the game would end up in C and 1 looses, and 1 therefore concludes that he could have prevented the C outcome by acting. In other words, it is within 1's power to convert C to B. Similarly for player 2 from A to B. This version of the game has one singleton change-of-mind equilibrium (that, thus, is also a Nash equilibrium): B.



Omnisight: The players have both hindsight and foresight, resulting in a C/P game with one change-of-mind equilibrium covering all outcomes (thus, no *pure* Nash equilibrium exists).



The C/P-game version of blink-and-you-lose makes it clear that game-theoretic analysis of strategic-form games, such as the one involved in the prisoner's dilemma, mandates subtle and non-elicited motives and capabilities on the part of the players. In the strategic-form version of blink-and-you-lose, for example, 'h' necessarily considers the lower-right cell as an alternative to the lower-left one although the latter is actually a final state of the modelled game. By contrast, the C/P-game versions can distinguish possible actions, such as in the foresight example, and what amounts to co-actions, e.g., "rectification" steps for non-actions, as in the hindsight example. C/P games can also, if so desired, be used to conduct a pure "strategic" analysis as considered by Nash, see Proposition 8. The point is that the explicit conversion relation of C/P games makes it clear and adjustable what capabilities are being considered.

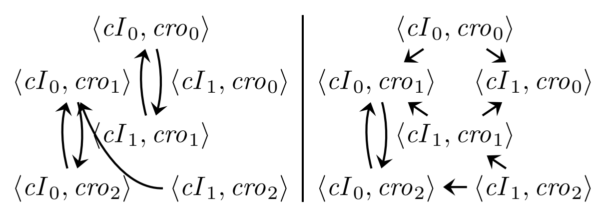
11 Rewriting Game Theory

We have dubbed applications of C/P games and change-of-mind equilibria *rewriting game theory* to stress our positive reading of the change-of-mind relation that Nash interpreted negatively, see Proposition 12. Graphs are the simplest and most abstract structure pursued in the field of rewriting, with focus on their dynamic aspects, equational theories, termination properties, and more (10).

We now present the highlights of two "foresight" uses of rewriting game theory. Generally speaking, they show that change-of-mind equilibria can be used to characterise the biologically meaningful parts of a purely chemically-conceived C/P game. Differently said, the identified change-of-mind equilibria will be functional units at the biochemical abstraction level. (The biologically-meaningful description relates to Theorem 19's sustainability and inevitability notions, while the functional-unit description relates to the theorem's atomicity notion.)

11.1 Gene Regulation

In (11), we use rewriting game theory to provide a foundation for Kauffman/Thomas-style gene-regulation analysis (12, 13). In the case of the standard example of 2-variable *bacteriophage lambda*, Kauffman's model produces the state-space graph on the left; to us, it is simply change-of-mind (with a slightly more elaborate C/P game underneath). The graph consists of *synchronous* updates of gene states.



By contrast, Thomas' model, right, is *asynchronous*, allowing each gene to update its own state in isolation, with some arrows indicating updates of *cro* and other arrows updates of cI.[6]

The relevance of Kaufmann/Thomas-style gene-regulation analysis is that, e.g., the singleton change-of-mind equilibrium in these graphs consisting of $\langle cI_1, cro_0\rangle$ is phage λ's *lysogenic* state that

[6] We see that Nash and Kauffman updating correspond to each other, while we can accommodate all approaches, depending on the choice of agents; natively, we correspond to Thomas.

"involves integration of the phage DNA into the bacterial chromosome [of its host] where it is passively replicated at each cell division — just as though it were a legitimate part of the bacterial genome" (14). Similarly, the change-of-mind equilibrium consisting of the cycle between $\langle cI_0, cro_1 \rangle$ and $\langle cI_0, cro_2 \rangle$ in the graphs is characteristic of phage λ's *lytic* state in which it actively uses its host's transcription mechanism to replicate itself (14).[7]

Rewriting game theory provides a unifying account of *static* and *dynamic steady states* of genes and uniformly accommodates Kauffman's and Thomas' models, as well as any hybrid. In addition, we prove equivalent two independent characterisations of dynamic steady states (as fixed-points (15) and as terminal SCCs (16)), see Lemmas 16 and 18. Rewriting game theory is a good foundation for gene-regulation analysis because it provides a general theory of possibly dynamic equilibria and because the technical means by which it accomplishes this, i.e., change-of-mind, has the same reading as the established models but for gene-independent reasons.

