Focused emulation of modal proof systems

Sonia Marin
with Dale Miller and Marco Volpe

Inria, LIX, École Polytechnique

Theory and Logic Group seminar
TU Wien
October 12, 2016

The quest

Modal logics:
program verification, artificial intelligence, distributed systems . . .

The quest

Modal logics:
program verification, artificial intelligence, distributed systems . . .

What if... one wants to have *automated* proof search for modal logics?

The quest

Modal logics:
program verification, artificial intelligence, distributed systems . . .

What if... one wants to have *automated* proof search for modal logics?

Their proof theory:
tableaux, sequents, hypersequents, nested sequents, labeled sequents . . .

The quest

We want to provide a general framework for:

1. comparing formalisms;
2. proof checking;
3. proof reconstruction and sharing.

The quest

We want to provide a general framework for:

1. comparing formalisms;
2. proof checking;
3. proof reconstruction and sharing.

The ProofCert approach:

proof in S → proof in LKF[a]

The quest

We want to provide a general framework for:

1. comparing formalisms;
2. proof checking;
3. proof reconstruction and sharing.

The ProofCert approach:

proof in S $\longrightarrow$ proof in LMF_* $\longrightarrow$ proof in LKF^a

- LMF_* : focused labeled framework for propositional modal logic
- LKF^a : focused framework for classical first-order logic

Modal logic

Formulas: $A ::= P \mid A \wedge A \mid A \vee A$

Logic K**:** Propositional Logic

Modal logic

Formulas: $A ::= P \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

Logic K**:** Propositional Logic $+ \Box(A \to B) \to (\Box A \to \Box B) + \mathit{nec}\ \dfrac{A}{\Box A}$

Modal logic

Formulas: $A ::= P \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

Logic K**:** Propositional Logic $+ \Box(A \to B) \to (\Box A \to \Box B) + nec \ \dfrac{A}{\Box A}$

Kripke semantics: Relational structures

Modal logic

Formulas: $A ::= P \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

Logic K**:** Propositional Logic $+ \Box(A \to B) \to (\Box A \to \Box B) + nec\ \dfrac{A}{\Box A}$

Kripke semantics: Relational structures

W : set of worlds;

R : binary relation on W;

V : valuation at each world.

Modal logic

Formulas: $A ::= P \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

Logic K**:** Propositional Logic + $\Box(A \to B) \to (\Box A \to \Box B)$ + $nec \, \dfrac{A}{\Box A}$

Kripke semantics: Relational structures

W : set of worlds;

R : binary relation on W;

V : valuation at each world.

$$\begin{array}{llll} \mathcal{M}, x \models \Box A & \text{iff} & \text{for all } y. & xRy \text{ implies } \mathcal{M}, y \models A \\ \mathcal{M}, x \models \Diamond A & \text{iff} & \text{there exists } y. & xRy \text{ and } \mathcal{M}, y \models A. \end{array}$$

Modal logic

Formulas: $A ::= P \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A$

Logic K**:** Propositional Logic + $\Box(A \to B) \to (\Box A \to \Box B)$ + $nec\ \dfrac{A}{\Box A}$

Kripke semantics: Relational structures

W : set of worlds;

R : binary relation on W;

V : valuation at each world.

Sequent system OS**:**

$$\text{id}\ \frac{}{\vdash \Gamma, P, \neg P} \quad \wedge\ \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B} \quad \vee\ \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \quad \Box_K\ \frac{\vdash \Gamma, A}{\vdash \Diamond\Gamma, \Box A, \Delta}$$

Labeled proof systems

Labeled deduction: encode semantical information in the syntax

Labeled proof systems

Labeled deduction: encode semantical information in the syntax

Two classes of formulas:

1. Labeled logical formulas $x : A$
2. Relational formulas xRy

Labeled proof systems

Labeled deduction: encode semantical information in the syntax

Two classes of formulas:

1. Labeled logical formulas $x : A$
2. Relational formulas xRy

- each label x refers to a world in the semantics
- an atomic relational symbol R refers to the accessibility relation

