Focused emulation of modal proof systems

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The quest

**Modal logics:**
program verification, artificial intelligence, distributed systems . . .
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What if... one wants to have automated proof search for modal logics?
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What if... one wants to have *automated* proof search for modal logics?

Their proof theory:
tableaux, sequents, hypersequents, nested sequents, labeled sequents . . .
The quest

We want to provide a general framework for:

1. comparing formalisms;
2. proof checking;
3. proof reconstruction and sharing.
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The ProofCert approach:
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1. comparing formalisms;
2. proof checking;
3. proof reconstruction and sharing.

The ProofCert approach:

- LMF$_*$ : focused labeled framework for propositional modal logic
- LKF$^a$ : focused framework for classical first-order logic
Modal logic

Formulas: \[ A ::= P \mid A \land A \mid A \lor A \]

Logic K: Propositional Logic
Modal logic

Formulas: \( A ::= P \mid A \land A \mid A \lor A \mid \Box A \mid \Diamond A \)

Logic K: Propositional Logic + \( \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \) + nec \( \frac{A}{\Box A} \)
Modal logic

**Formulas:** \( A ::= P \mid A \land A \mid A \lor A \mid \Box A \mid \Diamond A \)

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**Kripke semantics:** Relational structures
Modal logic

Formulas: $A ::= P \mid A \land A \mid A \lor A \mid \Box A \mid \Diamond A$

Logic $K$: Propositional Logic $+ \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) + \text{nec } \frac{A}{\Box A}$

Kripke semantics: Relational structures

$W$ : set of worlds;
$R$ : binary relation on $W$;
$V$ : valuation at each world.
Modal logic

**Formulas:** \( A ::= P \mid A \land A \mid A \lor A \mid \Box A \mid \Diamond A \)

**Logic K:** Propositional Logic + \( \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \) + nec \( \frac{A}{\Box A} \)

**Kripke semantics:** Relational structures

\( W \) : set of worlds;

\( R \) : binary relation on \( W \);

\( V \) : valuation at each world.

\( M, x \models \Box A \) iff for all \( y \). \( xRy \) implies \( M, y \models A \)

\( M, x \models \Diamond A \) iff there exists \( y \). \( xRy \) and \( M, y \models A \).
Modal logic

**Formulas:** $A ::= P | A \land A | A \lor A | \Box A | \Diamond A$

**Logic K:** Propositional Logic $+$ $\Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$ $+$ nec $\frac{A}{\Box A}$

**Kripke semantics:** Relational structures
- $W$: set of worlds;
- $R$: binary relation on $W$;
- $V$: valuation at each world.

**Sequent system OS:**

\[
\begin{align*}
\text{id} & \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \land B} \\
\land & \quad \frac{\vdash \Gamma, A \land B}{\vdash \Gamma, A \lor B} \\
\lor & \quad \frac{\vdash \Gamma, A \lor B}{\vdash \Gamma, \Box A} \\
\Box K & \quad \frac{\vdash \Box \Gamma, \Box A, \Delta}{\vdash \Diamond \Gamma}
\end{align*}
\]
Labeled proof systems

**Labeled deduction:** encode semantical information in the syntax
Labeled proof systems

**Labeled deduction**: encode semantical information in the syntax

**Two classes of formulas**:
1. Labeled logical formulas $x : A$
2. Relational formulas $xRy$
Labeled proof systems

**Labeled deduction:** encode semantical information in the syntax

**Two classes of formulas:**

1. Labeled logical formulas \( x : A \)
2. Relational formulas \( xRy \)

- each label \( x \) refers to a world in the semantics
- an atomic relational symbol \( R \) refers to the accessibility relation
A labeled proof system for modal logics (G3K)

\[
\begin{align*}
\text{id} & \quad \frac{P, \Gamma \vdash \Delta, P}{P, \Gamma \vdash \Delta} \\
L^\land & \quad \frac{A, B, \Gamma \vdash \Delta}{A \land B, \Gamma \vdash \Delta} \\
R^\land & \quad \frac{\Gamma \vdash \Delta, A \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \land B} \\
L^\lor & \quad \frac{A, \Gamma \vdash \Delta, B, \Gamma \vdash \Delta}{A \lor B, \Gamma \vdash \Delta} \\
R^\lor & \quad \frac{\Gamma \vdash \Delta, A, B}{R \vdash \Delta, A \lor B}
\end{align*}
\]