11.2 Signal Transduction Systems

In (17), we introduce a so-called *cascaded protein game* construction (using C/P games) and apply it to the sum-total 113 chemical reactions that are reported in 4 review articles and 4 online databases to be involved in MAPK cascades in mammalian cells. The constructed game has two change-of-mind equilibria. One turns out to be the ERK pathway, known as the signalling pathway responsible for cell growth, while the other is a combination of the JNK and p38 signalling pathways, known among other things for their cross-talk. The (C/P) agents in the constructed cascaded protein game are the enzymes that catalyse the involved reactions (for a standard increase in observed kinetics of 10^6 to 10^{12} times (18)). In this study, the inevitability and sustainability properties of our change-of-mind equilibria can be read directly to say that the identified signal transduction systems are good building blocks of a biological system. More, we show that, e.g., inhibition of the PP2A protein makes all the change-of-mind equilibria static, in a manner that is consistent with reported causes of tauopathies (such as Alzheimer's Diseases), specifically abnormal hyperphosphorylation of tau protein.

[7]The cycle between $\langle cI_0, cro_0 \rangle$ and $\langle cI_1, cro_1 \rangle$ is a known false positive of Kauffman's model.

12 Discussion

We have proved a new Nash theorem, Theorem 19, with three readings: *sustainability* of any equilibrium, *inevitability* of some equilibrium, and *atomicity* of the equilibria, e.g., relative to the life-science abstraction mechanism associated with *homeostasis* (dynamic equilibria over function) (19) and *autopoiesis* (dynamic equilibria over structure) (20). Summarising our development compared with Nash's, we have arbitrary vs array-structured games, arbitrary vs real-valued payoffs, Tarski's vs Brouwer's Fixed Point Theorems, quadratic vs PPAD-complete complexity, and discrete vs probabilistic equilibria. Qualitatively, Nash's theorem establishes a payoff-driven equilibrium notion while ours pertains to the dynamic concept of change-of-mind. Neither notion is uniformly preferable on quantitative grounds, i.e., in terms of a single measure on compromises. Our result allows for an extended range of technical applications of non-cooperative game theory, e.g., in life science, see Section 11. Dubbed rewriting game theory, we are currently pursuing other applications similar to those reported in Section 11, both in the life sciences more generally and elsewhere. The established complexity bound (i.e., quadratic in the size of $\mathcal{S}$) makes even large-scale studies feasible.

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A SCCs, Shrunken Graphs

- $\rightarrow \subseteq \mathcal{V} \times \mathcal{V}$ is a *graph* over *vertices* $\mathcal{V}$.

- $\rightarrow$'s *reflexive, transitive closure*, $\rightarrow^*$, is

$$\frac{v_1 \rightarrow v_2}{v_1 \rightarrow^* v_2} \qquad \frac{}{v \rightarrow^* v} \qquad \frac{v_1 \rightarrow^* v \quad v \rightarrow^* v_2}{v_1 \rightarrow^* v_2}$$

- v's strongly connected component (SCC) is

$$\lfloor v \rfloor \quad \triangleq \quad \{v' \mid v \rightarrow^* v' \wedge v' \rightarrow^* v\}$$

("_ is in $\lfloor - \rfloor$" is an equivalence relation.)

- The SCCs of a graph is $\lfloor \mathcal{V} \rfloor \triangleq \{\lfloor v \rfloor \mid v \in \mathcal{V}\}$.
- The *shrunken graph* of $\rightarrow \subseteq \mathcal{V} \times \mathcal{V}$ over $\lfloor \mathcal{V} \rfloor$ is

$$\lfloor v_a \rfloor \curvearrowright \lfloor v_b \rfloor \triangleq \lfloor v_a \rfloor \neq \lfloor v_b \rfloor \wedge v_a \rightarrow v_b$$

Theorem 20 (Tarjan (9, 21)) *Given a graph, $\rightarrow \subseteq \mathcal{V} \times \mathcal{V}$, the SCCs and their shrunken graph can be found in linear time in the sizes of $\rightarrow$ and $\mathcal{V}$.*

Proposition 21 *A shrunken graph is cycle-free.*

Proof If not, more would have been shrunken. □

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