A labeled proof system for modal logics (G3K)

$$\text{id}\ \frac{}{P, \Gamma \vdash \Delta, \quad P}$$

$$L\wedge\ \frac{A, \quad B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad R\wedge\ \frac{\Gamma \vdash \Delta, \quad A \quad \Gamma \vdash \Delta, \quad B}{\Gamma \vdash \Delta, \quad A \wedge B}$$

$$L\vee\ \frac{A, \Gamma \vdash \Delta \qquad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \qquad R\vee\ \frac{\Gamma \vdash \Delta, \quad A, \quad B}{\Gamma \vdash \Delta, \quad A \vee B}$$

[S. Negri, **Proof analysis in modal logic**, J. Philos. Logic 2005]

A labeled proof system for modal logics (G3K)

$$\text{id}\ \frac{}{x : P, \Gamma \vdash \Delta, x : P}$$

$$L\wedge\ \frac{x : A, x : B, \Gamma \vdash \Delta}{x : A \wedge B, \Gamma \vdash \Delta} \qquad R\wedge\ \frac{\Gamma \vdash \Delta, x : A \quad \Gamma \vdash \Delta, x : B}{\Gamma \vdash \Delta, x : A \wedge B}$$

$$L\vee\ \frac{x : A, \Gamma \vdash \Delta \quad x : B, \Gamma \vdash \Delta}{x : A \vee B, \Gamma \vdash \Delta} \qquad R\vee\ \frac{\Gamma \vdash \Delta, x : A, x : B}{\Gamma \vdash \Delta, x : A \vee B}$$

[S. Negri, **Proof analysis in modal logic**, J. Philos. Logic 2005]

A labeled proof system for modal logics (G3K)

$$\text{id}\ \frac{}{x : P, \Gamma \vdash \Delta, x : P}$$

$$L\wedge\ \frac{x : A, x : B, \Gamma \vdash \Delta}{x : A \wedge B, \Gamma \vdash \Delta} \qquad R\wedge\ \frac{\Gamma \vdash \Delta, x : A \quad \Gamma \vdash \Delta, x : B}{\Gamma \vdash \Delta, x : A \wedge B}$$

$$L\vee\ \frac{x : A, \Gamma \vdash \Delta \quad x : B, \Gamma \vdash \Delta}{x : A \vee B, \Gamma \vdash \Delta} \qquad R\vee\ \frac{\Gamma \vdash \Delta, x : A, x : B}{\Gamma \vdash \Delta, x : A \vee B}$$

$$L\Box\ \frac{y : A, x : \Box A, xRy, \Gamma \vdash \Delta}{x : \Box A, xRy, \Gamma \vdash \Delta} \qquad R\Box\ \frac{xRy, \Gamma \vdash \Delta, y : A}{\Gamma \vdash \Delta, x : \Box A}$$

$$L\Diamond\ \frac{xRy, y : A, \Gamma \vdash \Delta}{x : \Diamond A, \Gamma \vdash \Delta} \qquad R\Diamond\ \frac{xRy, \Gamma \vdash \Delta, x : \Diamond A, y : A}{xRy, \Gamma \vdash \Delta, x : \Diamond A}$$

In $R\Box$, y does not occur in the conclusion.

[S. Negri, **Proof analysis in modal logic**, J. Philos. Logic 2005]

Hocus-Focus

Focusing is a way to control non-determinism in proof search and ...

- Better organize the **structure** of derivations.
- Emphasis on: non-invertible vs. invertible rules.
- Propositional connectives have:
  - a positive version;
  - a negative version.

$$\vee^{+}\ \frac{\vdash \Theta, B_i}{\vdash \Theta, B_1 \vee B_2} \qquad \vee^{-}\ \frac{\vdash \Theta, B_1, B_2}{\vdash \Theta, B_1 \vee B_2}$$

- Polarization of a formula does not affect its **provability**.