[S. Negri, Proof analysis in modal logic, J. Philos. Logic 2005]
A labeled proof system for modal logics (G3K)

\[
\begin{align*}
\text{id} \quad & x : P, \Gamma \vdash \Delta, x : P \\
L \wedge & x : A, x : B, \Gamma \vdash \Delta \quad \Rightarrow \quad x : A \land B, \Gamma \vdash \Delta \\
R \wedge & \Gamma \vdash \Delta, x : A \quad \Rightarrow \quad \Gamma \vdash \Delta, x : A \land B \\
L \vee & x : A, \Gamma \vdash \Delta \quad \Rightarrow \quad x : A \lor B, \Gamma \vdash \Delta \\
R \vee & \Gamma \vdash \Delta, x : A, x : B \quad \Rightarrow \quad \Gamma \vdash \Delta, x : A \lor B
\end{align*}
\]

A labeled proof system for modal logics (G3K)

\[
\begin{align*}
\text{id} & \quad \frac{}{x : P, \Gamma \vdash \Delta, x : P} \\
L^\land & \quad \frac{x : A, x : B, \Gamma \vdash \Delta}{x : A \land B, \Gamma \vdash \Delta} \\
R^\land & \quad \frac{\Gamma \vdash \Delta, x : A \land B}{\Gamma \vdash \Delta, x : A, x : B} \\
L^\lor & \quad \frac{x : A, \Gamma \vdash \Delta \quad x : B, \Gamma \vdash \Delta}{x : A \lor B, \Gamma \vdash \Delta} \\
R^\lor & \quad \frac{\Gamma \vdash \Delta, x : A, x : B}{\Gamma \vdash \Delta, x : A \lor B} \\
L^\Box & \quad \frac{y : A, x : \Box A, xRy, \Gamma \vdash \Delta}{x : \Box A, xRy, \Gamma \vdash \Delta} \\
R^\Box & \quad \frac{xRy, \Gamma \vdash \Delta, y : A}{\Gamma \vdash \Delta, x : \Box A} \\
L^\Diamond & \quad \frac{xRy, y : A, \Gamma \vdash \Delta}{x : \Diamond A, \Gamma \vdash \Delta} \\
R^\Diamond & \quad \frac{xRy, \Gamma \vdash \Delta, x : \Diamond A, y : A}{xRy, \Gamma \vdash \Delta, x : \Diamond A}
\end{align*}
\]

In $R^\Box$, $y$ does not occur in the conclusion.

[S. Negri, Proof analysis in modal logic, J. Philos. Logic 2005]
Focusing is a way to control non-determinism in proof search and ...

- Better organize the **structure** of derivations.
- Emphasis on: **non-invertible** vs. **invertible** rules.
- Propositional connectives have:
  - a **positive** version;
  - a **negative** version.

\[ \vdash \Theta, B \quad \vdash \Theta, B_1 \lor B_2 \quad \vdash \Theta, B_1 \lor B_2 \]

- Polarization of a formula does not affect its **provability**.
What is a bipole?

store
\[ \vdash \Theta \uparrow \Gamma \]

release
\[ \vdash \Theta \downarrow A \]

decide
What is a bipole?