What is a bipole?

store

$\vdash \Theta \Uparrow \Gamma$

release

$\vdash \Theta \Downarrow A$

decide

What is a bipole?

store (a positive formula to possibly focus on later)

$\vdash \Theta \Uparrow \Gamma$ $\vee^-, \wedge^-, \forall$

release

$\vdash \Theta \Downarrow A$ $\vee^+, \wedge^+, \exists$

decide (on a positive formula to focus on)

What is a bipole?

store	(a positive formula to possibly focus on later)
$\vdash \Theta \Uparrow \Gamma$	**NEGATIVE PHASE (invertible)**
release	(change of phase)
$\vdash \Theta \Downarrow A$	**POSITIVE PHASE (non-invertible)**
decide	(on a positive formula to focus on)

A focused proof system for classical logic (LKF)

Negative introduction rules

$$t^- \ \frac{}{\vdash \Theta \Uparrow t^-, \Gamma} \qquad \wedge^- \ \frac{\vdash \Theta \Uparrow A, \Gamma \quad \vdash \Theta \Uparrow B, \Gamma}{\vdash \Theta \Uparrow A \wedge^- B, \Gamma} \qquad f^- \ \frac{\vdash \Theta \Uparrow \Gamma}{\vdash \Theta \Uparrow f^-, \Gamma} \qquad \vee^- \ \frac{\vdash \Theta \Uparrow A, B, \Gamma}{\vdash \Theta \Uparrow A \vee^- B, \Gamma}$$

$$\forall \ \frac{\vdash \Theta \Uparrow [y/x]B, \Gamma}{\vdash \Theta \Uparrow \forall x.B, \Gamma}$$

Positive introduction rules

$$t^+ \ \frac{}{\vdash \Theta \Downarrow t^+} \qquad \wedge^+ \ \frac{\vdash \Theta \Downarrow B_1 \quad \vdash \Theta \Downarrow B_2}{\vdash \Theta \Downarrow B_1 \wedge^+ B_2} \qquad \vee^+ \ \frac{\vdash \Theta \Downarrow B_i}{\vdash \Theta \Downarrow B_1 \vee^+ B_2} \qquad \exists \ \frac{\vdash \Theta \Downarrow [t/x]B}{\vdash \Theta \Downarrow \exists x.B}$$

Identity rules

$$\text{id} \ \frac{}{\vdash \neg P_a, \Theta \Downarrow P_a} \qquad \text{cut} \ \frac{\vdash \Theta \Uparrow B \quad \vdash \Theta \Uparrow \neg B}{\vdash \Theta \Uparrow \cdot}$$

Structural rules

$$store \ \frac{\vdash \Theta, C \Uparrow \Gamma}{\vdash \Theta \Uparrow C, \Gamma} \qquad release \ \frac{\vdash \Theta \Uparrow N}{\vdash \Theta \Downarrow N} \qquad decide \ \frac{\vdash P, \Theta \Downarrow P}{\vdash P, \Theta \Uparrow \cdot}$$

Labeled modal inference rules as bipoles

An **inference rule** in the labeled modal proof system G3K

corresponds to ($\Updownarrow$)

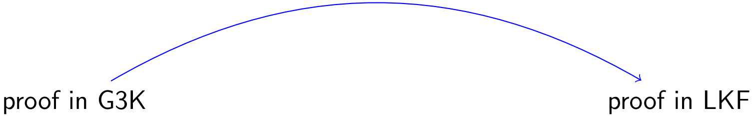
a **bipole** in the focused proof system LKF.

$$R\Box \; \frac{xRy, \mathcal{G} \vdash \Gamma, y : A}{\mathcal{G} \vdash \Gamma, x : \Box A}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\mathcal{G} \vdash \Gamma', \partial^+([\Box A]_x), \neg R(x,y), \partial^+[A]_y \Uparrow \cdot}{\mathcal{G} \vdash \Gamma', \partial^+([\Box A]_x), \neg R(x,y) \Uparrow \partial^+[A]_y}\,\mathit{store}}{\mathcal{G} \vdash \Gamma', \partial^+([\Box A]_x) \Uparrow \neg R(x,y), \partial^+[A]_y}\,\mathit{store}}{\mathcal{G} \vdash \Gamma', \partial^+([\Box A]_x) \Uparrow \neg R(x,y) \vee^- \partial^+[A]_y}\,\vee^-}{\mathcal{G} \vdash \Gamma', \partial^+([\Box A]_x) \Uparrow \forall y(\neg R(x,y) \vee^- \partial^+[A]_y)}\,\forall}{\mathcal{G} \vdash \Gamma', \partial^+([\Box A]_x) \Downarrow \forall y(\neg R(x,y) \vee^- \partial^+[A]_y)}\,\mathit{release}}{\mathcal{G} \vdash \Gamma', \partial^+([\Box A]_x) \Downarrow \partial^+(\forall y(\neg R(x,y) \vee^- \partial^+[A]_y))}\,\partial^+}{\mathcal{G} \vdash \Gamma', \partial^+([\Box A]_x) \Uparrow \cdot}\,\mathit{decide}$$