**store**  (a positive formula to possibly focus on later)

\[ \vdash \Theta \uparrow \Gamma \quad \land^- , \lor^- , \forall \]

**release**

\[ \vdash \Theta \downarrow A \quad \lor^+ , \land^+ , \exists \]

**decide**  (on a positive formula to focus on)
What is a bipole?

*store* (a positive formula to possibly focus on later)

\[ \vdash \Theta \uparrow \Gamma \]  \textbf{NEGATIVE PHASE (invertible)}

*release* (change of phase)

\[ \vdash \Theta \downarrow A \]  \textbf{POSITIVE PHASE (non-invertible)}

*decide* (on a positive formula to focus on)
A focused proof system for classical logic (LKF)

**Negative introduction rules**

\[
\begin{align*}
\vdash \Theta \uparrow t, \Gamma & \quad \wedge^- \quad \vdash \Theta \uparrow A, \Gamma, \vdash \Theta \uparrow B, \Gamma \\
\vdash \Theta \uparrow A \wedge^- B, \Gamma & \quad f^- \quad \vdash \Theta \uparrow \Gamma, \vdash \Theta \uparrow f, \Gamma \\
\vdash \Theta \uparrow A \vee^- B, \Gamma & \quad \forall \quad \vdash \Theta \uparrow [y/x]B, \Gamma \\
\vdash \Theta \uparrow \forall x.B, \Gamma & \end{align*}
\]

**Positive introduction rules**

\[
\begin{align*}
\vdash \Theta \downarrow t^+ & \quad \wedge^+ \quad \vdash \Theta \downarrow B_1, \vdash \Theta \downarrow B_2 \\
\vdash \Theta \downarrow B_1 \wedge^+ B_2 & \quad \forall^+ \quad \vdash \Theta \downarrow B_i, \vdash \Theta \downarrow B_1 \vee^+ B_2 \\
\vdash \Theta \downarrow \exists x.B & \end{align*}
\]

**Identity rules**

\[
\begin{align*}
id & \quad \vdash \neg P_a, \Theta \downarrow P_a \\
\text{cut} & \quad \vdash \Theta \uparrow B, \vdash \Theta \uparrow \neg B \\
\text{cut} & \quad \vdash \Theta \uparrow B, \vdash \Theta \uparrow \neg B \\
\end{align*}
\]

**Structural rules**

\[
\begin{align*}
\text{store} & \quad \vdash \Theta, C \uparrow \Gamma, \vdash \Theta \uparrow C, \Gamma \\
\text{release} & \quad \vdash \Theta \uparrow N, \vdash \Theta \downarrow N \\
\text{decide} & \quad \vdash P, \Theta \downarrow P, \vdash P, \Theta \uparrow.
\end{align*}
\]
Labeled modal inference rules as bipoles

An **inference rule** in the labeled modal proof system G3K

**corresponds to** (⇕)

a **bipole** in the focused proof system LKF.

\[
R \quad \frac{xRy, \ G \vdash \Gamma, y : A}{\ G \vdash \Gamma, x : \Box A}
\]

[D.Miller & M.Volpe, *Focused labeled proof systems for modal logic*, 2015]
A focused labeled proof system for modal logic (LMF)

- A restriction of LKF targeting the language of G3K.
- Quantifier rules only applied to the translation of □A or ◇A.
Negative introduction rules

\[
\begin{align*}
\text{init}_K \quad & \vdash x : \neg P_a, \Theta \downarrow x : P_a \\
\text{cut}_K \quad & \vdash \Theta \uparrow x : B \quad \vdash \Theta \uparrow x : \neg B
\end{align*}
\]

Positive introduction rules

\[
\begin{align*}
\text{store}_K \quad & \vdash \Theta, x : C \uparrow \Gamma \\
\text{release}_K \quad & \vdash \Theta \uparrow x : N \\
\text{decide}_K \quad & \vdash x : P, \Theta \downarrow x : P
\end{align*}
\]

Identity rules

\[
\begin{align*}
\text{init}_K \quad & \vdash x : \neg P_a, \Theta \downarrow x : P_a \\
\text{cut}_K \quad & \vdash \Theta \uparrow x : B \quad \vdash \Theta \uparrow x : \neg B
\end{align*}
\]

Structural rules
What happens with ordinary sequent systems?

This rule works at the same time on □s and ◇s.
What happens with ordinary sequent systems?

This rule works at the same time on $\Box$s and $\Diamond$s.