[D.Miller & M.Volpe, **Focused labeled proof systems for modal logic**, 2015]

A focused labeled proof system for modal logic (LMF)



- A restriction of LKF targeting the language of G3K.
- Quantifier rules only applied to the translation of $\Box A$ or $\Diamond A$.

Negative introduction rules

$$t^{-}{}_K \; \frac{}{\vdash \Theta \Uparrow x : t^{-}, \Gamma} \qquad f^{-}{}_K \; \frac{\vdash \Theta \Uparrow \Gamma}{\vdash \Theta \Uparrow x : f^{-}, \Gamma}$$

$$\wedge^{-}_K \; \frac{\vdash \Theta \Uparrow x : A, \Gamma \quad \vdash \Theta \Uparrow x : B, \Gamma}{\vdash \Theta \Uparrow x : A \wedge^{-} B, \Gamma} \qquad \vee^{-}_K \; \frac{\vdash \Theta \Uparrow x : A, x : B, \Gamma}{\vdash \Theta \Uparrow x : A \vee^{-} B, \Gamma} \qquad \Box_K \; \frac{\vdash \Theta, \neg xRy \Uparrow y : B, \Gamma}{\vdash \Theta \Uparrow x : \Box B, \Gamma}$$

Positive introduction rules

$$t^{+}{}_K \; \frac{}{\vdash \Theta \Downarrow x : t^{+}} \qquad \wedge^{+}_K \; \frac{\vdash \Theta \Downarrow x : B_1 \quad \vdash \Theta \Downarrow x : B_2}{\vdash \Theta \Downarrow x : B_1 \wedge^{+} B_2}$$

$$\vee^{+}_K, i \in \{1,2\} \; \frac{\vdash \Theta \Downarrow x : B_i}{\vdash \Theta \Downarrow x : B_1 \vee^{+} B_2} \qquad \Diamond_K \; \frac{\vdash \Theta, \neg xRy \Downarrow y : B}{\vdash \Theta, \neg xRy \Downarrow x : \Diamond B}$$

Identity rules

$$init_K \; \frac{}{\vdash x : \neg P_a, \Theta \Downarrow x : P_a} \qquad cut_K \; \frac{\vdash \Theta \Uparrow x : B \quad \vdash \Theta \Uparrow x : \neg B}{\vdash \Theta \Uparrow \cdot}$$

Structural rules

$$store_K \; \frac{\vdash \Theta, x : C \Uparrow \Gamma}{\vdash \Theta \Uparrow x : C, \Gamma} \qquad release_K \; \frac{\vdash \Theta \Uparrow x : N}{\vdash \Theta \Downarrow x : N} \qquad decide_K \; \frac{\vdash x : P, \Theta \Downarrow x : P}{\vdash x : P, \Theta \Uparrow \cdot}$$

What happens with ordinary sequent systems?

$$\Box_K \frac{\vdash \Gamma, A}{\vdash \Diamond\Gamma, \Box A, \Delta}$$

This rule works at the same time on $\Box$s and $\Diamond$s.

What happens with ordinary sequent systems?

$$\Box_K \frac{\vdash \Gamma, A}{\vdash \Diamond\Gamma, \Box A, \Delta}$$

This rule works at the same time on $\Box$s and $\Diamond$s.

Not A Bipole!

What happens with ordinary sequent systems?

$$\Box_K \frac{\vdash \Gamma, A}{\vdash \Diamond\Gamma, \Box A, \Delta}$$

This rule works at the same time on $\Box$s and $\Diamond$s.

Not A Bipole!

- Correspondence between ordinary and labeled sequents:
  - ordinary **classical rules** operate on a single world;
  - ordinary **modal rules** move from one world to another.

What happens with ordinary sequent systems?

$$\Box_K \frac{\vdash \Gamma, \mathbf{A}}{\vdash \Diamond\Gamma, \Box\mathbf{A}, \Delta}$$

$$\frac{\mathcal{G} \cup \{\mathbf{xRy}\} \vdash \Sigma, x : \Diamond\Gamma \Uparrow \mathbf{y} : \mathbf{A}}{\mathcal{G} \vdash \Sigma, x : \Diamond\Gamma \Uparrow \mathbf{x} : \Box\mathbf{A}}$$

What happens with ordinary sequent systems?

$$\Box_K \frac{\vdash \Gamma, \mathbf{A}}{\vdash \Diamond\Gamma, \Box\mathbf{A}, \Delta}$$

$$\frac{\mathcal{G} \cup \{\mathbf{xRy}\} \vdash \Sigma, x : \Diamond\Gamma \Uparrow \mathbf{y} : \mathbf{A}}{\mathcal{G} \vdash \Sigma, x : \Diamond\Gamma \Uparrow \mathbf{x} : \Box\mathbf{A}}$$

One **bipole** for the $\Box$-formula.

What happens with ordinary sequent systems?

$$R\Box \ \frac{\vdash \boldsymbol{\Gamma}, A}{\vdash \Diamond\boldsymbol{\Gamma}, \Box A, \Delta}$$

$$\frac{\mathcal{G} \cup \{xRy\} \vdash \Sigma, x : \Diamond\Gamma, y : A \Uparrow \mathbf{y} : \boldsymbol{\Gamma}}{\mathcal{G} \cup \{xRy\} \vdash \Sigma, x : \Diamond\Gamma, y : A \Downarrow \mathbf{x} : \Diamond\boldsymbol{\Gamma}}$$

What happens with ordinary sequent systems?

$$R\Box \frac{\vdash \mathbf{\Gamma}, A}{\vdash \Diamond\mathbf{\Gamma}, \Box A, \Delta}$$

$$\frac{\mathcal{G} \cup \{xRy\} \vdash \Sigma, x : \Diamond\Gamma, y : A \Uparrow \mathbf{y} : \mathbf{\Gamma}}{\mathcal{G} \cup \{xRy\} \vdash \Sigma, x : \Diamond\Gamma, y : A \Downarrow \mathbf{x} : \Diamond\mathbf{\Gamma}}$$

Multifocusing: the $\Diamond$s can be processed in parallel.

One **bipole** for the $\Diamond$-formulas.

The general framework LMF_*

Parameters of the framework: $*$ can be instantiated in a specific way by the following parameters (of the decide rule):

1. restrictions on the formulas on which multifocusing can be applied;
2. restrictions on the definition of the future σ of formulas in Ω;
3. restriction of the present $\mathcal{H}'$.

The general framework LMF_*

Parameters of the framework: $*$ can be instantiated in a specific way by the following parameters (of the decide rule):

1. restrictions on the formulas on which multifocusing can be applied;
2. restrictions on the definition of the future σ of formulas in Ω;
3. restriction of the present $\mathcal{H}'$.

By playing with polarization and parameters, one can obtain different systems.

The general framework LMF_*

Parameters of the framework: $*$ can be instantiated in a specific way by the following parameters (of the decide rule):

1. restrictions on the formulas on which multifocusing can be applied;
2. restrictions on the definition of the future σ of formulas in Ω;
3. restriction of the present $\mathcal{H}'$.

By playing with polarization and parameters, one can obtain different systems.

Theorem The framework LMF_* is sound and complete with respect to the logic K, for any polarization of formulas.

Conclusion

- We showed the case of K; but it works for geometric extensions.
- Emulation of modal focused systems (e.g., [Lellmann-Pimentel, 2015] or [Chaudhuri-Marin-Strassburger, 2016]).
- What about nested sequents?
  - Same polarization as for ordinary sequents.
  - No need for multifocusing.
  - No need for restrictions on futures.
  - The present is always the set of all labels.
- What about hypersequents?
  - the present is a multiset;
  - external structural rules as operations on such a present;
  - modal communication rules as a combination of relational and modal rules.
- Superpowers can be implemented in the augmented version of the focused system LKF used in the project ProofCert.