Not A Bipole!
What happens with ordinary sequent systems?

\[ \square_K \vdash \Gamma, A \quad \frac{\vdash \Gamma, A}{\vdash \Diamond \Gamma, \Box A, \Delta} \]

This rule works at the same time on $\Box$s and $\Diamond$s.

Not A Bipole!

- Correspondence between ordinary and labeled sequents:
  - ordinary **classical rules** operate on a single world;
  - ordinary **modal rules** move from one world to another.
What happens with ordinary sequent systems?

\[
\begin{align*}
\Box_K & \quad \vdash \Gamma, A \\
\vdash & \vdash \Diamond \Gamma, \Box A, \Delta
\end{align*}
\]

\[
\begin{align*}
G \cup \{xRy\} & \vdash \Sigma, x : \Diamond \Gamma \uparrow y : A \\
G & \vdash \Sigma, x : \Diamond \Gamma \uparrow x : \Box A
\end{align*}
\]
What happens with ordinary sequent systems?

One bipole for the $\Box$-formula.
What happens with ordinary sequent systems?

\[
\frac{\Gamma, A \vdash \Box \quad \Box \vdash \Gamma, A}{\vdash \Diamond \Gamma, \Box A, \Delta}
\]

\[
\frac{G \cup \{xRy\} \vdash \Sigma, x : \Diamond \Gamma, y : A \uparrow y : \Gamma}{G \cup \{xRy\} \vdash \Sigma, x : \Diamond \Gamma, y : A \downarrow x : \Diamond \Gamma}
\]
What happens with ordinary sequent systems?

\[
R \Box \frac{\vdash \Gamma, A}{\vdash \Diamond \Gamma, \Box A, \Delta}
\]

\[
G \cup \{xRy\} \vdash \Sigma, x : \Diamond \Gamma, y : A \uparrow y : \Gamma
\]

\[
G \cup \{xRy\} \vdash \Sigma, x : \Diamond \Gamma, y : A \downarrow x : \Diamond \Gamma
\]

**Multifocusing**: the $\Diamond$s can be processed in parallel.

One **bipole** for the $\Diamond$-formulas.
Parameters of the framework: * can be instantiated in a specific way by the following parameters (of the decide rule):

1. restrictions on the formulas on which multifocusing can be applied;
2. restrictions on the definition of the future $\sigma$ of formulas in $\Omega$;
3. restriction of the present $\mathcal{H}'$. 
The general framework $LMF_*$

**Parameters of the framework:** $*$ can be instantiated in a specific way by the following parameters (of the decide rule):

1. restrictions on the formulas on which *multifocusing* can be applied;
2. restrictions on the definition of the *future* $\sigma$ of formulas in $\Omega$;
3. restriction of the *present* $\mathcal{H}'$.

By playing with *polarization* and *parameters*, one can obtain different systems.
The general framework $LMF_*$

**Parameters of the framework:** $*$ can be instantiated in a specific way by the following parameters (of the decide rule):

1. restrictions on the formulas on which multifocusing can be applied;
2. restrictions on the definition of the future $\sigma$ of formulas in $\Omega$;
3. restriction of the present $\mathcal{H}'$.

By playing with polarization and parameters, one can obtain different systems.

**Theorem** The framework $LMF_*$ is sound and complete with respect to the logic $K$, for any polarization of formulas.
Conclusion

▶ We showed the case of K; but it works for geometric extensions.
▶ Emulation of modal focused systems (e.g., [Lellmann-Pimentel, 2015] or [Chaudhuri-Marin-Strassburger, 2016]).
▶ What about nested sequents?
  ▶ Same polarization as for ordinary sequents.
  ▶ No need for multifocusing.
  ▶ No need for restrictions on futures.
  ▶ The present is always the set of all labels.
▶ What about hypersequents?
  ▶ the present is a multiset;
  ▶ external structural rules as operations on such a present;
  ▶ modal communication rules as a combination of relational and modal rules.
▶ Superpowers can be implemented in the augmented version of the focused system LKF used in the project ProofCert.